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A (57,14,1) STRONGLY REGULAR GRAPH DOES NOT EXIST

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A (57,14,1) strongly regular graph does not exist

by

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ABSTRACT

We show that a strongly regular graph with parameters

$$n = 57, \quad k = 14, \quad \lambda = 1, \quad \mu = 4$$

((0,1)-eigenvalues: 1*14, 38*2, 18*(-5);
(1,-1)-eigenvalues: 1*28, 38*(-5), 18*9) does not exist.

KEY WORDS & PHRASES: *Strongly regular graph.*

1. TWO LEMMAS

LEMMA 1. *Let G be a strongly regular graph with parameters* n,k,λ,μ. *Let H be an induced subgraph with* N *points,* M *edges and degree sequence* $d_1,\ldots,d_N$. *Then*

$$(kN - 2M) - \left(\lambda M + \mu\left(\binom{N}{2} - M\right) - \sum_{i=1}^{N} \binom{d_i}{2}\right) \leq n - N$$

and equality holds iff exactly $(kN - 2M) - (n - N)$ *points in* G\H *are adjacent to precisely two points of* H, *while the remaining points in* G\H *are adjacent to precisely one point of* H.

PROOF. Let there be x_i points in G\H adjacent to i points of H. We have

$$\sum x_i = n - N,$$

$$\sum ix_i = kN - 2M,$$

$$\sum \binom{i}{2} x_i = \lambda M + \mu\left(\binom{N}{2} - M\right) - \sum_{i=1}^{N} \binom{d_i}{2}.$$

Since $\sum\binom{i}{2}x_i - \sum ix_i + \sum x_i = x_0 + \sum_{i=3}^{N}\binom{i-1}{2}x_i \geq 0$ this proves the lemma. □

LEMMA 2. *Let G be a strongly regular graph with parameters* n,k,λ,μ. *Let* s *be the smallest eigenvalue of the* (0,1)*-adjacency matrix of G, i.e., the negative root of the equation* $x^2 + (\mu-\lambda)x + \mu-k = 0$. *Then if* S *is a coclique in G we have*

$$V := |S| \leq \frac{n\cdot(-s)}{k-s}$$

and equality holds iff each point outside S *is adjacent to exactly*

$$K := \frac{k\cdot V}{n-V}$$

points in S. *In this case we find a* $2-(V,K,\mu)$ *design with point set* S *and blocks* $B_z = \{y \in S \mid y \text{ adjacent to } z\}$ *for* $z \in G\backslash S$.

PROOF. Let there be x_i points in G\S adjacent to i points of S. We have

$$\sum x_i = n - V,$$

$$\sum ix_i = k \cdot V,$$

$$\sum \binom{i}{2} x_i = \mu \cdot \binom{V}{2},$$

so that

$$\sum (i-K)^2 x_i = \mu V(V-1) + kV - \frac{k^2V^2}{n-V} \geq 0.$$

Writing $x = \frac{kV}{V-n}$ and simplifying (using $0 < V < n$) we see that this inequality is equivalent with

$$x^2 + (\mu \cdot \frac{n-1}{k} - k+1)x + \mu-k \leq 0$$

which is exactly the desired inequality (- note that the largest possible V corresponds to the smallest possible x, and that the middle coefficient equals $\mu-\lambda$ since $n = 1 + k + k(k-1-\lambda)/\mu$). □

2. THE NONEXISTENCE OF (57,14,1)

Let G be a strongly regular graph with parameters $n = 57$, $k = 14$ and $\lambda = 1$. Then $\mu = 4$ and the smallest eigenvalue of the (0,1)-adjacency matrix of G is $s = -5$. By Lemma 2 a coclique in G can have at most 15 points. We first derive a contradiction under the assumption that G contains a coclique of size 15, and then under the opposite assumption.

2.1. *G* has a 15-coclique

Let S be a 15-coclique in G. If we identify a point z not in S with the set $B_z = \{y \in S \mid y \sim z\}$ (where $\sim$ denotes adjacency) then the points of G are the points and blocks of a 2-(15,5,4) design $(S,\mathcal{B})$. Choose a block B_0, and investigate the intersection numbers

$$x_i := x_i(B_0) := \#\{B \in \mathcal{B} \mid |B \cap B_0| = i\}.$$

Obviously, since $\lambda,\mu \leq 4$ we have $x_5 = 1$, i.e., there are no repeated blocks.

Since $\lambda = 1$, each edge is in a unique triangle, and each point is incident with 7 triangles. Of the seven triangles incident with B_0, five contain a point of S and two consist of blocks only. But if a triangle consists of three blocks, these blocks must be mutually disjoint, because $\lambda = 1$. This proves $x_0 \geq 4$.

We have the equations

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 + x_4 &= 41, \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 5\cdot 13 = 65, \\ x_2 + 3x_3 + 6x_4 &= \binom{5}{2}\cdot 3 = 30. \end{aligned}$$

Consequently,

$$x_0 + x_3 + 3x_4 = 6.$$

Since $x_0 \geq 4$ it follows that $x_4 = 0$ and thus $x_0 + x_3 = 6$. But this soon leads to a contradiction:

Let B_0,B_1,B_2 and B_0,B_3,B_4 be two triangles containing B_0. Since intersections of size 4 do not occur we may suppose $|B_3 \cap B_1| = 3$, and then $|B_4 \cap B_2| = 3$.
Let B_1,B_5,B_7 be another triangle containing B_1. W.l.o.g. $|B_5 \cap B_0| = 3$.
Let B_2,B_6,B_8 be another triangle containing B_2. W.l.o.g. $|B_6 \cap B_0| = 3$.
Finally, let B_3,B,B' be another triangle containing B_3. W.l.o.g. $|B \cap B_0| = 3$.

B_0:	11111	00000	00000
B_1:	00000	11111	00000
B_2:	00000	00000	11111
B_3:	00000	11100	11000
B_4:	00000	00011	00111
B_5:	<3*1>	00000	<2*1>
B_6:	<3*1>	<2*1>	00000
B :	<3*1>	000..	00...

Since $x_3 \leq 2$ and $B_5 \neq B_6$, B must coincide with either B_5 or B_6. But then B and B_0 have at least five common neighbours: B_1 or B_2, B_3, and the three points in $B \cap B_0$. Contradiction, for $\lambda,\mu \leq 4$.

2.2. *G* does not contain a 15-coclique

LEMMA. *G does not contain a regular subgraph H with* 6 *points and valency* 3 *(i.e.,* $K_{3,3}$ *or the prism).*

<u>PROOF</u>. Apply Lemma 1 with $N = 6$, $M = 9$, $d_1 = \dots = d_6 = 3$.
We find $66-15 \leq 51$. Since equality holds, exactly 15 points outside H are connected with two points in H. If z is a point in $G\backslash H$ adjacent to two points of H, then let H_z be the graph induced by G on $H \cup \{z\}$. Again apply Lemma 1, now with $N = 7$, $M = 11$, $d_1 = 2$, $d_2 = d_3 = d_4 = d_5 = 3$, $d_6 = d_7 = 4$. We find $76-26 \leq 50$. Since equality holds again, no point in $G\backslash(H \cup \{z\})$ is adjacent to three points in $H \cup \{z\}$. It follows that if S is the set of 15 points adjacent to two points in H, then S is a 15-coclique. Contradiction. □

In the previous section we considered G as a 2-(15,5,4) design; now we shall consider G as a GD[4,3,2;14] group divisible design: Let ∞ be some fixed point, $\Gamma := \Gamma(\infty)$ the set of its neighbours and Δ the set of its non-neighbours. Then $|\Gamma| = 14$ and $|\Delta| = 42$. G induces on Γ a regular graph with valency $\lambda = 1$, so that we find seven disjoint pairs in Γ, the *groups*. For each point $z \in \Delta$ we find a *block* $B_z = \{x \in \Gamma \mid x \sim z\}$ of size $\mu = 4$. One verifies immediately that Γ with these groups and blocks is a group divisible design GD[4,3,2;14] (in HANANI's notation).

(A) *Let* T *be the union of two groups in* Γ. *The set* R *of the six points in* Δ *not joined to any point of* T *is a* 6*-coclique*.

<u>PROOF</u>. For $u \in R$, let $x_i := x_i(u) := \#\{z \in \Delta \mid z \sim u \text{ and } |\Gamma(z) \cap T| = i\}$. Then

$$x_0 + x_1 + x_2 = k - \mu = 10$$

and

$$x_1 + 2x_2 = \mu \cdot |T| = 16$$

so that $x_2 - x_0 = 6$. Suppose that $u, v \in R$ and $u \sim v$. Then $x_0 \geq 1$, so $x_2 \geq 7$ and hence both u and v have at least 7 neighbours in the set (of size 12) of points with two neighbours in T. But then they must have at least two common neighbours. Contradiction with $\lambda = 1$. □

(B) Let $U = U(B)$ be the union of the three groups that do not intersect B. Let $x_i := x_i(U) := \#\{z \in \Delta \mid |\Gamma(z) \cap U| = i\}$. Then

$$x_0 + x_1 + x_2 + x_3 = |\Delta| = 42,$$
$$x_1 + 2x_2 + 3x_3 = |U|\cdot(k-2) = 72,$$
$$x_2 + 3x_3 = 12\cdot(\mu-1) = 36,$$

so that $x_0+x_3 = 6$.

Let $y_i := y_i(B) := \#\{z \in \Delta \mid z \sim B \text{ and } |\Gamma(z)\cap U(B)| = i\}$. Then

$$y_0 + y_1 + y_2 + y_3 = k-\mu = 10$$

and

$$y_1 + 2y_2 + 3y_3 = \mu\cdot|U| = 24.$$

From (A) it follows that $y_0 = y_1 = 0$ and hence $y_2 = 6$, $y_3 = 4$. We can identify these four neighbours of B intersecting U in three points: they are the blocks B_p where $p \in N$ and $p\,B\,B_p$ is a triangle.
[For: suppose B_p intersects U in less than three points. Then there is a second group $\{r,s\}$ intersecting both B and B_p. Of course $r \in B \cap B_p$ is impossible since $\lambda = 1$, so we would have $r \in B$ and $s \in B_p$. But now we find a prism on the set $\{B,B_p,p,r,s,\infty\}$. Contradiction.]

There are 42 blocks, but only $\binom{7}{4} = 35$ sets of 4 groups. Therefore, there must be two blocks, say B' and B", intersecting the same four groups (i.e., $U = U(B') = U(B'')$). Now $x_0(U) \geq 2$ and $x_3(U) \geq y_3 = 4$, so $x_3(U) = y_3(B') = y_3(B'') = 4$: the four blocks intersecting U in three points are common neighbours of B' and B", so $B' \cap B'' = \emptyset$ since $\mu = 4$.
But for $p \in B'$ the block B'_p intersects $\Gamma\backslash U$ only in the point p, i.e., $B'_p \neq B''_q$ for $p \in B'$, $q \in B''$. Contradiction.

Hence no graph *G* exists.

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