stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE ZW 147/80 NOVEMBER
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER

THE UNIQUENESS OF THE STRONGLY REGULAR GRAPH ON 77 POINTS

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 05B30, 05C25

The uniqueness of the strongly regular graph on 77 points

by

A.E. Brouwer

ABSTRACT

We show the uniqueness of the strongly regular graph with parameters $v = 77$, $k = 16$, $\lambda = 0$, $\mu = 4$ by embedding it in the Higman-Sims graph as a second subconstituent.

KEY WORDS & PHRASES: *Strongly regular graph*

INTRODUCTION

There is a unique extension of the projective plane PG(2,4), namely S(3,6,22) (i.e. 3-(22,6,1)). (cf. [3].) It has parameters $v = 22$, $k = 6$, $\lambda_3 = 1$, $\lambda_2 = 5$, $\lambda_1 = 21$, $\lambda_0 = 77$ and any two blocks meet in either 0 or 2 points. Viewing the design as a quasi-symmetric 2-design we see that the graph with the blocks as vertices and pairs of disjoint blocks as edges is strongly regular. In fact this graph has parameters $v = 77$, $k = 16$, $\lambda = 0$, $\mu = 4$. It can be embedded in a graph on 100 vertices as follows: Take as vertices a symbol ∞, the 22 points and the 77 blocks of the design. Let ∞ be adjacent to all points of the design and let a point be adjacent to the blocks containing it and let blocks be adjacent if they are disjoint. This defines a strongly regular graph with parameters $v = 100$, $k = 22$, $\lambda = 0$, $\mu = 6$ called the Higman-Sims graph. Its automorphism group is twice the simple Higman-Sims group. (cf. [2].) We shall prove that any strongly regular graph with 77 vertices (and hence $k = 16$, $\lambda = 0$, $\mu = 4$ - this is the only possibility allowed by the known restrictions on parameters of strongly regular graphs) can be embedded as the set of nonneighbours of a given vertex in the Higman-Sims graph and hence is (isomorphic to) the graph described above.

As an illustration of the method let us first give a short proof of the unicity of a strongly regular graph with the parameters of the Higman-Sims graph. (cf. [1].)

PROPOSITION. *A strongly regular graph* G *with parameters* (100,22,0,6) *is isomorphic to the Higman-Sims graph.*

LEMMA. G *does not contain a* $K_{3,3}$.

PROOF. Let there be x_i vertices outside a $K_{3,3}$ adjacent to i vertices inside.
Counting points we see that $\sum_i x_i = 100-6 = 94$.
Counting edges we see that $\sum_i ix_i = 6.19 = 114$.
Counting paths of length two we see that $\sum_i \binom{i}{2} x_i = 6.3 = 18$.
Consequently $0 \le \frac{1}{2} \sum (i-1)(i-2)x_i = 18-114+94 = -2$, a contradiction. □

PROOF OF PROPOSITION. Select a vertex and call it ∞. Call its neighbours points and its nonneighbours blocks. Identify a block with the set of ($\mu=$) 6 points it is incident with. The preceding lemma shows that no two blocks have three points in common. From the parameters of the graph one sees immediately that the blocks form a design with parameters $v = 22$, $k = 6$, $\lambda_2 = 5$, $\lambda_1 = 21$, $\lambda_0 = 77$. Also, on the average each triple of vertices is covered once, and no triple is covered twice, so each triple is covered exactly once, i.e., $\lambda_3 = 1$. This proves that the design is $S(3,6,22)$. Adjacent blocks have no common points (since $\lambda = 0$), but in $S(3,6,22)$ each block is disjoint from 16 other blocks, so the adjacent blocks are exactly the disjoint blocks. This proves that G is the Higman-Sims graph. $\square$

THE 77-GRAPH

THEOREM. *Let* G *be a strongly regular graph with parameters* $v = 77$, $k = 16$, $\lambda = 0$, $\mu = 4$. *Then* G *is isomorphic to the complement of the block graph of* $S(3,6,22)$.

PROOF.

(i) G does not contain $K_{3,3}$ or $K_{4,4} - 3K_2$.

PROOF. Suppose G contains a subgraph with w vertices, E edges and N nonedges. Then as before we find

$$\sum x_i = 77 - w,\ \sum ix_i = 16w - 2E,\ \sum \binom{i}{2} x_i = 4N - \sum_{j=1}^{w} \binom{d_j}{2}$$

where $d_1, \ldots, d_w$ is the degree sequence of the subgraph. Again it follows that

$$77 - 17w + 2E + 4N - \sum \binom{d_j}{2} \geq 0.$$

But for $K_{3,3}$ we have $w = 6$, $E = 9$, $N = 6$ and degrees 3^6 and

$$77 - 17.6 + 2.9 + 4.6 - 6.\binom{3}{2} = -1.$$

Similarly for $K_{4,4} - 3K_2$ we have $w = 8$, $E = 13$, $N = 15$, degrees $4^2 3^6$, and

$$77 - 136 + 26 + 60 - 12 - 18 = -3. \quad \square$$

Fix a vertex of G and call it ∞; call its 16 neighbours points, and its 60 nonneighbours blocks. Identify a block with the set of four adjacent points. The first part of (i) says that two nonadjacent blocks intersect in at most two points. Since $\lambda = 0$, adjacent blocks are disjoint. The blocks form a 2-design 2-(16,4,3) with parameters $\lambda_2 = 3$, $\lambda_1 = 15$, $\lambda_0 = 60$. The derived design on a fixed point x is a 1- (15,3,3) design, and the second part of (i) says that this is the generalized quadrangle GQ(2,2). {As follows: the points of GQ are the points distinct from x; the lines of GQ are the $B\backslash\{x\}$ where B is a block containing x. Two lines intersect in at most one point, and if y is a point, L a line and $y \notin L$ then if there are two lines L_1, L_2 through y intersecting L in z_1, z_2 respectively, then we find a $K_{4,4} - 3K_2$ on the vertices ∞, L, L_1, L_2; x, y, z_2, z_1. Consequently the twelve lines non containing y intersecting the three lines containing y are all distinct, and we do have a generalized quadrangle.}

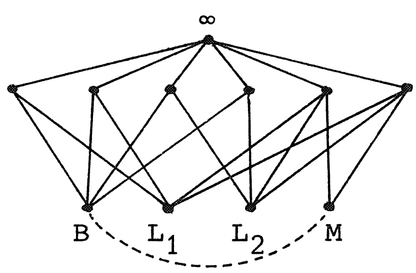
Now GQ(2,2) is unique and self-dual. One convenient representation for it is the following: let the points be the pairs from a fixed 6-set, and the lines the partitions of this 6-set into three pairs, with obvious incidence. From this representation one sees that there are six ovals (sets of five points, no two collinear) namely the six sets of pairs containing a given symbol. Any point is in two ovals and any two nonadjacent points determine a unique oval. By duality there are six spreads (parallel classes, sets of five lines partitioning the point set); any line is in two spreads and any two disjoint lines determine a unique spread.

The idea of the proof is to view the 2-(16,4,3) as a fragment of the 3-(22,6,1) design and to find the missing points and blocks. At this moment we can define the missing points. Fix a point x_0. The generalized quadrangle determined by x_0 contains 6 spreads. Let these be our new points. Call a line of the quadrangle incident with a spread if it is in it. This takes care of 15 of the blocks. We now have to find out how the remaining 45 blocks are related to the new points.

Let us first see what such a block looks like in the generalized quadrangle GQ.

(ii) There is a 1-1 correspondence between blocks B not containing x_0 and flags (y,M) in the GQ such that M is the unique line disjoint and nonadjacent to B and B is the symmetric difference of the lines L_1 and L_2 passing through y and distinct from M.

PROOF. A block B not containing x_0 is a 4-subset of GQ. Counting in the 2-(16,4,3) design we find that 12 blocks intersect B in a pair and 32 blocks intersect B in a single point so that 15 blocks are disjoint from B. On the other hand, B is adjacent to 12 blocks, so there are 3 blocks disjoint and nonadjacent to B. If two of these blocks had a point of intersection x, then in the corresponding GQ we have at least two lines disjoint from the 4-set B, and nonadjacent to B. Also, x and B have $\mu = 4$ common neighbours, i.e., in this GQ there are exactly four lines adjacent to (and hence disjoint from) B. This leaves at most 15-2-4 = 9 lines to intersect B, so that there are at least three lines L_i intersecting B in two points. The three pairs $L_i \cap B$ cannot be pairwise disjoint; suppose $L_1 \cap L_2 \cap B \neq \emptyset$, $L_1 \cap B = \{a,b\}$, $L_2 \cap B = \{a,c\}$. Then the second part of (i) applied to the graph induced by ∞, B, L_1, L_2; a,x,c,b yields a contradiction.
Consequently the three blocks disjoint and nonadjacent to B from a partition of the remaining 12 points, and exactly one is a line in the GQ determined by x_0. Reexamining the above argument we now see that exactly 10 lines intersect B, 8 in a single point, and two, L_1 and L_2 say, in a pair. The two pairs $B \cap L_1$ and $B \cap L_2$ are disjoint. The lines L_1 and L_2 cannot be disjoint, otherwise one of the three lines intersecting both L_1 and L_2 would contain two ponts of B. Hence L_1 and L_2 intersect in a point y. Let M be the third line through y.



All we have to show is that $B \not\sim M$. Consider the subgraph induced by ∞, the four points of B, x_0, y, B, L_1, L_2 and M. If $B \sim M$ then we find (using the notation of the proof of (i))

$$w = 11,\ E = 21,\ N = 34,\ \text{degrees } 6^1 5^1 4^4 3^5$$

so that

$$\sum x_i = 66,\ \sum i x_i = 134,\ \sum \binom{i}{2} x_i = 136-15-10-24-15 = 72,$$

hence

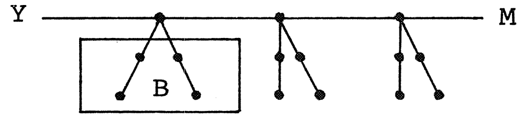
$$\sum \tfrac{1}{2}(i-2)(i-3)x_i = 3x_0+x_1+x_4+3x_5+\ \ldots = 2.$$

Consequently $x_1 \le 2$. But ∞ has 16 neighbours, 10 outside this subgraph and only two are adjacent to M, so $x_1 \ge 8$. Contradiction.
This proves that $B \not\sim M$. [And now there is a unique solution $x_0 = 0$, $x_1 = 8$, $x_2 = 46$, $x_3 = 12$.] (That the correspondence is 1-1 follows since there are 45 flags (y,M) in GQ.) □

Having found this unique line M corresponding to B, we make B adjacent to the same two new points M is adjacent to.

(iii) Considering the blocks as 6-sets (with 4 old and 2 new points each), we have $B \cap B' \in \{0,2\}$ and $B \cap B' = \emptyset$ iff $B \sim B'$ for all pairs B, B' of distinct blocks.

PROOF. This is clear if both B and B' are lines from GQ.
Suppose $x_0 \notin B$, $x_0 \in B'$. Let B correspond to (y,M). If $B' = M$ then the statement is clear. If $B' = L_i$ (i = 1 or 2) also. If B' intersects B in a single point then B' and M are in a unique spread so that B' and B share a unique new point.
Finally if B' is disjoint from B and different from M then B' and B have no new points in common (since B' is one of the four lines intersecting M in a point other than y) and $B' \sim B$.



Now let $x_0 \notin B$, $x_0 \notin B'$. Let B correspond to (y,M) and B' to (z,N). If B and B' have two new points in common (but $B \neq B'$) then $M = N$, $y \neq z$ and $B \cap B' = \emptyset$, and we have to show that $B \not\sim B'$. But if $M = \{x_0,y,z,u\}$ and L_5,L_6 are the lines through u different from M then L_5 and L_6 are common neighbours of B and B' so that $B \not\sim B'$. If B and B' have one new point in common then we have to show that they also have one old point in common. But "one new point in common" is equivalent to "M and N in a common parallel class", i.e., to M and N disjoint lines. Now y has a unique neighbour on N, say $N \cap L_1 = \{x,v\}$. If $v = z$ then $B \cap B' = L_1\backslash\{y,z\}$, a singleton. If $v \neq z$ then z has a neighbour $w \neq y$ on L_2 and $B \cap B' = \{w\}$.
Finally if B and B' do not have a new point in common, then M and N intersect in a point p. If $y = z = p$ and L is the third line through p then $B \cap B' = L\backslash\{p\}$. If $y \neq p$, $z \neq p$ then $|B \cap B'| = 2$ by the properties of a GQ. If $y \neq p$, $z = p$ or $y = p$, $z \neq p$ then $|B \cap B'| = 0$. In this case in fact $B \sim B'$: B has 16 neighbours, 4 are points, 4 are lines and the remaining 8 must be found in this case. But there are exactly 8 possibilities for B' here. □

All that is missing now are some new blocks that should cover the triples which are not covered yet. But these are uniquely determined: all triples containing both old and new points have been covered already (this follows by counting). Add a new block Ω consisting of the six new points. A 6-set through a point x containing only uncovered triples is an oval in the GQ corresponding to x, and there are exactly 6 of them. This shows that there are $16.6/6 = 16$ 6-sets containing only uncovered triples. Take these also as new blocks (they will be neighbours of Ω). This completes the construction of the S(3,6,22) design. Letting disjoint blocks be adjacent we find the Higman-Sims graph, and our original graph G consists of all non-neighbours of the vertex Ω. Since the Higman-Sims graph has a transitive group of automorphisms (this follows from the proof of the proposition) this shows that G is isomorphic to the graph on the non-neighbours of ∞, i.e., the complement of the block graph of $S(3,6,22)$. □

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