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ASYMPTOTIC POWER EFFICIENCY FOR A LOCATION AND SCALE PROBLEM

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ASYMPTOTIC POWER EFFICIENCY FOR A LOCATION AND SCALE PROBLEM[1]

by

YVES LEPAGE[2]

SUMMARY

The asymptotic power efficiency of the class of linear rank tests relative to the asymptotically most powerful rank test is derived for a general location and scale problem. The results are then specialised to the two-sample case and numerical evaluations are presented for two special tests.

KEY WORDS & PHRASES: *Asymptotic efficiency, rank tests, location and scale parameters, two-sample problems, combination of tests.*

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2) Université de Montréal; temporarily: Mathematisch Centrum.

1. INTRODUCTION

In this work, the asymptotic power efficiency of linear rank tests is studied for a location and scale problem. Section 2 contains the asymptotic power efficiency of linear rank tests with respect to the asymptotically most powerful rank test given by LEPAGE (1973) for a general location and scale problem. In section 3, the results are specialised to the two-sample case and bounds are found. Finally, some numerical evaluations are presented in section 4 for a linear rank test combining the Ansari-Bradley and Wilcoxon statistics and also for a linear rank test combining the quartile and median statistics.

2. GENERAL CASE

Let $N_\nu (\nu=1,2,\ldots)$ be a sequence of positive integers such that $N_\nu \to \infty$ when $\nu \to \infty$. For each ν, consider a sequence of random variables $X_{\nu 1},\ldots,X_{\nu N_\nu}$ and denote by $R_{\nu i}, i = 1,\ldots N_\nu$, the rank of $X_{\nu i}$ among $X_{\nu 1},\ldots,X_{\nu N_\nu}$.

Suppose that under H_ν, the random variables $X_{\nu 1},\ldots,X_{\nu N_\nu}$ are independently and identically distributed according to a continuous distribution and that under the alternatives K_ν, the joint density of $(X_{\nu 1},\ldots,X_{\nu N_\nu})$ is given by

$$(2.1) \qquad q_\nu = \prod_{i=1}^{N_\nu} e^{-c_{\nu i}} f(e^{-c_{\nu i}} x_i - d_{\nu i})$$

with $c_\nu = (c_{\nu 1},\ldots,c_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}, d_\nu = (d_{\nu 1},\ldots,d_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$ and a known density f in the class C of absolutely continuous density functions on $\mathbb{R}$ such that

$$(2.2) \qquad I(f) = \int_0^1 \phi^2(u,f)du < \infty \ , \ I_1(f) = \int_0^1 \phi_1^2(u,f)du < \infty$$

where if F is the distribution function corresponding to f,

$$(2.3) \qquad \phi(u,f) = \frac{f'(F^{-1}(u))}{f(F^{-1}(u)} \text{ and } \phi_1(u,f) = -1 - F^{-1}(u)\frac{f'(F^{-1}(u))}{f'(F^{-1}(u))} ,$$

$0 < u < 1$. Note that if $f \in C$,

$$(2.4)\qquad \int_0^1 \phi(u,f)du = \int_0^1 \phi_1(u,f)du = 0.$$

Further, for $f \in C$ and $K \in \mathbb{R}$, define

$$(2.5)\qquad I(f,K) = \int_0^1 \phi^2(u,f,K)du$$

where

$$(2.6)\qquad \phi(u,f,K) = \phi_1(u,f) + K\phi(u,f)\ , \qquad 0 < u < 1.$$

The linear rank statistics considered are of the form

$$(2.7)\qquad S_\nu = \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)a_\nu(R_{\nu i})$$

with $\gamma_\nu = (\gamma_{\nu 1},\ldots,\gamma_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$, $\bar{\gamma}_\nu = \sum_{i=1}^{N_\nu} \gamma_{\nu i} / N_\nu$ and $a_\nu(1),\ldots,a_\nu(N_\nu)$ the values of a score function $a_\nu(\cdot)$. We will assume that the sequence of score functions $a_\nu(\cdot), \nu = 1,2,\ldots,$ is generated by some square integrable function $\phi(u)$, $0 < u < 1$, in the sense that

$$(2.8)\qquad \lim_{\nu\to\infty} \int_0^1 (a_\nu(1+[uN_\nu])-\phi(u))^2du = 0$$

with $[uN_\nu]$ denoting the largest integer not exceeding uN_ν.

From Theorem 4.3 and Corollary 4.1 of LEPAGE (1973), we know that if

$$(2.9)\qquad \lim_{\nu\to\infty} \max_{1\le i\le N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 = 0,$$

$$(2.10)\qquad c_{\nu i} - \bar{c}_\nu \neq 0, \qquad i = 1,\ldots,N_\nu, \quad \nu = 1,2,\ldots,$$

$$(2.11)\qquad \lim_{\nu\to\infty} \max_{1\le i\le N_\nu} \left(\frac{d_{\nu i}-\bar{d}_\nu \exp(-c_{\nu i}+\bar{c}_\nu)}{c_{\nu i}-\bar{c}_\nu} - K\right)^2 = 0 \quad \text{for some } K \in \mathbb{R},$$

and

$$(2.12)\qquad \lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 I(f,K) = b^2 \quad \text{where } 0 < b^2 < \infty,$$

the test based on

$$S_\nu^\circ = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)a_\nu^\circ(R_{\nu i}) \ , \tag{2.13}$$

where the sequence of score functions $a_\nu^\circ(\cdot), \nu = 1,2,\ldots,$ is generated by $\phi(u,f,K)$, $0 < u < 1$, with critical region

$$S_\nu^\circ \geq k_{1-\alpha}b, \tag{2.14}$$

where $k_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standardized normal distribution, is an asymptotically most powerful test for H_ν versus q_ν given by (2.1) at level α.

In the following theorem, the asymptotic power efficiency of the S_ν-test with respect to the S_ν°-test is given in the sense of HÁJEK & ŠIDÁK (1967), p.267.

<u>Theorem 2.1</u>. *Consider testing* H_ν *versus* q_ν *given by* (2.1). *Under conditions* (2.9) *through* (2.12) *and*

$$\lim_{\nu\to\infty} \frac{\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu)}{\left[\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \cdot \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2\right]^{\frac{1}{2}}} = \rho_2 \ , \tag{2.15}$$

the asymptotic power efficiency of the S_ν*-test with respect to the* S_ν°*-test, denoted* e, *is given by*

$$e = \rho_1^2 \, \rho_2^2 \tag{2.16}$$

where

$$\rho_1 = \frac{\int_0^1 \phi(u)\phi(u,f,K)du}{\left[\int_0^1 (\phi(u)-\bar{\phi})^2 du \cdot \int_0^1 \phi^2(u,f,K)du\right]^{\frac{1}{2}}} \tag{2.17}$$

with $\bar{\phi} = \int_0^1 \phi(u)du.$

Proof. Let $\Phi(\cdot)$ denote the distribution function of a standardized normal random variable. According to Theorem 4.3 of LEPAGE (1973), the asymptotically most powerful test S_ν° yields the asymptotic power.

$$(2.18) \qquad 1 - \Phi(k_{1-\alpha}-b),$$

whereas the S_ν-test yields, from Theorem 3.2 of LEPAGE (1973), the asymptotic power

$$(2.19) \qquad 1 - \Phi(k_{1-\alpha}-\rho_1\rho_2 b).$$

Thus, the result is immediate. □

It is tacitly assumed that $\rho_1\rho_2 \geq 0$ since if $\rho_1\rho_2 < 0$, the S_ν-test is less powerful than the test with critical function constantly equal to α regardless of the observations and their ranks.

3. TWO-SAMPLE CASE

Let (m_ν, n_ν), $\nu = 1,2,\ldots$, be a sequence of pairs of positive integers such that $N_\nu = m_\nu + n_\nu \to \infty$ when $\nu \to \infty$. For each ν, define

$$(3.1) \qquad c_{\nu i} = \begin{cases} \Delta_1 (m_\nu n_\nu / N_\nu)^{-\frac{1}{2}} & \text{if } i = 1,\ldots,m_\nu, \\ 0 & \text{if } i = m_\nu+1,\ldots,N_\nu, \end{cases}$$

and,

$$(3.2) \qquad d_{\nu i} = \begin{cases} \Delta_2 (m_\nu n_\nu / N_\nu)^{-\frac{1}{2}} & \text{if } i = 1,\ldots,m_\nu, \\ 0 & \text{if } i = m_\nu+1,\ldots,N_\nu, \end{cases}$$

where $\Delta = (\Delta_1,\Delta_2) \in \mathbb{R}^2$. Also, put

$$(3.3) \qquad \gamma_{\nu i} = \begin{cases} 1 & \text{if } i = 1,\ldots,m_\nu, \\ 0 & \text{if } i = m_\nu+1,\ldots,N_\nu. \end{cases}$$

The statistics (2.7) can now be rewritten as

$$(3.4) \qquad S_\nu = \sum_{i=1}^{m_\nu} a_\nu(R_{\nu i}) - \frac{m_\nu}{N_\nu} \sum_{i=1}^{N_\nu} a_\nu(i).$$

From Theorem 5.2 of LEPAGE (1973), we know that if $\Delta_1 \neq 0$, $\min(m_\nu,n_\nu)\to\infty$ when $\nu \to \infty$ and the sequence of score functions $a^\circ_{\nu,\Delta}(\cdot), \nu = 1,2,\ldots$, is generated by $\phi(u,f,\Delta_2/\Delta_1)$, $0 < u < 1$, the test based on

$$(3.5) \qquad S_{\nu,\Delta} = \sum_{i=1}^{N_\nu} a^\circ_{\nu,\Delta}(R_{\nu i})$$

with critical region

$$(3.6) \qquad (m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}(\Delta_1/|\Delta_1|)S^\circ_{\nu,\Delta} \geq k_{1-\alpha} I^{\frac{1}{2}}(f,\Delta_2/\Delta_1)$$

is an asymptotically most powerful test, at level α, for H_ν versus

$$(3.7) \qquad q_{\nu,\Delta} = \prod_{i=1}^{m_\nu+1} e^{-(\Delta_1(m_\nu n_\nu/N_\nu))^{-\frac{1}{2}}} f(e^{-(\Delta_1(m_\nu n_\nu/N_\nu))^{-\frac{1}{2}}} x_i - \Delta_2(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}) \prod_{i=m_\nu+1}^{N_\nu} f(x_i).$$

The asymptotic power efficiency obtained in the preceding section is now given for the S_ν-test and the $S^\circ_{\nu,\Delta}$-test given respectively by (3.4) and (3.5)

<u>Theorem 3.1</u>. *Consider testing* H_ν *versus* $q_{\nu,\Delta}$ *given by* (3.7). *Then, if* $\Delta_1 \neq 0$ *and* $\min(m_\nu,n_\nu) \to \infty$ *when* $\nu \to \infty$, *the asymptotic power efficiency of the* S_ν*-test with respect to the* $S^\circ_{\nu,\Delta}$*-test is given by*

$$(3.8)\qquad e = \frac{\left(\int_0^1 \phi(u)\phi(u,f,\Delta_2/\Delta_1)du\right)^2}{\int_0^1 (\phi(u)-\bar{\phi})^2 du \cdot \int_0^1 \phi^2(u,f,\Delta_2/\Delta_1)du} .$$

Proof. First note that conditions (2.9) through (2.12) are fulfilled for $K = \Delta_2/\Delta_1$. Also, by easy algebraic manipulations, we have in view of (3.1) and (3.3) that

$$(3.9)\qquad \lim_{\nu\to\infty} \frac{\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu)}{\left[\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \cdot \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2\right]^{\frac{1}{2}}} = 1.$$

Thus, the result follows from theorem 2.1. □

On account of the last paragraph of section 2, we have assumed that

$$(3.10)\qquad \int_0^1 \phi(u)\phi(u,f,\Delta_2/\Delta_1)du \geq 0.$$

In view of the definition of the asymptotically most powerful test $S^\circ_{\nu,\Delta}$, a natural class of competitors can be given by

$$(3.11)\qquad S_{\nu,\Delta} = \sum_{i=1}^{N_\nu} a_{\nu,\Delta}(R_{\nu i})$$

where the sequence of score functions $a_{\nu,\Delta}(\cdot), \nu = 1,2,\ldots$, is generated by

$$(3.12)\qquad \phi(u,f_1,f_2,\Delta_2/\Delta_1) = \phi_1(u,f_1) + \Delta\phi(u,f_2) , \qquad 0 < u < 1,$$

where f_1 and f_2 belong to C and $\Delta = \Delta_2/\Delta_1$. From theorem 3.1, one obtains that the asymptotic power efficiency of the $S_{\nu,\Delta}$-test with respect to the $S^\circ_{\nu,\Delta}$-test is given by

$$(3.13)\qquad e = \frac{\left(\int_0^1 (\phi_1(u,f_1)+\Delta_2/\Delta_1\phi(u,f_2))(\phi_1(u,f)+\Delta_2/\Delta_1\phi(u,f))du\right)^2}{\left(\int_0^1 (\phi_1(u,f_1)+\Delta_2/\Delta_1\phi(u,f_2))^2 du\right)\left(\int_0^1 (\phi_1(u,f)+\Delta_2/\Delta_1\phi(u,f))^2 du\right)} .$$

Thus, if we assume that

$$(3.14)\qquad \int_0^1 \phi(u,f)\phi_1(u,f)du = \int_0^1 \phi(u,f_2)\phi_1(u,f_1)du = \int_0^1 \phi(u,f_2)\phi_1(u,f)du = $$
$$= \int_0^1 \phi(u,f)\phi_1(u,f_1)du = 0 ,$$

we can write

$$(3.15)\qquad e = \frac{\left(\int_0^1 \phi_1(u,f)\phi_1(u,f_1)du+\Delta_2^2/\Delta_1^2 \int_0^1 \phi(u,f)\phi(u,f_2)du\right)^2}{(I_1(f_1)+\Delta_2^2/\Delta_1^2\, I(f_2))(I_1(f)+\Delta_2^2/\Delta_1^2\, I(f))}$$

It should be observed that as a function of Δ_2/Δ_1, the preceding expression for e is symmetric with respect to the origin. Furthermore, when $\Delta_2/\Delta_1 = 0$, we have $e = e_S(f,f_1)$ where

$$(3.16)\qquad e_S(f,f_1) = \frac{\left(\int_0^1 \phi_1(u,f)\phi_1(u,f_1)du\right)^2}{I_1(f)\; I_1(f_1)} = \frac{(J_1(f,f_1))^2}{I_1(f)I_1(f_1)}$$

is the asymptotic power efficiency of the test based on $\sum_{i=1}^{m_\nu} a_{1\nu}(R_{\nu i})$ where the sequence of score $a_{1\nu}(\cdot), \nu = 1,2,\ldots,$ is generated by $\phi_1(u,f_1)$, $0<u<1$, with respect to the asymptotically most powerful rank test for contiguous scale alternatives for a density f. Also, when $\Delta_2/\Delta_1 \to \pm\infty$, we have $e = e_L(f,f_2)$ where

$$(3.17)\qquad e_L(f,f_2) = \frac{\left(\int_0^1 \phi(u,f)\phi(u,f_2)du\right)^2}{I(f)\; I(f_2)} = \frac{(J(f,f_2))^2}{I(f)\; I(f_2)}$$

is the asymptotic power efficiency of the test based on $\sum_{i=1}^{m_\nu} a_\nu(R_{\nu i})$ where the sequence of score $a_\nu(\cdot), \nu = 1,2,\ldots,$ is generated by $\phi(u,f_2)$, $0 < u < 1$, with respect to the asymptotically most powerful rank test for contiguous location alternatives for a density f.

Let

$$r(f,f_1,f_2) = (I(f)I_1(f_1)+I(f_2)I_1(f))J(f,f_2) - 2I(f)I(f_2)J_1(f,f_1) \tag{3.18}$$

and,

$$s(f,f_1,f_2) = 2I_1(f)I_1(f_1)J(f,f_2) - (I(f)I_1(f_1)+I(f_2)I_1(f))J_1(f,f_1). \tag{3.19}$$

In the following theorem, the power efficiency e given by (3.15) is studied as a function of Δ_2/Δ_1.

Theorem 3.2. *Suppose that* $\Delta_1 \neq 0$, $\min(m_\nu,n_\nu) \to \infty$ *when* $\nu \to \infty$, *condition* (3.14) *is satisfied and*

$$J(f,f_2) \geq 0 \quad \textit{and} \quad J_1(f,f_1) \geq 0. \tag{3.20}$$

(i) *If*

$$r(f,f_1,f_2) \leq 0 \quad \textit{and} \quad e_L(f,f_2) \leq e_S(f,f_1) \tag{3.21}$$

or

$$s(f,f_1,f_2) \geq 0 \quad \textit{and} \quad e_L(f,f_2) \geq e_S(f,f_1) \tag{3.22}$$

then, the asymptotic power efficiency e *of the* $S_{\nu,\Delta}$*-test with respect to the* $\overset{\circ}{S}_{\nu,\Delta}$*-test is bounded in the following way:*

$$\min(e_L(f,f_2),e_S(f,f_1)) \leq e \leq \max(e_L(f,f_2),e_S(f,f_1)). \tag{3.23}$$

Furthermore, for $0 \leq \Delta_2/\Delta_1 < \infty$, e *is monotone (non-decreasing or non-increasing).*

(ii) *If*

$$r(f,f_1,f_2) > 0 \quad \textit{and} \quad e_L(f,f_2) \le e_S(f,f_1) \tag{3.24}$$

or

$$s(f,f_1,f_2) < 0 \quad \textit{and} \quad e_L(f,f_2) \ge e_S(f,f_1) \tag{3.25}$$

then, the asymptotic power efficiency e *of the* $S_{\nu,\Delta}$*-test with respect to the* $S^{\circ}_{\nu,\Delta}$*-test is bounded in the following way:*

$$e_{\circ} \le e \le \max(e_L(f,f_2), e_S(f,f_1)) \tag{3.26}$$

where $e_{\circ}$ *is the value of* e *given by* (3.15) *for*

$$\Delta_2^2/\Delta_1^2 = -\frac{s(f,f_1,f_2)}{r(f,f_1,f_2)} = \ell_0^2 \tag{3.27}$$

Furthermore, for $0 \le \Delta_2/\Delta_1 \le \ell_0$, e *is non-increasing and for* $\ell_0 \le \Delta_2/\Delta_1 < \infty$, e *is non-decreasing. Also,* $e_0 = 0$ *if and only if* $e_L(f,f_2) = e_S(f,f_1)$.

Proof. Let $\ell = \Delta_2/\Delta_1$ and denote by $e(\ell)$ the expression of e given by (3.15). The derivative of $e(\ell)$ with respect to ℓ can be written as

$$e'(\ell) = \frac{2\ell(J_1(f,f_1)+J(f,f_2)\ell^2)(s(f,f_1,f_2)+r(f,f_1,f_2)\ell^2)}{(I_1(f)+I(f)\ell^2)^2(I_1(f_1)+I(f_2)\ell^2)^2} \tag{3.28}$$

Also, note that since $I(f)I_1(f_1) + I(f_2)I_1(f) \ge 2(I(f)I(f_2)I_1(f)I_1(f_1))^{\frac{1}{2}}$, we have that

$$r(f,f_1,f_2) \ge 0 \text{ if } e_L(f,f_2) \ge e_S(f,f_1) \tag{3.29}$$

and

$$s(f,f_1,f_2) \le 0 \text{ if } e_L(f,f_2) \le e_S(f,f_1). \tag{3.30}$$

Thus, in view of (3.20), (3.29), (3.30) and condition (i), we deduce that for $0 \le \ell < \infty$, $e'(\ell)$ 0 or $e'(\ell) \le 0$. Consequnetly, the result of part (i) follows from (3.16) and (3.17).

In the case condition (ii) holds, $e'(\ell_0) = 0$ and, from (3.29) and (3.30), we deduce that for $0 \le \ell \le \ell_0$, $s(f,f_1,f_2) + r(f,f_1,f_2)\ell^2 \le 0$ and for $\ell_0 \le \ell < \infty$, $s(f,f_1,f_2) + r(f,f_1,f_2)\ell^2 \ge 0$. Hence, the result of part (ii) follows from (3.16), (3.17) and (3.20). □

In many practical situations, the value of Δ_2/Δ_1 is unknown and consequently, the class of tests $S_{\nu,\Delta}$ given by (3.11) cannot be used. Instead, we will consider the class of tests $S_{\nu,\Delta}$ where Θ is any real number. From theorem 3.1, one obtains that the asymptotic power efficiency of the $S_{\nu,\theta}$-test with respect to the $S^{\circ}_{\nu,\Delta}$-test is given by

$$(3.31)\qquad e = \frac{\left(\int_0^1 (\phi_1(u,f_1)+\theta\phi(u,f_2))(\phi_1(u,f)+\Delta_2/\Delta_1\phi(u,f))du\right)^2}{\left(\int_0^1 (\phi_1(u,f_1)+\theta\phi(u,f_2))^2\left(\int_0^1 (\phi_1(u,f)+\Delta_2/\Delta_1\phi(u,f))^2 du\right)} .$$

Under condition (3.14), one can write

$$(3.32)\qquad e = \frac{\left(\int_0^1 \phi_1(u,f)\phi_1(u,f_1)du+\theta\Delta\int_0^1 \phi(u,f)\phi(u,f_2)du\right)^2}{(I_1(f_1)+\theta^2 I(f_2))(I_1(f)+\Delta^2 I(f))}$$

with $\Delta = \Delta_2/\Delta_1$. It should be observed that if $\theta = \Delta$, we get back relation (3.15). Furthermore, when $\theta = 0$, we obtain

$$(3.33)\qquad e = \frac{J_1^2(f,f_1)}{I_1(f_1)(I_1(f)+\Delta^2 I(f))}$$

and thus, $e_S(f,f_1)$ if $\Delta = 0$. Also, when $\theta = \pm\infty$, we obtain

$$(3.34)\qquad e = \frac{\Delta^2 J^2(f,f_2)}{I(f_2)(I_1(f)+\Delta^2 I(f))}$$

and thus, $e_L(f,f_2)$ if $\Delta = \pm\infty$. The values of e given by (3.33) and (3.34) will be denoted by e_0^* and e_∞^* respectively. For

$$(3.35)\qquad \theta = \Delta\,\frac{J(f,f_2)I_1(f_1)}{J_1(f,f_1)I(f_2)} = \theta^*,$$

the value of e given by (3.32) can be written as

$$(3.36)\qquad e^* = \frac{J_1^2(f,f_1)I(f_2)+\Delta^2 J(f,f_2)I_1(f_1)}{I(f_2)I_1(f)(I_1(f)+\Delta I(f))} = e_0^* + e_\infty^* .$$

In the following theorem, the power efficiency e given by (3.32) is studied as a function of θ for Δ fixed.

Theorem 3.3. *Suppose that* $\Delta = \Delta_2/\Delta_1$ *is fixed* $(\Delta_1 \neq 0)$, $\min(m_\nu, n_\nu) \to \infty$ *when* $\nu \to \infty$, condition (3.14) is satisfied and

$$(3.37)\qquad J(f,f_2) > 0 \quad \textit{and} \quad J_1(f,f_1) > 0.$$

(i) *For* $\theta\Delta > 0$, *the asymptotic power efficiency* e *of the* $S_{\nu,\theta}$*-test with respect to the* $S^\circ_{\nu,\Delta}$*-test is bounded in the following way:*

$$(3.38)\qquad 0 < \min(e_0^*, e_\infty^*) \le e \le e_0^* + e_\infty^*.$$

Furthermore, for $\theta < \theta^*$, e *is strictly increasing and for* $\theta > \theta^*$, e *is strictly decreasing.*

(ii) *For* $\theta\Delta < 0$, *the asymptotic power efficiency* e *of the* $S_{\nu,\theta}$*-test with respect to the* $S^\circ_{\nu,\Delta}$*-test is bounded in the followsing way:*

$$(3.39)\qquad 0 \le e \le \max(e_0^*, e_\infty^*).$$

Furthermore, if

$$(3.40)\qquad \theta^{**} = -\frac{J_1(f,f_1)}{\Delta J(f,f_2)},$$

e *is strictly decreasing for* $\theta < \theta^{**}$, *and strictly increasing for* $\theta > \theta^{**}$.

Proof. Denote by $e(\theta)$ the expression of e given by (3.32). The derivative of $e(\theta)$ with respect to θ can be written as

(3.41) $$e'(\theta) = \frac{2(J_1(f,f_1)+\theta\Delta J(f,f_2))(\Delta J(f,f_2)I_1(f_1)-\theta J_1(f,f_1)I(f_2))}{(I_1(f_1)+\theta^2 I(f_2))^2(I_1(f)+\Delta^2 I(f))}$$

Thus, $e'(\theta) = 0$ if and only if $\theta = \theta^*$ or θ^{**}. Since $e(\theta^*) = e^*$ and $e(\theta^{**}) = 0$, one can easily deduce in view of (3.37) that if $\theta\Delta > 0$ and $\theta < \theta^*$ or if $\theta\Delta < 0$ and $\theta > \theta^{**}$, $e'(\theta) < 0$ and, if $\theta\Delta > 0$ and $\theta > \theta^*$ or if $\theta\Delta < 0$ and $\theta > \theta^{**}$, $e'(\theta) > 0$. Consequently, the proof is complete. □

In the preceding theorem, if $J(f,f_2) > 0$ and $J_1(f,f_1) = 0$, one can easily verify from relation (3.41) that

(3.42) $$0 \le e \le e^*_\infty$$

and, for $\theta \in (-\infty,0]$, e is strictly decreasing and for $\theta \in [0,\infty)$, e is strictly increasing. Similarly, if $J(f,f_2) = 0$ and $J_1(f,f_1) > 0$, we have

(3.43) $$0 \le e \le e^*_0$$

and, for $\theta \in (-\infty,0]$, e is strictly increasing and for $\theta \in [0,\infty)$, e is strictly decreasing.

4. NUMERICAL EVALUATIONS

A particulary interesting class of tests $S_{\nu,\Delta}$ is given by combining the Ansari-Bradley statistic (see ANSARI & BRADLEY (1960) and the Wilcoxon-Mann-Whitney statistic (see MANN & WHITNEY (1947)). Consequently, in view HÁJEK & ŠIDÁK (1967), p.87 and 95, let f_1 be the double quadratic density, $f_1(x) = \frac{1}{2}(1+|x|)^{-2}$, and f_2 be the logistic density, $f_2(x) = e^{-x}(1+e^{-x})^{-2}$, and define

(4.1) $$S_{\nu,\Delta} = \frac{1}{N_\nu+1}\left[4\sum_{i=1}^{m_\nu} |R_{\nu i}-(N_\nu+1)/2| - m_\nu(N_\nu+1)+\Delta(2\sum_{i=1}^{m_\nu} R_{\nu i}-m_\nu(N+1))\right]$$

where $\Delta = \Delta_2/\Delta_1$.

If f is the normal density, one obtains that the asymptotic power efficiency e of the $S_{\nu,\Delta}$-test with respect to the $S^{\circ}_{\nu,\Delta}$-test is given by

$$(4.2) \qquad e = \frac{3(\sqrt{\pi}\,\Delta_2^2/\Delta_1^2+2)^2}{\pi^2(\Delta_2^2/\Delta_1^2+1)(\Delta_2^2/\Delta_1^2+2)}$$

and is thus strictly increasing from $6/\pi^2$ ($\approx$.61) to $3/\pi$ ($\approx$.95) as Δ_2/Δ_1 varies from 0 to ∞. Similarly, if f is the Cauchy density,

$$(4.3) \qquad e = \frac{6(\pi\ \Delta_2^2/\Delta_1^2+4)^2}{\pi^4(\Delta_2^2/\Delta_1^2+1)^2}$$

and is strictly decreasing from $96/\pi^4$ ($\approx$.99) to $6/\pi^2$ as Δ_2/Δ_1 varies from o to ∞. If f is the double exponential density, we get

$$(4.4) \qquad e = 3/4$$

independently of Δ_2/Δ_1. When f is the logistic density,

$$(4.5) \qquad e = \frac{3(\Delta_2^2/\Delta_1^2+4\ln 2-1)^2}{(\Delta_2^2/\Delta_1^2+1)(3\Delta_2^2/\Delta_1^2+\pi^2+3)},$$

this function is strictly decreasing from $3(\ln 2-1)/(\pi^2+3)$ ($\approx$.73) to .719 for $0 \le \Delta_2/\Delta_1 \le .675$ and then, is strictly increasing from .719 to 1 for $.675 \le \Delta_2/\Delta_1 < \infty$. Finally, if f is the double quadratic density,

$$(4.6) \qquad e = \frac{\Delta_2^2/\Delta_1^2+1}{4\Delta_2^2/\Delta_1^2+1}$$

and is consequently strictly decreasing from 1 to 1/4 for $0 \le \Delta_2^2/\Delta_1^2 < \infty$.

In table 1, the asymptotic power efficiencies given by (4.2), (4.3), (4.4), (4.5) and (4.6) are respectively evaluated for different values of Δ_2/Δ_1.

Δ_2/Δ_1 \ f	0	.15	.25	.5	.75	1	2.5	5	10	∞
Normal	.61	.61	.62	.65	.68	.72	.87	.93	.95	.95
Cauchy	.99	.98	.96	.90	.84	.79	.66	.62	.61	.61
Double exponential	.75	.75	.75	.75	.75	.75	.75	.75	.75	.75
Logistic	.73	.73	.73	.72	.72	.73	.84	.94	.98	1.00
Double quadratic	1.00	.94	.85	.63	.48	.40	.28	.26	.25	.25

Table 1. Asymptotic power efficiency of the combined Ansari-Bradley and Wilcoxon-Mann-Whitney $S_{\nu,\Delta}$-test with respect to the $\overset{\circ}{S}_{\nu,\Delta}$-test.

Another class of tests $S_{\nu,\Delta}$ which are easy to apply, is given by combining the quartile statistic (see HAJEK & ŠIDAK (1967), p.96-97) and the median statistic (see HAJEK & ŠIDAK (1967), p.88). Thus, let $f_1(x) = 1$ for $|x| \le 1/4$, $1/(16x^2)$ for $|x| > 1/4$ and f_2 be the double exponential density and define

$$(4.7) \qquad S_{\nu,\Delta} = \frac{2}{N_\nu+1}\left[\sum_{i=1}^{m_\nu} \text{sign}(|R_{\nu i}-(N_\nu+1)/2|-(N_\nu+1)/4)+\Delta \sum_{i=1}^{m_\nu} \text{sign}(R_{\nu i}-(N_\nu+1)/2)\right]$$

where sign(x) = −1 for $x < 0$, 0 for $x = 0$ and 1 for $x > 0$ and $\Delta = \Delta_2/\Delta_1$.

If f is the normal density, the asymptotic power efficiency e of the $S_{\nu,\Delta}$-test with respect to the $\overset{\circ}{S}_{\nu,\Delta}$-test is given by

$$(4.8) \qquad e = \frac{2(\Delta_2^2/\Delta_1^2+2\gamma e^{-\frac{1}{2}\gamma^2})^2}{\pi(\Delta_2^2/\Delta_1^2+1)(\Delta_2^2/\Delta_1^2+2)}$$

where $\Phi(\gamma) = .75$ and $\Phi(\cdot)$ is the distribution function of a standardized normal random variable; this function is strictly increasing from $8[(2\pi)^{-\frac{1}{2}}\gamma e^{-\frac{1}{2}\gamma^2}]^2$ (≈.37) to $2/\pi$ (≈.64) as Δ_2/Δ_1 varies from 0 to ∞. If f is the Cauchy, we get

(4.9) $$e = 8/\pi^2 \approx .81.$$

If f is the double exponential density, we get

(4.10) $$e = \frac{(\Delta_2^2/\Delta_1^2+\ln 2)^2}{(\Delta_2^2/\Delta_1^2+1)^2}$$

which is strictly increasing from ln2(≈.48) to 1 as Δ_2/Δ_1 varies from o to ∞. When f is the logistic density,

(4.11) $$e = \frac{9(2\Delta_2^2/\Delta_1^2+3\ln 3)^2}{16(\Delta_2^2/\Delta_1^2+1)(3\Delta_2^2/\Delta_1^2+\pi^2+3)}$$

and the function is strictly decreasing from .4748 to .4745 for $0 \le \Delta_2/\Delta_1 \le .26$ and then, strictly increasing from .4745 to 3/4 for $.26 \le \Delta_2/\Delta_1 < \infty$. Finally, when f is the double quadratic density,

(4.12) $$e = \frac{3(2\Delta_2^2/\Delta_1^2+1)^2}{4(\Delta_2^2/\Delta_1^2+1)(4\Delta_2^2/\Delta_1^2+1)}$$

and is consequently strictly decreasing from 3/4 to 2/3 for $0 \le \Delta_2/\Delta_1 \le .703$ and then, strictly increasing from 2/3 to 3/4 for $.703 \le \Delta_2/\Delta_1 < \infty$.

In table 2, the asymptotic power efficiencies given by (4.8), (4.9), (4.10), (4.11) and (4.12) are evaluated for different values of Δ_2/Δ_1.

f \ Δ_2/Δ_1	0	.15	.25	.5	.75	1	2.5	5	10	∞
Normal	.37	.37	.38	.40	.43	.46	.57	.62	.63	.64
Cauchy	.81	.81	.81	.81	.81	.81	.81	.81	.81	.81
Double exponential	.48	.49	.51	.57	.65	.72	.92	.98	.99	1.00
Logistic	.475	.475	.474	.476	.483	.50	.61	.70	.74	.75
Double quadratic	.75	.74	.72	.68	.67	.68	.73	.74	.75	.75

Table 2. Asymptotic power efficiency of the combined quartile and median $S_{\nu,\Delta}$-test with respect to the $\overset{\circ}{S}_{\nu,\Delta}$-test.

In the case $\Delta = \Delta_2/\Delta_1$ is unknown, the $S_{\nu,\theta}$-test given by (4.1) with θ any real number can be used. If f is the normal density, one obtains that the asymptotic power efficiency of the $S_{\nu,\theta}$-test with respect to the $S^{\circ}_{\nu,\Delta}$-test is given by

$$(4.13) \qquad e = \frac{3(\sqrt{\pi}\,\Delta\theta+2)^2}{\pi^2(\Delta^2+2)(\theta^2+1)}$$

with

$$(4.14) \qquad e_0^* = \frac{12}{\pi^2(\Delta^2+2)} \quad \text{and} \quad e_\infty^* = \frac{3\Delta^2}{\pi(\Delta^2+2)} .$$

If f is the Cauchy density, we get

$$(4.15) \qquad e = \frac{6(\pi\ \Delta\theta+4)^2}{\pi^4(\Delta^2+1)(\theta^2+1)}$$

with

$$(4.16) \qquad e_0^* = \frac{96}{\pi^4(\Delta^2+1)} \quad \text{and} \quad e_\infty^* = \frac{6\Delta^2}{\pi^2(\Delta^2+1)} .$$

If f is the double exponential, we get

$$(4.17) \qquad e = \frac{3(\Delta\theta+1)^2}{4(\Delta^2+1)(\theta^2+1)}$$

with

$$(4.18) \qquad e_0^* = \frac{3}{4(\Delta^2+1)} \quad \text{and} \quad e_\infty^* = \frac{3\Delta^2}{4(\Delta^2+1)} .$$

It should be observed that in this case, $e_0^* + e_\infty^* = 3/4$ independently of $\Delta = \Delta_2/\Delta_1$. When f is the logistic density,

$$(4.19) \qquad e = \frac{3(\Delta\theta+4\ln 2-1)^2}{(\Delta^2+1)(3\theta^2+\pi^2+3)}$$

with

(4.20) $$e_0^* = \frac{3(4\ln 2-1)^2}{3\Delta^2+\pi^2+3} \quad \text{and} \quad e_\infty^* = \frac{3\Delta^2}{3\Delta^2+\pi^2+3} .$$

Finally, when f is the double quadratic density,

(4.21) $$e = \frac{(\Delta\theta+1)^2}{(4\Delta^2+1)(\theta^2+1)}$$

with

(4.22) $$e_0^* = \frac{1}{4\Delta^2+1} \quad \text{and} \quad e_\infty^* = \frac{\Delta^2}{4\Delta^2+1} .$$

In table 3, the bounds e_0^* and e_∞^* given by (4.14), (4.16), (4.18), (4.20) and (4.22) are evaluated for different values of $|\Delta| = |\Delta_2/\Delta_1|$.

f \ $\lvert\Delta\rvert$	0	.15	.25	.5	.75	1	2.5	5	10	∞
Normal	.61 0	.60 .01	.59 .03	.54 .11	.47 .21	.41 .32	.15 .72	.05 .88	.01 .94	0 .95
Cauchy	.99 0	.96 .01	.93 .04	.79 .12	.63 .22	.49 .30	.14 .52	.04 .58	.01 .60	0 .61
Double exponential	.75 0	.73 .02	.71 .04	.60 .15	.48 .27	.375 .375	.10 .65	.03 .72	.01 .74	0 .75
Logistic	.73 0	.73 .01	.72 .01	.69 .06	.65 .12	.59 .19	.30 .59	.11 .85	.03 .96	0 1.00
quadratic	1.00 0	.92 .02	.80 .05	.50 .13	.31 .17	.20 .20	.04 .24	.01 .25	0 .25	0 .25

Table 3. Components e_0^* and e_∞^* of the bounds on the asymptotic power efficiency of the combined Ansari-Bradley and Wilcoxon-Mann-Whitney $S_{\nu,\theta}$-test with respect to the $S^\circ_{\nu,\Delta}$-test when Δ is unknown.

Similarly, the $S_{\nu,\theta}$-test given by (4.7) with any real number θ, can also be used when $\Delta = \Delta_2/\Delta_1$ is unknown. If f is the normal density, the asymptotic power efficiency of the $S_{\nu,\theta}$-test with respect to the $S^\circ_{\nu,\Delta}$-test is given by

$$(4.23)\qquad e = \frac{(2\Delta\theta+4\gamma e^{-\frac{1}{2}\gamma^2})^2}{2\pi(\Delta^2+2)(\theta^2+1)}$$

with

$$(4.24)\qquad e_0^* = \frac{8\gamma^2 e^{-\gamma^2}}{\pi(\Delta^2+2)} \quad\text{and}\quad e_\infty^* = \frac{2\Delta^2}{\pi(\Delta^2+2)} .$$

If f is the Cauchy density, we get

$$(4.25)\qquad e = \frac{8(\Delta\theta+1)^2}{\pi^2(\Delta^2+1)(\theta^2+1)}$$

with

$$(4.26)\qquad e_0^* = \frac{8}{\pi^2(\Delta^2+1)} \quad\text{and}\quad e_\infty^* = \frac{8\Delta^2}{\pi^2(\Delta^2+1)} .$$

It should be observed that, independently of $\Delta = \Delta_2/\Delta_1$, $e_0^* + e_\infty^* = e^* = 8/\pi^2$ ($\approx$.81). When f is the double exponential density,

$$(4.27)\qquad e = \frac{(\Delta\theta+1)^2}{(\Delta^2+1)(\theta^2+1)}$$

with

$$(4.28)\qquad e_0^* = \frac{(\ln 2)^2}{\Delta^2+1} \quad\text{and}\quad e_\infty^* = \frac{\Delta^2}{\Delta^2+1} .$$

When f is the logistic density

$$(4.29)\qquad e = \frac{9(2\Delta\theta+3\ln 3)^2}{16(3\Delta^2+\pi^2+3)(\theta^2+1)}$$

with

$$(4.30)\qquad e_0^* = \frac{81(\ln 3)^2}{16(3\Delta^2+\pi^2+3)} \quad\text{and}\quad e_\infty^* = \frac{9\Delta^2}{4(3\Delta^2+\pi^2+3)} .$$

Finally, when f is the double quadratic density,

$$(4.31)\qquad e = \frac{3(2\Delta\theta+1)^2}{4(4\Delta^2+1)(\theta^2+1)}$$

with

$$(4.32)\qquad e_0^* = \frac{3}{4(4\Delta^2+1)} \qquad \text{and} \qquad e_\infty^* = \frac{3\Delta^2}{4\Delta^2+1}\ .$$

In this case, $e_0^* + e_\infty^* = e^* = .75$ independently of $\Delta = \Delta_2/\Delta_1$.

In table 4, the elements of the bounds e_0^* and e_∞^* given by (4.24), (4.26), (4.28), (4.30) and (4.32) are evaluated for different values of $|\Delta| = |\Delta_2/\Delta_1|$.

f \ $\|\Delta\|$	0	.15	.25	.5	.75	1	2.5	5	10	∞
Normal	.37 0	.36 .01	.37 .02	.33 .07	.29 .14	.25 .21	.09 .48	.03 .59	.01 .62	0 .64
Cauchy	.81 0	.79 .02	.76 .05	.65 .16	.52 .29	.405 .405	.11 .70	.03 .78	.01 .80	0 .81
Double exponential	.48 0	.47 .02	.45 .06	.38 .20	.31 .36	.24 .50	.07 .86	.02 .96	.005 .99	0 1.00
Logistic	.475 0	.47 .004	.47 .01	.45 .04	.42 .09	.39 .14	.19 .44	.07 .64	.02 .72	0 .75
Double quadratic	.75 0	.69 .06	.60 .15	.375 .375	.23 .52	.15 .60	.03 .72	.01 .74	.002 .748	0 .75

Table 4. Components e_0^* and e_∞^* of the bounds on the asymptotic power efficiency of the combined quartile and median $S_{\nu,\theta}$-test with respect to the $S^\circ_{\nu,\Delta}$-test when Δ is unknown.

In the preceding section, it has been mentioned that when $\Delta\theta > 0$ and Δ is unknown, the maximum power efficiency of the $S_{\nu,\theta}$-test with respect to the $S^\circ_{\nu,\Delta}$-test is achieved at $\theta = \theta^*$ given by (3.35). In practice, one can try to approach this maximum by estimating Δ and using the $S_{\nu,\theta}$-test with the corresponding θ^*. In table 5, the value θ^* is giving as a function of Δ.

Test / f	Combined Ansari-Bradley and Wilcoxon Mann-Whitney	Combined quartile and median
Normal	$\frac{\Delta\sqrt{\pi}}{2}$	$2\Delta\gamma e^{-\frac{1}{2}\gamma^2}$
Cauchy	$\frac{\Delta\pi}{4}$	Δ
Double exponential	Δ	$\frac{\Delta}{\ln 2}$
Logistic	$\frac{\Delta}{4\ln 2-1}$	$\frac{2\Delta}{3\ln 3}$
Double quadratic	Δ	2Δ

Table 5. Value of θ^* in function of Δ.

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