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DISCRETIZATION OF HYPERBOLIC DIFFERENTIAL EQUATIONS WITH PERIODIC SOLUTIONS

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1. Introduction

Many problems in fluid dynamics are described by partial differential equations of the form

$$(1.1)\quad \partial w/\partial t = (A\partial/\partial x_1 + B\partial/\partial x_2 + C\partial/\partial x_3)w + g,$$

where the matrices A, B and C, and the function g depend on t, x and w. In this contribution, we shall be interested in the special case where it is known in advance that the exact solution is dominated by components of the form

$$(1.2)\quad v(t)\exp(if.x),\ x := (x_1,x_2,x_3),\ f := (f_1,f_2,f_3),$$

and where the frequency vector f lies in a given region.
Often, the functions v(t) are also complex exponentials of the form

$$v_0 \exp(if_4 t).$$

A widely used approach for solving (1.1) numerically is the application of the method of lines. Let us define difference operators of order s:

$$D_q = \partial/\partial x_q + O(\Delta x_q^s),\ q = 1,\ 2,\ 3.$$

Then, firstly (1.1) is replaced by the equation

$$(1.3)\quad dW/dt = (AD_1 + BD_2 + CD_3)W + g.$$

and secondly, by restricting x to a set of grid points, the resulting system of ordinary differential equations(ODEs) is solved numerically. Thus, we are faced with the task (i) to construct efficient difference operators which can be tuned to the dominant frequencies, and (ii) to choose an appropriate ODE solver for integrating (1.3). For instance, since generally many grid points are involved, a storage economic ODE solver is required. Furthermore, when v(t) is periodic, it would be desirable if this extra information can be exploited.

In Section 2, time integrators for periodic solutions will be dicussed; Section 3 presents a special space discretization formula, and in Section 4 numerical experiments are reported.

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2. Time integrators for periodic solutions

Gautschi [5] was presumably the first who proposed ODE solvers that take into account the periodicity of the solution. Several modifications and improvements of this early method have been proposed; we mention the papers [1,7,12,13]. All these methods are of the linear multistep type and are suitable to integrate both linear and nonlinear systems.

In the class of one-step methods, Liniger and Willoughby [11] applied the ideas of Gautschi and introduced the concept of exponential fitting for the accurate representation of stiff components in the numerical solution. This technique can be adapted to represent the periodic components in the numerical solution with greater accuracy [8]. Unfortunately, the exponential fitting technique, when applied to one-step methods, requires the equation to be linear or almost linear.

Recently, there has been an increased interest in methods that generate small phase errors when computing periodic solutions (cf. [2,3,4,9,10,14,15,16]). Such methods, although constructed with application to dynamic systems in mind, are also suitable for the long-term time integration of hyperbolic equations with periodic solutions. However, just as the exponentially fitted methods, these small phase lag methods are derived on the basis of the linear test equation and loose their nice properties when nonlinear terms enter into the equation.

The methods proposed in the literature can be divided into two categories: methods that reduce the phase lag corresponding to the homogeneous solution components and methods that reduce the phase lag corresponding to the inhomogeneous solution component (cf. [10]). The 'homogeneous' phase lag consists of an initial error independent of t and an accumulated error that increases linearly with t; the 'inhomogeneous' phase lag does not depend upon t.

Below we present four explicit Runge-Kutta methods which reduce the accumulated phase error of the homogeneous solution components. These methods require only a few arrays for storage, so that they are suitable for the time integration of large linear systems of semi-discrete hyperbolic equations [9].

$$\begin{array}{c|cccc} 1/5 & 1/5 & & & \\ 1/3 & 0 & 1/3 & & \\ 1/2 & 0 & 0 & 1/2 & \\ \hline & 0 & 0 & 0 & 1 \end{array}$$

$p=2,\ q=6,\ r=3,\ b=2.66$

$$\begin{array}{c|cccc} 32/85 & 32/85 & & & \\ 8/15 & 1/4 & 17/60 & & \\ 2/3 & 1/4 & 0 & 5/12 & \\ \hline & 1/4 & 0 & 0 & 3/4 \end{array}$$

$p=3,\ q=6,\ r=3,\ b=2.66$

$$\begin{array}{c|ccccc} 1/8 & 1/8 & & & & \\ 8/35 & 0 & 8/35 & & & \\ 1/3 & 0 & 0 & 1/3 & & \\ 1/2 & 0 & 0 & 0 & 1/2 & \\ \hline & 0 & 0 & 0 & 0 & 1 \end{array}$$

$p=2,\ q=8,\ r=3,\ b=3.38$

$$\begin{array}{c|ccccc} 128/429 & 128/429 & & & & \\ 256/495 & 1/4 & 429/2380 & & & \\ 8/15 & 1/4 & 0 & 17/60 & & \\ 2/3 & 1/4 & 0 & 0 & 5/12 & \\ \hline & 1/4 & 0 & 0 & 0 & 3/4 \end{array}$$

$p=3,\ q=8,\ r=3,\ b=3.38$

Here, p, q and r denote the orders of accuracy, of phase lag and of dissipation, respectively; b denotes the imaginary stability boundary. A comparison of the first and second method reveals that they only differ in their order of accuracy and the complexity of the generating diagram; this is also true in the second pair of methods. Notice that the first of each pair is extremely simple to implement on a computer. Finally, we observe that conventional Runge-Kutta methods have $q=r=p$.

3. Spatial discretizations

Let E denote the forward shift operator over one mesh size, then we define the symmetric difference operators

$$(3.1) \qquad D_q = \Delta x_q^{-1} \sum_{j=1}^{k} c_j \,(E_q^{+j} - E_q^{-j}), \; q = 1, 2, 3.$$

The weights in these expressions are free to adapt the difference operator to the dominant solution components.

The following theorem gives the truncation error, introduced by the spatial discretization, in the case of periodic solutions:

Theorem 3.1. Let

$$L(y) := \partial y/\partial t - (AD_1 + BD_2 + CD_3)y - g$$

$$w := \sum_{r=1}^{R} v^{(r)}(t) \exp(if^{(r)}.x)$$

If w satisfies the equation (1.1) then its truncation error is given by

$$L(w) := i \sum_{r=1}^{R} [f_1^{(r)} a_1^{(r)} A + f_2^{(r)} a_2^{(r)} B + f_3^{(r)} a_3^{(r)} C] \; v^{(r)}(t) \exp(if^{(r)}.x),$$

$$a_q^{(r)} := 1 - 2 \sum_{j=1}^{k} c_j \sin(jf_q^{(r)} \Delta x_q) \,/\, f_q^{(r)} \Delta x_q \;.$$

This theorem suggests the minimization of the functions

$$(3.2) \qquad a_q(z) := 1 - 2 \sum_{j=1}^{k} c_j \sin(jz)/z, \; q = 1, 2, 3$$

in the frequency intervals corresponding to the 3 spatial directions. Such a minimization determines the weights in the difference operators (cf. [8]).

Alternatively, we can eliminate as many given dominant Fourier components from the truncation error as there are free weights in the difference operators (3,1), just by assigning zeros to the coefficient functions (3.2) at those points on the z-axis that correspond to the given dominant frequencies. For instance, in the case of two given dominant frequencies we may choose in (3.1) k=2 and

$$c_1 = \frac{z_1}{2\sin(z_1)} - 2c_2\cos(z_1),$$

$$(3.3) \qquad c_2 = \frac{z_1/\sin(z_1) - z_2/\sin(z_2)}{4[\cos(z_1) - \cos(z_2)]} \;,$$

$$z_j = \Delta x_q.(j\text{-th dominant frequency}).$$

We observe that introducing zero-valued frequencies into (3.3) leads to the convential fourth-order space discretization.

4. Numerical experiments

We confine our experiments to the integration of a scalar, one-dimensional equation of the form

$$(4.1)\quad \partial w/\partial t = A(t,x,w)\partial w/\partial x + g(t,x,w), \quad 0 \leq t \leq T$$

with periodic boundary conditions at $x = 0$ and $x = 2\pi$. The initial condition is taken from the prescribed exact solution.

The spatial derivative is approximated by using (3.1) and (3.3). The two frequencies to be eliminated from the truncation error are specified in the tables of results. In order to demonstrate the effect of the space discretization, we use for the time integration the standard fourth-order Runge-Kutta method with a relatively small step. By choosing

$$\Delta t = 1/10, \quad \Delta x = \pi/10,$$

the time discretization error will be insignificant in all experiments presented here.

The accuracy obtained is measured by

$$-\log(\text{maximal absolute error at } t=T),$$

i.e., by the minimal number of correct digits at the end point.

Table 4.1. A=1, g=0,
w=sin(t+x)+.5cos(2t+2x), T=5

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(2,2)
Correct significant digits	0.45	0.67	4.16	0.47

Table 4.2. A=1, g=0,
w=sin(t+x)+.5cos(2t+2x)+.033sin(3t+3x), T=5

Eliminated frequencies	(0,0)	(1,2)	(1,3)	(2,3)
Correct significant digits	0.45	1.19	0.27	-0.09

Table 4.3. A=1, g=0,
w=sin(sin(t+x)), T=5

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(1,3)
Correct significant digits	1.07	1.12	1.13	3.99

Table 4.4. A=1, g=0,
w=tan(sin(t+x)), T=5

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(1,3)
Correct significant digits	0.61	0.53	0.53	2.50

Table 4.5. A=w*w, g=(1-w*w)cos(sin(t+x))cos(t+x),
w=sin(sin(t+x)), T=5

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(1,3)
Correct significant digits	1.58	1.61	1.71	3.30

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