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(DEPARTMENT OF PURE MATHEMATICS)

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Ovoids and fans in the generalized quadrangle GQ(4,2)

by

A.E. Brouwer & H.A. Wilbrink

ABSTRACT

We determine all ovoids and fans (partitions into ovoids) of the (unique) generalized quadrangle GQ(4,2).

KEY WORDS & PHRASES: *ovoid, fan, rosette, generalized quadrangle, near n-gon*

INTRODUCTION

As shown by SHULT & YANUSHKA, near polygons tend to be full of generalized quadrangles (called *quads*). Given a point x and a quad Q the set of points in Q closest to x either is a singleton (in which case x is said to be of *classical type* w.r.t. Q) or an ovoid (and now x is called of *ovoid type*). The occurrence of points of ovoid type imposes certain restrictions on the quad. First of all, clearly, the quad should contain ovoids. But not all quads do:

PROPOSITION. *Suppose that a generalized quadrangle* GQ(s,t) *possesses an ovoid (i.e., a coclique of size* 1+st). *Then* $t \le s(s-1)$ *or* $s = 1$.

PROOF. The pointgraph has eigenvalues $s(t+1)$, $s-1$ and $-(t+1)$ with multiplicities 1, $s(s+1)t(t+1)/(s+t)$ and $s^2(st+1)/(s+t)$ respectively. But Cvetković showed that the size of a coclique is at most the number of non-positive eigenvalues, so if $s > 1$ then $st+1 \le s^2(st+1)/(s+t)$, i.e., $t \le s^2 - s$. □

Secondly, when there is a line ℓ such that all of its points have the same distance to Q and are of ovoid type, then these $|\ell|$ ovoids partition Q. But not all generalized quadrangles possessing ovoids also admit a partition into ovoids:

- it is trivial to verify that any two of the six ovoids of GQ(2,2) have a point in common;
- SHULT & SHAD showed that neither of the two GQ(3,3)'s can be partitioned into ovoids;
- by PAYNE, there is only one generalized quadrangle GQ(4,4). If we take the singular points and totally singular lines in PG(4,4) under an orthogonal polarity as a model then we find that the 120 ovoids are exactly the hyperplane sections of the set of singular points that are nondegenerate elliptic quadrics. Consequently any two ovoids intersect in 1 or 5 points, and GQ(4,4) cannot be partitioned into ovoids.

Finally, when there is a line ℓ such that there is a unique point x on it closest to Q, and x is of classical type while all other points of ℓ are of ovoid type, then these $|\ell|-1$ ovoids partition the set of nonneighbours of πx in Q, where πx is the point in Q closest to x. We do not know of generalized quadrangles possessing ovoids but devoid of such 'semifans'.

Thus, motivated by the study of near polygons, we are led to investigate ovoids, fans and semifans in generalized quadrangles. In this note we determine all of these in the case of GQ(4,2). The reader is not supposed to have any knowledge of near polygons, but should be familiar with unitary geometry.

0. UNITALS

Let σ be a unitary polarity on a projective plane $PG(2,q^2)$. The q^3+1 absolute points carry a $2-(q^3+1,q+1,1)$ design, called a unital. Any point P outside the unital U determines a parallel class in the design: the q^2-q secants through P and the block $U \wedge P^{\perp}$. In the sequel we shall use that for $q = 2$ there are no other parallel classes:

<u>OBSERVATION</u>. Let $\{\ell_1,\ell_2,\ell_3\}$ be a parallel class in U ($\cong AG(2,3)$), the unital in PG(2,4). If P is the point of intersection of ℓ_1 and ℓ_2 then $\ell_3 = U \wedge P^{\perp}$.

The twelve nonisotropic points in PG(2,4) fall into four triples: each parallel class in U is determined by three nonisotropic points. Let us call these *special* triples.

1. OVOIDS

Consider the generalized quadrangle $GQ(q^2,q)$ associated with $U(4,q^2)$: the points are the isotropic points (so $v = (q^2+1)(q^3+1)$) and the lines are the totally isotropic lines (and $b = (q+1)(q^3+1)$).Lines have size $s+1 = q^2+1$ and there are $t+1 = q+1$ lines on a point. We are interested in ovoids and fans in this generalized quadrangle. An *ovoid* is a subset of the generalized quadrangle meeting every line in exactly one point. (Fans are the subject of the next section.) In our case an ovoid has $1+st = q^3+1$ points. We know of two easy constructions.

A. <u>Plane ovoids</u>

Let X = {isotropic points} be our pointset. For any nontangent plane π the set $\pi \cap X$ is an ovoid. (For: any line meets π and π does not contain any totally isotropic lines.) In this way we obtain $q^3(q^2+1)(q-1)$ ovoids, the *plane* ovoids.

B. Modification of ovoids

Let O be any ovoid and ℓ projective line meeting O in $q+1$ points. Then $(O\backslash\ell) \cup (X \cap \ell^{\perp})$ is an ovoid. (For: $\ell^{\perp}$ meets X again in $q+1$ points, and the $q+1$ lines of the generalized quadrangle on a given point of ℓ are just the lines joining it to points of $X \cap \ell^{\perp}$, so that ℓ and $\ell^{\perp}$ block the same lines.)

Starting from plane ovoids repeated applications of this procedure yields large classes of ovoids. The case $q = 2$ is somewhat special, however. Here we can describe all possible ovoids.

C. The case $q = 2$

PROPOSITION. $GQ(4,2)$ *contains* 200 *ovoids falling into two types*:

a) *there are* 40 *plane ovoids*

b) *there are* 160 *ovoids of the form* $X \cap (\ell_1 \cup \ell_2 \cup \ell_3)$, *where the* ℓ_i *are lines passing through a nonisotropic point* P *such that their intersections with* $P^{\perp}$ *form a special triple.*

PROOF. Let us call ovoids of type b) *tripods* with center P.

CLAIM.

(i) Starting with a plane ovoid, our modification produces a tripod.

(ii) Every tripod is obtained three times in this way.

(iii) Modifying a tripod yields a plane ovoid again.

The proof is simple exercise. In this way we find 200 distinct ovoids, but by exhaustive search it follows that there are no others. □

2. FANS

A *fan* in a generalized quadrangle is a collection of $s+1$ ovoids partitioning the point set. In the case of $GQ(q^2,q)$ no fan can consist of plane ovoids only. Surprisingly enough however it is always possible to find fans. Let us treat the case $q = 2$ first.

A. The case $q = 2$

PROPOSITION. $GQ(4,2)$ *contains* 520 *fans falling into two types:*

a) *there are* 40 *fans consisting of the plane ovoid on* $P^{\perp}$ *and the four tripods with center* P, *where* P *is one of the* 40 *nonisotropic points.*

b) *there are* 480 *fans constructed as follows: let* P *be one of the* 40 *nonisotropic points, and let* Q *be one of the* 12 *nonisotropic points orthogonal to* P. *Take the plane ovoid on* $P^{\perp}$, *the tripod on* P *containing* $PQ \cap X$ *and the three tripods on* Q *not containing* $PQ \cap X$.

PROOF. It is an easy exercise to check that the indicated sets of five ovoids are indeed fans. By exhaustive search it follows that there are no others. □

Note that each fan contains exactly one plane ovoid.

B. A construction for all q

It is possible to generalize type a) of the previous proposition. Fix a nonisotropic point P, choose S isotropic in $\pi_0 := P^{\perp}$, and let ℓ be a tangent through S in π_0. Then $\ell \backslash \{S\} = \{P_i \mid 1 \le i \le q^2\}$. Set $\pi_i = P_i^{\perp}$. Let $O_0 = \pi_0 \cap X$ and $O_i = (\pi_i \cap X)\backslash\pi_0 \cup ((\pi_i \cap \pi_0)^{\perp} \cap X)$. Then O_i is an ovoid $(0 \le i \le q^2)$.

CLAIM. $\{O_i \mid 0 \le i \le q^2\}$ is a fan.

PROOF. Easy exercise. □

For $q = 2$ this gives us the 40 fans of type a) - each is founc 9 times: different choices of S determine the same fan. For $q > 2$ we can recognize π_i from O_i so that we get $(q^3+1)q^3(q^2+1)(q-1)$ different fans.

3. SEMIFANS

A *semifan* in a generalized quadrangle $GQ(s,t)$ is a collection of s ovoids all containing some fixed point P and partitioning the s^2t points nonadjacent to P. It is easy to construct semifans in our $GQ(q^2,q)$:

A. Semifans of plane ovoids

Fix an isotropic point P and a tangent line ℓ through P. Then

$$F = \{S^{\perp} \cap X \mid S \in \ell\backslash\{P\}\}$$

is a semifan with center P.

B. Semifans of tripods

When q = 2 we can completely classify the semifans with a given center P :

- there are 2 semifans of plane ovoids, as described under A;
- there are 24 semifans of tripods, constructed as follows:
  Let $Q \in X$ such that PQ is a totally isotropic line (there are 12 choices for Q). Let ℓ be a tangent line through Q. (There are 2 choices for ℓ.) Then the collection of tripods centered at a point of $\ell\backslash\{Q\}$ and containing P is a semifan with center P.
  (PROOF. We have the right number of ovoids, so it suffices to check that no two intersect in a point other than P. But if $P_1, P_2 \in \ell\backslash\{Q\}$ and the tripods centered at P_i and containing P intersect in a point R then the third isotropic point, S say, on the line PR is orthogonal to P_1 and P_2 and hence to Q. But $Q^{\perp} \cap X$ is the line PQ and cannot contain S. Contradiction. □)

By exhaustive search it follows that there are no other semifans for q = 2.

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