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ON THE NONEXISTENCE OF CERTAIN PLANAR SPACES

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On the nonexistence of certain planar spaces

by

A.E. Brouwer

ABSTRACT

In certain cases we show the nonexistence of planar spaces all of whose planes are isomorphic to a given linear space L. In particular we settle five of the six possibilities for L suggested by DOYEN.

KEY WORDS & PHRASES: *linear space, planar space*

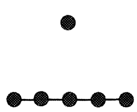
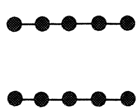
0. INTRODUCTION

During the "Séminaire de Mathématiques supérieures: Configurations et applications non-statistiques" held at the university of Montréal (780612-780707) Jean DOYEN gave a series of lectures about linear spaces. One of the questions he posed was whether in a nontrivial planar space in which all planes are isomorphic to a given linear space L all lines have the same cardinality (unless L is a degenerate projective plane); he also gave the six smallest linear spaces L for which this was not settled yet.

I showed the impossibility of five of these six cases (by various counting arguments); in this note a rather strong result on the parameters (implying the previous results) and some miscellaneous results on special classes of linear spaces are given. Other results on this subject were found by Douglas LEONARD during the seminar.

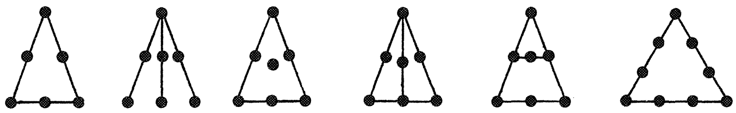
1. PLANAR SPACES

A planar space is a set with two kinds of distinguished subsets, called lines and planes, such that any two points determine a unique line, and any three points not on a line determine a unique plane. Examples are for instance (truncations of) classical spaces (PG(d,n) and AG(d,n); d ≥ 3). A planar space is trivial when it has only one plane or no plane at all. A planar space is called an L-space when each of its planes is isomorphic to the linear space L. The only known example of a nontrivial L-space containing lines of different sizes is

L =  , P = 

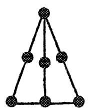
one where L is a degenerate projective plane: one line of size n and n lines of size two on n+1 points, and the planar space P is union of two disjoint lines of size n, all other lines having size two, and planes are chosen in the obvious way. BUEKENHOUT & DE HERDER showed that given a linear space L

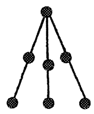
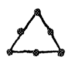
with lines of at least two sizes, there are only finitely many L-spaces. During his lectures DOYEN gave the six smallest possibilities for nontrivial L-spaces with varying line sizes (lines of size 2 are not drawn):

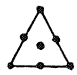
L: 

v:	126	27	27	47	47	1444
p:	19065	90	90	517	517	6951264

Here v and p denote the number of points and planes of the planar space P. I shall show the impossibility of five of the six cases, leaving only the case

L = 

Let us start with the simplest case, L =  . {That v = 27, p = 90 is the only possibility for an L-space is seen by writing down the condition that three points not on a line determine a unique plane. In this case we find $\binom{v}{4} = p\ (8v-21)$, hence $8v(8v-8)(8v-16)(8v-24) \equiv 0 \pmod{8v-21}$, i.e., $(8v-21) \mid 3^2.5.7.13$. Only two solutions have $p > 1$: $p = 11$, $v = 14$ and $p = 90$, $v = 27$. But the first is impossible because of HANANI's inequality $p \geq v$.} Suppose an L-space P exists, and fix a point $x_0 \in P$. Let x_0 be in u lines of size 3. Since v = 27 it follows that $u \leq 13$, and u = 13 is impossible because otherwise each line of size 3 must contain x_0 (L does not contain a configuration ), i.e. each plane contains x_0, contradiction. If $5 < u < 13$ then let L_0 be a line of size 2 through x_0. For each line L_i of size 3 through x_0 we find a plane $<L_0,L_i>$ $(1 \leq i \leq u)$, and these planes are mutually distinct. But L_0 is in only $\frac{v-2}{7-2} = 5$ distinct planes. Contradiction. Next observe that the planes with 'top' x_0 determine a Steiner triple system $S(2,3,u)$ on the lines of size 3 through x_0, so that $u \equiv 1$ or $3 \pmod 6$ and hence $u \leq 3$. But this means that x_0 is 'top' of at most one plane! Since x_0 was arbitrary we have 90 planes but at most 27 tops, a contradiction. Douglas LEONARD proved more generally that L cannot look like a fan: one point only in lines of size at least three, all other points in at most one such line.

Next examine the case L = . The parameters are the same as in the previous case: $v = 27$, $p = 90$. Distinguish three kinds of points in L: its 'centre' (in no lines of size 3), its 'side's (in one line of size 3) and its 'corner's (in two lines of size 3). Suppose an L-space P exists and choose $x_0 \in P$, fixed for the moment. Let x_0 be centre in q_0 planes, side in q_1 planes and corner in q_2 planes. Count triples (x_0,ℓ_2,π) with $x_0 \in \ell_2 \subset \pi$, ℓ_2 a line of size two and π a plane in P:

$$\#(x_0,\ell_2,\pi) = 6q_0 + 4q_1 + 2q_2 \text{ and likewise } \#(x_0,\ell_3,\pi) = q_1 + 2q_2.$$

A line of size two is in $\frac{v-2}{5} = 5$ planes, and a line of size three is in $\frac{v-3}{4} = 6$ planes. Hence

$$\#(x_0,\ell_2) = \frac{1}{5}(6q_0+4q_1+2q_2) \text{ and } \#(x_0,\ell_3) = \frac{1}{6}(q_1+2q_2) =: t.$$

The point x is joined to 26 other points, hence

$$\frac{1}{5}(6q_0+4q_1+2q_2) + \frac{1}{3}(q_1+2q_2) = 26,$$

i.e.,

$$18q_0 + 17q_1 + 16q_2 = 390.$$

From

$$8q_1 + 16q_2 = 48t \quad \text{(by the definition of t)}$$

it follows that

$$6q_0 + 3q_1 = 130 - 16t, \text{ so that } t \equiv 1 \pmod 3 \text{ and } t \le 8.$$

We find three solutions; using $q_2 = \binom{t}{2}$ (for: two lines of size 3 through x determine a plane):

(i) $t = 1$, $q_0 = 16$, $q_1 = 6$, $q_2 = 0$,
(ii) $t = 4$, $q_0 = 5$, $q_1 = 12$, $q_2 = 6$,
(iii) $t = 7$, $q_0 = 3$, $q_1 = 0$, $q_2 = 21$.

This means that P may contain three kinds of points with incidence numbers given by (i), (ii) or (iii). Let there be A points $x \in P$ of the first kind, B of the second kind and C of the third kind. Then

$$A + B + C = 27 \quad \text{(count points)},$$
$$A + 4B + 7C = 135 \quad \text{(count pairs } (x,\ell_3)\text{; note that } \mathcal{P} \text{ contains 45 lines of size three)},$$
$$6B + 21C = 270 \quad \text{(count pairs } (x,\pi) \text{ where } x \text{ is a corner of } \pi).$$

This system has the unique solution A = -3, B = 24, C = 6, a contradiction.

The other three spaces L die in a similar way (in fact the argument is shorter for them since they contain only two kinds of points so that we have no variable q_0; it turns out that there is not even an integral solution for t, q_1, q_2). We skip the detailed arguments since these contain no new ideas and the non-existence of L-spaces for each of the five possibilities for L follows immediately from more general results presented in the next section.

1A. Planar spaces (continued)

Visiting Vanløse (Copenhagen) for two weeks after the seminar in Montreal I found some general results from which the statements in the previous section can be derived immediately.

Let us first generalize the concept of planar space slightly. In counting arguments one can distinguish lines of different lengths but sometimes it is desirable to distinguish between lines of the same length. Therefore define a *colored linear space* as a linear space where each line has some color and lines of the same color have the same length (but not necessarily conversely). Of course each linear space can be viewed as a colored linear space by taking the length of a line as its color. (The drawing convention for linear spaces corresponds to giving lines of size two the color invisible). If L is a colored linear space then an L-space is a planar space with colored lines such that each of its planes is a copy of L.

A planar space is called *proper* if it contains three points not on a line. If $\mathcal{P}$ is a proper L-space then L contains more than one line. In the sequel we shall always assume that all planar spaces under consideration are proper.

A planar space is called *nontrivial* if it contains more than one plane.

THEOREM 1. *Let L be a colored linear space and P a nontrivial L-space. Let P have* v *points and* p *planes. Let L have* u *points and* n_i *lines of color* i*, each of size* k_i*. Then each point of P is incident with the same number* $\frac{pu}{v}$ *of planes, and for each color* i *with the same number* $\frac{pn_ik_i(u-k_i)}{v(v-k_i)}$ *of lines of color* i.

PROOF. Fix a point $x_0 \in P$. Fix a color ('red') and set $k = k_{red}$, $n = n_{red}$. Let a be the number of red lines through x_0 and q the number of planes through x_0. Let there be q_j planes π through x_0 such that in π the point x_0 belongs to j red lines.

We have (counting pairs (x_0,π) with $x_0 \in \pi$):

$$q = \sum_j q_j,$$

and (counting triples (x_0,ℓ,π) with $x_0 \in \ell \subset \pi$, ℓ a red line):

$$a \cdot \frac{v-k}{u-k} = \sum_j j \cdot q_j,$$

and (counting triples (x_0,ℓ,π) with ℓ a red line not containing x_0, π the plane determined by x_0 and ℓ):

$$np\,\frac{u-k}{v-k} - a = \sum_j (n-j)q_j.$$

Adding the last two equations and using the first yields

$$a = n\left(q - p\,\frac{u-k}{v-k}\right)\frac{u-k}{v-u}\ .$$

This is valid for every color, so writing k_i, n_i and a_i again we find (since x_0 is joined to the v-1 other points)

$$v-1 = \sum_i a_i(k_i-1) = \sum_i n_i\left(q - p\,\frac{u-k_i}{v-k_i}\right)\frac{u-k_i}{v-u}(k_i-1).$$

This is a linear equation in q, and q has coefficient $\Sigma_i\, n_i \cdot \frac{u-k_i}{v-u} \cdot (k_i-1) \neq 0$ hence is determined uniquely. But this means we can express q in v, p, u, n_i, k_i so that q does not depend on the point x_0 chosen. Now counting pairs (x,π) with $x \in \pi$ we find

$$vq = pu,$$

i.e.,

$$q = \frac{pu}{v}.$$

Substituting this in the expression for a yields

$$a = np\left(\frac{u}{v} - \frac{u-k}{v-k}\right)\frac{u-k}{v-u} = \frac{npk(u-k)}{v(v-k)} \,.$$

proving the theorem. □

<u>COROLLARY</u>. *The numbers* $\frac{pu}{v}$ *and* $\frac{pnk(u-k)}{v(v-k)}$ *are integers (for each color).*

Using the notation defined in the above proof we have moreover

<u>PROPOSITION 1</u>. *Fix a color ('red'). The numbers* q_j *defined above satisfy*

(i) $\sum_j q_j = \frac{pu}{v}$

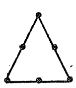
(ii) $\sum_j j\cdot q_j = a \cdot \frac{v-k}{u-k} = \frac{npk}{v}$

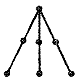
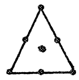
(iii) $\sum_j \binom{j}{2} q_j = \binom{a}{2}$

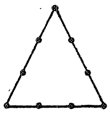
where $a = \frac{pnk(u-k)}{v(v-k)}$.

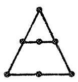
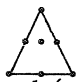
<u>PROOF</u>. (i) and (ii) were already derived. (iii) follows by counting quadruples (x_0,ℓ,ℓ',π) where ℓ and ℓ' are two distinct red lines contained in π and intersecting in x_0. □

As an application we may look again at the six examples given by Doyen:

1. L =  , $u = 6$, $v = 126 = 7.18$, $p = 19065 = 3.5.31.41$, $\frac{pu}{v} \notin \mathbb{N}$ hence no such L-space exists.

2. $L =$  or $L =$  , $u = 7$, $v = 27$, $p = 90$, $\frac{pu}{v} \notin \mathbb{N}$ hence no such L-spaces exist.

3. $L =$  , $u = 9$, $v = 1444 = 4.19^2$, $p = 6951264 = 2^5.3.19.37.103$ and again $\frac{pu}{v} \notin \mathbb{N}$.

4. $L =$  or $L =$  , $u = 7$, $v = 47$, $p = 517$, $k = 3$, $n = 4$, $k' = 2$, $n' = 9$, $\frac{pu}{v} = 77$, $\frac{pnk(u-k)}{v(v-k)} = 12$, $\frac{pn'k'(n-k')}{v(v-k')} = 22$ does not give a contradiction. Solving the system

$$\begin{aligned} q_1 + q_2 + q_3 &= 77 \\ q_1 + 2q_2 + 3q_3 &= 132 \\ q_2 + 3q_3 &= 66 \end{aligned}$$

we find $q_1 = q_2 = 33$, $q_3 = 11$. But clearly in the first case $q_3 = 0$. Hence only the second case might be possible.

PROPOSITION 2. *A solution of the system of equations given in proposition 1 is given by* $\tilde{q}_j = \frac{u_j}{u} q$ *where* $q = \frac{pu}{v}$ *and* u_j *is the number of points in* L *incident with* j *red lines. If* u_j *is nonzero for at most three different values of* j *then this is the unique solution, i.e. we have* $q_j = \frac{u_j}{u} q = u_j \cdot \frac{p}{v}$.

PROOF.

(i) $\sum \tilde{q}_j = \frac{p}{v} \sum u_j = \frac{pu}{v}$. (Count points in L.)

(ii) $\sum j\, \tilde{q}_j = \frac{p}{v} \sum j u_j = \frac{p}{v} \cdot nk$ (Count pairs (x,ℓ) with $x \in \ell$, ℓ a red line in L)

(iii) $v \sum \binom{j}{2} \tilde{q}_j = p \sum \binom{j}{2} u_j = v\binom{a}{2}$ (Count triples (x,ℓ,ℓ') with $x \in \ell \cap \ell'$, $\ell \neq \ell'$, ℓ and ℓ' red lines in P)

The unicity in case of not more than three variables follows from the fact that the determinant on the coefficients is essentially a Vandermonde determinant:

$$\begin{vmatrix} 1 & 1 & 1 \\ j_1 & j_2 & j_3 \\ \binom{j_1}{2} & \binom{j_2}{2} & \binom{j_3}{2} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ j_1 & j_2 & j_3 \\ j_1^2 & j_2^2 & j_3^2 \end{vmatrix} \neq 0. \qquad \square$$

Note that in order to verify that the $\tilde{q}_j$ satisfied equation (iii) we used the existence of P. This gives a rather strong condition on the parameters: usually the $\tilde{q}_j$ do in fact not satisfy (iii), as in the first half of example 4 above ($\tilde{q}_1 = 22$, $\tilde{q}_2 = 55$, $\tilde{q}_3 = 0$).

From theorem 1 it follows that for any two colors that occur in L there is a pair of intersecting lines in L with these colors (because there is such a pair in P). We also get strong results on the parameters of P when in L all lines with a given color are disjoint.

PROPOSITION 3. *Let L be a colored linear space in which all red lines are mutually disjoint. Let P be a non-trivial L-space. Then the red lines form a partition of P and*

$$p = \frac{v(v-k)}{nk(u-k)}$$

where $n > 0$ is the number of red lines in L and k is the length of a red line.

PROOF. Each point of P is incident with $a = \frac{pnk(u-k)}{v(v-k)}$ red lines by theorem 1. But no two red lines can intersect in P since they do not intersect in L. Hence $a = 1$. $\square$

Note that this equation for p implies that v must be small: regarding L as fixed and p as a function of v we have that p grows like v^3 but here p is only quadratic in v. In fact if the index i runs through the colors except red then we have

$$v - 1 = k - 1 + \sum a_i(k_i - 1)$$

so

$$v - k = \sum \frac{v-k}{nk(u-k)} n_i k_i \frac{u-k_i}{v-k_i} (k_i - 1),$$

hence

$$nk(u-k) = \sum n_i k_i (k_i - 1) \frac{u-k_i}{v-k_i} .$$

Letting

$$A := \frac{\Sigma n_i k_i (k_i - 1) \frac{u-k_i}{v-k_i}}{\Sigma n_i k_i (k_i - 1)} .$$

we find

$$\begin{aligned} nk(u-k) &= A \sum n_i k_i (k_i - 1) \\ &= A(u(u-1) - nk(k-1)) \end{aligned}$$

so that

$$A = \frac{nk(u-k)}{u(u-1) - nk(k-1)} .$$

But A is a (weighted) average of the numbers $\frac{u-k_i}{v-k_i}$, hence if $k' = \min k_i$ and $k'' = \max k_i$ then

$$\frac{v-k'}{u-k'} \leq \frac{u(u-1) - nk(k-1)}{nk(u-k)} \leq \frac{v-k''}{u-k''}$$

and (using $\frac{v}{u} \leq \frac{v-k'}{u-k'}$ and $n \geq 1$):

$$v \leq \frac{u(u+k-1)}{nk} ,$$

$$p \leq \frac{u^2(u+k-1)^2}{n^3 k^3 (u-k)} .$$

THEOREM 2. *Let L be a linear space colored with two colors (red and green) such that the red lines are mutually disjoint. Let P be a non-trivial L-space. Let L contain* $n > 0$ *red lines of length* m *and let the green lines have length* k. *Then either* $P = PG(3,q)$, $L = PG(2,q)$, $p = v = q^3 + q^2 + q + 1$, $u = q^2 + q + 1$, $k = m = q + 1$, *L has one red line and in P the red lines form a spread* ($q^2 + 1$ *disjoint lines), or* $P =$, $L =$, $p = v = 2m$, $u = m+1$, $k = 2$, $m \geq 2$, *L has one red line and P is union of two disjoint red lines.*

PROOF. Let L be a colored linear space with u points and n red lines (of size m). Then L has $(u(u-1)-nm(m-1))/k(k-1)$ green lines (of size k). Let P be a non-trivial L space with v points and p planes. Observing that in a planar space any line of length ℓ is contained in $\frac{v-\ell}{u-\ell}$ planes we find that P contains $np\frac{u-m}{v-m}$ red lines and $\frac{u(u-1)-nm(m-1)}{k(k-1)} \cdot p \cdot \frac{u-k}{v-k}$ green lines. Counting pairs gives the equation

$$v(v-1) = p((u(u-1)-nm(m-1))\frac{u-k}{v-k} + nm(m-1)\frac{u-m}{v-m})$$

or,

$$v(v-1)(v-k)(v-m) = p((v-m)(u-k)(u(u-1)-nm(m-1)) + (v-k)(u-m)nm(m-1)).$$

By proposition 3 the red lines partition P and

$$p = \frac{v(v-m)}{nm(u-m)}\ .$$

Substituting this in the previous equation yields

$$(v-1)(v-k)(u-m)nm = (v-m)(u-k)(u(u-1)-nm(m-1)) + (m-1)(v-k)(u-m)nm,$$

i.e.,

$$(v-k)(u-m)nm = (u-k)(u(u-1) - nm(m-1)),$$

or,

$$\frac{v-k}{u-k} = \frac{(u-m)(u+m-1) - (n-1)m(m-1)}{nm(u-m)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

When P does not contain a red and green line which are mutually skew then we have the second possibility of theorem. For: if $u \geq m+2$ then fix a red line ℓ_0 in P and let π and π' be two distinct planes containing ℓ_0. Choose $x \in \pi\backslash\ell_0$ and two distinct points $y_1, y_2 \in \pi'\backslash\ell_0$. The lines xy_1 and xy_2 are both skew to ℓ_0 and thus both red. But red lines do not intersect, a contradiction. This proves $u < m+2$. Since L cannot consist of a single red line

we must have $u = m+1$. Now clearly $k = 2$ and L is a projective plane in the broad sense: red picture: , green picture: . We know already that P is partitioned by the red lines. But if there were three red lines then we could find a green triangle in P, but such a configuration does not occur in a plane. Hence there are exactly two red lines, and $v = 2m$.
When P does contain a red line which is skew to a green line then this green line is in at least m distinct planes, so $\frac{v-k}{u-k} \geq m$. Now (*) yields $m \leq \frac{v-k}{u-k} \leq \frac{u+m-1}{nm}$, i.e.,

$$(1) \qquad u \geq nm^2 - m+1.$$

But $\frac{v-k}{u-k}$ is an integer, so (*) implies that $(u-m)|(n-1)m(m-1)$ and when $n > 1$ it then follows that

$$(2) \qquad u \leq m + (n-1)m(m-1).$$

But (1) and (2) contradict each other, and consequently $n = 1$. The equality (*) now passes into

$$\frac{v-k}{u-k} = \frac{u+m-1}{m}.$$

It follows that $u \equiv 1 \pmod m$, say $u = qm + 1$ for some integer q. We just treated the case $q = 1$ hance may suppose now that $q \geq 2$. Now

$$\frac{v-m}{u-m} = q + 1 + \frac{qm-qk}{qm - m+1} = q + 2 - \frac{qk-m+1}{qm-m+1} .$$

If $m > k$ then it follows that $\frac{v-m}{u-m} > q + 1$ and $\frac{v-m}{u-m} < q + 1 + \frac{q}{q-1}$, i.e. $\frac{v-m}{u-m} = q + 2$. Consequently $m = qk + 1$, $q = (m-1)/k$, $k(u-1) = m(m-1)$,

$$k^2(v-1) = (m-1)(m^2 - m - k^2 + k + km).$$

Since $m|v$ we find $-k^2 \equiv -(-k^2+k) \pmod m$, i.e., $k(2k-1) \equiv 0 \pmod m$. But $m = qk+1$ implies $(m,k) = 1$ and thus $2k-1 \equiv 0 \pmod m$ which is impossible because $m \geq 2k+1$. Therefore we must have $m \leq k$. But from the existence of L it follows that $m-1 \equiv 0 \pmod{k-1}$ [for: L contains points which are only incident with green lines, hence $u-1 \equiv 0 \pmod{k-1}$, and L also contains points which are incident with some green lines and (the unique) red line, hence $u-m \equiv 0 \pmod{k-1}$ and subtracting these congruences yields $m-1 \equiv 0 \pmod{k-1}$] and hence $m \geq k$. It follows that $m = k$: all lines of L have the same size. Now $v = q^2k+q+1$. Fix a plane π_0 in P and a point $x_0 \in \pi_0$ and count the number of planes intersecting π_0 in the point x_0 only. Through x_0 pass $\frac{pu}{v} = \frac{u}{k}(q+1)$ planes; through each of the $\frac{u-1}{k-1}$ lines through x_0 and π_0 pass $\frac{v-k}{u-k} - 1 = q$ planes different from π_0 so that there are

$$s := \frac{u}{k}(q+1) - \frac{u-1}{k-1}\cdot q - 1$$

planes intersecting π_0 exactly in x_0. But then

$$k(k-1)s = u(k-q) - k^2 + k - 1$$

and it follows that $k > q$. From $k|v$ we see that $q \equiv -1 \pmod{k}$ and hence $q = k-1$. Now $m = k = q+1$, $u = q^2+q+1$, $v = q^3+q^2+q+1$, L is PG(2,q), P is PG(3,q) which is the first alternative of the theorem. $\square$