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THE POINCARÉ SERIES OF THE POLYNOMIALS
INVARIANT UNDER SU_2 IN ITS IRREDUCIBLE
REPRESENTATION OF DEGREE ≤ 17

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The Poincaré series of the polynomials invariant under SU_2 in its irreducible representation of degree ≤ 17

by

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ABSTRACT

Recently T.A. Springer established a formula for the Poincaré series of the invariant polynomials of the compact Lie group SU_2 corresponding to its irreducible representation of given degree $d+1$. The formula expresses the Poincaré series as a rational function that is presented as a sum of several other straightforwardly computable rational functions. By use of a computer these functions have been determined for all $d \leq 16$.

INTRODUCTION

Springer's formula (cf. [3]) for the Poincaré series $P_d(z)$ of the SU_2-invariant polynomials associated with its $(d+1)$-dimensional irreducible representation reads as follows.

$$P_d(z) = \sum_{0 \le j < \frac{d}{2}} (-1)^j \phi_{d-2j}\left(\frac{(1-z^2)z^{j(j+1)}}{(j,z^2)!(d-j,z^2)!}\right) \quad,$$

where for $n \in \mathbb{N}$ the operator ϕ_n transforms a rational function f in z to the rational function $\phi_n f$ determined by

$$(\phi_n f)(z^n) = \frac{1}{n}\sum_{j=1}^{n} f(e^{2\pi ij/n}z),$$

and where $(n,z)!$ is the polynomial function given by $(n,z)! = (1-z)(1-z^2)\ldots$
$\ldots(1-z^n)$.

About the Calculation

Clearly, if $f = \Sigma_{j=0}^N f_j z^j$ is a polynomial function in z, then we have

$$(1) \qquad (\phi_n f) = \sum_{j=0}^{\lfloor N/n \rfloor} f_{jn} z^j,$$

where $\lfloor N/n \rfloor$ is the greatest integer $\le N/n$.

Suppose that fg is the product of two rational functions one of which, say g, is a function in z^n, in other words there exists a function h in z such that $h(z^n) = g(z)$. Then

$$(2) \qquad \phi_n(fg) = h\phi_n(f)$$

As the denominator of the ϕ_{d-2j}-argument in a summand of $P_d(z)$ is a product of factors of the form $1 - z^m$, we may, after suitable multiplication of denominator and numerator by common factors, assume that the denominator is a product of factors having the form $1-z^{(d-2j)m}$. Application of (2) reduces the summand in $P_d(z)$ to the quotient of the ϕ_{d-2j}-image of a polynomial by

a product of terms of the form $1-z^m$.

Using (1), the numerator is readily computed, whereafter we have obtained a summand that is expressed as a rational function with integral coefficients. Addition of these summands (while keeping track of the signs $(-1)^j$ that appear in the formula) establishes the results.

RESULTS. The $P_d(z)$ for $d \leq 16$ are listed in the table below in a so-called representative form (cf. [4]). The interested reader may find the individual summands in the appendix.

TABLE

The Poincaré series of the invariants under SU_2 in its irreducible representation of degree $d+1 \leq 17$.

$$d = 2 \qquad \frac{1}{1-z^2}$$

$$d = 3 \qquad \frac{1}{1-z^4}$$

$$d = 4 \qquad \frac{1}{(1-z^2)(1-z^3)}$$

$$d = 5 \qquad \frac{1+z^{18}}{(1-z^4)(1-z^8)(1-z^{12})}$$

$$d = 6 \qquad \frac{1+z^{15}}{(1-z^2)(1-z^4)(1-z^6)(1-z^{10})}$$

$$d = 7 \qquad \frac{1+2z^8+4z^{12}+4z^{14}+5z^{16}+9z^{18}+6z^{20}+9z^{22}+8z^{24}+9z^{26}+6z^{28}+9z^{30}+5z^{32}+4z^{34}+4z^{36}+2z^{40}+z^{48}}{(1-z^4)(1-z^8)(1-z^{12})(1-z^{12})(1-z^{20})}$$

$$d = 8 \qquad \frac{1+z^8+z^9+z^{10}+z^{18}}{(1-z^2)(1-z^3)(1-z^4)(1-z^5)(1-z^6)(1-z^7)}$$

d = 9

$$\frac{\begin{aligned}&1+z^4+5z^8+4z^{10}+17z^{12}+20z^{14}+47z^{16}+61z^{18}+97z^{20}+120z^{22}+\\&165z^{24}+189z^{26}+223z^{28}+241z^{30}+254z^{32}+254z^{34}+241z^{36}+\\&223z^{38}+189z^{40}+165z^{42}+120z^{44}+97z^{46}+61z^{48}+47z^{50}+20z^{52}+\\&17z^{54}+4z^{56}+5z^{58}+z^{62}+z^{66}\end{aligned}}{(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})}$$

d = 10

$$\frac{\begin{aligned}&1+2z^6+4z^8+4z^9+7z^{10}+8z^{11}+15z^{12}+15z^{13}+20z^{14}+27z^{15}+29z^{16}+\\&35z^{17}+40z^{18}+44z^{19}+47z^{20}+55z^{21}+52z^{22}+57z^{23}+56z^{24}+57z^{25}+\\&52z^{26}+55z^{27}+47z^{28}+44z^{29}+40z^{30}+35z^{31}+29z^{32}+27z^{33}+20z^{34}+\\&15z^{35}+15z^{36}+8z^{37}+7z^{38}+4z^{39}+4z^{40}+2z^{42}+z^{48}\end{aligned}}{(1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^8)(1-z^9)(1-z^{10})(1-z^{14})}$$

d = 11

$$\frac{\begin{aligned}&1+z^4+10z^8+12z^{10}+57z^{12}+95z^{14}+233z^{16}+394z^{18}+745z^{20}+\\&1173z^{22}+1925z^{24}+2814z^{26}+4139z^{28}+5669z^{30}+7664z^{32}+\\&9821z^{34}+12390z^{36}+14953z^{38}+17692z^{40}+20204z^{42}+22537z^{44}+\\&24378z^{46}+25746z^{48}+26413z^{50}+26413z^{52}+25746z^{54}+24378z^{56}+\\&22537z^{58}+20204z^{60}+17692z^{62}+14953z^{64}+12390z^{66}+9821z^{68}+\\&7664z^{70}+5669z^{72}+4139z^{74}+2814z^{76}+1925z^{78}+1173z^{80}+745z^{82}+\\&394z^{84}+233z^{86}+95z^{88}+57z^{90}+12z^{92}+10z^{94}+z^{98}+z^{102}\end{aligned}}{\begin{aligned}&(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18})\\&(1-z^{20})\end{aligned}}$$

d = 12

$$\frac{\begin{aligned}&1+z^3+z^4+3z^6+5z^7+7z^8+11z^9+21z^{10}+25z^{11}+40z^{12}+55z^{13}+75z^{14}+\\&96z^{15}+130z^{16}+156z^{17}+194z^{18}+228z^{19}+267z^{20}+294z^{21}+330z^{22}+\\&345z^{23}+364z^{24}+365z^{25}+364z^{26}+345z^{27}+330z^{28}+294z^{29}+\\&267z^{30}+228z^{31}+194z^{32}+156z^{33}+130z^{34}+96z^{35}+75z^{36}+55z^{37}+\\&40z^{38}+25z^{39}+21z^{40}+11z^{41}+7z^{42}+5z^{43}+3z^{44}+z^{46}+z^{47}+z^{50}\end{aligned}}{\begin{aligned}&(1-z^2)(1-z^4)(1-z^5)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9)\\&(1-z^{11})\end{aligned}}$$

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$d = 13$

$$1+z^4+19z^8+32z^{10}+156z^{12}+340z^{14}+916z^{16}+1913z^{18}+4115z^{20}+$$
$$7741z^{22}+14544z^{24}+25118z^{26}+42489z^{28}+68114z^{30}+106498z^{32}+$$
$$159697z^{34}+233971z^{36}+331412z^{38}+458878z^{40}+617966z^{42}+$$
$$814543z^{44}+1047676z^{46}+1321155z^{48}+1629760z^{50}+1972851z^{52}+$$
$$2341055z^{54}+2728439z^{56}+3120893z^{58}+3509075z^{60}+3876056z^{62}+$$
$$4210210z^{64}+4496206z^{66}+4723183z^{68}+4879836z^{70}+4960463z^{72}+$$
$$4960463z^{74}+4879836z^{76}+4723183z^{78}+4496206z^{80}+4210210z^{82}+$$
$$3876056z^{84}+3509075z^{86}+3120893z^{88}+2728439z^{90}+2341055z^{92}+$$
$$1972851z^{94}+1629760z^{96}+1321155z^{98}+1047676z^{100}+814543z^{102}+$$
$$617966z^{104}+458878z^{106}+331412z^{108}+233971z^{110}+159697z^{112}+$$
$$106498z^{114}+68114z^{116}+42489z^{118}+25118z^{120}+14544z^{122}+$$
$$7741z^{124}+4115z^{126}+1913z^{128}+916z^{130}+340z^{132}+156z^{134}+$$
$$32z^{136}+19z^{138}+z^{142}+z^{146}$$

$$\overline{(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18})}$$
$$(1-z^{20})(1-z^{22})(1-z^{24})$$

$d = 14$

$$1+z^4+5z^6+3z^7+18z^8+22z^9+55z^{10}+77z^{11}+158z^{12}+227z^{13}+$$
$$396z^{14}+579z^{15}+908z^{16}+1286z^{17}+1904z^{18}+2609z^{19}+3655z^{20}+$$
$$4865z^{21}+6502z^{22}+8386z^{23}+10800z^{24}+13480z^{25}+16765z^{26}+$$
$$20342z^{27}+24480z^{28}+28872z^{29}+33748z^{30}+38709z^{31}+43994z^{32}+$$
$$49174z^{33}+54374z^{34}+59258z^{35}+63866z^{36}+67854z^{37}+71317z^{38}+$$
$$73958z^{39}+75790z^{40}+76724z^{41}+76724z^{42}+75790z^{43}+73958z^{44}+$$
$$71317z^{45}+67854z^{46}+63866z^{47}+59258z^{48}+54374z^{49}+49174z^{50}+$$
$$43994z^{51}+38709z^{52}+33748z^{53}+28872z^{54}+24480z^{55}+20342z^{56}+$$
$$16765z^{57}+13480z^{58}+10800z^{59}+8386z^{60}+6502z^{61}+4865z^{62}+$$
$$3655z^{63}+2609z^{64}+1904z^{65}+1286z^{66}+908z^{67}+579z^{68}+396z^{69}+$$
$$227z^{70}+158z^{71}+77z^{72}+55z^{73}+22z^{74}+18z^{75}+3z^{76}+5z^{77}+z^{79}+$$
$$z^{83}$$

$$\overline{(1-z^2)(1-z^4)(1-z^6)(1-z^6)(1-z^7)(1-z^8)(1-z^9)(1-z^{10})}$$
$$(1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})$$

$d = 15$

$$1+2z^4+z^6+32z^8+78z^{10}+378z^{12}+1027z^{14}+3124z^{16}+7672z^{18}+$$
$$18676z^{20}+40784z^{22}+86197z^{24}+170002z^{26}+323704z^{28}+$$
$$586665z^{30}+1029080z^{32}+1737391z^{34}+2849501z^{36}+4530000z^{38}+$$
$$7019920z^{40}+10596388z^{42}+15636688z^{44}+22556344z^{46}+$$
$$31885923z^{48}+44179876z^{50}+60104681z^{52}+80312694z^{54}+$$
$$105538484z^{56}+136437227z^{58}+173688963z^{60}+217804048z^{62}+$$
$$269239746z^{64}+328183158z^{66}+394684584z^{68}+468435861z^{70}+$$
$$548927416z^{72}+635239454z^{74}+726230157z^{76}+820357306z^{78}+$$
$$915894836z^{80}+1010799973z^{82}+1102948881z^{84}+1190050285z^{86}+$$
$$1269884526z^{88}+1340258093z^{90}+1399218002z^{92}+1445037255z^{94}+$$
$$1476376889z^{96}+1492284743z^{98}+1492284743z^{100}+$$
$$1476376889z^{102}+1445037255z^{104}+1399218002z^{106}+$$
$$1340258093z^{108}+1269884526z^{110}+1190050285z^{112}+$$
$$1102948881z^{114}+1010799973z^{116}+915894836z^{118}+$$
$$820357306z^{120}+726230157z^{122}+635239454z^{124}+548927416z^{126}+$$
$$468435861z^{128}+394684584z^{130}+328183158z^{132}+269239746z^{134}+$$
$$217804048z^{136}+173688963z^{138}+136437227z^{140}+105538484z^{142}+$$
$$80312694z^{144}+60104681z^{146}+44179876z^{148}+31885923z^{150}+$$
$$22556344z^{152}+15636688z^{154}+10596388z^{156}+7019920z^{158}+$$
$$4530000z^{160}+2849501z^{162}+1737391z^{164}+1029080z^{166}+$$
$$586665z^{168}+323704z^{170}+170002z^{172}+86197z^{174}+40784z^{176}+$$
$$18676z^{178}+7672z^{180}+3124z^{182}+1027z^{184}+378z^{186}+78z^{188}+$$
$$32z^{190}+z^{192}+2z^{194}+z^{198}$$

$$(1-z^4)(1-z^8)(1-z^{10})(1-z^{12})(1-z^{12})(1-z^{14})(1-z^{16})(1-z^{18})$$
$$(1-z^{20})(1-z^{22})(1-z^{24})(1-z^{26})(1-z^{28})$$

$$d = 16 \quad \frac{\begin{matrix} 1+z^3+z^4+2z^5+8z^6+11z^7+30z^8+59z^9+112z^{10}+204z^{11}+389z^{12}+ \\ 655z^{13}+1146z^{14}+1896z^{15}+3067z^{16}+4823z^{17}+7477z^{18}+11167z^{19}+ \\ 16506z^{20}+23764z^{21}+33593z^{22}+46576z^{23}+63588z^{24}+85049z^{25}+ \\ 112273z^{26}+145668z^{27}+186298z^{28}+234773z^{29}+292016z^{30}+ \\ 357883z^{31}+433493z^{32}+518058z^{33}+611720z^{34}+713572z^{35}+ \\ 822997z^{36}+937742z^{37}+1057289z^{38}+1178403z^{39}+1299472z^{40}+ \\ 1417560z^{41}+1530508z^{42}+1634576z^{43}+1728646z^{44}+1808808z^{45}+ \\ 1873856z^{46}+1921520z^{47}+1951002z^{48}+1960390z^{49}+1951002z^{50}+ \\ 1921520z^{51}+1873856z^{52}+1808808z^{53}+1728646z^{54}+1634576z^{55}+ \\ 1530508z^{56}+1417560z^{57}+1299472z^{58}+1178403z^{59}+1057289z^{60}+ \\ 937742z^{61}+822997z^{62}+713572z^{63}+611720z^{64}+ \\ 518058z^{65}+433493z^{66}+357883z^{67}+292016z^{68}+234773z^{69}+ \\ 186298z^{70}+145668z^{71}+112273z^{72}+85049z^{73}+63588z^{74}+ \\ 46576z^{75}+33593z^{76}+23764z^{77}+16506z^{78}+11167z^{79}+7477z^{80}+ \\ 4823z^{81}+3067z^{82}+1896z^{83}+1146z^{84}+655z^{85}+389z^{86}+204z^{87}+ \\ 112z^{88}+59z^{89}+30z^{90}+11z^{91}+8z^{92}+2z^{93}+z^{94}+z^{95}+z^{98} \end{matrix}}{\begin{matrix} (1-z^2)(1-z^4)(1-z^5)(1-z^6)(1-z^6)(1-z^7)(1-z^7)(1-z^8)(1-z^9) \\ (1-z^{10})(1-z^{11})(1-z^{12})(1-z^{13})(1-z^{15}) \end{matrix}}$$

REMARKS.

(i) The series $P_d(z)$ for $d \le 10$ and $d = 12$ appear in [4]. They agree with the contents of the table. Results for $d \le 12$ are also available in [1,2], but we have not compared them with ours. The expressions of $P_d(z)$ for $13 \le d \le 16$ seem to be new.

(ii) The expression given for $P_d(z)$ in the appendix has the property that the numerator and denominator are not simultaneously divisible by a polynomial of the form $(1-z^m)/(1-z^\ell)$ (ℓ divisor of m) for m such that $1-z^m$ occurs as an explicitly given factor in the denominator. The representative form however is easily obtained from the given one.

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```
D=2;     J=0
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z2)

*****THE POINCARE SERIES FOR D=2*****:
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z2)



D=3;     J=0
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z4)(1-Z2)

D=3;     J=1
NUMERATOR:
+1Z2
DENOMINATOR:
(1-Z2)(1-Z4)

*****THE POINCARE SERIES FOR D=3*****:
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z4)



D=4;     J=0
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z1)(1-Z3)(1-Z2)

D=4;     J=1
NUMERATOR:
+1Z1
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)

*****THE POINCARE SERIES FOR D=4*****:
NUMERATOR:
+1Z0
DENOMINATOR:
(1-Z2)(1-Z3)



D=5;     J=0
NUMERATOR:
+1Z0+1Z2+4Z4+5Z6+7Z8+4Z10+3Z12
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z2)
```

D=5;   J=1
NUMERATOR:
$+2Z2+2Z4+3Z6+1Z8+1Z10$
DENOMINATOR:
$(1-Z2)(1-Z4)(1-Z2)(1-Z8)$

D=5;   J=2
NUMERATOR:
$+1Z6$
DENOMINATOR:
$(1-Z2)(1-Z4)(1-Z4)(1-Z6)$

*****THE POINCARE SERIES FOR D=5*****:
NUMERATOR:
$+1Z0-1Z6+1Z12$
DENOMINATOR:
$(1-Z4)(1-Z6)(1-Z8)$

D=6;   J=0
NUMERATOR:
$+1Z0+1Z2+2Z3+2Z4+1Z5+2Z6$
DENOMINATOR:
$(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)$

D=6;   J=1
NUMERATOR:
$+1Z1+1Z2+1Z3+1Z5$
DENOMINATOR:
$(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z5)$

D=6;   J=2
NUMERATOR:
$+1Z3$
DENOMINATOR:
$(1-Z1)(1-Z2)(1-Z2)(1-Z3)(1-Z4)$

*****THE POINCARE SERIES FOR D=6*****:
NUMERATOR:
$+1Z0-1Z2-1Z3+1Z6+1Z7-1Z9$
DENOMINATOR:
$(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z5)$

D=7;   J=0
NUMERATOR:
$+1Z0+3Z2+18Z4+45Z6+97Z8+156Z10+236Z12+288Z14+335Z16+329Z18+304Z20+235Z22+174Z24+100Z26+54Z28+20Z30+6Z32$
DENOMINATOR:
$(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)$

D=7;   J=1
NUMERATOR:
$+5Z2+16Z4+38Z6+60Z8+88Z10+93Z12+103Z14+85Z16+69Z18+41Z20+21Z22+5Z24+1Z26$
DENOMINATOR:
$(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)$

D=7;   J=2
NUMERATOR:

+1Z2+2Z4+9Z6+11Z8+15Z10+15Z12+14Z14+7Z16+6Z18+1Z20
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z2)(1-Z8)(1-Z10)

D=7;   J=3
NUMERATOR:
+1Z12
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z4)(1-Z6)(1-Z8)

*****THE POINCARE SERIES FOR D=7*****:
NUMERATOR:
+1Z0-1Z6+2Z8-1Z10+5Z12+2Z14+6Z16+2Z18+5Z20-1Z22+2Z24-1Z26+1Z32
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)

D=8;   J=0
NUMERATOR:
+1Z0+3Z2+4Z3+7Z4+7Z5+9Z6+8Z7+9Z8+6Z9+5Z10+3Z11+2Z12
DENOMINATOR:
(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)

D=8;   J=1
NUMERATOR:
+2Z1+3Z2+5Z3+8Z4+10Z5+10Z6+12Z7+10Z8+8Z9+7Z10+4Z11+1Z12+1Z13
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)

D=8;   J=2
NUMERATOR:
+1Z2+1Z3+1Z4+1Z6
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)

D=8;   J=3
NUMERATOR:
+1Z6
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z2)(1-Z3)(1-Z4)(1-Z5)

*****THE POINCARE SERIES FOR D=8*****:
NUMERATOR:
+1Z0-1Z2-1Z3+1Z5+1Z6-1Z11-1Z12+1Z14+1Z15-1Z17
DENOMINATOR:
(1-Z2)(1-Z2)(1-Z3)(1-Z3)(1-Z4)(1-Z5)(1-Z7)

D=9;   J=0
NUMERATOR:
+1Z0+6Z2+49Z4+160Z6+414Z8+818Z10+1451Z12+2223Z14+3164Z16+4099Z18+5021Z20+5680Z22+6105Z24+6092Z26+5773Z28+5077Z30+4235Z32+3244Z34+2341Z36
+1513Z38+893Z40+438Z42+188Z44+53Z46+11Z48
DENOMINATOR:
(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z2)

D=9;   J=1
NUMERATOR:
+11Z2+67Z4+256Z6+670Z8+1452Z10+2629Z12+4268Z14+6160Z16+8247Z18+10101Z20+11636Z22+12411Z24+12531Z26+11742Z28+10395Z30+8488Z32+6490Z34+449

7Z36+2857Z38+1581Z40+766Z42+294Z44+86Z46+13Z48+1Z50
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)(1-Z16)

D=9;   J=2
NUMERATOR:
+3Z2+20Z4+88Z6+214Z8+447Z10+729Z12+1089Z14+1417Z16+1726Z18+1882Z20+1917Z22+1754Z24+1499Z26+1143Z28+812Z30+486Z32+262Z34+101Z36+32Z38+4Z4
0
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)(1-Z14)

D=9;   J=3
NUMERATOR:
+1Z4+2Z6+9Z8+11Z10+15Z12+15Z14+14Z16+7Z18+6Z20+1Z22
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)

D=9;   J=4
NUMERATOR:
+1Z20
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z4)(1-Z6)(1-Z8)(1-Z10)

*****THE POINCARE SERIES FOR D=9*****:
NUMERATOR:
+1Z0+1Z4-1Z6+5Z8+3Z10+18Z12+15Z14+44Z16+43Z18+82Z20+76Z22+122Z24+107Z26+147Z28+119Z30+147Z32+107Z34+122Z36+76Z38+82Z40+43Z42+44Z44+15Z46
+18Z48+3Z50+5Z52-1Z54+1Z56+1Z60
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)

D=10;   J=0
NUMERATOR:
+1Z0+1Z1+8Z2+21Z3+50Z4+88Z5+161Z6+244Z7+372Z8+510Z9+678Z10+835Z11+1012Z12+1137Z13+1256Z14+1304Z15+1320Z16+1257Z17+1178Z18+1028Z19+881Z20
+704Z21+546Z22+390Z23+277Z24+169Z25+104Z26+53Z27+27Z28+9Z29+4Z30
DENOMINATOR:
(1-Z2)(1-Z3)(1-Z4)(1-Z1)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z2)

D=10;   J=1
NUMERATOR:
+3Z1+6Z2+15Z3+25Z4+39Z5+55Z6+72Z7+83Z8+98Z9+103Z10+104Z11+98Z12+90Z13+73Z14+60Z15+42Z16+29Z17+17Z18+9Z19+2Z20+1Z21
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)(1-Z9)

D=10;   J=2
NUMERATOR:
+1Z1+2Z2+9Z3+15Z4+26Z5+35Z6+52Z7+60Z8+74Z9+77Z10+80Z11+72Z12+69Z13+52Z14+42Z15+29Z16+19Z17+8Z18+5Z19+1Z20
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)(1-Z8)

D=10;   J=3
NUMERATOR:
+1Z3+2Z5+2Z6+3Z7+2Z8+2Z9+1Z10+2Z11+1Z12
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)(1-Z7)

D=10;   J=4
NUMERATOR:
+1Z10

DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)

*****THE POINCARE SERIES FOR D=10*****:
NUMERATOR:
+1Z0-1Z2-1Z3+3Z6+2Z8+2Z9+4Z10+5Z12-1Z13+1Z14+3Z16-4Z17+4Z18-3Z19-1Z21+1Z22-6Z23-4Z25-2Z26-2Z27-3Z29+1Z32+1Z33-1Z35
DENOMINATOR:
(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)

D=11;   J=0
NUMERATOR:
+1Z0+1Z2+15Z4+86Z6+351Z8+1110Z10+2973Z12+69185Z14+145026Z16+277592Z18+493864Z20+822988Z22+1297562Z24+1944356Z26+2786068Z28+3828074
Z30+5063386Z32+6458374Z34+7964750Z36+9506816Z38+11002659Z40+12353603Z42+13472597Z44+14273622Z46+14702066Z48+14719445Z50+14330077Z52+1355
8279Z54+12467535Z56+11131643Z58+9647262Z60+8103600Z62+6592623Z64+5183387Z66+3933427Z68+2871889Z70+2013147Z72+1348408Z74+860020Z76+518372
Z78+293571Z80+154219Z82+74416Z84+32197Z86+12274Z88+3912Z90+1004Z92+177Z94+20Z96
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z2)

D=11;   J=1
NUMERATOR:
+2Z2+201Z4+982Z6+3164Z8+8046Z10+17042Z12+31863Z14+53525Z16+83060Z18+119965Z20+163632Z22+211257Z24+260380Z26+306277Z28+345814Z30+374338Z
32+390158Z34+390706Z36+377030Z38+349333Z40+311295Z42+265496Z44+216902Z46+168610Z48+124653Z50+86697Z52+56499Z54+33846Z56+18424Z58+8780Z60
+3566Z62+1127Z64+251Z66+27Z68+1Z70
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z2)(1-Z20)

D=11;   J=2
NUMERATOR:
+8Z2+94Z4+569Z6+2102Z8+6071Z10+14204Z12+28962Z14+52249Z16+86119Z18+130619Z20+185535Z22+247579Z24+313406Z26+376697Z28+432357Z30+473515Z32
+496598Z34+497895Z36+478268Z38+438986Z40+385509Z42+322512Z44+257075Z46+193851Z48+138040Z50+91681Z52+56530Z54+31633Z56+15913Z58+6887Z60+2
508Z62+685Z64+133Z66+11Z68
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)(1-Z16)(1-Z18)

D=11;   J=3
NUMERATOR:
+9Z4+61Z6+262Z8+770Z10+1863Z12+3709Z14+6596Z16+10459Z18+15366Z20+20812Z22+26538Z24+31595Z26+35635Z28+37787Z30+38081Z32+36148Z34+32587Z36
+27549Z38+22018Z40+16373Z42+11408Z44+7254Z46+4228Z48+2143Z50+946Z52+329Z54+87Z56+12Z58+1Z60
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)(1-Z14)(1-Z16)

D=11;   J=4
NUMERATOR:
+3Z8+6Z10+19Z12+29Z14+50Z16+67Z18+84Z20+99Z22+97Z24+84Z26+74Z28+54Z30+38Z32+20Z34+12Z36+2Z38+1Z40
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)

D=11;   J=5
NUMERATOR:
+1Z30
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)

*****THE POINCARE SERIES FOR D=11*****:
NUMERATOR:
+1Z0+1Z4-1Z6+1Z8+11Z10+58Z12+85Z14+222Z16+336Z18+660Z20+951Z22+1589Z24+2154Z26+3188Z28+4080Z30+5510Z32+6633Z34+8310Z36+9443Z38+11059Z40
+11894Z42+13094Z44+13319Z46+13852Z48+13319Z50+13094Z52+11894Z54+11059Z56+9443Z58+8310Z60+6633Z62+5510Z64+4080Z66+3188Z68+2154Z70+1589Z72
+951Z74+660Z76+336Z78+222Z80+85Z82+58Z84+11Z86+10Z88-1Z90+1Z92+1Z96

DENOMINATOR:
(1-Z4)(1-Z6)(1-78)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)

D=12;   J=0
NUMERATOR:
+1Z0+1Z1+10Z2+26Z3+55Z4+106Z5+181Z6+276Z7+393Z8+520Z9+647Z10+764Z11+865Z12+933Z13+972Z14+976Z15+943Z16+876Z17+784Z18+667Z19+541Z20+413Z2
1+296Z22+194Z23+117Z24+63Z25+29Z26+11Z27+4Z28
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z2)(1-Z5)(1-Z1)(1-Z7)(1-Z4)(1-Z3)(1-Z5)(1-Z11)(1-Z2)

D=12;   J=1
NUMERATOR:
+5Z1+20Z2+66Z3+168Z4+372Z5+717Z6+1279Z7+2106Z8+3268Z9+4792Z10+6716Z11+8991Z12+11595Z13+14397Z14+17288Z15+20080Z16+22633Z17+24729Z18+2626
8Z19+27094Z20+27179Z21+26482Z22+25099Z23+23082Z24+20624Z25+17854Z26+14979Z27+12135Z28+9498Z29+7136Z30+5149Z31+3540Z32+2314Z33+1417Z34+81
4Z35+424Z36+201Z37+81Z38+27Z39+5Z40+1Z41
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z1)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z2)(1-Z11)

D=12;   J=2
NUMERATOR:
+1Z1+7Z2+17Z3+37Z4+68Z5+112Z6+161Z7+221Z8+278Z9+332Z10+369Z11+396Z12+394Z13+382Z14+340Z15+293Z16+234Z17+179Z18+120Z19+80Z20+43Z21+22Z22+
7Z23+3Z24
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)(1-Z9)(1-Z5)

D=12;   J=3
NUMERATOR:
+1Z2+2Z3+9Z4+15Z5+26Z6+35Z7+52Z8+60Z9+74Z10+77Z11+80Z12+72Z13+69Z14+52Z15+42Z16+29Z17+19Z18+8Z19+6Z20+1Z21
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)(1-Z8)(1-Z3)

D=12;   J=4
NUMERATOR:
+1Z5+2Z7+2Z8+3Z9+2Z10+2Z11+1Z12+2Z13+1Z14
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)(1-Z7)(1-Z4)

D=12;   J=5
NUMERATOR:
+1Z15
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)

*****THE POINCARE SERIES FOR D=12*****:
NUMERATOR:
+1Z0-1Z2+1Z4+2Z6+4Z7+4Z8+4Z9+10Z10+10Z11+15Z12+20Z13+25Z14+26Z15+35Z16+35Z17+38Z18+37Z19+38Z20+28Z21+26Z22+13Z23+6Z24-6Z25-13Z26-26Z27-2
8Z28-38Z29-37Z30-38Z31-35Z32-35Z33-26Z34-25Z35-20Z36-15Z37-10Z38-10Z39-4Z40-4Z41-4Z42-2Z43-1Z45+1Z47-1Z49
DENOMINATOR:
(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z11)

D=13;   J=0
NUMERATOR:
+1Z0+23Z2+376Z4+2870Z6+15123Z8+60700Z10+201500Z12+574964Z14+1458029Z16+3350844Z18+7098549Z20+14019844Z22+26071711Z24+45976366Z26+7736784
5Z28+124821533Z30+193880853Z32+290870261Z34+422704305Z36+596397419Z38+818633011Z40+1094992432Z42+1429391191Z44+1823198363Z46+2274785102Z
48+2778812117Z50+3326200519Z52+3903870722Z54+4495366115Z56+5081193883Z58+5640128954Z60+6150055700Z62+6589717995Z64+6939720382Z66+7184275
874Z68+7311905549Z70+7316702593Z72+7198328171Z74+6962478228Z76+6620024603Z78+6186658269Z80+5681349032Z82+5125446905Z84+4540871714Z86+394

9131842Z88+3369733043Z90+2319539516Z92+2311771083Z94+1855943097Z96+1457563177Z98+1118631012Z100+837893344Z102+611697729Z104+434483358Z10
6+2996901817108+200259819Z110+129294239Z112+803799367Z114+479332207Z116+272816927118+147364887120+749667772122+3560403Z124+1559005Z126+6203
50Z128+219297Z130+670907Z132+16915Z134+3323Z136+440Z138+33Z140
DENOMINATOR:
(1-Z4)(1-Z6)(1-Z8)(1-710)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z2)

D=13;   J=1
NUMERATOR:
+4Z2+64Z4+484Z6+2404Z78+91109Z10+282918Z12+756462Z14+1794004Z16+3861266Z18+7659540Z20+14180466Z22+24722428Z24+40897719Z26+64559801Z28
+97718840Z30+142356964Z32+200254644Z34+272702924Z36+360324529Z38+462769288Z40+578640027Z42+705295968Z44+838996271Z46+974894735Z48+110744
9465Z50+1230600907Z52+1336384239Z54+1425186900Z56+1486402715Z58+1518598931Z60+1520028679Z62+1490553557Z64+1431884069Z66+1347206811Z68+12
41135432Z70+1119113130Z72+987183163Z74+851328821Z76+717234070Z78+589748950Z80+472786093Z82+369031893Z84+280044645Z86+206220207Z88+147052
452Z90+101270775Z92+67152619Z94+42708038Z96+25935191Z98+14949591Z100+8123262Z102+4121867Z104+1931020Z106+821760Z108+311266Z110+101665Z11
2+27437Z114+5667Z115+793Z118+54Z120+1Z122
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z2)(1-Z24)

D=13;   J=2
NUMERATOR:
+19Z2+323Z4+2450Z6+11080Z8+37352Z10+100760Z12+232740Z14+474244Z16+877302Z18+1496115Z20+2387579Z22+3595701Z24+5153833Z26+7064297Z28+93056
79Z30+11812566Z32+14493758Z34+17216780Z36+19838851Z38+22195506Z40+24139617Z42+25530689Z44+26275954Z46+26313847Z48+25648539Z50+24321968Z5
2+22436202Z54+20115136Z56+17517472Z58+14795406Z60+12105258Z52+9571389Z64+7298860Z66+5347939Z68+3752916Z70+2504759Z72+1581316Z74+933925Z7
6+510391Z78+253052Z80+111544Z82+41978Z84+12923Z86+2917Z88+431Z90+26Z92
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z2)(1-Z20)(1-Z22)

D=13;   J=3
NUMERATOR:
+1Z2+53Z4+499Z6+2788Z8+10959Z10+34202Z12+89031Z14+202328Z16+410514Z18+760221Z20+1301267Z22+2085019Z24+3150499Z26+4524096Z28+6200838Z30+8
150008Z32+102972221Z34+12542590Z36+14748164Z38+16771885Z40+18459458Z42+19686236Z44+20346716Z46+20393381Z48+19815348Z50+18667267Z52+170353
13Z54+15053430Z56+12862036Z58+10616110Z60+8445590Z62+6464802Z64+4744696Z66+3328278Z68+2218624Z70+1397724Z72+824281Z74+450860Z76+225129Z77
8+100988Z80+39537Z82+13098Z84+3436Z86+661Z88+74Z90+3Z92
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)(1-Z16)(1-Z18)(1-Z20)

D=13;   J=4
NUMERATOR:
+1Z4+13Z6+113Z8+500Z10+1726Z12+4698Z14+11102Z16+22854Z18+42814Z20+73244Z22+116948Z24+174808Z26+247443Z28+332087Z30+425609Z32+521008Z34+6
12010Z36+689659Z38+747365Z40+779577Z42+782912Z44+756152Z46+703365Z48+628585Z50+540083Z52+444497Z54+350481Z56+263309Z58+188354Z60+127225Z
62+80987Z64+47864Z66+26192Z68+12890Z70+5680Z72+2119Z74+665Z76+150Z78+25Z80+1Z82
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)(1-Z14)(1-Z16)(1-Z18)

D=13;   J=5
NUMERATOR:
+1Z10+2Z12+13Z14+28Z16+58Z18+101Z20+174Z22+245Z24+347Z26+438Z28+532Z30+591Z32+646Z34+638Z36+620Z38+552Z40+473Z42+369Z44+288Z46+191Z48+12
4Z50+71Z52+38Z54+13Z56+7Z58+1Z60
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)

D=13;   J=6
NUMERATOR:
+1Z42
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)

*****THE POINCARE SERIES FOR D=13*****:
NUMERATOR:
+1Z0-1Z2+2Z4-2Z6+2Z8+11Z10+145Z12+195Z14+721Z16+1192Z18+2923Z20+4818Z22+9726Z24+15392Z26+27097Z28+41017Z30+65481Z32+94216Z34+139755Z36+
191657Z38+267221Z40+350745Z42+463798Z44+583878Z46+737277Z48+892483Z50+1080368Z52+1260687Z54+1467752Z56+1653141Z58+1855934Z60+2020122Z62+

2190088Z64+2306118Z66+2417065Z68+2462771Z70+2497692Z72+2452771Z74+2417065Z76+2306118Z78+2190088Z80+2020122Z82+1855934Z84+1653141Z86+1467
752Z88+1260687Z90+1080368Z92+892483Z94+737277Z96+583878Z98+463798Z100+350745Z102+267221Z104+191657Z106+139755Z108+94216Z110+65481Z112+41
0177Z114+270977Z116+15392Z118+9726Z120+4818Z122+2923Z124+1192Z126+721Z128+195Z130+145Z132+11Z134+21Z136-2Z138+2Z140-1Z142+1Z144
DENOMINATOR:
(1-72)(1-Z8)(1-Z10)(1-Z12)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)

D=14;   J=0
NUMERATOR:
+1Z0+3Z1+28Z2+113Z3+390Z4+1078Z5+2717Z6+6047Z7+12570Z8+24150Z9+43995Z10+75811Z11+125325Z12+198563Z13+304184Z14+450521Z15+648585Z16+90783
8Z17+1240034Z18+1653185Z19+2156821Z20+2754202Z21+3448978Z22+4236090Z23+5110360Z24+6055854Z25+7057225Z26+8087789Z27+9123203Z28+1012901072
9+11076417Z30+11928639Z31+12658848Z32+13235243Z33+13639593Z34+13851565Z35+13867020Z36+13680851Z37+13305035Z38+12750170Z39+12042142Z40+11
203717Z41+10269685Z42+9268538Z43+8236946Z44+7202463Z45+6196750Z46+5240713Z47+4356569Z48+3555293Z49+2848037Z50+2235868Z51+1719873Z52+1293
490Z53+950952Z54+681331Z55+475649Z56+322161Z57+211652Z58+134005Z59+81788Z60+47603Z61+26467Z62+13806Z63+6793Z64+3034Z65+1258Z66+444Z67+14
1Z68+33Z69+7Z70
DENOMINATOR:
(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z1)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)(1-Z2)

D=14;   J=1
NUMERATOR:
+7Z1+28Z2+94Z3+243Z4+529Z5+1020Z6+1782Z7+2865Z8+4304Z9+6098Z10+8223Z11+10621Z12+13220Z13+15916Z14+18607Z15+21163Z16+23453Z17+25339Z18+26
703Z19+27438Z20+27497Z21+26869Z22+25608Z23+23798Z24+21567Z25+19045Z26+16370Z27+13663Z28+11044Z29+8605Z30+6431Z31+4578Z32+3081Z33+1937Z34
+1128Z35+595Z36+278Z37+112Z38+37Z39+7Z40+1Z41
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z2)(1-Z5)(1-Z1)(1-Z7)(1-Z4)(1-Z3)(1-Z5)(1-Z11)(1-Z2)(1-Z13)

D=14;   J=2
NUMERATOR:
+3Z1+20Z2+102Z3+312Z4+845Z5+1901Z6+3926Z7+7310Z8+12836Z9+21045Z10+32964Z11+49133Z12+70572Z13+97463Z14+130525Z15+169144Z16+213385Z17+2616
61Z18+313191Z19+365390Z20+416937Z21+464601Z22+506927Z23+540826Z24+565378Z25+578240Z26+579723Z27+568682Z28+546755Z29+514225Z30+473792Z31+
426634Z32+376034Z33+323460Z34+271919Z35+222605Z36+177718Z37+137675Z38+103694Z39+75410Z40+53054Z41+35765Z42+23168Z43+14186Z44+8261Z45+444
7Z46+2236Z47+989Z48+397Z49+122Z50+33Z51+4Z52
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z2)(1-Z3)(1-Z4)(1-Z1)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z2)(1-Z11)(1-Z12)

D=14;   J=3
NUMERATOR:
+3Z2+12Z3+37Z4+96Z5+194Z6+356Z7+593Z8+917Z9+1314Z10+1803Z11+2332Z12+2907Z13+3466Z14+3997Z15+4423Z15+4764Z17+4931Z18+4959Z19+4809Z20+4534
Z21+4102Z22+3612Z23+3042Z24+2480Z25+1924Z26+1438Z27+1001Z28+670Z29+407Z30+230Z31+111Z32+52Z33+15Z34+5Z35
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)(1-Z9)(1-Z5)(1-Z11)

D=14;   J=4
NUMERATOR:
+3Z4+6Z5+20Z6+32Z7+59Z8+98Z9+161Z10+204Z11+292Z12+340Z13+432Z14+460Z15+535Z16+529Z17+564Z16+514Z19+507Z20+425Z21+387Z22+293Z23+246Z24+16
5Z25+125Z26+70Z27+48Z28+21Z29+12Z30+2Z31+1Z32
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)(1-Z8)(1-Z3)(1-Z10)

D=14;   J=5
NUMERATOR:
+1Z8+2Z9+2Z10+2Z11+5Z12+5Z13+7Z14+7Z15+7Z15+5Z17+6Z18+4Z19+5Z20+3Z21+2Z22+1Z24
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)(1-Z7)(1-Z4)(1-Z9)

D=14;   J=6
NUMERATOR:
+1Z21
DENOMINATOR:

(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)

*****THE POINCARE SERIES FOR D=14*****:
NUMERATOR:
+1Z0+1Z4-1Z5+5Z6+3Z7+18Z8+21Z9+56Z10+72Z11+155Z12+209Z13+375Z14+523Z15+636Z16+1131Z17+1695Z18+2234Z19+3132Z20+4029Z21+5371Z22+6691Z23+85
66Z24+10348Z25+12736Z26+14971Z27+17792Z28+20306Z29+2340CZ30+25973Z31+29023Z32+31385Z33+34068Z34+35858Z35+37893Z36+38831Z37+39932Z38+3989
0Z39+39932Z40+38831Z41+37893Z42+35858Z43+34058Z44+31395Z45+29023Z46+25973Z47+23400Z48+20306Z49+17789Z50+14971Z51+12736Z52+10348Z53+8566Z
54+6691Z55+5371Z56+4029Z57+3132Z58+2234Z59+1695Z60+1131Z61+836Z62+523Z63+375Z64+209Z65+155Z66+72Z67+56Z68+21Z69+18Z70+3Z71+5Z72-1Z73+1Z7
4+1Z78
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z5)(1-Z6)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)

D=15;   J=0
NUMERATOR:
+1Z0+3Z2+78lZ4+684976+39203Z8+163746Z10+548461Z12+1541113Z14+3776418Z16+8263874Z18+15485049Z20+30410166Z22+52520437Z24+85693905Z26+1331
20549Z28+198046537Z30+283613598Z32+392520604Z34+526860269Z36+687748361Z38+875197714Z40+1087755068Z42+1322493470Z44+1574771430Z46+1838469
934Z48+2105978316Z50+2368728082Z52+261740740UZ54+2842723878Z56+303572J855Z58+3188583260Z60+3294853794Z62+3350094491Z64+33518714550Z66+330
0179988Z68+3197160663Z70+3047256919Z72+2856673194Z74+2633273183Z76+2385833516Z78+2123757504Z80+1856262169Z82+1592085664Z84+1338814362Z86
+1102767476Z88+888601135Z90+699435601Z92+536745422Z94+400674476Z96+290114572Z98+203081623Z10C+136863828Z102+88380313Z104+54350326Z106+31
602454Z108+17208650Z110+8678387Z112+3990534Z114+1643369Z116+589934Z118+1789222Z12C+4333Z122+7847Z124+900Z126+54Z128
DENOMINATOR:
(1-Z4)(1-Z2)(1-Z6)(1-Z2)(1-Z4)(1-Z14)(1-Z16)(1-Z6)(1-Z4)(1-Z22)(1-Z8)(1-Z26)(1-Z28)(1-Z2)

D=15;   J=1
NUMERATOR:
+7Z2+1665Z4+166877Z6+10548678+5050342Z10+192887312+6247203Z14+17744574Z16+45292581Z18+105731137Z20+228818638Z22+463891390Z24+88841661072
6+16180665252Z28+28181030069Z30+471484230 8Z32+76055563172134+118715739217236+179728435882338+25455901838Z40+37940223493Z42+53100591875Z44+726
40319855Z46+97253260012Z48+127579383219Z50+164151439424Z52+207341021126Z54+257301673609Z56+313920660166Z58+376775891173Z60+445118804407Z
62+517827740590Z64+593493232023Z66+670378977844Z68+746517181198Z70+819774233145Z72+887948109083Z74+948867977037Z76+100C507687035778+1041
096864849Z80+1069172159326Z82+1083751285539Z84+1094295176842Z86+1C7C782617643Z88+1043706829951Z90+1004042139671Z92+953194507389Z94+89291
8474434Z96+825227267729Z98+75228225301Z2100+676288956616Z102+599387745408Z104+523653994440Z106+450565899045Z108+381850023561Z11C+3185416
435033Z112+261424136933Z114+210943308396Z116+167235461438Z118+130164900680Z120+99375706159Z122+74344822260Z124+54439223137Z126+3896697171
0Z128+27224481000Z130+18533315340Z132+12269277555Z134+7880479361Z136+4897659431Z138+29359232672140+16912050005Z142+9319313647Z144+48861329
3Z146+2421312017Z148+1124878922Z150+464873002Z152+191393272154+67982332156+2122958Z158+5636152160+1212062162+1948072164+2039Z166+99Z168+1Z17
0
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z2)(1-Z28)

D=15;   J=2
NUMERATOR:
+4lZ2+10662Z4+1159026+7608428+363753Z10+1379223Z12+4395902Z14+12205126Z16+30316741Z18+68631733Z20+143630206Z22+281208604Z24+519112010Z26+
910117301Z28+1524180259Z30+2449590437Z32+3792955851Z34+5676733192Z36+8235059285Z38+11606377290Z40+15924599364Z42+21307234966Z44+27843484
461Z46+35580507268Z48+44512542767Z50+54569328702Z52+65611166426Z54+77425165807Z56+89729542913Z58+102180707614Z60+114388696219Z62+1259343
07635764+136393172282766+145358481676768+152467540967Z70+157422870783Z72+160013467539Z74+160126870092776+157758647431778+153010743226Z80
+146086584877Z82+137275641621784+126936445225Z86+115471972939Z88+103306826824Z90+90860882384Z92+78528325677Z94+66656932734Z96+5553517973
4Z98+45381832594Z100+36343823283Z102+284968519622104+218527710532106+163683194972108+119579329222110+85058395593Z112+58793874982114+39399
0561ZZ116+2552725650Z118+1593954219Z120+955541976Z122+547400729Z124+298016833Z126+153122694Z128+73627335Z130+32769393Z132+13315934Z134+4
84795772136+154307372138+413839Z140+88861Z142+1393Z144+1379Z146+56Z148
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z2)(1-Z24)(1-Z26)

D=15;   J=3
NUMERATOR:
+5Z2+2207Z4+241676+148367Z8+62287Z10+203960Z12+552988Z14+1300192Z16+2721380Z18+5184445Z20+9120288Z22+15002712Z24+23277924Z26+34327595Z28+4
8369794Z30+65429688Z32+85243577Z34+107277737Z36+130677146Z38+154365863Z40+177056486Z42+197423306Z44+214149522Z46+226130832Z48+2325115047Z
50+232857429752+227123077754+215738312756+199479169Z58+179463527Z60+156956218Z62+133321168Z64+109824373Z66+87598645Z68+67499441Z70+50121
282Z72+35735298Z74+24365053Z76+15794543Z78+9670505Z80+5539535Z82+2936941Z84+1418300Z86+612433Z88+229516Z90+71982Z92+17605Z94+3053Z96+294
Z98+9Z100

DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z2)(1-Z20)(1-Z22)(1-Z8)

D=15; J=4
NUMERATOR:
+10Z4+187Z6+1661Z8+9135Z10+38048Z12+127672Z14+365451Z16+917298Z18+2071378Z20+4272481Z22+8165350Z24+14597354Z26+24628480Z28+39464288Z30+6
040869SZ32+88707199Z34+125445704Z36+171322365Z38+226544195Z40+290597061Z42+362234328Z44+439349796Z46+519137053Z48+598120187Z50+672499705
Z52+738309854Z54+791877981Z56+830004661Z58+850403981Z60+851769818Z62+834055611Z64+798325132Z66+746802175Z68+682507592Z70+609127929Z72+53
0548308Z74+453666214Z76+372967180Z78+300412503Z80+235169296Z82+178647944Z84+131424903Z86+93421234Z88+63970094Z90+42054684Z92+26421779Z94
+15784859Z96+8904938Z98+4708429Z100+2308297Z102+1037244Z104+419581Z106+149883Z108+45661Z110+11422Z112+2144Z114+274Z116+15Z118
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z2)(1-Z16)(1-Z18)(1-Z20)(1-Z22)

D=15; J=5
NUMERATOR:
+1Z6+13Z8+113Z10+500Z12+1726Z14+4698Z16+11102Z18+22854Z20+42814Z22+73244Z24+116948Z26+174808Z28+247443Z30+332087Z32+425609Z34+521003Z36+
612010Z38+689569Z40+747865Z42+779577Z44+782912Z46+756152Z48+703365Z50+628585Z52+540083Z54+444497Z56+350481Z58+263309Z60+188354Z62+127225
Z64+80987Z66+47864Z68+26192Z70+12390Z72+5680Z74+2119Z76+665Z78+150Z80+25Z82+1Z84
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z2)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z4)

D=15; J=6
NUMERATOR:
+1Z14+2Z16+13Z18+28Z20+58Z22+101Z24+174Z26+245Z28+347Z30+438Z32+532Z34+591Z36+646Z38+638Z40+620Z42+552Z44+473Z46+369Z48+288Z50+191Z52+12
4Z54+71Z56+38Z58+13Z60+7Z62+1Z64
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z4)(1-Z2)(1-Z8)(1-Z10)(1-Z4)(1-Z14)(1-Z16)(1-Z6)

D=15; J=7
NUMERATOR:
+1Z56
DENOMINATOR:
(1-Z2)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z4)(1-Z6)(1-Z8)(1-Z10)(1-Z12)(1-Z14)(1-Z16)

*****THE POINCARE SERIES FOR D=15*****:
NUMERATOR:
+1Z0-1Z2+3Z4-2Z6+34Z8+44Z10+334Z12+693Z14+2431Z16+5241Z18+13435Z20+27349Z22+58848Z24+111154Z26+212550Z28+374115Z30+654965Z32+1082426Z34+
1767075Z36+2762925Z38+4256995Z40+6339393Z42+9297295Z44+13259049Z46+18626874Z48+25553002Z50+34551679Z52+45761015Z54+59777469Z56+76659758Z
58+97029205Z60+120774843Z62+148464903Z64+179718255Z66+214966329Z68+253469532Z70+295457984Z72+339781570Z74+386448587Z76+433908719Z78+4819
86117Z80+528138567Z82+574135025Z84+615915260Z86+653969266Z88+686288827Z90+712929175Z92+732108080Z94+744268809Z96+748015934Z98+744268809Z
100+732108080Z102+712929175Z104+686288827Z106+653969266Z108+615915260Z110+574135025Z112+528138567Z114+481986117Z116+433908719Z118+386448
587Z120+339781570Z122+295457884Z124+253469532Z126+214966329Z128+179718255Z130+148464903Z132+120774843Z134+97029205Z136+76659758Z138+5977
7469Z140+45761015Z142+34551679Z144+25553002Z146+18626874Z148+13259049Z150+9297295Z152+6339393Z154+4256995Z156+2762925Z158+1767075Z160+10
82426Z162+654965Z164+374115Z166+212550Z168+111154Z170+58848Z172+27349Z174+13435Z176+5241Z178+2431Z180+693Z182+334Z184+44Z186+34Z188-2Z19
0+3Z192-1Z194+1Z196
DENOMINATOR:
(1-Z2)(1-Z8)(1-Z10)(1-Z12)(1-Z12)(1-Z14)(1-Z16)(1-Z18)(1-Z20)(1-Z22)(1-Z24)(1-Z26)(1-Z28)

D=16; J=0
NUMERATOR:
+1Z0+4Z1+36Z2+153Z3+525Z4+1506Z5+3804Z6+8604Z7+17890Z8+34529Z9+62623Z10+107566Z11+176183Z12+276513Z13+417905Z14+610234Z15+863775Z16+1188
295Z17+1592457Z18+2082748Z19+2663199Z20+3333959Z21+4090910Z22+4925705Z23+5824926Z24+6770242Z25+7739567Z26+8706680Z27+9643362Z28+10519949
Z29+11307229Z30+11977441Z31+12506691Z32+12874964Z33+13068579Z34+13080029Z35+12908993Z36+12561771Z37+12051864Z38+11397891Z39+10623842Z40+
9756819Z41+8826081Z42+7861059Z43+6890686Z44+5940938Z45+5034920Z46+4191230Z47+3423931Z48+2742076Z49+2150389Z50+1648887Z51+1234307Z52+9002
68Z53+638390Z54+438915Z55+291773Z56+186791Z57+114700Z58+67195Z59+37323Z60+19482Z61+9484Z62+4230Z63+1705Z64+602Z65+182Z66+41Z67+8Z68
DENOMINATOR:
(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z1)(1-Z9)(1-Z5)(1-Z11)(1-Z3)(1-Z13)(1-Z7)(1-Z15)(1-Z2)

D=16;   J=1
NUMERATOR:
+1Z1+79Z2+402Z3+1517Z4+4773Z5+13001Z6+3179Z7+70958Z8+147188Z9+286483Z10+528281Z11+928867Z12+1566654Z13+2545270Z14+3999324Z15+6095453Z1
6+9035971Z17+13056351Z18+18424614Z19+25431928Z20+34387351Z21+45600410Z22+59369612Z23+75959365Z24+95584689Z25+118383340Z26+144403348Z27+1
73575948Z28+205708867Z29+240468763Z30+277386338Z31+315849844Z32+355128926Z33+394381597Z34+432692894Z35+469097175Z36+502627550Z37+5323430
91Z38+557385451Z39+577002474Z40+590597620Z41+597745644Z42+598226247Z43+592020018Z44+579325932Z45+560535078Z46+536226434Z47+507126269Z48+
474086578Z49+438032199Z50+399935033Z51+360758159Z52+321430462Z53+282801486Z54+245624127Z55+210522607Z56+177991787Z57+148380521Z58+121903
99859+98644627Z60+78573486Z61+61560874762+47404566Z63+35843599Z64+25584988Z65+19317953Z66+13734402Z67+9538097Z68+6458804Z69+4255029Z70+
2720475Z71+1682677772+1003433Z73+574154Z74+313678Z75+162438Z76+79121Z77+35809Z78+14882Z79+5541Z80+1810Z81+491Z82+104Z83+13Z84+1Z85
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)(1-Z2)(1-Z15)

D=16;   J=2
NUMERATOR:
+4Z1+36Z2+140Z3+425Z4+1069Z5+2328Z6+4554Z7+8168Z8+13599Z9+21271Z10+31506Z11+44480Z12+60195Z13+78438Z14+98764Z15+120563Z16+143018Z17+1651
79Z18+186011Z19+204451Z20+219464Z21+230189Z22+235960Z23+236404Z24+231508Z25+221576Z26+207211Z27+189281Z28+168774Z29+146756Z30+124284Z31+
102320Z32+81689Z33+63071Z34+46915Z35+33472Z36+22793Z37+14714Z38+8931Z39+5052Z40+2621Z41+1225Z42+503Z43+173Z44+43Z45+8Z46
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z1)(1-Z1)(1-Z2)(1-Z5)(1-Z1)(1-Z7)(1-Z4)(1-Z3)(1-Z5)(1-Z11)(1-Z2)(1-Z13)(1-Z7)

D=16;   J=3
NUMERATOR:
+9Z2+62Z3+283Z4+954Z5+2703Z6+6571Z7+14429Z8+28991Z9+54280Z10+95604Z11+160000Z12+255586Z13+392668Z14+581551Z15+833770Z16+1160136Z17+15708
07Z18+2072980Z19+2671551Z20+3356270Z21+4152587Z22+5019504Z23+5951135Z24+5924585Z25+7913422Z26+8885860Z27+9808874Z28+10647655Z29+11370122
Z30+11945814Z31+12351438Z32+12563707Z33+12588905Z34+12410181Z35+12041460Z36+11497621Z37+10802645Z38+9984325Z39+9075639Z40+8110100Z41+712
2278Z42+6143090Z43+5201421Z44+4319875Z45+3516539Z46+2802773Z47+2185046Z48+1663606Z49+1235321Z50+892730Z51+626603Z52+425881Z53+279513Z54+
176283Z55+106418Z56+61051Z57+33052Z58+16680Z59+7770Z60+3244Z61+1198Z62+367Z63+88Z64+12Z65+1Z66
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z2)(1-Z3)(1-Z4)(1-Z1)(1-Z6)(1-Z7)(1-Z8)(1-Z9)(1-Z2)(1-Z11)(1-Z12)(1-Z13)

D=16;   J=4
NUMERATOR:
+3Z3+12Z4+37Z5+96Z6+194Z7+356Z8+593Z9+917Z10+1314Z11+1803Z12+2332Z13+2907Z14+3466Z15+3997Z15+4423Z17+4764Z18+4931Z19+4959Z20+4809Z21+453
4Z22+4102Z23+3612Z24+3042Z25+2480Z26+1924Z27+1438Z28+1001Z29+670Z30+407Z31+230Z32+111Z33+52Z34+15Z35+5Z36
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z1)(1-Z3)(1-Z1)(1-Z5)(1-Z3)(1-Z7)(1-Z2)(1-Z9)(1-Z5)(1-Z11)(1-Z3)

D=16;   J=5
NUMERATOR:
+1Z5+2Z6+13Z7+28Z8+62Z9+115Z10+209Z11+329Z12+520Z13+738Z14+1035Z15+1363Z16+1757Z17+2150Z18+2594Z19+2981Z20+3368Z21+3661Z22+3907Z23+4009Z
24+4055Z25+3938Z26+3763Z27+3466Z28+3128Z29+2707Z30+2313Z31+1877Z32+1498Z33+1132Z34+838Z35+576Z36+395Z37+241Z38+144Z39+76Z40+39Z41+13Z42+
7Z43+1Z44
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z1)(1-Z4)(1-Z5)(1-Z2)(1-Z7)(1-Z8)(1-Z3)(1-Z10)(1-Z11)

D=16;   J=6
NUMERATOR:
+1Z11+2Z12+2Z13+2Z14+5Z15+5Z16+7Z17+7Z18+7Z19+5Z20+6Z21+4Z22+5Z23+3Z24+2Z25+1Z27
DENOMINATOR:
(1-Z1)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)(1-Z1)(1-Z3)(1-Z2)(1-Z5)(1-Z3)(1-Z7)(1-Z4)(1-Z9)(1-Z5)

D=16;   J=7
NUMERATOR:
+1Z28
DENOMINATOR:
(1-Z1)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z2)(1-Z3)(1-Z4)(1-Z5)(1-Z6)(1-Z7)(1-Z8)(1-Z9)

*****THE POINCARE SERIES FOR D=15*****:
NUMERATOR:
+1Z0+1Z3+1Z4+2Z5+8Z6+11Z7+30Z8+59Z9+112Z10+204Z11+389Z12+655Z13+1146Z14+1896Z15+3067Z16+4823Z17+7477Z18+11167Z19+16506Z20+23764Z21+33593

$Z22+46576Z23+63588Z24+85049Z25+112273Z26+145668Z27+186298Z28+234773Z29+292016Z30+357883Z31+433493Z32+518058Z33+611720Z34+713572Z35+82299$
$7Z36+93774Z37+1057289Z38+1178403Z39+1299472Z40+1417560Z41+1530508Z42+1634576Z43+1728646Z44+1808808Z45+1873856Z46+1921520Z47+1951002Z48+$
$1960390Z49+1951002Z50+1921520Z51+1873856Z52+1808808Z53+1728646Z54+1634576Z55+1530508Z56+1417560Z57+1299472Z58+1178403Z59+1057289Z60+9377$
$42Z61+822997Z62+713572Z63+611720Z64+518058Z65+433493Z66+357883Z67+292016Z68+234773Z69+186298Z70+145668Z71+112273Z72+85049Z73+63588Z74+46$
$576Z75+33593Z76+23764Z77+16505Z78+11167Z79+7477Z80+4823Z81+3067Z82+1896Z83+1146Z84+655Z85+389Z86+204Z87+112Z88+59Z89+30Z90+11Z91+8Z92+2Z$
$93+1Z94+1Z95+1Z98$

DENOMINATOR:
$(1-Z2)(1-Z4)(1-Z5)(1-Z6)(1-Z6)(1-Z7)(1-Z7)(1-Z8)(1-Z9)(1-Z10)(1-Z11)(1-Z12)(1-Z13)(1-Z15)$