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ON DUAL PAIRS OF ANTICHAINS

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On Dual Pairs of Antichains

by

A. E. Brouwer

ABSTRACT.

In this note we prove that for each n different from 1 and 3 there exists an antichain on n points with period 2 under the operator ↑c max .

KEY WORDS: *Antichains*.

INTRODUCTION.

Let $X_n := \{1,2,\dots,n\}$. An antichain on n points is a collection of subsets of X_n such that no inclusion holds between any two of them. If A is a collection of subsets of X_n then we define

$$A^{\uparrow} = \{B \subset X_n \mid \exists A \in A : A \subset B\}$$

$$A^{c} = \{B \subset X_n \mid B \notin A\}$$

$$A^{\underline{c}} = \{B \subset X_n \mid X_n \setminus B \in A\}$$

$$A^{max} = \{B \in A \mid B \subsetneq A \Rightarrow A \notin A\}$$

[Note that the definition of $\uparrow$, c and $\underline{c}$ depends on n.]

The operator $\uparrow$ c max permutes antichains and has an inverse $\downarrow$ c min (see[1]). In BROUWER & SCHRIJVER [1] the question was raised to determine the possible lengths of the orbits under this operator for given n. For $n < 4$ this is quickly done (here we write 13 instead of $\{1,3\}$ etc.):

$n = 0$	one orbit of length 2 :	$\emptyset$, $\{\emptyset\}$.
$n = 1$	one orbit of length 3 :	$\emptyset$, $\{1\}$, $\{\emptyset\}$.
$n = 2$	one orbit of length 4 :	$\emptyset$, $\{12\}$, $\{1,2\}$, $\{\emptyset\}$;
	and one orbit of length 2 :	$\{1\}$, $\{2\}$.
$n = 3$	four orbits of length 5 :	$\emptyset$, $\{123\}$, $\{12,13,23\}$, $\{1,2,3\}$, $\{\emptyset\}$;
		$\{1\}$, $\{23\}$, $\{12,13\}$, $\{1,23\}$, $\{2,3\}$ and
		two others starting with $\{2\}$ resp. $\{3\}$.

The lengths occurring are for $n = 4$: 2, 3 and 6
$n = 5$: 2, 3, 7, 16 and 27
$n = 6$: more than sixty possible lengths are known; an exhaustive search for all possibilities has not yet been feasible.

In general, the question of which periods (=lengths of orbits) occur for a certain n seems to be untractable. Therefore we attacked this problem from the other side and ask for all n such that a given period p is possible for an antichain on n points. This note solves the easiest nontrivial special case: $p = 2$.

THEOREM. *A dual pair of antichains on* n *points exists iff* $n \neq 1$ *and* $n \neq 3$. *(Here two antichains* $\mathcal{A}$ *and* $\mathcal{B}$ *are called dual if* $\mathcal{A} = \mathcal{B}^{\uparrow c \max}$ *and* $\mathcal{B} = \mathcal{A}^{\uparrow c \max}$ *or equivalently if* $\mathcal{A} = \mathcal{B}^{\uparrow c \max} = \mathcal{B}^{\downarrow c \min}$.*)*

Previously the following special cases of this theorem were known:

a) n even (see[1], addendum)

b) n = 5, the configuration $\mathcal{A} = \{12,23,34,45,51\}$ (see [1], p.10)

c) n = 7, the configuration $\mathcal{A} = \{124,125$ and cyclic permutations of these$\}$ (found independently by the PDP8/I computer (executing a heuristic search program), two students of Prof. VAN LINT: B.J. BRAAMS & H.D.L. HOLLMANN and A.E. BROUWER & A. SCHRIJVER. [Here $|\mathcal{A}| = 14$ and $|\mathcal{B}| = 21$.]

d) n = 9, a configuration consisting of 4-tuples found by A. SCHRIJVER with geometrical arguments about the affine plane with 9 points. Also a configuration consisting of triples found by A.E. BROUWER and reproduced in the appendix. This example showed that an antichain on n points with period 2 need not consist entirely of either $[\frac{n}{2}]$ - or of $[\frac{n+1}{2}]$-tuples.

e) n = 11, existence proof by M.M. KRIEGER [2]; many (structureless) examples found by the PDP8/I.

The experience with a conversational program on the PDP8/I computer showed that for larger n pairs of dual antichains exist in abundance. It appears that comparatively few m-sets are needed to cover all larger and all smaller sets; if to both $\mathcal{A}$ and $\mathcal{B}$ enough m-sets are assigned, the remaining m-sets can be assigned arbitrarily, always obtaining a dual pair of antichains. [Here and in the sequel $m = [\frac{n}{2}]$.]

For example, if n = 4 and $\{12,34\} \subset \mathcal{A}$, $\{13,24\} \subset \mathcal{B}$, then the pairs 14 and 23 can be assigned arbitrarily, giving rise to the two different orbits of length 2: $\{12,34\}$, $\{13,14,23,24\}$ and $\{12,23,34\}$, $\{13,14,24\}$.

PROOF OF THE THEOREM

Define $\mathcal{Y}_{n,k} := \{A \subset X_n \mid |A| = k\}$.

First we give a characterization of antichains with period 2 contained in some $\mathcal{Y}_{n,k}$:

PROPOSITION 1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be antichains on* n *points contained in* $\mathcal{Y}_{n,k}$. *Then* $\mathcal{A}$ *and* $\mathcal{B}$ *form a pair of dual antichains iff*:

(i) $\mathcal{A} \cup \mathcal{B} = \mathcal{Y}_{n,k}$

and *(ii)* $\mathcal{Y}_{n,k+1} \subset \mathcal{A}^{\uparrow}$ *and* $\mathcal{Y}_{n,k+1} \subset \mathcal{B}^{\uparrow}$

and *(iii)* $\mathcal{Y}_{n,k-1} \subset \mathcal{A}^{\downarrow}$ *and* $\mathcal{Y}_{n,k-1} \subset \mathcal{B}^{\downarrow}$.

PROOF. Obvious. □

It is easy to find dual antichains on an even number of points:

PROPOSITION 2. *Let* $n = 2m$,

$$\mathcal{A}_0 := \{A \in \mathcal{Y}_{n,m} \mid |A \cap X_m| \text{ even}\},$$

$$\mathcal{B}_0 := \{A \in \mathcal{Y}_{n,m} \mid |A \cap X_m| \text{ odd}\}.$$

Then $\mathcal{B}_0 = \mathcal{A}_0^{\uparrow c\, \max} = \mathcal{A}_0^{\downarrow c\, \min}$.

PROOF. Obvious. □

To get a solution for odd n we need some freedom in the choice of solutions for even n:

PROPOSITION 3. *Let* $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *be as in proposition* 1. *Let* $\mathcal{C}$ *be either* $\mathcal{A}_0$ *or* $\mathcal{B}_0$, *and let* $C \in \mathcal{C}$. *If* C *is different from* X_m *and* $X_n \backslash X_m$ *then*

$$(\mathcal{C} \backslash \{C\})^{\uparrow} = \mathcal{C}^{\uparrow} \backslash \{C\}$$

and

$$(\mathcal{C} \backslash \{C\})^{\downarrow} = \mathcal{C}^{\downarrow} \backslash \{C\}.$$

Therefore $\mathcal{C} \backslash \{C\}$ *has period* 2.

PROOF. By symmetry it is enough to prove $(\mathcal{C}\setminus\{C\})^{\uparrow} = \mathcal{C}^{\uparrow}\setminus\{C\}$. Suppose $F = C \cup \{f\}$ and $F \notin (\mathcal{C}\setminus\{C\})^{\uparrow}$. Then for each $c \in C$ the numbers $|((C\setminus\{c\}) \cup \{f\}) \cap X_m|$ and $|C \cap X_m|$ have different parity. If $f \in X_m$ this means that $C = X_n\setminus X_m$, and if $f \notin X_m$ then it follows that $C = X_m$; but these are just the excluded cases. □

The freedom allowed by this proposition allows the construction of solutions for odd n. (Remember the notation $\mathcal{A}^{\underline{c}} = \{A^c \mid A \in \mathcal{A}\}$).

PROPOSITION 4. *Let* $n = 2m$, $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *as before.*

(i) *If* $m = 2k$ *then* $\mathcal{A}_0 = \mathcal{A}_0^{\underline{c}}$ *and* $\mathcal{B}_0 = \mathcal{B}_0^{\underline{c}}$. *If* C *is any element of* $\mathcal{B}_0$ *then* $\mathcal{A}_1 := \mathcal{A}_0 \cup \{C\}$ *and* $\mathcal{B}_1 := \mathcal{B}_0\setminus\{C\}$ *form a dual pair of antichains in* $\mathcal{Y}_{n,m}$ *such that* $C \in \mathcal{A}_1$ *and* $X_n\setminus C \in \mathcal{B}_1$. [Note: such a C exists iff k > 0.]

(ii) *If* $m = 2k+1$ *then* $\mathcal{A}_0 = \mathcal{B}_0^{\underline{c}}$ *and* $\mathcal{B}_0 = \mathcal{A}_0^{\underline{c}}$. *If* C *is any element of* $\mathcal{A}_0$ *different from* $X_n\setminus X_m$ *then* $\mathcal{A}_1 := \mathcal{A}_0\setminus\{C\}$ *and* $\mathcal{B}_1 := \mathcal{B}_0 \cup \{C\}$ *form a dual pair of antichains in* $\mathcal{Y}_{n,m}$ *such that both* C *and* $X_n\setminus C$ *are in* $\mathcal{B}_1$. [Note: such a C exists iff k > 0.]

Now define in both cases:

$$\mathcal{A} := \{A \in \mathcal{Y}_{n+1,m} \mid (n+1 \in A \text{ and } |A\cap C| \text{ even}) \text{ or } A \in \mathcal{A}_1\}$$

and $$\mathcal{B} := \{A \in \mathcal{Y}_{n+1,m} \mid (n+1 \in A \text{ and } |A\cap C| \text{ odd}) \text{ or } A \in \mathcal{B}_1\}.$$

Then $\mathcal{A}$ *and* $\mathcal{B}$ *form a dual pair of antichains on* $n + 1$ *points.*

PROOF. The statements about $\mathcal{A}_1$ and $\mathcal{B}_1$ follow from proposition 3 and the fact that if $m = 2k$ then both X_m and $X_n\setminus X_m$ lie in $\mathcal{A}_0$, while if $m = 2k+1$ then $X_m \in \mathcal{B}_0$ and $X_n\setminus X_m \in \mathcal{A}_0$. The cardinalities of $\mathcal{A}_0$ and $\mathcal{B}_0$ are

$$|\mathcal{A}_0| = \sum_{i \text{ even}} \binom{m}{i}^2 = \begin{cases} \frac{1}{2}\binom{2m}{m} & \text{(m odd)} \\ \frac{1}{2}\binom{2m}{m} + (-1)^{m/2} \cdot \frac{1}{2}\binom{m}{m/2} & \text{(m even)} \end{cases}$$

and

$$|\mathcal{B}_0| = \sum_{i \text{ odd}} \binom{m}{i}^2 = \begin{cases} \frac{1}{2}\binom{2m}{m} & (m \text{ odd}) \\ \frac{1}{2}\binom{2m}{m} - (-1)^{m/2} \cdot \frac{1}{2}\binom{m}{m/2} & (m \text{ even}), \end{cases}$$

hence $|\mathcal{B}_0| > 0$ iff $k > 0$ and $|\mathcal{A}_0| > 1$ iff $k > 0$.

Next we have to prove that both $\mathcal{A}^{\uparrow}$ and $\mathcal{B}^{\uparrow}$ cover $\mathcal{Y}_{n+1,m+1}$. Let $F = m + 1$. If $n + 1 \notin F$ then F is covered already by $\mathcal{A}_1^{\uparrow}$ and $\mathcal{B}_1^{\uparrow}$. If $n + 1 \in F$ and F intersects both C and $X_n \setminus C$ then F contains m-subsets containing n + 1 and intersecting C with prescribed parity. If $F = X_n \setminus C \cup \{n+1\}$ then $F \in \mathcal{B}^{\uparrow}$ since $X_n \setminus C \in \mathcal{B}_1$, and $F \in \mathcal{A}^{\uparrow}$ since for each $c \notin C$: $(X_n \setminus C \cup \{c\})) \cup \{n+1\} \in \mathcal{A}$. [Note that 0 is even.]

Finally, if $F = C \cup \{n+1\}$ then F contains m-subsets C and $(C \setminus \{c\}) \cup \{n+1\}$; one belonging to $\mathcal{A}$ and the other to $\mathcal{B}$ (depending on the parity of m). If m is even then $C \in \mathcal{A}_1$ and $(C \setminus \{c\}) \cup \{n+1\} \in \mathcal{B}$, and if m is odd then $C \in \mathcal{B}_1$ and $(C \setminus \{c\}) \cup \{n+1\} \in \mathcal{A}$.

Finally we have to prove that both $\mathcal{A}^{\downarrow}$ and $\mathcal{B}^{\downarrow}$ cover $\mathcal{Y}_{n+1,m-1}$. Let $|F| = m - 1$. If $n + 1 \notin F$ then F is covered already by $\mathcal{A}_1^{\downarrow}$ and $\mathcal{B}_1^{\downarrow}$. If $n + 1 \in F$ then F is contained in m-supersets containing n + 1 and intersecting C with prescribed parity. □

PROOF OF THE THEOREM. The previous propositions provide antichains with period 2 for all n except n = 1 and n = 3. On the other hand, we saw in the introduction that for these values of n only the periods 3 resp. 5 occur. □

REMARKS AND QUESTIONS

The above construction produces a partition of all m-tuples into two parts in such a way that never all m-tuples contained in a (m+1)-tuple lie in the same part. Therefore n must be less than the Ramsey number $N(m+1, m+1; m)$.

COROLLARY. *For all* m:

$$N(m+1,m+1;\, m) \geq 2m + 1;$$

If $m \geq 2$ *then*

$$N(m+1,m+1;\, m) \geq 2m + 2.$$

[This may be compared with the values $N(1,1;0) = 1$, $N(2,2;1) = 3$, $N(3,3;2) = 6$, $N(4,4;3) \geq 13$.]

Conversely, this agreement provides an upper bound for the numbers n such that there exists a pair of dual antichains, each contained in $\mathcal{Y}_{n,k}$ for some k. (For example: such a pair can be contained in $\mathcal{Y}_{n,2}$ only for $n \leq 5$.) In other words, one cannot have arbitrarily large pairs of dual antichains consisting entirely of pairs or of triples etc. On the other hand, as is shown by the example of triples for $n = 9$, it is possible to have antichains of period 2 in $\mathcal{Y}_{n,k}$ for $k < [n/2]$.

The question remains whether there exist antichains with period 2 not contained in some $\mathcal{Y}_{n,k}$. The propositions below give some results in this direction.

PROPOSITION 5. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be a pair of dual antichains.*
Then $\max\{|A| \mid A \in \mathcal{A}\} = \max\{|B| \mid B \in \mathcal{B}\}$
and $\min\{|A| \mid A \in \mathcal{A}\} = \min\{|B| \mid B \in \mathcal{B}\}$.

PROOF. Let $|A_0| = \max\{|A| \mid A \in \mathcal{A}\}$, where $A_0 \in \mathcal{A}$.
If $a \in A_0$ then $A_0 \in \mathcal{A} = \mathcal{B}^{\downarrow c\, \min}$ implies $A_0 \setminus \{a\} \notin \mathcal{B}^{\downarrow c}$, i.e. $A_0 \setminus \{a\} \in \mathcal{B}^{\downarrow}$. But $A_0 \setminus \{a\} \notin \mathcal{B}$ since $\mathcal{A} \cup \mathcal{B}$ is an antichain, so $\max\{|B| \mid B \in \mathcal{B}\} \geq \max\{|A| \mid A \in \mathcal{A}\}$. By symmetry we are through. □

PROPOSITION 6. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be a pair of dual antichains on* n *points, and set* $k := \min\{|A| \mid A \in \mathcal{A}\} = \min\{|B| \mid B \in \mathcal{B}\}$. *Then*

$$(\mathcal{A} \cap \mathcal{Y}_{n,k})^{\uparrow} \setminus (\mathcal{A} \cap \mathcal{Y}_{n,k}) = (\mathcal{B} \cap \mathcal{Y}_{n,k})^{\uparrow} \setminus (\mathcal{B} \cap \mathcal{Y}_{n,k}).$$

PROOF. Obvious. □

REFERENCES

[1] BROUWER, A.E. & A. SCHRIJVER, *On the Period of an Operator Defined on Antichains*, Math. Centre report ZW 24/74.

[2] KREIGER, M.M., *On Permutations of Antichains in Boolean Lattices: An Application of Ramsey's Theorem*, Reprint Computer Science Dept., Univ. of California, Los Angeles.

APPENDIX

Reproduction of three antichains on nine points consisting of triples only. (The format is described in [1])

```
.R ANTIQ

ANTIQ AB-V03

N=9
#A
ABD/BCE/CDF/DEG/EFA/FGB/GAC/ABE/BCF/CDG/DEA/EFB/FGC/GAD/ABH/ABI/CDH/CD
I/EFH/EFI/BGH/BGI/BDH/BDI/DFH/DFI/AFH/AFI/AHI/CHI/EHI/GHI/AGI/CEI/BFI/
FGH/ACH/EGH/
#L
GHI/FGH/EHI/EGH/EFI/EFH/DFI/DFH/DEG/CHI/CFG/CEI/CDI/CDH/CDG/CDF/
BGI/BGH/BFI/BFG/BEF/BDI/BDH/BCF/BCE/AHI/AGI/AFI/AFH/AEF/ADG/ADE/
ACH/ACG/ABI/ABH/ABE/ABD/
#G
1:  FHI/FGI/EGI/EFG/DHI/DGI/DGH/DFG/DEI/DEH/DEF/CGI/CGH/CFI/CFH/
CEH/CEG/CEF/CDE/BHI/BFH/BEI/BEH/BEG/BDG/BDF/BDE/BCI/BCH/BCG/BCD/
AGH/AFG/AEI/AEH/AEG/ADI/ADH/ADF/ACI/ACF/ACE/ACD/ABG/ABF/ABC/
2:  GHI/FGH/EHI/EGH/EFI/EFH/DFI/DFH/DEG/CHI/CFG/CEI/CDI/CDH/CDG/
CDF/BGI/BGH/BFI/BFG/BEF/BDI/BDH/BCF/BCE/AHI/AGI/AFI/AFH/AEF/ADG/
ADE/ACH/ACG/ABI/ABH/ABE/ABD/

#T
PERIOD:   2
#A
DGH/
#T
PERIOD:   2
#A
ACE/
#T
PERIOD:   2
#S
ON
#G
1:  44    (0   0   0   44  0   0   0   0   0   0)
2:  40    (0   0   0   40  0   0   0   0   0   0)

#
.
```