SA

stichting
mathematisch
centrum

MC

AFDELING MATHEMATISCHE STATISTIEK SW 20/73 OCTOBER

Y. LEPAGE

SA

# ASYMPTOTICALLY OPTIMUM RANK TESTS FOR CONTIGUOUS LOCATION AND SCALE ALTERNATIVES

Prepublication

2e boerhaavestraat 49 amsterdam

*Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.*

*The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.*

AMS(MOS) subject classification scheme (1970) : 62G10, 62G30

Contents

# Asymptotically optimum rank tests for contiguous location and scale alternatives *)

Yves Lepage **)

## Abstract

The problem of testing identity of distribution against alternatives containing both location and scale parameters is studied. Conditions are given to obtain contiguous location and scale alternatives and, for those alternatives, an asymptotically most powerful rank test is found. The results are then specialised to the two-sample case.

## 1. Introduction

In the paper of Hájek (1962) and the book of Hájek and Šidák (1967), the problem of testing the null hypothesis of randomness versus contiguous location alternatives or contiguous scale alternatives was treated. In each case, an asymptotically most powerful rank test is found. In this paper, the problem of testing the null hypothesis of randomness versus contiguous location and scale alternatives is considered. The approach adopted follows that of Hájek and Šidák (1967) and many of our proofs are similar to theirs.

Section 2 contains the basic notations and tools that will be needed. In section 3, conditions are given to provide contiguous location and scale alternatives and the asymptotic distribution of linear rank statistics under such contiguous alternatives is found. In section 4, the notion of asymptotic sufficiency is explored to deduce a rank test asymptotically most powerful among all tests while in section 5 all the results are specialised to the two-sample case. Sections 6, 7 and 8 contain the proof of the results of respectively sections 3, 4 and 5.

---

*) This work is part of the author's Ph.D. dissertation written at the Université de Montréal under the direction of Professor Constance van Eeden and, it was partially supported by the National Research Council of Canada, Grant No. A-8555. The manuscript was completed while the author was visiting the Mathematisch Centrum, Amsterdam. The paper is not for review; it has been submitted for publication in a journal.

**) Université de Montréal; temporarily: Mathematisch Centrum.

## 2. Notations and conditions

Let $N_\nu$ ($\nu=1,2,\ldots$) be a sequence of positive integers such that $N_\nu \to \infty$ when $\nu \to \infty$. For each $\nu$, consider a sequence of random variables $X_{\nu 1},\ldots, X_{\nu N_\nu}$ and denote by $R_{\nu i}$ the rank of $X_{\nu i}$ among $X_{\nu 1},\ldots, X_{\nu N_\nu}$.

Suppose that under $H_\nu$, the random variables $X_{\nu 1},\ldots, X_{\nu N_\nu}$ are independently and identically distributed according to a continuous distribution and that under $K_\nu$, the joint density of $(X_{\nu 1},\ldots,X_{\nu N_\nu})$ is given by

$$(2.1) \qquad q_\nu = \prod_{i=1}^{N_\nu} e^{-c_{\nu i}} f(e^{-c_{\nu i}} x_i - d_{\nu i})$$

with $c_\nu = (c_{\nu 1},\ldots,c_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$, $d_\nu = (d_{\nu 1},\ldots,d_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$ and a known density f in the class C of absolutely continuous density functions on $\mathbb{R}$ such that

$$(2.2) \qquad I(f) = \int_0^1 \phi^2(u,f)du < \infty, \; I_1(f) = \int_0^1 \phi_1^2(u,f)du < \infty$$

and

$$(2.3) \qquad \int_0^1 \phi(u,f)du = \int_0^1 \phi_1(u,f)du = 0$$

where if F(x) is the distribution function corresponding to f(x),

$$(2.4) \qquad \phi(u,f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))} \text{ and } \phi_1(u,f) = -1 - F^{-1}(u)\frac{f'(F^{-1}(u))}{f(F^{-1}(u))},$$

$0 < u < 1$.

Let $\bar{c}_\nu = \sum_{i=1}^{N_\nu} c_{\nu i}/N_\nu$, $\bar{d}_\nu = \sum_{i=1}^{N_\nu} d_{\nu i}/N_\nu$, $c_\nu^0 = (c_{\nu 1}-\bar{c}_\nu,\ldots,c_{\nu N_\nu}-\bar{c}_\nu)$ and $d_\nu^0 = (d_{\nu 1}-\bar{d}_\nu,\ldots,d_{\nu N_\nu}-\bar{d}_\nu)$. We now define some sets of conditions for the vectors $c_\nu$ and $d_\nu$.

Condition A.

(i) $\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 = 0.$

(ii) For $\nu = 1,2,\ldots,\ c_{\nu i} - \bar{c}_\nu \ne 0\ (i=1,\ldots,N_\nu)$.

(iii) There exists a real number K such that

$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} (e_{\nu i}(c_{\nu i}-\bar{c}_\nu)^{-1}-K)^2 = 0$ where

$e_{\nu i} = d_{\nu i} - \bar{d}_\nu.\exp(-c_{\nu i}+\bar{c}_\nu),\ i = 1,\ldots,\ N_\nu.$

It is easily seen that condition A implies $\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} e_{\nu i}^2 = 0$.

For $K \in \mathbb{R}$ and $f \in C$, define

$$(2.5)\qquad I(f,K) = \int_0^1 \phi^2(u,f,K)\,du$$

where

$$(2.6)\qquad \phi(u,f,K) = K\phi(u,f) + \phi_1(u,f),\ 0 < u < 1.$$

Condition B.

(i) Condition A is satisfied.

(ii) For $f \in C$, $\lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \,.\, I(f,K) = b^2$ where $0 < b^2 < \infty$.

Consider a sequence of subsets $M_\nu$ of $\mathbb{R}^{N_\nu} \times \mathbb{R}^{N_\nu}$. We will define for the vectors $(c_\nu,d_\nu) \in M_\nu$, an analogue of conditions A and B by the following statement.

Condition M.

(i) $\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \max_{1\le i\le N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 = 0.$

(ii) For each $(c_\nu,d_\nu) \in M_\nu$, $c_{\nu i} - \bar{c}_\nu \ne 0\ (i=1,\ldots,N_\nu;\ \nu=1,2,\ldots)$.

(iii) There exists a real number K such that

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \max_{1\le i\le N_\nu} (e_{\nu i}(c_{\nu i}-\bar{c}_\nu)^{-1}-K)^2 = 0.$$

(iv) For $f \in C$, if $\theta_\nu^2 = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \cdot I(f,K)$, $\sup_{(c_\nu,d_\nu)\in M_\nu} \theta_\nu^2 \le M < \infty$ for all $\nu$.

The linear rank statistics considered are of the form

$$S_\nu = \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)a_\nu(R_{\nu i}) \tag{2.7}$$

with $\gamma_\nu = (\gamma_{\nu 1},\ldots,\gamma_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$, $\bar{\gamma}_\nu = \sum_{i=1}^{N_\nu} \gamma_{\nu i}/N_\nu$ and $a_\nu(1),\ldots, a_\nu(N_\nu)$ the values of a score function $a_\nu(.)$. The usual regularity condition on the vectors of constants $\gamma_\nu$ is represented by

Condition D.

(i) For $\nu = 1,2,\ldots$, $\sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 > 0$.

(ii) $\lim_{\nu\to\infty} [\sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 / \max_{1\le i\le N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2] = \infty$.

We will say that a sequence of score functions $a_\nu(.)$, $\nu = 1,2,\ldots$, is generated by a real valued function $\phi(u)$, $0 < u < 1$, if

(i) $\int_0^1 \phi^2(u)du < \infty$ and $\int_0^1 (\phi(u)-\bar{\phi})^2 du > 0$ where $\bar{\phi} = \int_0^1 \phi(u)du$.

(ii) $\lim_{\nu\to\infty} \int_0^1 (a_\nu(1+[uN_\nu])-\phi(u))^2 du = 0$ with $[uN_\nu]$ denoting the largest integer not exceeding $uN_\nu$.

In Hájek and Šidák (1967) (p. 158, 164-165), one can find methods for constructing score functions that are generated by a given function $\phi(u)$.

Further, for an ordered sample $U_\nu^{(1)} < \ldots < U_\nu^{(N_\nu)}$ from the uniform distribution on $[0,1]$, we will let

$$a_\nu(i,f) = E\phi(U_\nu^{(i)},f) \text{ and } a_{1\nu}(i,f) = E\phi_1(U_\nu^{(i)},f), \tag{2.8}$$

$i = 1, \ldots, N_\nu$; then, one can easily show that if $f \in C$ and $K \in \mathbb{R}$, the sequence of score functions

(2.9) $$a_\nu(.,f,K) = Ka_\nu(.,f) + a_{1\nu}(.,f),$$

$\nu = 1,2,\ldots$, is generated by $\phi(u,f,K)$, $0 < u < 1$.

Finally, $\Phi(.)$ will denote the standardized normal distribution function and $k_{1-\alpha}$, the $(1-\alpha)$-quantile of the standardized normal distribution. By convention, for $\sigma^2 = 0$, we will let

(2.10) $$\Phi(x/\sigma) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0. \end{cases}$$

## 3. Asymptotic distribution under contiguous alternatives

Under $H_\nu$, it is well known from Hájek (1962) or Hájek and Šidák (1967), p. 163, that if condition D is satisfied and $a_\nu(.)$, $\nu = 1,2,\ldots$, are generated by a function $\phi(u)$, $0 < u < 1$, then, the statistics $S_\nu$ given by (2.7) are asymptotically normal $(0,\sigma_\nu^2)$ with

$$(3.1) \qquad \sigma_\nu^2 = \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 \cdot \int_0^1 (\phi(u)-\bar{\phi})^2 du.$$

For the alternatives $K_\nu$ defined by (2.1), the following results will be proved in section 6.

*Theorem 3.1.* Suppose that a sequence of vectors $c_\nu$ and $d_\nu$ satisfies condition B. Then, $K_\nu$ are contiguous to $H_\nu$.

*Theorem 3.2.* If $a_\nu(.)$, $\nu = 1,2,\ldots$, are generated by a function $\phi(u)$, $0 < u < 1$, if conditions D and A are satisfied and if for $\nu = 1,2,\ldots$, $\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \leq b^2 (0 \leq b^2 < \infty)$ then under $K_\nu$, the statistics $S_\nu$ given by (2.7) are asymptotically normal $(\mu_\nu,\sigma_\nu^2)$ with

$$(3.2) \qquad \mu_\nu = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu) \cdot \int_0^1 \phi(u)\phi(u,f,K)du$$

and $\sigma_\nu^2$ given by (3.1).

Beran (1970) has found the asymptotic distribution of linear rank statistics under contiguous alternatives indexed by a q-dimensional parameter. Although his results are more general, the conditions under which they hold are non comparable with the conditions obtained here for the special case of the location and scale parameters. For example, if $N_\nu$ is a multiple of 4 ($\nu=1,2,\ldots$) and we define

$$(3.3) \qquad c_{\nu i} = \begin{cases} 0 \text{ if } 1 \leq i \leq N_\nu/2 \\ (N_\nu)^{-\frac{1}{2}} \text{ if } N_\nu/2 < i \leq 3N_\nu/4 \\ -(N_\nu)^{-\frac{1}{2}} \text{ if } 3N_\nu/4 < i \leq N_\nu \end{cases}$$

and,

$$(3.4) \qquad d_{\nu i} = \begin{cases} -(N_\nu)^{-\frac{1}{2}} \text{ if } 1 \leq i \leq N_\nu/2 \\ (N_\nu)^{-\frac{1}{2}} \text{ if } N_\nu/2 < i \leq 3N_\nu/4 \\ 0 \text{ if } 3N_\nu/4 < i \leq N_\nu \end{cases}$$

($\nu$=1,2,...), one can easily verify that condition (3.20) of Beran is satisfied while our condition A is not. On the other hand, the double-exponential density function belongs to our class C but it fails to satisfy Beran's condition A.

## 4. Asymptotic sufficiency and asymptotic optimality

The definition of asymptotically sufficient for distinguishing between $H_\nu$ and $K_\nu$, given by Hájek and Šidák (1967), p.243-245, can be reformulated for the problem considered here in the following way.

*Definition 4.1.* The vectors of ranks $R_\nu = (R_{\nu 1},\ldots,R_{\nu N_\nu})$ is asymptotically sufficient for distinguishing between $H_\nu$ and

(4.1) $$K_\nu = \{q_\nu : (c_\nu,d_\nu) \in M_\nu\}$$

where $q_\nu$ is given by (2.1) and $M_\nu$ is a subset of $\mathbb{R}^{N_\nu} \times \mathbb{R}^{N_\nu}$, if

(i) there are densities $p_\nu = p_\nu(x_1,\ldots,x_{N_\nu};\bar{c}_\nu,\bar{d}_\nu) \in H_\nu$ and rank statistics $h_\nu = h_\nu(r_{\nu 1},\ldots,r_{\nu N_\nu};c_\nu^0,d_\nu^0)$ such that for $(c_\nu,d_\nu) \in M_\nu$, the functions

$$q_\nu^0 = p_\nu \cdot h_\nu$$

are densities ($\nu$=1,2,...).

(ii) $\lim\limits_{\nu\to\infty} \sup\limits_{(c_\nu,d_\nu) \in M_\nu} \| q_\nu - q_\nu^0 \| = 0$ where $\| p-q\|$ denotes the $L_1$-distance of two probability densities:

$$\| p-q\| = \int |p-q|\,d\mu$$

with $\mu$ being a $\sigma$-finite measure with respect to which the densities are defined.

The following results will be proved in section 7.

*Theorem 4.1.* If the sequence $M_\nu$ satisfies condition M, the vector of ranks $R_\nu$ is asymptotically sufficient for distinguishing between $H_\nu$ and $K_\nu$ where $K_\nu$ is given by (4.1).

*Theorem 4.2.* Consider testing $H_\nu$ versus $K_\nu$ given by (4.1) and, assume that the sequence $M_\nu$ satisfies condition M. Denote by $\beta(\alpha,H_\nu,K_\nu)$ the power of the maximin most powerful test, and by $\bar{\beta}(\alpha,H_\nu,K_\nu)$ the power of the maximin most powerful rank test. Then,

$$\lim_{\nu\to\infty} [\beta(\alpha,H_\nu,K_\nu) - \bar{\beta}(\alpha,H_\nu,K_\nu)] = 0, \quad 0 \le \alpha \le 1. \tag{4.2}$$

From theorem 4.2, the asymptotically maximin most powerful test for $H_\nu$ versus $K_\nu$ can be found among the tests based on ranks. The theorem, however, does not specify this test. For the special case where for $\nu = 1,2,\ldots$, the subset $M_\nu$ contains a unique pair of vectors $(c_\nu,d_\nu)$, the following theorem 4.3 provides an alternate proof of the result of theorem 4.2 and specifies the asymptotically most powerful test explicitly.

*Theorem 4.3.* Suppose that the sequences of vectors $c_\nu$ and $d_\nu$ satisfy condition B. Then, the test based on

$$S_\nu^0 = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)a_\nu(R_{\nu i},f,K) \tag{4.3}$$

with critical region $S_\nu^0 \ge k_{1-\alpha}b$ is an asymptotically most powerful test for $H_\nu$ versus $q_\nu$ at level $\alpha$. Furthermore, the asymptotic power is given by $1 - \Phi(k_{1-\alpha}-b)$.

*Corollary 4.1.* The results of theorem 4.3 still hold if the score functions $a_\nu(\cdot,f,K)$ are replaced by score functions $a_\nu(.)$ generated by $\phi(u,f,K)$, $0 < u < 1$.

*Corollary 4.2.* In theorem 4.3 and corollary 4.1, the densities $q_\nu$ can be replaced by

$$q_{\nu,\omega} = \prod_{i=1}^{N_\nu} e^{-(c_{\nu i}+\omega)} f(e^{-(c_{\nu i}+\omega)} x_i-d_{\nu i}) \tag{4.4}$$

where $\omega \in \mathbb{R}$ is unknown and, the test based on $S_\nu^0$ is then an asymptotically uniformly most powerful test for $H_\nu$ versus $\{q_{\nu,\omega} : \omega \in \mathbb{R}\}$ at level $\alpha$.

If we let $d_{\nu i} = 0$ ($i=1,\ldots,N_\nu$ and $\nu=1,2,\ldots$) in theorem 4.3, we obtain the solution of Hájek and Šidák (1967), p.250-251, for scale alternatives. Their solution for location alternatives can also be obtained by transposing the expressions of sections 3 and 4 in terms of $(d_{\nu i}-\bar{d}_\nu)$ instead of $(c_{\nu i}-\bar{c}_\nu)$ and then, setting $c_{\nu i} = 0$ ($i=1,\ldots,N_\nu$ and $\nu=1,2,\ldots$).

## 5. Two-sample case

Let $(m_\nu, n_\nu)$, $\nu = 1,2,\ldots$, be a sequence of pairs of positive integers such that $N_\nu = m_\nu + n_\nu \to \infty$ when $\nu \to \infty$. For each $\nu$, define

$$(5.1) \qquad c_{\nu i} = \begin{cases} \Delta_1 (m_\nu n_\nu / N_\nu)^{-1/2} & \text{if } i = 1,\ldots, m_\nu \\ 0 \text{ if } i = m_\nu+1,\ldots, N_\nu & \end{cases}$$

$$d_{\nu i} = \begin{cases} \Delta_2 (m_\nu n_\nu / N_\nu)^{-1/2} & \text{if } i = 1,\ldots, m_\nu \\ 0 \text{ if } i = m_\nu+1,\ldots, N_\nu & \end{cases}$$

where $\Delta = (\Delta_1, \Delta_2) \in \mathbb{R}^2$. The density (2.1) can now be rewritten as

$$(5.2) \qquad q_{\nu,\Delta} = \prod_{i=1}^{m_\nu} \exp(-\Delta_1 (m_\nu n_\nu/N_\nu)^{-1/2}) f(\exp(-\Delta_1 (m_\nu n_\nu/N_\nu)^{-1/2}) x_i - \Delta_2 (m_\nu n_\nu/N_\nu)^{-1/2}) \prod_{i=m_\nu+1}^{N_\nu} f(x_i)$$

where f is a density function in C. In the following theorem, the asymptotic distribution, under $q_{\nu,\Delta}$, of statistics of the form (2.7) is given.

*Theorem 5.1.* Let $a_\nu(.)$, $\nu = 1,2,\ldots$, be a sequence of score functions generated by a function $\phi(u)$, $0 < u < 1$, and $\gamma_{\nu i} = 1$ if $i = 1,\ldots, m_\nu$ or, $= 0$ if $i = m_\nu+1,\ldots, N_\nu$ $(\nu=1,2,\ldots)$. Then, if $\Delta_1 \neq 0$ and $\min(m_\nu, n_\nu) \to \infty$ when $\nu \to \infty$, the statistics $(m_\nu n_\nu/N_\nu)^{-1/2} S_\nu$ where $S_\nu$ is given by (2.7) are, under $q_{\nu,\Delta}$, asymptotically normal with mean

$$(5.3) \qquad \int_0^1 \phi(u)(\Delta_2 \phi(u,f) + \Delta_1 \phi_1(u,f))du$$

and variance

$$(5.4) \qquad \int_0^1 (\phi(u) - \bar{\phi})^2 du.$$

The asymptotically optimum tests for $H_\nu$ versus $q_{\nu,\Delta}$ are given in the following theorems.

*Theorem 5.2.* Suppose that $\min(m_\nu, n_\nu) \to \infty$ when $\nu \to \infty$. Then, the test based on

$$(5.5) \qquad S_{\nu,\Delta} = \sum_{i=1}^{m_\nu} a_\nu(R_{\nu i}, f, \Delta_2/\Delta_1)$$

with critical region

$$(5.6) \qquad (m_\nu n_\nu/N_\nu)^{-1/2}(\Delta_1/|\Delta_1|)S_{\nu,\Delta} \geqq k_{1-\alpha} I^{1/2}(f, \Delta_2/\Delta_1)$$

is an asymptotically most powerful test for $H_\nu$ versus $q_{\nu,\Delta}$ where $\Delta_1 \neq 0$, at level $\alpha$. Furthermore, the asymptotic power is given by $1 - \Phi(k_{1-\alpha} - |\Delta_1| I^{1/2}(f, \Delta_2/\Delta_1))$.

*Theorem 5.3.* Suppose that $\min(m_\nu, n_\nu) \to \infty$ when $\nu \to \infty$ and let

$$(5.7) \qquad S'_{\nu,\Delta} = \sum_{i=1}^{m_\nu} a_\nu(R_{\nu i}, f, \ell).$$

The test based on $S'_{\nu,\Delta}$ with critical region

$$(5.8) \qquad (m_\nu n_\nu/N_\nu)^{-1/2} S'_{\nu,\Delta} \geq k_{1-\alpha} I^{1/2}(f, \ell)$$

is an asymptotically uniformly most powerful $\alpha$ level test for $H_\nu$ versus $\{q_{\nu,\Delta} : \Delta_1 > 0, \Delta_2/\Delta_1 = \ell\}$.
The test based on $S'_{\nu,\Delta}$ with critical region

$$(5.9) \qquad (m_\nu n_\nu/N_\nu)^{-1/2} S'_{\nu,\Delta} \leq k_\alpha I^{1/2}(f, \ell)$$

is an asymptotically uniformly most powerful $\alpha$ level test for $H_\nu$ versus $\{q_{\nu,\Delta} : \Delta_1 < 0, \Delta_2/\Delta_1 = \ell\}$.

*Corollary 5.1.* In theorems 5.2 and 5.3, the densities $q_{\nu,\Delta}$ can be replaced by

(5.10) $$q'_{\nu,\Delta} = \prod_{i=1}^{m_\nu} \exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}})f(\exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}})(x_i-\Delta_2(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}})) \prod_{i=m_\nu+1}^{N_\nu} f(x_i).$$

*Corollary 5.2.* In theorems 5.2 and 5.3, if the densities $q_{\nu,\Delta}$ are replaced by

(5.11) $$q_{\nu,\Delta,\omega} = \prod_{i=1}^{m_\nu} \exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}-\omega)f(\exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}-\omega) x_i-\Delta_2(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}) \prod_{i=m_\nu+1}^{N_\nu} e^{-\omega}f(e^{-\omega}x_i)$$

where $\omega \in \mathbb{R}$ is unknown, then the test based on $S_{\nu,\Delta}$ with critical region given by (5.6) is an asymptotically uniformly most powerful $\alpha$ level test for $H_\nu$ versus $\{q_{\nu,\Delta,\omega} : \Delta_1 \neq 0,\ \omega \in \mathbb{R}\}$, the test based on $S'_{\nu,\Delta}$ with critical region given by (5.8) is an asymptotically uniformly most powerful $\alpha$ level test for $H_\nu$ versus $\{q_{\nu,\Delta,\omega} : \Delta_1 > 0,\ \Delta_2/\Delta_1 = \ell,\ \omega \in \mathbb{R}\}$ and the test based on $S'_{\nu,\Delta}$ with critical region given by (5.9) is an asymptotically uniformly most powerful $\alpha$ level test for $H_\nu$ versus $\{q_{\nu,\Delta,\omega} : \Delta_1 < 0,\ \Delta_2/\Delta_1 = \ell,\ \omega \in \mathbb{R}\}$.

*Corollary 5.3.* The results of theorems 5.2, 5.3 and corollaries 5.1, 5.2 still hold if the score functions a $(.,f,\ell)$ are replaced by score functions $a_\nu(.)$ generated by $\phi(u,f,\ell)$, $0 < u < 1$.

## 6. Proof of the results of section 3

Define for $i = 1,\ldots, N_\nu$ and $\nu = 1,2,\ldots$ the real functions

$$k_{\nu i}(x) = \frac{\exp(-1/2(c_{\nu i}-\bar{c}_\nu))s(\exp(-c_{\nu i}+\bar{c}_\nu)-e_{\nu i}) - s(x-e_{\nu i})}{c_{\nu i} - \bar{c}_\nu},$$

$$(6.1) \qquad l_{\nu i}(x) = \frac{s(x-e_{\nu i}) - s(x)}{c_{\nu i} - \bar{c}_\nu},$$

$$h_{\nu i}(x) = k_{\nu i}(x) + l_{\nu i}(x)$$

with $s(x) = [f(x)]^{1/2}$ where $f(x)$ is a density function in C. For the proof of theorem 3.1, the following lemmas are needed.

*Lemma 6.1.* Suppose that the sequences of vectors $c_\nu$ and $d_\nu$ satisfy condition A. Then,

$$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} \int_{-\infty}^{\infty} (h_{\nu i}(x) + \tfrac{1}{2}s(x) + (x+K)s'(x))^2 dx = 0.$$

*Proof.* Observe first that $\max_{1\le i\le N_\nu} \int_{-\infty}^{\infty} h_{\nu i}^2(x)dx < \infty$, $\nu = 1,2,\ldots$, and

$$(6.2) \qquad I(f,K) = 4 \int_{-\infty}^{\infty} (-\tfrac{1}{2}s(x) - (x+K)s'(x))^2 dx < \infty.$$

Also, since $s(x)$ is absolutely continuous, we have for almost all x

$$(6.3) \qquad \lim_{\substack{h_1\to 0\\ h_2\to 0}} s(e^{h_1}x+h_2) = s(x) \quad \text{and} \quad \lim_{y\to x} \frac{s(y) - s(x)}{y - x} = s'(x).$$

From condition A and (6.3), we deduce that for almost all x

$$(6.4) \qquad \begin{aligned} &\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} k_{\nu i}(x) = -\tfrac{1}{2}s(x) - xs'(x), \\ &\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} l_{\nu i}(x) = -Ks'(x), \\ &\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} h_{\nu i}(x) = -\tfrac{1}{2}s(x) - (x+K)s'(x). \end{aligned}$$

Furthermore, by the Cauchy-Schwarz inequality, we have

$$k_{\nu i}^2(x) = [\frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{c_{\nu i}-\bar{c}_\nu}(-\tfrac{1}{2}e^{-\frac{1}{2}t}s(e^{-t}x-e_{\nu i})-e^{-\frac{3t}{2}}xs'(e^{-t}x-e_{\nu i}))dt]^2$$

(6.5)

$$\leq \frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{c_{\nu i}-\bar{c}_\nu}(-\tfrac{1}{2}e^{-\frac{1}{2}t}s(e^{-t}x-e_{\nu i})-e^{-\frac{3t}{2}}xs'(e^{-t}x-e_{\nu i}))^2dt$$

and,

$$l_{\nu i}^2(x) = [\frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{e_{\nu i}}(-s'(x-t))dt]^2$$

(6.6)

$$\leq \frac{e_{\nu i}}{c_{\nu i}-\bar{c}_\nu}\int_0^{e_{\nu i}}(-s'(x-t))^2dt$$

so that by Tonelli's theorem

$$\int_{-\infty}^{\infty}k_{\nu i}^2(x)dx \leq \frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{c_{\nu i}-\bar{c}_\nu}\int_{-\infty}^{\infty}(-\tfrac{1}{2}e^{-\frac{1}{2}t}s(e^{-t}x-e_{\nu i})-e^{-\frac{3}{2}t}xs'(e^{-t}x-e_{\nu i}))^2dxdt$$

(6.7)

$$= \int_{-\infty}^{\infty}(-\tfrac{1}{2}s(x)-(x+e_{\nu i})s'(x))^2dx$$

and,

$$\int_{-\infty}^{\infty}l_{\nu i}^2(x)dx \leq \frac{e_{\nu i}}{(c_{\nu i}-\bar{c}_\nu)^2}\int_0^{e_{\nu i}}\int_{-\infty}^{\infty}(-s'(x-t))^2dxdt$$

(6.8)

$$= \frac{e_{\nu i}^2}{(c_{\nu i}-\bar{c}_\nu)^2}\int_{-\infty}^{\infty}(-s'(x))^2dx.$$

We can thus conclude from (6.4), (6.7) and (6.8) by means of theorems II.4.2 and V.1.3 of Hájek and Šidák (1967) that

(6.9) $$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} \int_{-\infty}^{\infty} (k_{\nu i}(x) + \tfrac{1}{2}s(x) + xs'(x))^2 dx = 0$$

and

(6.10) $$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} \int_{-\infty}^{\infty} (l_{\nu i}(x) + Ks'(x))^2 dx = 0.$$

Consequently, the result follows. □

For a density function $f \in C$ and a sequence of vectors $c_\nu$ and $d_\nu$ satisfying condition A, define for $\nu = 1,2,\ldots$ the statistics

(6.11) $$T_\nu = -\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)[1 + (e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu+K)\frac{f'(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)}{f(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)}],$$

(6.12) $$J_\nu = 2\sum_{i=1}^{N_\nu} [(\frac{e^{-c_{\nu i}}f(e^{-c_{\nu i}}X_{\nu i}-d_{\nu i})}{e^{-\bar{c}_\nu}f(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)})^{\frac{1}{2}} - 1],$$

and

(6.13) $$L_\nu = \prod_{i=1}^{N_\nu} L_{\nu i}$$

where for $i = 1,\ldots, N_\nu$

(6.14) $$L_{\nu i} = \frac{e^{-c_{\nu i}}f(e^{-c_{\nu i}}X_{\nu i}-d_{\nu i})}{e^{-\bar{c}_\nu}f(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)}.$$

*Lemma 6.2.* Suppose that the sequences of vectors $c_\nu$ and $d_\nu$ satisfy condition B. Then, we have

$$\lim_{\nu\to\infty} E(J_\nu) = -\tfrac{1}{4}b^2 \quad \text{and} \quad \lim_{\nu\to\infty} Var(J_\nu-T_\nu) = 0$$

under $\bar{P}_\nu$ where $\bar{P}_\nu$ is the probability measure corresponding to the density

$$(6.15) \qquad \bar{p}_\nu = \prod_{i=1}^{N_\nu} e^{-\bar{c}_\nu} f(e^{-\bar{c}_\nu} x_i - \bar{d}_\nu).$$

*Proof*. Obviously

$$(6.16) \qquad E(J_\nu) = - \sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 \int_{-\infty}^{\infty} h_{\nu i}^2(x)dx$$

and,

$$(6.17) \qquad \mathrm{Var}(J_\nu - T_\nu) \leq E(J_\nu - T_\nu)^2$$

$$= 4 \sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 \int_{-\infty}^{\infty} (h_{\nu i}(x) + \tfrac{1}{2}s(x) + (x+K)s'(x))^2 dx.$$

Thus, by lemma 6.1 and part (ii) of condition B, the lemma is established. □

*Lemma 6.3*. Suppose that the sequences of vectors $c_\nu$ and $d_\nu$ satisfy condition A. Then, for arbitrary $\varepsilon > 0$,

$$\lim_{\nu\to\infty} \max_{1\leq i\leq N_\nu} \bar{P}_\nu(|L_{\nu i} - 1| > \varepsilon) = 0$$

where $\bar{P}_\nu$ is given by (6.15).

*Proof*. We have by part (i) of condition A and lemma 6.1 that under $\bar{P}_\nu$,

$$(6.18) \qquad \lim_{\nu\to\infty} \max_{1\leq i\leq N_\nu} E(\sqrt{L_{\nu i}} - 1)^2 = 0.$$

Thus, by the Markov inequality and corollary 5.1.2 of Billingsley (1968), the lemma is established. □

*Proof of theorem 3.1*. From lemma 6.2 and since that under $\bar{P}_\nu$

$$(6.19) \qquad E(T_\nu) = 0 \quad \text{and} \quad \lim_{\nu\to\infty} \mathrm{Var}(T_\nu) = b^2,$$

it follows that under $\bar{P}_\nu$

(6.20) $$\lim_{\nu\to\infty} E(J_\nu - T_\nu + \tfrac{1}{4}b^2)^2 = 0.$$

By theorem V.1.2 of Hájek and Šidák (1967) we have $T_\nu$ asymptotically normal $(0,b^2)$ under $\bar{P}_\nu$ and by (6.20) we have then that $J_\nu$ are asymptotically normal $(-\frac{1}{4}b^2,b^2)$ under $\bar{P}_\nu$. This entails with lemma 6.3 and Le Cam's second lemma (see Hájek and Šidák (1967), p.205) that

(6.21) $$\lim_{\nu\to\infty} \bar{P}_\nu(|\ln L_\nu - J_\nu + \tfrac{1}{2}b^2| > \varepsilon) = 0$$

for arbitrary $\varepsilon > 0$ and, $\ln L_\nu$ asymptotically normal $(-\frac{1}{2}b^2,b^2)$ under $\bar{P}_\nu$. Consequently, since $\bar{P}_\nu \in H_\nu$, the corollary of Le Cam's first lemma (see Hájek and Šidák (1967), p.204) completes the proof. □

For $i = 1,\dots, N_\nu$ and $\nu = 1,2,\dots,$ we introduce the random variables

(6.22) $$U_{\nu i} = F(e^{-\bar{c}_\nu} X_{\nu i} - \bar{d}_\nu)$$

where F is the distribution function of a density $f \in C$. Under $\bar{P}_\nu$, the random variables $U_{\nu 1},\dots, U_{\nu N_\nu}$ are independently uniformly distributed on [0,1]. The next two lemmas are needed in the proof of theorem 3.2.

*Lemma 6.4.* Let $a_\nu(.)$, $\nu = 1,2,\dots,$ be a sequence of score functions generated by a function $\phi(u)$, $0 < u < 1$, and assume that the sequence of vectors $\gamma_\nu$ satisfies condition D. Then, for $S_\nu$ given by (2.7) and

(6.23) $$T_\nu^\phi = \sum_{i=1}^{N_\nu} (\gamma_{\nu i} - \bar{\gamma}_\nu)\phi(U_{\nu i}),$$

we have for arbitrary $\varepsilon > 0$

$$\lim_{\nu\to\infty} \bar{P}_\nu(|S_\nu - T_\nu^\phi| > \varepsilon) = 0$$

where $\bar{P}_\nu$ is given by (6.15).

The proof of this lemma is similar to the arguments of Hájek and Šidák (1967), p.160-161 and 164-165.

*Lemma 6.5.* Let $a_\nu(.)$, $\nu = 1,2,\dots$, be a sequence of score functions generated by a function $\phi(u)$, $0 < u < 1$, and suppose that the sequences of vectors $c_\nu$ and $d_\nu$ satisfy condition B. Assume also that the sequence of vectors $\gamma_\nu$ satisfies condition D and,

$$(6.24) \qquad \lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu) = b_{12}.$$

Then, for $\bar{P}_\nu$, $T_\nu^\phi$ and $T_\nu$ given respectively by (6.15), (6.23) and (6.11), we have that under $\bar{P}_\nu$, $(T_\nu^\phi, T_\nu)$ are asymptotically jointly normal with mean vector (0,0) and covariance matrix $\begin{pmatrix} \sigma^2 & \sigma_{12} \\ \sigma_{12} & b^2 \end{pmatrix}$ where

$$(6.25) \qquad \sigma^2 = \int_0^1 (\phi(u)-\bar{\phi})^2 du$$

and,

$$(6.26) \qquad \sigma_{12} = b_{12} \int_0^1 \phi(u)\phi(u,f,K)du.$$

*Proof.* Since from (6.11) and (6.22), we can write

$$(6.27) \qquad T_\nu = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)\phi(U_{\nu i},f,K),$$

the proof of this lemma is obtained by arguments similar to Hájek and Šidák (1967), p.217-218. □

*Proof of theorem 3.2.* Without loss of generality one can suppose that

$$(6.28) \qquad \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 = 1, \quad \nu = 1,2,\dots .$$

Then, from condition D, it follows that

$$(6.29) \qquad \lim_{\nu\to\infty} \max_{1\le i\le N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 = 0.$$

It is sufficient to prove the theorem under the additional assumptions:

(6.30) $$\lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu) = b_{12}$$

and,

(6.31) $$\lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu).I(f,K) = b_1^2 \text{ with } 0 \le b_1^2 < \infty.$$

Indeed, if the theorem were false, there would exists a subsequence of $\{\nu\}$ with the property that for all its subsequences the theorem would fail to hold. However, every subsequence has a further subsequence such that (6.30) and (6.31) hold. That the theorem is true under the assumptions (6.28), (6.29), (6.30) and (6.31) can be seen as follows.

Suppose first $b_1^2 > 0$. From (6.20), (6.21) and lemma 6.4, we have that under $\bar{P}_\nu$, $(S_\nu, \ln L_\nu)$ has the same asymptotic distribution as $(T_\nu^\phi, T_\nu - \frac{1}{2}b_1^2)$. Thus, from lemma 6.5, it follows that under $\bar{P}_\nu$, $(S_\nu, \ln L_\nu)$ is asymptotically jointly normal with mean vector (0,0) and covariance matrix $\begin{pmatrix} \sigma^2 & \sigma_{12} \\ \sigma_{12} & b_1^2 \end{pmatrix}$ where $\sigma^2$ and $\sigma_{12}$ are respectively given by (6.25) and (6.26). By Le Cam's third lemma (see Hájek and Šidák (1967), p.208), we conclude that $S_\nu$ are asymptotically normal $(\sigma_{12}, \sigma^2)$ under $K_\nu$.

The case $b_1^2 = 0$ follows from the remarks of Hájek and Šidák (1967), p.210 and 219. This completes the proof. □

## 7. Proof of the results of section 4

Before presenting the proof of Theorem 4.1, it is usefull to give the next two lemmas.

*Lemma 7.1.* If the sequence $M_\nu$ satisfies condition M, then, for $\bar{p}_\nu$ and $S_\nu^0$ given by respectively (6.15) and (4.3), we have

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \bar{P}_\nu(|q_\nu/\bar{p}_\nu - \exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2)| > \varepsilon) = 0 \tag{7.1}$$

for arbitrary $\varepsilon > 0$.

*Proof.* We shall first show that (7.1) is implied by

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \bar{P}_\nu(|\ln(q_\nu/\bar{p}_\nu) - S_\nu^0 + \tfrac{1}{2}\theta_\nu^2| > \varepsilon) = 0 \tag{7.2}$$

for arbitrary $\varepsilon > 0$.

Since $E(q_\nu/\bar{p}_\nu) \leq 1$, the Markov inequality gives us that for every $\eta > 0$, there exist $\delta = \delta(\eta) > 0$ such that

$$\bar{P}_\nu(q_\nu/\bar{p}_\nu \geq \delta) \leq \eta. \tag{7.3}$$

Let $\alpha = \ln(1+\varepsilon/\delta(\eta))$ with $\varepsilon > 0$. From

$$\begin{aligned} &\bar{P}_\nu(\{|\ln(q_\nu/\bar{p}_\nu) - S_\nu^0 + \tfrac{1}{2}\theta_\nu^2| < \alpha\} \cap \{q_\nu/\bar{p}_\nu \leq \delta(\eta)\}) \\ &\qquad \leq \bar{P}_\nu(|q_\nu/\bar{p}_\nu - \exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2)| \leq \varepsilon) \end{aligned} \tag{7.4}$$

we may conclude that for every $\varepsilon > 0$ and $\eta > 0$,

$$\begin{aligned} &\bar{P}_\nu(|\ln(q_\nu/\bar{p}_\nu) - S_\nu^0 + \tfrac{1}{2}\theta_\nu^2| < \alpha) \\ &\qquad \leq \bar{P}_\nu(|q_\nu/\bar{p}_\nu - \exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2)| \leq \varepsilon) + \eta. \end{aligned} \tag{7.5}$$

Then, by taking $\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu}$ on each side of (7.5) and noting (7.2), we get (7.1).

Now, (7.1) may be proved by reasoning similar to Hājek and Šidāk (1967), p.246, i.e. by assuming that (7.1) is false thus, that (7.2) is false and then drawing a contradictory subsequence by making use of (6.20), (6.21) and the fact that $T_\nu - S_\nu^0 \to 0$ in $\bar{P}_\nu$-probability as $\nu \to \infty$, which can be proved as in Hājek and Šidāk (1967), p.161. □

*Lemma 7.2.* Suppose that the sequence $M_\nu$, $\nu = 1,2,\ldots$, satisfies condition M and let $\{\nu_k\}$ be a strictly increasing subsequence of $\{\nu\}$. Then, for the sequence $C_\nu$ define by $C_\nu = k$ if $\nu_k \le \nu < \nu_{k+1}$, we have

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \left| 1 - \int_{-C_\nu}^{C_\nu} \exp(x-\tfrac{1}{2}\theta_\nu^2)\,d\bar{P}_\nu(S_\nu^0 \le x) \right| = 0$$

where $\bar{P}_\nu$ and $S_\nu^0$ are respectively given by (6.15) and (4.3).

*Proof.* From part (iv) of condition M, we may deduce

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \left| \int_{-C_\nu}^{C_\nu} \exp(x-\tfrac{1}{2}\theta_\nu^2)\,d\Phi(x/\theta_\nu) - 1 \right|$$

(7.6)

$$\le \lim_{\nu\to\infty} \sup_{0\le\theta_\nu^2\le M} \left| \Phi((C_\nu - M)/\theta_\nu) - \Phi((-C_\nu - M)/\theta_\nu) - 1 \right| = 0.$$

Assume now the existence of an $\varepsilon_0 > 0$ and a subsequence $\{\nu_j\} \subset \{\nu_k\}$ such that

$$(7.7)\qquad \left| \int_{-C_{\nu_j}}^{C_{\nu_j}} \exp(x-\tfrac{1}{2}\theta_{\nu_j}^2)\,d\bar{P}_{\nu_j}(S_{\nu_j}^0 \le x) - \int_{-C_{\nu_j}}^{C_{\nu_j}} \exp(x-\tfrac{1}{2}\theta_{\nu_j}^2)\,d\Phi(x/\theta_{\nu_j}) \right| > \varepsilon_0.$$

But, since the sequence $M_{\nu_j}$, $j = 1,2,\ldots$, satisfies condition M, the sequence $\{\nu_j\}$ contains a subsequence $\{\nu_\ell\}$ such that

$$(7.8)\qquad \lim_{\ell\to\infty} \theta_{\nu_\ell}^2 = b^2,\ 0 \le b^2 < \infty.$$

And, from Hājek and Šidāk (1967), p.163, we have that under $\bar{P}_{\nu_\ell}$ the statistics

$S^0_{\nu_\ell}$ are asymptotically normal $(0,\theta^2_{\nu_\ell})$. Let

$$(7.9) \qquad h^0_{\nu_\ell}(x) = \begin{cases} \exp(x-\frac{1}{2}\theta^2_{\nu_\ell}) \text{ if } |x| \leq C_{\nu_\ell}, \\ 0 \text{ if } |x| > C_{\nu_\ell} \end{cases}$$

and denote by E the set of x such that $h^0_{\nu_\ell}(x_\ell) \to \exp(x-\frac{1}{2}b^2)$ for some sequence $x_\ell$ approaching x. Since the complement of E is empty and the random variables $h^0_{\nu_\ell}(S^0_{\nu_\ell})$ are uniformly integrable, we conclude from Billingsley (1968), p.32-34, that

$$(7.10) \qquad \lim_{\ell\to\infty} \left| \int_{-C_{\nu_\ell}}^{C_{\nu_\ell}} \exp(x-\tfrac{1}{2}\theta^2_{\nu_\ell})\, d\bar{P}_{\nu_\ell}(S^0_{\nu_\ell} \leq x) - 1 \right| = 0.$$

Thus, by combining (7.10) with (7.6) we contradict (7.7). The proof is finished. □

*Proof of theorem 4.1.* Let $p_\nu = \bar{p}_\nu$ where $\bar{p}_\nu$ is given by (6.15) and, define

$$(7.11) \qquad h_\nu = \begin{cases} B_\nu \exp(S^0_\nu - \frac{1}{2}\theta^2_\nu) \text{ if } |S^0_\nu| \leq C_\nu, \\ 0 \quad \text{if } |S^0_\nu| > C_\nu \end{cases}$$

where

$$(7.12) \qquad B_\nu = \left[\int_{-C_\nu}^{C_\nu} \exp(x-\tfrac{1}{2}\theta^2_\nu) dP_\nu(S^0_\nu \leq x)\right]^{-1},$$

$S^0_\nu$ is given by (4.3), $P_\nu$ is the probability measure corresponding to $p_\nu$ and $C_\nu$, $\nu = 1,2,\ldots,$ is a sequence of reals such that $C_\nu > 0$ and $C_\nu \to \infty$ when $\nu \to \infty$. This sequence will be specified later.

Obviously, $p_\nu \in H_\nu$ and $h_\nu$ is a rank statistics depending on the vectors $c^0_\nu$ and $d^0_\nu$ only since $S^0_\nu$ as the same property. Furthermore, the functions $q^0_\nu = p_\nu . h_\nu$ provide densities for $(c_\nu, d_\nu) \in M_\nu$ $(\nu=1,2,\ldots)$. Consequently, it remains to show that

$$(7.13) \qquad \lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \| q_\nu - q^0_\nu \| = 0.$$

Since $\|p-q\| = 2\int_{\{q<p\}} (1-q/p)dP$, we have

$$(7.14) \qquad \|q_\nu - q_\nu^0\| = 2\int_{\{q_\nu/p_\nu<h_\nu\}} (h_\nu - \exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2))dP_\nu$$

$$+ 2\int_{\{q_\nu/p_\nu<h_\nu\}} (\exp(S_\nu - \tfrac{1}{2}\theta_\nu^2) - q_\nu/p_\nu)dP_\nu .$$

For each $(c_\nu, d_\nu) \in M_\nu$, the absolute value of the first of the last two integrals is bounded by $|1-B_\nu^{-1}|$ and, the second integral is bounded by $\alpha_\nu \exp(C_\nu) + \varepsilon_\nu$ where $\varepsilon_\nu$ $(\nu=1,2,\ldots)$ is a sequence of positive real numbers and

$$(7.15) \qquad \alpha_\nu = P_\nu(\exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2) - q_\nu/p_\nu > \varepsilon_\nu).$$

Consequently,

$$(7.16) \qquad \|q_\nu - q_\nu^0\| \le 2 \sup_{(c_\nu,d_\nu)\in M_\nu} |1-B_\nu^{-1}| + 2e^{C_\nu} \sup_{(c_\nu,d_\nu)\in M_\nu} \alpha_\nu + 2\varepsilon_\nu .$$

Let $\varepsilon > 0$ be given. From lemma 7.1, for every integer k, there exists $\nu_k$ such that for $\nu > \nu_k$

$$(7.17) \qquad e^k \sup_{(c_\nu,d_\nu)\in M_\nu} P_\nu(\exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2) - q_\nu/p_\nu > 1/k) < 1/k.$$

We can assume that the sequence $\nu_k$, $k = 1,2,\ldots$, is strictly increasing and then, define

$$(7.18) \qquad C_\nu = k \text{ for } \nu_k \le \nu < \nu_{k+1}.$$

From lemma 7.2, there exists $\nu_1(\varepsilon)$ such that for $\nu > \nu_1(\varepsilon)$

$$(7.19) \qquad \sup_{(c_\nu,d_\nu)\in M_\nu} |1-B_\nu^{-1}| < \varepsilon/6.$$

For $\varepsilon_\nu = 1/k$ if $\nu_k \le \nu < \nu_{k+1}$, (7.17) becomes

$$(7.20) \qquad e^{C_\nu} \sup_{(c_\nu,d_\nu)\in M_\nu} \alpha_\nu < 1/C_\nu \text{ for } \nu_k \leq \nu < \nu_{k+1},\ k = 1,2,\ldots$$

Thus, there exists $\nu_2(\varepsilon)$ such that for $\nu > \nu_2(\varepsilon)$

$$(7.21) \qquad e^{C_\nu} \sup_{(c_\nu,d_\nu)\in M_\nu} \alpha_\nu < \varepsilon/6 \text{ and } \varepsilon_\nu < \varepsilon/6.$$

From (7.19) and (7.21), (7.13) is deduce and the proof is complete. □

*Proof of theorem 4.2.* Using theorem 4.1, the result follows from Hājek and Šidāk (1967), p.243-244, with $a = (\bar{c}_\nu, \bar{d}_\nu)$ and $b = (c^0_\nu, d^0_\nu)$. □

*Proof of theorem 4.3.* In view of theorems 3.1 and 3.2, the proof is similar to Hājek and Šidāk (1967), p.251. □

*Proof of corollary 4.1.* From theorem 3.2, the asymptotic power of the test is $1 - \Phi(k_{1-\alpha} - b)$ and thus, the result follows. □

*Proof of corollary 4.2.* Let $c'_{\nu i} = c_{\nu i} + \omega$, $i = 1, \ldots, N_\nu$. The sequence of vectors $c'_\nu$ and $d_\nu$ satisfies condition B with the same K. Thus, since $c_{\nu i} - \bar{c}_\nu = c'_{\nu i} - \bar{c}'_\nu$, the result is immediate. □

## 8. Proof of the results of section 5

*Proof of theorem 5.1.* It may be shown, by easy algebraic transformations, that the sequence of vectors $c_\nu$ and $d_\nu$, defined by (5.1), satisfies condition B with

$$K = \Delta_2/\Delta_1 \text{ and } b^2 = \Delta_1^2\, I(f,\Delta_2/\Delta_1). \tag{8.1}$$

Also, from Hájek and Šidák (1967), p.162, condition D is verified. Consequently, the result is deduced from theorem 3.2.□

*Proof of theorem 5.2.* The direct application of (8.1) in theorem 4.3 permits us to conclude the result.□

The other results are deduced from theorem 5.2.

*Acknowledgements.* I would like to mention my sincere gratitude to Professor Constance van Eeden for suggesting me this problem and also, for her constructive comments during the preparation of the manuscript.

REFERENCES

Beran, R.J. (1970). Linear rank statistics under alternatives indexed by a vector parameter. Ann. Math. Statist. 41 1896-1905.

Billingsley, P. (1968). Convergence of Probability Measures. Wiley, New York.

Hájek, J. (1962). Asymptotically most powerful rank-order tests. Ann. Math. Statist. 33 1124-1147.

Hájek, J. and Šidák, Z. (1967). Theory of Rank Tests. Academic Press, New York.