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The M/G/1 queue with randomly alternating services

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The M/G/1 Queue with Randomly Alternating Services

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In this report the M/G/1 queue with randomly alternating services is analysed with respect to its queue-length process. The queue-length distribution is obtained and explicit expressions have been derived for the first moments of various interesting performance measures.

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note on notations and referencing

Throughout this report, a bold letter denotes a stochastic variable. References to formulas are given according to the following rule: A reference to, say, relation (3.1) (the first numbered relation of Section 3) in Chapter II is denoted by (3.1) in that chapter and by II.(3.1) in another chapter. For the appendices the roman numeral, indicating the chapter, is replaced by a letter, indicating the appendix. Similar rules apply for references to sections, theorems etc. The name of an author, followed by a number between brackets, refers to the list of references.

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Chapter I

Introduction

A local area network is a computer network that is completely situated within a limited area, usually with a range of 0.1 Km. to 10 Km., such as for instance a university campus or an office building. For an introduction to LAN's cf. Tanenbaum [17]. A LAN configuration typically consists of a number, say N, of workstations (terminals, PC's) interconnected by a transmission medium (coaxial cable, optical fibre). For various reasons, such as routing problems and the high costs of wiring, this type of network often has a ring or bus topology: The workstations share a single communication channel. (cf. Figure 1)

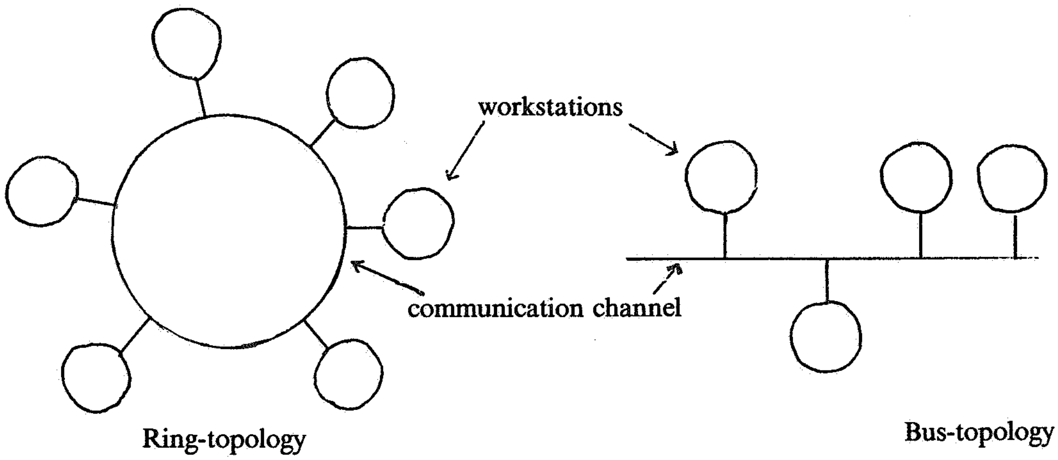
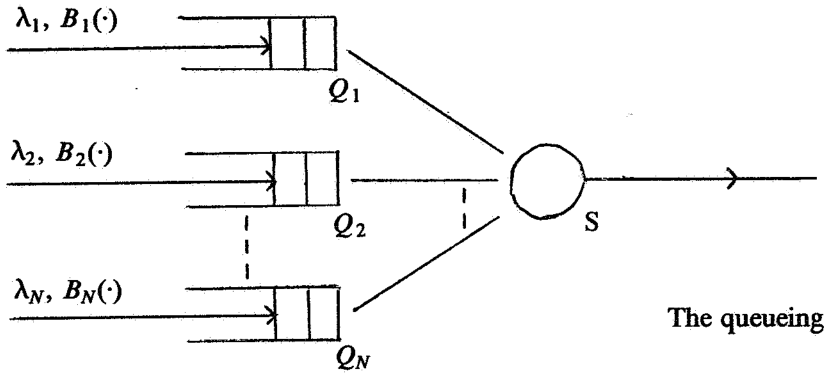


FIGURE 1

If station i wants to send a message to station j it has to use the single communication channel. If another station wants to send at the same time there might arise a conflict if there is no controlling mechanism present on the channel to avoid collisions. Often "token-passing" is used as such a controlling mechanism: In a token-passing system, a permission token to access a shared transmission medium is passed on (often in a cyclic manner) among the stations attached.

For a performance analysis of such a system various performance measures (like the waiting times of the workstations until access) have to be considered. Hereto a mathematical queueing model of the system is introduced. The queueing model of the network described above might be a single-server multi-queue system with a certain service discipline: A server (the communication channel of the network) is shared by customers (the workstations) and the service discipline specifies how this sharing is accomplished, i.e., the service discipline represents the controlling mechanism. We shall describe the queueing model in some more detail below:



The queueing model

FIGURE 2

The model under consideration consists of N queues $Q_1, \ldots, Q_N$ (with infinite buffer capacities) and one server S. Each queue has its own Poissonian arrival stream of customers, with, for Q_i, arrival intensity λ_i, $i = 1, \ldots, N$. The arrival processes are assumed to be independent of each other. Customers who arrive at queue i (type i customers) require from the server a service time with distribution $B_i(\cdot)$. It is assumed that the service times are independent and apart from their type also independent of the arrival processes.

Sofar we have not discussed the manner in which the server serves the incoming customers. A class of service disciplines that is often encountered in multi-queue single-server systems is the class of the *cyclic* service disciplines: After the server has visited queue i (mod N) the next queue to be visited will be queue i+1 (mod N). There are roughly three types of cyclic service to be distinguished:

1. *Exhaustive service* (also called polling, or alternating priority): When the server visits a queue, he serves its customers until that queue is empty.
2. *Gated service:* Only those customers who are found at the instant at which the server visits the queue, are served in the current round; those who arrive during this service period are reserved for service in the next round.
3. *Non-exhaustive service* (also called chaining, or limited service): When the server visits a queue, he serves at most a fixed number of customers, K, if there are at least K customers present at the instant at which the server visits the queue; if not, the server serves the queue until it empties, where arrivals during the service period are served in the current round if the number K has not yet been reached.

For (1) and (2) an exact mathematical analysis is possible, whereas for (3) such an analysis has only been found for the case N = 2, K = 1.

In the present report we analyse a new service discipline (as proposed by Cohen [6]): When the server has served a customer the next customer to be served is with probability α_i (i = 1,...,N) the first customer of Q_i. When indeed Q_i is chosen and Q_i is empty the server chooses again until a non-empty queue is found. If the system is empty the server waits for the first arrival in the system. It is assumed that the switching times are negligible (switching time: the time it takes the server to switch from one queue to another).

It will turn out that an exact mathematical analysis of the resulting model is, for N = 2, in some cases possible. Furthermore this service discipline has the nice feature of flexibility: By adapting the probabilities α_i (i = 1,...,N), with which the server chooses Q_i, it is possible to (sometimes approximately) model other, related systems, which gives insight into the manner in which related disciplines behave. As an example of this we could, for N = 2, take $\alpha_1 = 1$ and $\alpha_2 = 0$. Then the model is (from a performance point of view) identical to an M/G/1 non break-in priority model with two priority levels.

Throughout this report we shall refer to the multi-queue single-server model with the service discipline described above as to an "M/G/1 queue with random allocation". The report is devoted to a mathematical analysis of this M/G/1 queue. Its organisation is as follows:
In Chapter II we shall analyse the queue-length process of the M/G/1 queue for $N = 2$. In Chapter III some expressions for various interesting performance measures such as mean queue lengths, mean waiting times and mean sojourn times are derived. Chapter IV will be devoted to a discussion whether we can use the model for $N = 2$ to approximate means of performance measures in a model with random allocation and N ($N > 2$) queues. These results may be used to obtain more insight into the influence of various service disciplines on the performance of (LAN) ring systems.

Note on related literature:
The analysis in Chapter II is mainly based on the study of Cohen [6]. Cohen shows that the problem of the determination of the joint queue-length distribution can be transformed into a Riemann boundary value problem. The method to effectuate this has recently been developed by Cohen and Boxma (cf. Cohen and Boxma [7]) and is the result of a number of researches initiated in the studies of Fayolle and Iasnogorodski (cf. Fayolle and Iasnogorodski [9]). For more detailed references concerning related work in this area and a general introduction to two-dimensional birth-and-death processes, of which the considered queue-length process is a special case, see Cohen and Boxma [7].

Chapter II

Analysis of the queue-length process

1. Model description

In this section we describe the queueing model under consideration in more detail. The system consists of two queues, Q_1 and Q_2, served by a single service facility. The two queues have infinite capacities. Each queue has its own arrival stream of customers; the arrival processes are independent Poisson processes, with arrival rates λ_1 and λ_2 respectively. The total arrival rate is λ and we define

$$r_i := \frac{\lambda_i}{\lambda}, \quad i = 1,2, \tag{1.1}$$

as the fraction of type-i customers (i.e., customers which arrive at Q_i). Type-i customers require from the server a service time with distribution $B_i(\cdot)$ with first and second moments β_i and $\beta_i^{(2)}$ and Laplace-Stieltjes Transform (LST) $\beta_i(\cdot)$; all service times are independent of each other, and apart from their type also independent of the arrival processes.

The utilisation at Q_i, a_i, is defined as

$$a_i := \lambda_i \beta_i, \quad i = 1,2. \tag{1.2}$$

The total utilisation, a, of the service facility is defined as

$$a := a_1 + a_2. \tag{1.3}$$

The service discipline considered will be called Random Allocation (RA) from now on and is defined as follows:
If the server has completed a service and both queues are not empty then the next customer to be served is the first customer of Q_i with probability α_i,

$$\alpha_1 + \alpha_2 = 1, \quad 0 < \alpha_1 < 1,\ 0 < \alpha_2 < 1, \tag{1.4}$$

if just one of the queues is empty, the server proceeds with a customer from the non-empty queue; if both queues are empty the server waits for the first arrival in the system. No switching times are incorporated. See fig. 3.

Denote by $\mathbf{z}_n^{(i)}$, $n=0,1,2,\ldots$; $i=1,2$, the number of type-i customers left behind in the system after the completion of the n^{th} service. It will be shown that the stochastic vector process $(\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)})$ is a discrete-time Markov chain with a two-dimensional discrete state space. A large part of this study concerns the analysis of this Markov chain. It is effectuated by introducing the generating function of the joint distribution of the stochastic variables $\mathbf{z}_n^{(1)}$ and $\mathbf{z}_n^{(2)}$. This generating function satisfies a functional equation. It will be shown that the analysis of this functional equation can be reduced to the analysis of a Riemann-Hilbert type boundary value problem by a method developed by Cohen and Boxma [7].

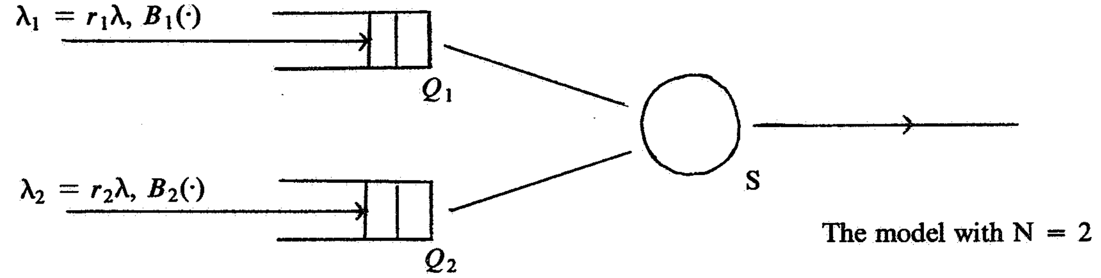


FIGURE 3

2. FORMULATION OF THE MATHEMATICAL PROBLEM

In this section the inherent mathematical problem for the queueing model described in section 1 will be formulated. The queueing process is considered at departure epochs because this embedded process defines a Markov chain. Let for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$\Phi(r;p_1,p_2) = \sum_{n=0}^{\infty} r^n E\{p_1^{\mathbf{z}_n^{(1)}} p_2^{\mathbf{z}_n^{(2)}} \mid \mathbf{z}_0^{(1)} = \mathbf{z}_0^{(2)} = 0\}. \tag{2.1}$$

We shall derive recurrence relations for the series $\{\mathbf{z}_n^{(i)}\}_{n\geqslant 0}$, $i=1,2$, and then we shall prove that the generating function $\Phi(r;p_1,p_2)$ satisfies a functional equation and possesses regularity properties.
We first give a definition:

DEFINITION 2.1
For $i=1,2$; $j=1,2$, let $\boldsymbol{\nu}_n^{(i,j)}$, $n=1,2,...,$ denote the number of type-i customers arriving during the $(n+1)^{th}$ service time if this service time is of type j, i.e., has distribution $B_j(\cdot)$.

For the generating function of $\boldsymbol{\nu}_n^{(1,j)}$ and $\boldsymbol{\nu}_n^{(2,j)}$, $j=1,2$, we have the following lemma:

LEMMA 2.1
For $|p_1|\leqslant 1, |p_2|\leqslant 1$, $n = 1,2,...;\ j = 1,2,$

$$E\{p_1^{\boldsymbol{\nu}_n^{(1,j)}} p_2^{\boldsymbol{\nu}_n^{(2,j)}}\} = \beta_j\{\lambda(1-r_1p_1-r_2p_2)\}. \tag{2.2}$$

PROOF OF LEMMA 2.1:
Note that the number of customers that arrive during a service time is independent of the past interarrival time measured at the moment that this service starts since the interarrival times are negative exponentially distributed. Given that a service time has duration τ the number of arriving type-1 customers and the number of type-2 customers are independent and have a Poisson distribution with parameters λ_1 and λ_2 respectively. This implies the following formula for the joint distribution of $(\boldsymbol{\nu}_n^{(1,j)},\boldsymbol{\nu}_n^{(2,j)})$, $j=1,2,...$:

$$Pr\{\boldsymbol{\nu}_n^{(1,j)} = k_1, \boldsymbol{\nu}_n^{(2,j)} = k_2\} = \int_{\tau=0}^{\infty} \frac{(\lambda_1\tau)^{k_1}}{k_1!}\frac{(\lambda_2\tau)^{k_2}}{k_2!} e^{-\lambda_1\tau} e^{-\lambda_2\tau} dB_j(\tau), \tag{2.3}$$

for $k_1,k_2 = 0,1,2,...$.

Now it follows that for $|p_1|\leq 1$, $|p_2|\leq 1$:

$$E\{p_1^{\nu_n^{(1,j)}}p_2^{\nu_n^{(2,j)}}\} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} p_1^{k_1}p_2^{k_2}\int_{\tau=0}^{\infty}\frac{\lambda_1\tau^{k_1}}{k_1!}\frac{\lambda_2\tau^{k_2}}{k_2!}e^{-\lambda\tau}dB_j(\tau). \tag{2.4}$$

Of course we may change the order of summation and integration here and obtain, for $|p_1|\leq 1$, $|p_2|\leq 1$:

$$E\{p_1^{\nu_n^{(1,j)}}p_2^{\nu_n^{(2,j)}}\} = \int_{\tau=0}^{\infty} e^{p_1\lambda_1\tau}e^{p_2\lambda_2\tau}e^{-\lambda\tau}dB_j(\tau). \tag{2.5}$$

Because Re $(1 - r_1p_1 - r_2p_2) > 0$, it is immediately seen that the right-hand side of (2.5) is equal to the LST of the distribution $B_j(\cdot)$ with argument $\lambda(1 - r_1p_1 - r_2p_2)$. This concludes the proof of Lemma 2.1 .

From the model description in the preceding section it follows that, with $\mathbf{z}_0^{(1)} = \mathbf{z}_0^{(2)} = 0$:

$$\text{if } \mathbf{z}_n^{(1)} > 0,\ \mathbf{z}_n^{(2)} > 0,\ n = 1,2,\dots : \tag{2.6}$$

$$\mathbf{z}_{n+1}^{(1)} = \mathbf{z}_n^{(1)} - 1 + \boldsymbol{\nu}_{n+1}^{(1,1)} \text{ and } \mathbf{z}_{n+1}^{(2)} = \mathbf{z}_n^{(2)} + \boldsymbol{\nu}_{n+1}^{(2,1)} \textit{ with probability } \alpha_1 ,$$

$$\mathbf{z}_{n+1}^{(1)} = \mathbf{z}_n^{(1)} + \boldsymbol{\nu}_{n+1}^{(1,2)} \text{ and } \mathbf{z}_{n+1}^{(2)} = \mathbf{z}_n^{(2)} - 1 + \boldsymbol{\nu}_{n+1}^{(2,2)} \textit{ with probability } \alpha_2 ;$$

$$\text{if } \mathbf{z}_n^{(1)} = 0,\ \mathbf{z}_n^{(2)} > 0,\ n = 1,2,\dots, : \tag{2.7}$$

$$\mathbf{z}_{n+1}^{(1)} = \boldsymbol{\nu}_{n+1}^{(1,2)} \text{ and } \mathbf{z}_{n+1}^{(2)} = \mathbf{z}_n^{(2)} - 1 + \boldsymbol{\nu}_{n+1}^{(2,2)} ;$$

$$\text{if } \mathbf{z}_n^{(1)} > 0,\ \mathbf{z}_n^{(2)} = 0,\ n = 1,2,\dots : \tag{2.8}$$

$$\mathbf{z}_{n+1}^{(1)} = \mathbf{z}_n^{(1)} - 1 + \boldsymbol{\nu}_{n+1}^{(1,1)} \text{ and } \mathbf{z}_{n+1}^{(2)} = \boldsymbol{\nu}_{n+1}^{(2,1)} ;$$

$$\text{if } \mathbf{z}_n^{(1)} = \mathbf{z}_n^{(2)} = 0,\ n = 1,2,\dots : \tag{2.9}$$

$$\mathbf{z}_{n+1}^{(1)} = \boldsymbol{\nu}_{n+1}^{(1,1)} \text{ and } \mathbf{z}_{n+1}^{(2)} = \boldsymbol{\nu}_{n+1}^{(2,1)} \quad \textit{with probability } r_1,$$

$$\mathbf{z}_{n+1}^{(1)} = \boldsymbol{\nu}_{n+1}^{(1,2)} \text{ and } \mathbf{z}_{n+1}^{(2)} = \boldsymbol{\nu}_{n+1}^{(2,2)} \quad \textit{with probability } r_2.$$

REMARK 2.1
It is immediately verified that the probability distribution of the vector $(\mathbf{z}_n^{(1)},\mathbf{z}_n^{(2)})$ is uniquely (recursively) determined by (2.6),...,(2.9) for every n, $n=1,2,\dots$ (with $\mathbf{z}_0^{(1)} = \mathbf{z}_0^{(2)} = 0$).

Because the vectors $(\boldsymbol{\nu}_n^{(1,j)},\boldsymbol{\nu}_n^{(2,j)})$, $j=1,2$, are independent of the vector $(\mathbf{z}_m^{(1)},\mathbf{z}_m^{(2)})$, $m<n$, for every n, $n=1,2,\dots$, it follows from relations (2.6),...,(2.9) that the process $\{ (\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)}),\ n=0,1,2,\dots \}$ possesses the Markov property. From remark 2.1, (2.6),...,(2.9) and lemma 1.1 it is readily seen that this Markov chain has the state space $\{0,1,2,\dots\} \times \{0,1,2,\dots\}$, that it is irreducible, aperiodic and that it has stationary transition probabilities.
We now derive a functional equation for $\Phi(r,p_1,p_2)$ (cf. (2.1)). Put,

$$x := \lambda(1-r_1p_1-r_2p_2). \tag{2.10}$$

We shall consider the expression

$$E\{p_1^{\mathbf{z}_{n+1}^{(1)}}p_2^{\mathbf{z}_{n+1}^{(2)}}\ |\mathbf{z}_0^{(1)} = \mathbf{z}_0^{(2)} = 0\}, \tag{2.11}$$

for $|p_1|\leq 1$, $|p_2|\leq 1$, $n=0,1,2,\dots$.

Denote with (A) the indicator function of the event A. It follows from relations (2.6),...,(2.9) (for convenience suppressing the conditional event $\mathbf{z}_0^{(1)} = \mathbf{z}_0^{(2)} = 0$ in the expectations below) that for $|p_1|\leq 1$, $|p_2|\leq 1$, $n = 0,1,2,\dots$,

$$E\{p_1^{\mathbf{z}_{n+1}^{(1)}}p_2^{\mathbf{z}_{n+1}^{(2)}}\} = \alpha_1 E\{p_1^{\mathbf{z}_n^{(1)}-1+\boldsymbol{\nu}_{n+1}^{(1,1)}}p_2^{\mathbf{z}_n^{(2)}+\boldsymbol{\nu}_{n+1}^{(2,1)}}(\mathbf{z}_n^{(1)} > 0,\mathbf{z}_n^{(2)} > 0)\} \tag{2.12}$$

$$
\begin{aligned}
&+\ \alpha_2 E\{p_1^{\mathbf{z}_n^{(1)}+\boldsymbol{\nu}_{n+1}^{(1,2)}} p_2^{\mathbf{z}_n^{(2)}-1+\boldsymbol{\nu}_{n+1}^{(2,2)}}(\mathbf{z}_n^{(1)}>0,\ \mathbf{z}_n^{(2)}>0)\} \\
&+\ E\{p_1^{\mathbf{z}_n^{(1)}-1+\boldsymbol{\nu}_{n+1}^{(1,1)}} p_2^{\boldsymbol{\nu}_{n+1}^{(2,1)}}(\mathbf{z}_n^{(1)}>0,\ \mathbf{z}_n^{(2)}=0)\} \\
&+\ E\{p_1^{\boldsymbol{\nu}_{n+1}^{(1,2)}} p_2^{\mathbf{z}_n^{(2)}-1+\boldsymbol{\nu}_{n+1}^{(2,2)}}(\mathbf{z}_n^{(1)}=0,\ \mathbf{z}_n^{(2)}>0)\} \\
&+\ r_1 E\{p_1^{\boldsymbol{\nu}_{n+1}^{(1,1)}} p_2^{\boldsymbol{\nu}_{n+1}^{(2,1)}}(\mathbf{z}_n^{(1)}=0,\mathbf{z}_n^{(2)}=0)\} \\
&+\ r_2 E\{p_1^{\boldsymbol{\nu}_{n+1}^{(1,2)}} p_2^{\boldsymbol{\nu}_{n+1}^{(2,2)}}(\mathbf{z}_n^{(1)}=0,\mathbf{z}_n^{(2)}=0)\} \\
&=\ \{\frac{\alpha_1}{p_1}\beta_1(x)+\frac{\alpha_2}{p_2}\beta_2(x)\}E\{p_1^{\mathbf{z}_n^{(1)}} p_2^{\mathbf{z}_n^{(2)}}\} \\
&+\ \alpha_2\{\frac{\beta_1(x)}{p_1}-\frac{\beta_2(x)}{p_2}\}E\{p_1^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} \\
&+\ \alpha_1\{\frac{\beta_1(x)}{p_2}-\frac{\beta_2(x)}{p_1}\}E\{p_2^{\mathbf{z}_n^{(2)}}(\mathbf{z}_n^{(1)}=0)\} \\
&+\ \{r_1\beta_1(x)+r_2\beta_2(x)-\alpha_1\frac{\beta_2(x)}{p_2}-\alpha_2\frac{\beta_1(x)}{p_1}\}E\{(\mathbf{z}_n^{(1)}=\mathbf{z}_n^{(2)}=0)\}.
\end{aligned}
$$

Multiplying both sides of (2.12) with $p_1p_2r^{n+1}$ and summing over n, we obtain (cf.(2.1)):

$$
\begin{aligned}
&\Big[p_1p_2-r\{\alpha_1p_2\beta_1(x)+\alpha_2p_1\beta_2(x)\}\Big]\Phi(r;p_1,p_2)=p_1p_2 \qquad (2.13)\\
&+\ r\{p_2\beta_1(x)-p_1\beta_2(x)\}\{\alpha_2\Phi(r;p_1,0)-\alpha_1\Phi(r;0,p_2)\} \\
&+\ r\{p_1p_2(r_1\beta_1(x)+r_2\beta_2(x))-\alpha_1p_1\beta_2(x)-\alpha_2p_2\beta_1(x)\}\Phi(r;0,0).
\end{aligned}
$$

Relation (2.13) represents the functional equation for the generating function $\Phi(r;p_1,p_2)$.

From some well-known properties of generating functions the following lemma easily follows (cf. appendix A, section 2):

LEMMA 2.2

(i) for fixed p_1,p_2 *with* $|p_1|\leqslant 1, |p_2|\leqslant 1$, $\Phi(r;p_1,p_2)$ *is a regular function of* r *in the unit disk* $|r|<1$,

(ii) for fixed $|r|<1$ *the function* $\Phi(r;p_1,p_2)$ *is:*

(1) for fixed p_2 *with* $|p_2|\leqslant 1$ *regular in* $|p_1|<1$, *continuous in* $|p_1|\leqslant 1$,

(2) for fixed p_1 *with* $|p_1|\leqslant 1$ *regular in* $|p_2|<1$, *continuous in* $|p_2|\leqslant 1$.

The expression between square brackets in the left-hand side of (2.13) is called the *kernel* of the functional equation:

$$K(r;p_1,p_2):=p_1p_2-r\{\alpha_1p_2\beta_1(x)+\alpha_2p_1\beta_2(x)\}. \qquad (2.14)$$

The analysis of $K(r;p_1,p_2)$ is the starting point for the determination of the function $\Phi(r;p_1,p_2)$ satisfying (2.13) and the conditions stated in lemma 2.2, because if for a pair (q_1,q_2) with $|q_1|\leqslant 1$, $|q_2|\leqslant 1$ this kernel vanishes then the right-hand side of equation (2.13) must be zero because of the stated regularity properties of $\Phi(r;p_1,p_2)$ (cf. lemma 2.2). This provides us with a relation between the functions $\Phi(r;p_1,0)$ and $\Phi(r;0,p_2)$, from which these functions, and hence $\Phi(r;p_1,p_2)$, will be determined. We shall examine the kernel, in particular with respect to its zeros, in the following sections.

3. Analysis of the kernel

In this section we analyse the kernel $K(r;p_1,p_2)$ of equation (2.13). Consider the kernel for

$$p_1 = gs, \quad p_2 = gs^{-1}, \quad |s|=1, \ |g|\leqslant 1. \tag{3.1}$$

Consequently we have:

$$K(r;gs,gs^{-1}) = g^2 - rg\{\alpha_1 s^{-1}\beta_1(x)+\alpha_2 s\beta_2(x)\}, \tag{3.2}$$

with, cf.(2.10),

$$x = \lambda(1-r_1p_1-r_2p_2) = \lambda(1-g(r_1s+r_2s^{-1})). \tag{3.3}$$

LEMMA 3.1

For $|r|<1$:

(i) The kernel $K(r;gs,gs^{-1})$ has in $|g|\leqslant 1$ exactly two zeros, of which one is identically zero. Denote the other zero by:

$$g = g(r,s). \tag{3.4}$$

For $|s|=1$:

(ii) $g(r,s) = -g(r,-s)$.

PROOF OF LEMMA 3.1: Obviously, for fixed $|s|=1$, $g=0$ is a zero of $K(r;gs,gs^{-1})$, $|g|\leqslant 1$. For $|g|=1$ it is seen that, for $|s|=1$,

$$|r\{\alpha_1 s^{-1}\beta_1(x)+\alpha_2 s\beta_2(x)\}| \leqslant |r|\{\alpha_1|\beta_1(x)|+\alpha_2|\beta_2(x)|\} \leqslant |r|<1 = |g|. \tag{3.5}$$

Note that $\beta_j\{\lambda(1-g(r_1s+r_2s^{-1}))\}$ is regular in g for $|g|<1$, with s, $|s|=1$ fixed and continuous for $|g|\leqslant 1$. By applying Rouche's theorem (cf. Titchmarsh [18]) to the contour $|g|=1$ it is seen that $g^{-1}K(r;gs,gs^{-1})$ has for every fixed s with $|s|=1$ a unique zero in $|g|\leqslant 1$, its multiplicity is always one.

The validity of the second statement may be seen by considering the equation

$$g(r,s) = r\left[\alpha_1 s^{-1}\beta_1\{\lambda(1-g(r,s)(r_1s+r_2s^{-1}))\}+\alpha_2 s\beta_2\{\lambda(1-g(r,s)(r_1s+r_2s^{-1}))\}\right]. \tag{3.6}$$

Define, for $|r|<1$:

$$S_1(r) := \{p_1 : p_1 = g(r,s)s, \ |s|=1\}, \tag{3.7}$$
$$S_2(r) := \{p_2 : p_2 = g(r,s)s^{-1}, |s|=1\}.$$

For reasons that will become apparent in the sequel we have to prove that the contours $S_1(r)$ and $S_2(r)$ are simply connected. To accomplish this we are forced to introduce a number of simplifications. Firstly it will be assumed in the following that both types of customers have the same service-time distribution, i.e., $B_1(\cdot) = B_2(\cdot)$, so for Re $\rho\geqslant 0$:

$$\beta(\rho) := \beta_1(\rho) = \beta_2(\rho). \tag{3.8}$$

Now, cf.(3.2), for $|s|=1$,

$$g^{-1}K(r;gs,gs^{-1}) = g - r(\alpha_1 s^{-1}+\alpha_2 s)\beta\{\lambda(1-g(r_1s+r_2s^{-1}))\}. \tag{3.9}$$

Denote by $\mathbf{n}$ the number of customers served in a busy period of an M/G/1 queueing model with arrival rate λ and service-time distribution with LST $\beta(\rho)$. It is proved in appendix A, section 2, that, for $a = \lambda\beta < 1$:

$$g(r,s) = E\{r^{\mathbf{n}}\left[r_1s+r_2s^{-1}\right]^{\mathbf{n}-1}\left[\alpha_1s^{-1}+\alpha_2s\right]^{\mathbf{n}}\}, \ |s|=1, \tag{3.10}$$

is the unique zero in $|g|\leqslant 1$ of the function in (3.9).

The problem now arises that it is not possible to prove (because it is not true), for general α_i and r_i, that the contours $S_1(r)$ and $S_2(r)$ are simply connected. In fact, even if we restrict ourself to the case where

$$\beta(\rho) = \frac{1}{1+\beta\rho},$$

i.e., the service-time distribution is negative exponential, α_i and r_i can be found such that $S_1(r)$ and $S_2(r)$ are not simple (cf. appendix B).

Hence it follows from the preceding that we have to restrict ourself to the study of special cases where we are able to prove that the contours (3.7) are simply connected. In the following we will assume:

ASSUMPTION (3.1) :
$\alpha_1 = r_1$, $\alpha_2 = r_2$ with $r_1 > r_2$.

In the sequel we shall point out why we have excluded the case $\alpha_1 = r_1 = 0.5$, $\alpha_2 = r_2 = 0.5$.

Taking into account Assumption 3.1, we have for $|s|=1$, cf.(3.2):

$$g^{-1}K(r;gs,gs^{-1}) = g - r(r_1s^{-1}+r_2s)\beta\{\lambda(1-g(r_1s+r_2s^{-1}))\}. \tag{3.11}$$

Note that, for $|s|=1$:

$$r_1s^{-1} + r_2s = \overline{(r_1s+r_2s^{-1})}. \tag{3.12}$$

From (3.10) it follows that for this case

$$g(r,s) = (r_1s^{-1}+r_2s)E\{r^{\mathbf{n}}|r_1s+r_2s^{-1}|^{2\mathbf{n}-2}\}, \tag{3.13}$$

is the unique zero of (3.11) in $|g|\leqslant 1$.

REMARK 3.1
The generality of the discussion is hardly influenced by taking r real and nonnegative, because (cf. Lemma 2.2) if $\Phi(r;p_1,p_2)$ is known for $0 < r < 1$, it can be found for $|r|<1$ by analytic continuation.

We take in the following r real and nonnegative.

LEMMA 3.2
Under Assumption 3.1, the contours $S_1(r)$ and $S_2(r)$ are both simple and smooth.

PROOF OF LEMMA 3.2:
We have (cf.(3.7), (3.15)):

$$\begin{aligned} S_1(r) &= \{p_1: p_1=(r_1+r_2s^2)E\{r^{\mathbf{n}}\ |r_1s+r_2s^{-1}|^{2\mathbf{n}-2}\},\ |s|=1\}, \\ S_2(r) &= \{p_2: p_2=(r_1s^{-2}+r_2)E\{r^{\mathbf{n}}\ |r_1s+r_2s^{-1}|^{2\mathbf{n}-2}\},\ |s|=1\}. \end{aligned} \tag{3.14}$$

Let,

$$\begin{aligned} U_1 &:= \{u_1: u_1=r_1+r_2s^2,\ |s|=1\} \\ &= \{u_1: u_1=r_1+r_2e^{2i\phi},\ 0\leqslant\phi\leqslant 2\pi\}, \\ U_2 &:= \{u_2: u_2=r_1s^{-2}+r_2,\ |s|=1\} \\ &= \{u_2: u_2=r_1e^{-2i\phi}+r_2,\ 0\leqslant\phi\leqslant 2\pi\}. \end{aligned} \tag{3.15}$$

It is immediately clear that U_1 and U_2 are smooth, simple contours. Further if s traverses the unit circle once anticlockwise then U_1 is traversed twice anticlockwise and U_2 is traversed twice clockwise.

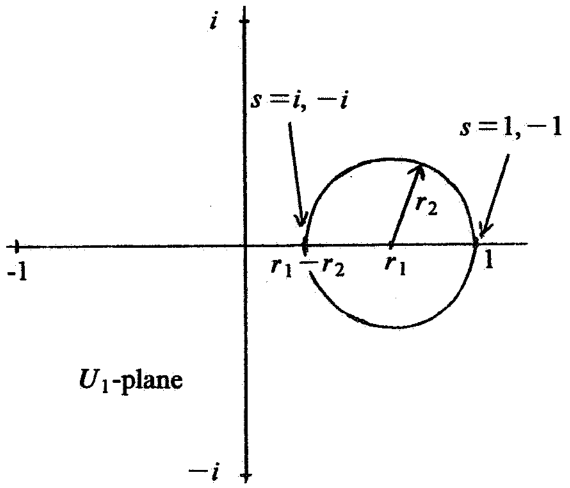


FIGURE 4 U_1 - contour

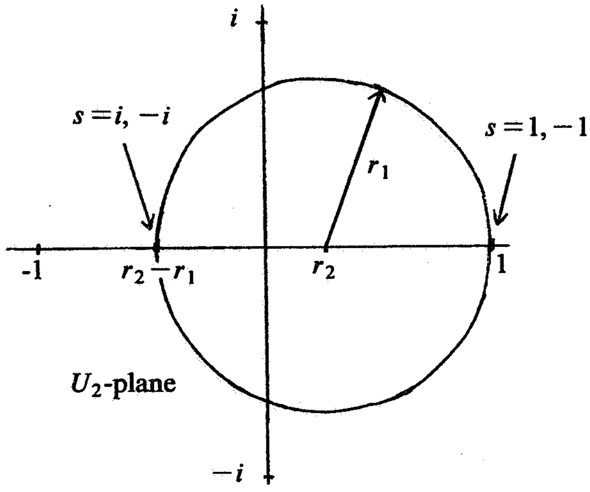


FIGURE 5 U_2 - contour

Note that we have, for $|s|=1$:

$$0 < |r_1-r_2| = \big|\,|r_1 s| - |r_2 s^{-1}|\,\big| \leqslant |r_1 s + r_2 s^{-1}| \leqslant |r_1 s| + |r_2 s^{-1}| = r_1+r_2 = 1. \tag{3.16}$$

It follows easily that, because $\mathbf{n} \geqslant 1$ with probability 1,

$$0 < (r_1-r_2)^2 \leqslant E\{r^{\mathbf{n}}\, |r_1 s + r_2 s^{-1}|^{2\mathbf{n}-2}\} \leqslant 1. \tag{3.17}$$

Consequently,

$$S_1(r) \subset \{p_1 \colon \operatorname{Re} p_1 > 0\}. \tag{3.18}$$

Because for $s = i, -i$,

$$(r_1 s^{-2} + r_2) E\{r^{\mathbf{n}}\, |r_1 s + r_2 s^{-1}|^{2\mathbf{n}-2}\} = (r_2-r_1) E\{r^{\mathbf{n}} (r_1-r_2)^{2\mathbf{n}-2}\} < 0, \tag{3.19}$$

we have that

$$p_2 = 0 \notin S_2(r). \tag{3.20}$$

It is now immediately verified that $S_1(r)$ and $S_2(r)$ are both simple, smooth contours, and if s traverses the unit circle once then $S_1(r)$ is traversed twice anticlockwise and $S_2(r)$ is traversed twice clockwise.

Let $S_i^+(r)$ denote the interior of $S_i(r)$ and $S_i^-(r)$ the exterior of $S_i(r)$, $i=1,2$. Note that

$$p_1 = 0 \in S_1^-(r),\ p_2 = 0 \in S_2^+(r). \tag{3.21}$$

4. Parametrisation of the kernel

Throughout this and the next sections the following assumptions will be made (unless explicitly stated otherwise):

$$\beta(\rho) := \beta_1(\rho) = \beta_2(\rho); \tag{4.1}$$

$$\alpha_1 = r_1,\ \alpha_2 = r_2;$$

$$r_1 > r_2.$$

The functional equation for the function $\Phi(r;p_1,p_2)$ is in this case given by, cf. (2.13): for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, $0<r<1$,

$$K(r;p_1,p_2)\Phi(r;p_1,p_2) = r\beta(x)(p_2-p_1)\{r_2\Phi(r;p_1,0)-r_1\Phi(r;0,p_2)\} \tag{4.2}$$

$$+\ r\beta(x)\{p_1p_2-r_1p_1-r_2p_2\}\Phi(r;0,0) + p_1p_2,$$

where x is given by (2.10) and $K(r;p_1,p_2)$, cf. (2.14), by:

$$K(r;p_1,p_2) = p_1p_2 - \{r_1p_2+r_2p_1\}r\beta(x). \tag{4.3}$$

We first give a brief outline of the basic ideas in this section:
In the previous section it has been shown, that, for fixed r with $0 < r < 1$ and every s with $|s|=1$, a function $g(r,s)$ exists such that (p_1,p_2) with

$$p_1 := g(r,s)s,\ p_2 := g(r,s)s^{-1}, \tag{4.4}$$

is a zero pair of the kernel (4.3).
For these functions (4.4) the following boundary value problem will be considered.

Determine in the z-plane a simply connected Jordan contour $L(r)$ and a real function $\lambda(r,z)$, $z \in L(r)$ such that:

PROBLEM 4.1

(1) $g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)}$ is the boundary value of a function $p_1(r,z)$ which is regular for $z \in L^+(r)$ and continuous for $z \in L(r) \bigcup L^+(r)$;
(2) $g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)}$ is the boundary value of a function $p_2(r,z)$ which is regular for $z \in L^-(r)$ and continuous for $z \in L(r) \bigcup L^-(r)$.

If this problem possesses a solution then (p_1,p_2) with

$$p_1 = p_1(r,z),\ p_2 = p_2(r,z),\quad z \in L(r), \tag{4.5}$$

is a zero pair of the kernel (4.3). Consequently, because of Lemma 2.2 and (4.2) we have with (4.5):

$$r\beta(x)(p_2-p_1)\{r_2\Phi(r;p_1,0)-r_1\Phi(r;0,p_2)\} + p_1p_2 + \tag{4.6}$$

$$+\ r\beta(x)\{p_1p_2-r_1p_1-r_2p_2\}\Phi(r;0,0) = 0.$$

It will appear in the sequel that $\Phi(r;0,0)$ can be found by comparison with an ordinary M/G/1 queueing model and hence (4.6) provides us with a relation between two unknown functions, $\Phi(r;p_1,0)$ and $\Phi(r;0,p_2)$, on the contour $L(r)$. This relation will be exploited in Section 5.

Some notation: for $t_0 \in L(r)$,

$$p_1^+(r,t_0) := \lim_{\substack{z\to t_0\\ z\in L^+(r)}} p_1(r,z),\quad p_2^-(r,t_0) := \lim_{\substack{z\to t_0\\ z\in L^-(r)}} p_2(r,z). \tag{4.7}$$

We consider the following problem.

PROBLEM 4.2

To construct in the z-plane a simply connected Jordan contour $L(r)$ and a pair of mappings $p_1(r,z)$, $z \in L(r) \bigcup L^+(r)$, $p_2(r,z)$, $z \in L(r) \bigcup L^-(r)$, such that:

(1) $p_1(r,z)$ is regular and univalent for $z \in L^+(r)$, continuous for $z \in L(r) \bigcup L^+(r)$,
$p_2(r,z)$ is regular and univalent for $z \in L^-(r)$, continuous for $z \in L(r) \bigcup L^-(r)$;

(2) $p_1(r,z)$ maps $L^+(r)$ conformally onto $S_1^+(r)$;
$p_2(r,z)$ maps $L^-(r)$ conformally onto $S_1^-(r)$;

(3) $p_1^+(r,z), p_2^-(r,z)$, $z \in L(r)$, is a zero pair of the kernel (4.3);

(4) $p_1(r,0) = c_1$ *), $\frac{\delta}{\delta z} p_1(r,z)|_{z=0} > 0$ for a $c_1 \in S_1^+(r)$,
$p_2(r,\infty) = 0$, $0 < d := \lim_{|z| \to \infty} |zp_2(r,z)| < \infty$.

*) Note that it is always possible to choose the origin of the z-plane so that it belongs to $L^+(r)$. Further it is possible to choose $z = 1 \in L(r)$.

In Cohen and Boxma [7], p.162, the following theorem is proven:

THEOREM 4.1

For $0 < r < 1$ there exist a pair of functions $p_1(r,z)$, $p_2(r,z)$ and a Jordan contour $L(r)$ satisfying (4.9) 1,...,4; $L(r)$ is an analytic contour.

The following section will be concerned with the determination of $L(r)$ and the mappings $p_1(r,\cdot)$ and $p_2(r,\cdot)$.

5. THE INTEGRAL EQUATIONS

For $z \in L(r)$, $(p_1^+(r,z), p_2^-(r,z))$ is a zero pair of the kernel (4.3) for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$ with $p_1^+(r,z) \in S_1(r)$, $p_2^-(r,z) \in S_2(r)$. Hence we may write, cf. (3.7), for $z \in L(r)$:

$$p_1^+(r,z) = g(r,e^{i\lambda(r,z)})e^{i\lambda(r,z)}, \qquad (5.1)$$

$$p_2^-(r,z) = g(r,e^{i\lambda(r,z)})e^{-i\lambda(r,z)},$$

with for every fixed $r \in (0,1)$,

$$\lambda(r,\cdot) : L(r) \to [0,\pi], \qquad (5.2)$$

$$\lambda(r,1) = 0.$$

From (5.1) it is seen that for $z \in L(r)$:

$$(1) \quad \log p_1^+(r,z) + \log \frac{zp_2^-(r,z)}{d} = \log \frac{g^2(r,e^{i\lambda(r,z)})}{d} + \log z, \qquad (5.3)$$

$$(2)\quad \log p_1^+(r,z) - \log \frac{zp_2^-(r,z)}{d} = 2i\lambda(r,z) - \log z + \log d.$$

From (3.15) and (4.1) it can be seen that for fixed $r \in (0,1)$,

$$\underset{\{s:|s|=1\}}{ind}\ g(r,s) = \underset{\{s:|s|=1\}}{ind}\ (r_1 s^{-1} + r_2 s) = -1, \tag{5.4}$$

(for the concept of "the index of a function on a contour" cf. App. A).
From (5.4) it follows that for fixed $r \in (0,1)$,

$$\underset{z\in L(r)}{ind}\ g(r,e^{i\lambda(r,z)}) = \underset{0\leqslant\phi\leqslant\pi}{ind}\ g(r,e^{i\phi}) = -\tfrac{1}{2}. \tag{5.5}$$

And so,

$$\underset{z\in L(r)}{ind}\ g(r,e^{i\lambda(r,z)})z^{\frac{1}{2}} = 0. \tag{5.6}$$

Hence the arguments of the r.h.s in (5.3) have a zero increase if z traverses $L(r)$ once.
It is clear from (5.1) and (5.4) that

$$\underset{z\in L(r)}{ind}\ p_1(r,z) = 0, \quad \underset{z\in L(r)}{ind}\ zp_2(r,z) = 0. \tag{5.7}$$

Write (cf. (5.3)), for $z \in L(r)$,

$$\exp\{2i\lambda(r,z) - \log z\} = \frac{p_1^+(r,z)}{zp_2^-(r,z)}. \tag{5.8}$$

It is now easily seen from the definition of $\lambda(r,\cdot)$ that $\lambda(r,z)$ is strictly monotonic on L(r).

For $r \in (0,1)$, $S_1(r)$ and $L(r)$ are both analytic contours, this implies that $p_1(r,z)$ is regular for $z \in L(r)$, so that the existence of $\frac{d}{ds}g(r,s)$, $|s|=1$, implies that $\lambda(r,z)$ should have on $L(r)$ a derivative with respect to the arc coordinate on $L(r)$. Consequently, for fixed $r \in (0,1)$: (cf. Muskhelishvili [15], p.13)

$$g(r,e^{i\lambda(r,z)}) \textit{ and } \lambda(r,z) \textit{ satisfy on } L(r) \textit{ a Holder condition.} \tag{5.9}$$

Relation (5.3)(1) together with the conditions (4.8)(1) and the existence of the limit formulate a boundary value problem with boundary $L(r)$. This type of boundary value problem is discussed a.o. in Cohen and Boxma [7], Section I.1.7. From the results in that section it is immediately clear that

$$(1)\quad \log p_1(r,z) = \frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log \frac{g^2(r,e^{i\lambda(r,\zeta)})}{d} + \log \zeta]\,\frac{d\zeta}{\zeta - z},\quad z \in L^+(r), \tag{5.10}$$

$$(2)\quad \log \frac{zp_2^-(r,z)}{d} = -\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log \frac{g^2(r,e^{i\lambda(r,\zeta)})}{d} + \log \zeta]\,\frac{d\zeta}{\zeta - z},\quad z \in L^-(r).$$

From the fact that we have chosen $z = 1 \in L(r)$, (5.2) and (5.10)(1) it follows by applying the Plemelj-Sokhotski formulas:

$$\log p_1^+(r,1) = \tfrac{1}{2}\log \frac{g^2(r,1)}{d} + \frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log \frac{g^2(r,e^{i\lambda(r,\zeta)})}{d} + \log \zeta]\,\frac{d\zeta}{\zeta - 1}. \tag{5.11}$$

Hence, because of (5.1) and (5.2) we obtain the following expression:

$$\log d = \frac{1}{\pi i}\int_{\zeta\in L(r)} \log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}\,\frac{d\zeta}{\zeta - 1}. \tag{5.12}$$

Inserting (5.12) into (5.10) and applying Cauchy's theorem yields:

$$\log p_1(r,z) = \frac{2}{2\pi i}\int_{\zeta\in L(r)} [\log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}]\{\frac{1}{\zeta - z} - \frac{1}{\zeta - 1}\}\,d\zeta,\ z \in L^+(r), \tag{5.13}$$

$$\log zp_2(r,z) = -\frac{2}{2\pi i}\int_{\zeta\in L(r)} [\log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}]\{\frac{1}{\zeta-z}-\frac{1}{\zeta-1}\}\,d\zeta,\ z\in L^-(r).$$

Or, equivalently,

$$p_1(r,z) = \exp\{\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}]\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\},\ z\in L^+(r), \tag{5.14}$$

$$p_2(r,z) = z^{-1}\exp\{-\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}]\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\},\ z\in L^-(r).$$

It follows by applying the Plemelj-Sokhotski formulas that for $z\in L(r)$:

$$p_1^+(r,z) = g(r,e^{i\lambda(r,z)})z^{\frac{1}{2}}\exp\{\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}]\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\}, \tag{5.15}$$

$$p_2^-(r,z) = g(r,e^{i\lambda(r,z)})z^{-\frac{1}{2}}\exp\{-\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}]\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\}.$$

Substitution of the relations (5.15) into (5.3)(3) yields the following singular integral equation: for $z\in L(r)$, with $0<r<1$ fixed,

$$\exp\{i\lambda(r,z)\} = z^{\frac{1}{2}}\exp\{\frac{1}{2\pi i}\int_{\zeta\in L(r)} [\log\{g(r,e^{i\lambda(r,\zeta)})\zeta^{\frac{1}{2}}\}]\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\}. \tag{5.16}$$

We obtain similar expressions by starting from (5.3)(2) as a boundary condition:

$$p_1(r,z) = g(r,1)\exp\{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{2i\lambda(r,\zeta)-\log\zeta\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\},\ z\in L^+(r), \tag{5.17}$$

$$p_2(r,z) = g(r,1)z^{-1}\exp\{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{2i\lambda(r,\zeta)-\log\zeta\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\},\ z\in L^-(r).$$

$$g(r,e^{i\lambda(r,z)}) = g(r,1)z^{-\frac{1}{2}}\exp\{\frac{1}{2\pi i}\int_{\zeta\in L(r)} \{2i\lambda(r,\zeta)-\log\zeta\}\{\frac{\zeta+z}{\zeta-z}-\frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\},\ z\in L(r).$$

Because $p_1^+(r,z)$ maps the contour $L(r)$ one-to-one onto the contour $S_1(r)$, and because $p_1(r,z)$ as given by (5.14) is regular for $z\in L(r)\bigcup L^+(r)$, it follows from the principle of corresponding boundaries (cf. App. A) that $p_1(r,z)$ as given by (5.14) maps $L^+(r)$ conformally onto $S_1^+(r)$. Similarly for $p_2(r,z)$.

As in Cohen and Boxma [7], Section II.3.6, p.176, it is seen that (5.16) and (5.18) are equivalent and hence that also (5.14) and (5.17) are equivalent. Further as in Cohen and Boxma [7] it may be seen that relation (5.16) represents an integral equation for the determination of $L(r)$ and $\lambda(r,z)$, $z\in L(r)$.

6. The Riemann boundary value problem

In this section we formulate and solve a Riemann boundary value problem of the type as formulated in Cohen and Boxma [7], Section I.2.1.

Let $0 < r < 1$ be fixed. Because for $z \in L(r)$, $(p_1^+(r,z), p_2^-(r,z))$ is a zero pair of the kernel (4.3) it should hold for $z \in L(r)$ (cf. (4.7)):

$$r\{p_1^+(r,z)p_2^-(r,z) - r_1p_1^+(r,z) - r_2p_2^-(r,z)\}\beta\{\lambda(1-r_1p_1^+(r,z)-r_2p_2^-(r,z))\}\Phi(r;0,0) \tag{6.1}$$
$$+ r(p_2^-(r,z)-p_1^+(r,z))\beta\{\lambda(1-r_1p_1^+(r,z)-r_2p_2^-(r,z))\}\{r_2\Phi(r;p_1^+(r,z),0) - r_1\Phi(r;0,p_2^-(r,z))\}$$
$$+ p_1^+(r,z)p_2^-(r,z) = 0.$$

By noting that,

$$p_1^+(r,z)p_2^-(r,z) = r\{r_1p_2^-(r,z)+r_2p_1^+(r,z)\}\beta\{\lambda(1-r_1p_1^+(r,z)-r_2p_2^-(r,z))\}, \tag{6.2}$$

we write (6.1) as

$$r_2(p_1^+(r,z)-p_2^-(r,z))\Phi(r;p_1^+(r,z),0) + r_1(p_2^-(r,z)-p_1^+(r,z))\Phi(r;0,p_2^-(r,z)) = \tag{6.3}$$
$$\{p_1^+(r,z)p_2^-(r,z) - r_1p_1^+(r,z) - r_2p_2^-(r,z)\}\Phi(r;0,0) + r_1p_2^-(r,z) + r_2p_1^+(r,z).$$

Using (5.1) we obtain:

$$\Phi(r;p_1^+(r,z),0) = \frac{r_1}{r_2}\Phi(r;0,p_2^-(r,z)) + \frac{1}{r_2}H(r,z), \tag{6.4}$$

where

$$H(r,z) = \frac{\{g(r,e^{i\lambda(r,z)}) - r_1e^{i\lambda(r,z)} - r_2e^{-i\lambda(r,z)}\}\Phi(r;0,0) + r_1e^{-i\lambda(r,z)} + r_2e^{i\lambda(r,z)}}{e^{i\lambda(r,z)} - e^{-i\lambda(r,z)}}. \tag{6.5}$$

Note that $\frac{r_1}{r_2}$ is never zero on $L(r)$, that

$$\underset{z\in L(r)}{ind}\ \frac{r_1}{r_2} = 0, \tag{6.6}$$

and that $\frac{r_1}{r_2}$ satisfies (trivially) a Holder condition on $L(r)$. Because the numerator and the denominator of (6.5) both satisfy a Holder condition on $L(r)$ (cf. (5.9)) and the denominator is never zero on $L(r)$ except for $z = 1$, the function

$$\frac{1}{r_2}H(r,z),$$

satisfies a Holder condition on $L(r)$.

First note that the maximum modulus principle (cf. Nehari [16]) implies that:

$$|p_1(r,z)| < 1 \text{ for } z \in L^+(r), \tag{6.7}$$
$$|p_2(r,z)| < 1 \text{ for } z \in L^-(r),$$

because $p_1(r,z)$ is regular in $L^+(r)$, continuous in $L(r) \bigcup L^+(r)$ and $|p_1(r,z)| < 1$ for $z \in L(r)$, analogously for $p_2(r,z)$.

From (6.7) and the definition of $\Phi(r;p_1,p_2)$ it follows that:

$$\Phi(r;p_1(r,z),0) \textit{ should be regular for } z \in L^+(r), \tag{6.8}$$
$$\textit{continuous for } z \in L(r) \bigcup L^+(r),$$

$$\lim_{\substack{\zeta \to z \\ \zeta \in L^+(r)}} \Phi(r;p_1(r,\zeta),0) = \Phi(r;p_1^+(r,z),0), \quad z \in L(r);$$

$$\Phi(r;0,p_2^-(r,z)) \textit{ should be regular } \text{for } z \in L^-(r), \tag{6.9}$$

$$\textit{continuous } \text{for } z \in L(r) \cup L^-(r),$$

$$\lim_{\substack{\zeta \to z \\ \zeta \in L^-(r)}} \Phi(r;0,p_2(r,\zeta)) = \Phi(r;0,p_2^-(r,z)), \quad z \in L(r).$$

Note that, according to (4.8)(4),

$$\lim_{|z| \to \infty} \Phi(r;0,p_2^-(r,z)) = \Phi(r;0,0). \tag{6.10}$$

The conditions (6.4), (6.8) and (6.9) formulate the Riemann boundary value problem on the contour $L(r)$ announced in the beginning of this section. We may and do apply the results of Cohen and Boxma [7], Sections I.2.1-I.2.4 and obtain:[1]

$$r_2\Phi(r;p_1(r,z),0) = \frac{1}{2\pi i} \int_{\zeta \in L(r)} H(r,\zeta) \frac{d\zeta}{\zeta - z} + r_1\Phi(r;0,0), \quad z \in L^+(r), \tag{6.11}$$

$$r_1\Phi(r;0,p_2(r,z)) = \frac{1}{2\pi i} \int_{\zeta \in L(r)} H(r,\zeta) \frac{d\zeta}{\zeta - z} + r_1\Phi(r;0,0), \quad z \in L^-(r),$$

where $H(r,\cdot)$ is given by (6.5).
$\Phi(r;0,0)$ can be determined from the observation that

$$\Phi(r;0,0) = \sum_{n=0}^{\infty} r^n E\{(\mathbf{z}_n^{(1)}=0, \mathbf{z}_n^{(2)}=0)\} = \sum_{n=0}^{\infty} r^n Pr\{\mathbf{z}_n^{(1)} + \mathbf{z}_n^{(2)} = 0\}, \tag{6.12}$$

where $\{\mathbf{z}_n^{(1)} + \mathbf{z}_n^{(2)}, n=0,1,2,...\}$ is the queue-length process, measured at departure epochs, of an M/G/1 queueing system with arrival rate λ and service-time distribution with LST $\beta(\rho)$. It may be proved that

$$\Phi(r;0,0) = \frac{1}{1 - E\{r^{\mathbf{n}}\}}, \tag{6.13}$$

with $\mathbf{n}$ the number of customers served in a busy period of the M/G/1 queueing system described above.

In Cohen [5] it is proven that

$$E\{r^{\mathbf{n}}\} = \sum_{n=1}^{\infty} \int_{t=0}^{\infty} r^n e^{-\lambda t} \frac{(\lambda t)^{n-1}}{n!} \, dB^{n*}(t). \tag{6.14}$$

1. By applying Liouville's theorem it is possible to derive the results (6.11) more straightforwardly, cf, Cohen [4].

7. The determination of $\Phi(r;p_1,p_2)$ for $p_1 \in S_1^+(r)$, $p_2 \in S_2^+(r)$

According to the discussion at the end of Section 5, $p_1(r,z)$ maps $L^+(r)$ conformally onto $S_1^+(r)$. Denote by,

$$z = p_{10}(r,p_1), \quad p_1 \in S_1^+(r), \tag{7.1}$$

the inverse mapping, that is the conformal map of $S_1^+(r)$ onto $L^+(r)$. Analogously, denote by

$$z = p_{20}(r,p_2), \quad p_2 \in S_2^+(r), \tag{7.2}$$

the conformal map of $S_2^+(r)$ onto $L^-(r)$.
Because $S_1(r)$ and $S_2(r)$ are smooth contours and $L(r)$ is also smooth, the theorem of corresponding boundaries (cf. App. A) implies that $p_{10}(r,\cdot)$ maps $S_1(r)$ onto $L(r)$, and $p_{20}(r,\cdot)$ maps $S_2(r)$ onto $L(r)$.

From the preceding section, the following is now immediately clear:
for $0<r<1, p_1 \in S_1^+(r), p_2 \in S_2^+(r)$,

$$\begin{aligned} r_2\Phi(r;p_1,0) &= \frac{1}{2\pi i}\int_{\zeta\in L(r)} H(r,\zeta)\frac{d\zeta}{\zeta-p_{10}(r,p_1)} + r_1\Phi(r;0,0),\\ r_1\Phi(r;0,p_2) &= \frac{1}{2\pi i}\int_{\zeta\in L(r)} H(r,\zeta)\frac{d\zeta}{\zeta-p_{20}(r,p_2)} + r_1\Phi(r;0,0),\end{aligned} \tag{7.3}$$

with $\Phi(r;0,0)$ given by (6.13).
By substitution of (7.3) and (6.13) into (2.13) it follows:
for $0<r<1, p_1 \in S_1^+(r), p_2 \in S_2^+(r)$,

$$\begin{aligned}\Phi(r;p_1,p_2) = {} & \frac{r\beta\{\lambda(1-r_1p_1-r_2p_2)\}(p_2-p_1)}{p_1p_2-\{r_1p_2+r_2p_1\}r\beta\{\lambda(1-r_1p_1-r_2p_2)\}}\times\\ & \frac{1}{2\pi i}\int_{\zeta\in L(r)} H(r,\zeta)\{\frac{1}{\zeta-p_{10}(r,p_1)}-\frac{1}{\zeta-p_{20}(r,p_2)}\}\,d\zeta\; + \\ & r\{p_1p_2-r_1p_2-r_2p_1\}\beta\{\lambda(1-r_1p_1-r_2p_2)\}\Phi(r;0,0) + p_1p_2.\end{aligned} \tag{7.4}$$

We should now use the technique of analytic continuation to obtain an expression for $\Phi(r;p_1,p_2)$ that holds for $|p_1| \leq 1$, $|p_2| \leq 1$. However, since the main interests in this report are the (first) moments of the queue-length distribution, such an analytic continuation will not be carried out here. It will appear in the sequel that the (first) moments can be computed from the results in Section 6.
In the next sections the stationary distribution of the queue-length process is investigated.

8. The stationary distribution - I

As pointed out in Section 2 the imbedded Markov chain $\{(\mathbf{z}_n^{(1)},\mathbf{z}_n^{(2)}),\ n=0,1,2,...\}$ is irreducible and aperiodic. It will be shown under which conditions this Markov chain consists of positive recurrent, null recurrent or transient states (cf. Chung [3]) respectively.

For $n=0,1,2,...$, for $k_1,k_2=0,1,2,...$, denote by:

$$p_{k_1k_2}^{(n)} := Pr\{\mathbf{z}_n^{(1)}=k_1,\ \mathbf{z}_n^{(2)}=k_2\ |\mathbf{z}_0^{(1)}=0,\ \mathbf{z}_0^{(2)}=0\}, \tag{8.1}$$

the conditional probability that at the epoch of the n^{th} service completion k_j customers of type "j", j=1,2, are left behind in the system, given that the system is empty at $t=0$.
The following theorem is easily verified (cf. Cohen [5]):

Theorem 8.1
For $k_1,k_2 = 0,1,2,\ldots$, *the limits*

$$\lim_{n\to\infty} p^{(n)}_{k_1,k_2}, \tag{8.2}$$

exist and are independent of the initial state. If the Markov chain $\{(\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)}), n=0,1,2,\ldots\}$ *is ergodic then*

$$u_{k_1,k_2} := \lim_{n\to\infty} p^{(n)}_{k_1,k_2} > 0, \tag{8.3}$$

for every $k_1,k_2 = 0,1,2,\ldots$, *and*

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} u_{k_1,k_2} = 1, \tag{8.4}$$

otherwise $u_{k_1,k_2} = 0$ *for every* $k_1,k_2 = 0,1,2,\ldots$.

Denote by $(\mathbf{z}_1, \mathbf{z}_2)$ a stochastic vector with for every $k_1,k_2 = 0,1,2,\ldots$,

$$Pr\{\mathbf{z}_1=k_1, \mathbf{z}_2=k_2\} = u_{k_1,k_2}. \tag{8.5}$$

We define for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$ the generating function of the joint distribution of $(\mathbf{z}_1, \mathbf{z}_2)$ by:

$$\Phi(p_1,p_2) := \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} p_1^{k_1}p_2^{k_2}u_{k_1,k_2}. \tag{8.6}$$

From Theorem 8.1 and a well-known Abelian theorem for generating functions (cf. Titchmarsh [18]) it follows,

Lemma 8.1
If the states of the Markov chain $\{(\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)}), n=0,1,2,\ldots\}$ *are positive recurrent then for* $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$\lim_{r\uparrow 1}(1-r)\Phi(r;p_1,p_2) = \Phi(p_1,p_2), \tag{8.7}$$

and if they are not then for $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, $p_1\neq 1$, $p_2\neq 1$,

$$\lim_{r\uparrow 1}(1-r)\Phi(r;p_1,p_2) = 0, \tag{8.8}$$

both limits being independent of the initial state.

Because all states of an irreducible Markov chain are of the same type it is sufficient to investigate the state (0,0), i.e., we have to consider

$$\lim_{r\uparrow 1}(1-r)\Phi(r;0,0), \tag{8.9}$$

in order to obtain the conditions for which the Markov chain is ergodic.
From (6.13) it follows that

$$\begin{aligned}\lim_{r\uparrow 1}(1-r)\Phi(r;0,0) = \lim_{r\uparrow 1}\frac{1-r}{1-E\{r^{\mathbf{n}}\}} &= 1-\lambda\beta \quad \text{if } \lambda\beta<1,\\ &= 0 \quad\quad \text{if } \lambda\beta\geqslant 1,\end{aligned} \tag{8.10}$$

(cf. Cohen [5]). Hence the Markov chain $\{(\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)}), n=0,1,2,\ldots\}$ is ergodic if and only if $\lambda\beta < 1$.

Remark 8.1
Obviously, the distribution of the busy period of the present model is identical with that of the

process $\{\mathbf{z}_n^{(1)} + \mathbf{z}_n^{(2)}, n=0,1,2,...\}$. This process is the queue-length process of an M/G/1 queueing system with arrival rate λ and service-time distribution with LST $\beta(\rho)$. It is well known that this process has a busy period with finite first moment if and only if $\lambda\beta < 1$. Consequently, the process $\{(\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)}), n=0,1,2,...\}$ possesses a unique stationary distribution if and only if $\lambda\beta < 1$.

From Lemma 8.1 and (7.4) we should now be able to obtain the generating function (8.6) by investigating $(1-r)\Phi(r;p_1,p_2)$ for $r \uparrow 1$. The evaluation of this limit however, poses some serious difficulties, not in in the least because of the occurrence of r in the integration contour $L(r)$. For a treatment of this limit in a similar case cf. Blanc [1]. In this report we shall take a different approach which will be pointed out in the next section.

9. The stationary distribution - II

Another way to find the (generating function of the) stationary distribution is to derive it directly, in an analogous manner as for the time-dependent case. In this section the major differences are indicated and at the end the results are presented.
Throughout this section the restrictions (4.1) are assumed to hold.
By $(\mathbf{z}_1, \mathbf{z}_2)$ the stochastic vector with distribution the stationary distribution of the process $\{(\mathbf{z}_n^{(1)}, \mathbf{z}_n^{(2)}), n=0,1,2,...\}$ is denoted. If this process is stationary then the function

$$E\{p_1^{\mathbf{z}_n^{(1)}} p_2^{\mathbf{z}_n^{(2)}} \mid \mathbf{z}_0^{(1)}=0, \mathbf{z}_0^{(2)}=0\}, \tag{9.1}$$

is independent of n and of the initial state. Hence, denoting by $\Phi(p_1,p_2)$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, the stationary distribution it follows from (2.12):

$$\begin{aligned}\{p_1p_2-(r_1p_2+r_2p_1)\beta(x)\}\Phi(p_1,p_2) &= (p_2-p_1)\beta(x)\{r_2\Phi(p_1,0)-r_1\Phi(0,p_2)\} \\ &+ \{p_1p_2-(r_1p_1+r_2p_2)\}\beta(x)\Phi(0,0),\end{aligned} \tag{9.2}$$

where x is given by (2.10). As it is seen from (8.10) we have

$$\Phi(0,0) = 1-\lambda\beta. \tag{9.3}$$

This is also immediately clear by comparing with the M/G/1 queueing model discussed in Section 6. As can be seen from (9.2) the kernel of the functional equation is in this case given by:

$$K(p_1,p_2) := p_1p_2-(r_1p_2+r_2p_1)\beta\{\lambda(1-r_1p_1-r_2p_2)\}. \tag{9.4}$$

Concerning the analysis of the kernel (9.4) we state the following analogon of Lemma 3.1

Lemma 9.1

(1) If $\lambda\beta < 1$ the kernel $K(gs,gs^{-1})$ has in $|g|\leqslant 1$ exactly two zeros, of which one is identically zero. Denote the other zero by

$$g = g(s), \tag{9.5}$$

it is given by:
for $|s|=1$,

$$g(s) = (r_1s^{-1}+r_2s)E\{\ |r_1s+r_2s^{-1}|^{2n-2} \tag{9.6}$$

(2) for $|s|=1$:

$$g(s) = -g(-s),\ g(\bar{s}) = \overline{g(s)}. \tag{9.7}$$

proof of lemma 9.1
It is clear that, for fixed $|s|=1$, $g=0$ is a zero of $K(gs,gs^{-1})$, $|g|\leqslant 1$. By a direct proof or noting

Figures 4 - 5 it is seen that for $|s|=1,\ s\neq-1,+1,\ |g|=1$,

$$|(r_1s^{-1}+r_2s)\beta\{\lambda(1-g(r_1s+r_2s^{-1}))\}| \leqslant |r_1s^{-1}+r_2s| < 1 = |g|. \tag{9.8}$$

Hence, as in Section 3, it can be seen by applying Rouche's theorem that $g^{-1}K(gs,gs^{-1})$, $|s|=1,\ s\neq1,-1$ has in $|g|\leqslant1$ exactly one zero, with multiplicity one.
By applying Takacs' lemma (cf. Cohen [5]) to the equation in g:

$$g = \beta\{\lambda(1-g)\}, \tag{9.9}$$

it follows that if $s = 1$ then $g = 1$ is the only zero of $g^{-1}K(gs,gs^{-1})$ in $|g|\leqslant1$ and that $\lambda\beta < 1$ implies that it is a simple zero. Analogously, if $s = -1$ then $g = -1$ is the only zero of $g^{-1}K(gs,gs^{-1})$ in $|g|\leqslant1$ and its multiplicity is one if $\lambda\beta < 1$. This proves the first statement of (9.9) (1). The second statement of (9.9) (1) has an analogous proof as in Lemma 3.1 whereas the validity of the statements (9.9) (2) is readily verified.

Define

$$\begin{aligned} S_1 &:= \{p_1 : p_1=g(s)s,\ |s|=1\}, \\ S_2 &:= \{p_2 : p_2=g(s)s^{-1},\ |s|=1\}, \end{aligned} \tag{9.10}$$

with $g(s)$ defined by (9.6).
In precisely the same manner as for Lemma 3.2 we can prove:

LEMMA 9.2
The contours S_1 and S_2 are simply connected and smooth.

Furthermore we have,

$$p_1 = 0 \in S_1^-,\ p_2 = 0 \in S_2^+. \tag{9.11}$$

Note that it is readily verified from (9.6) that the contours S_1 and S_2 are analytic contours for $|s|=1,\ s\neq1,-1$.

REMARK 9.1
In Appendix B several graphs have been plotted of the contours S_1 and S_2 for various values of the parameters.

The boundary value problem reads in this case:

construct in the z-plane a simply connected Jordan contour L and a pair of mappings $p_1(z),\ z \in L \bigcup L^+,\ p_2(z),\ z \in L \bigcup L^-$, such that

(9.12)

(*i*) $p_1(z)$ *is regular* and *univalent* for $z \in L^+$,
continuous for $z \in L \bigcup L^+$,

(*ii*) $p_2(z)$ *is regular* and *univalent* for $z \in L^-$,
continuous for $z \in L \bigcup L^-$;

(*ii*) $p_1(z)$ *maps* L^+ *conformally onto* S_1^+,
$p_2(z)$ *maps* L^- *conformally onto* S_2^+;

(*iii*) $p_1^+(z),\ p_2^-(z),\ z \in L$ *is a zero pair of the kernel* (9.4);

(*iv*) $p_1(0) > 0^{*)}$, $p_2(\infty) = 0,\ 0 < d := \lim_{|z|\to\infty} |zp_2(z)| < \infty$;

$$p_1(1) = 1^{*)}.$$

*) Note that it is always possible to choose the origin of the z-plane so that it belongs to L^+. Further it is possible to choose $1 \in L$.

The remaining formulas are completely analogous to the formulas derived in the preceding sections. We shall just give the analogons of the most important expressions here.

$$p_1(z) = \exp\{\frac{1}{2\pi i}\int_{\zeta\in L} [\log\{g(e^{i\lambda(\zeta)})\zeta^{1/2}\}]\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\}, \quad z \in L^+, \tag{9.13}$$

$$p_2(z) = z^{-1}\exp\{-\frac{1}{2\pi i}\int_{\zeta\in L} [\log\{g(e^{i\lambda(\zeta)})\zeta^{1/2}\}]\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\}, \quad z \in L^-.$$

The relation for the determination of L and $\lambda(z)$, $z \in L$:

$$\exp\{i\lambda(z)\} = z^{1/2}\exp\{\frac{1}{2\pi i}\int_{\zeta\in L} [\log\{g(e^{i\lambda(\zeta)})\zeta^{1/2}\}]\{\frac{\zeta+z}{\zeta-z} - \frac{\zeta+1}{\zeta-1}\}\frac{d\zeta}{\zeta}\}, \quad z \in L. \tag{9.14}$$

The following Riemann boundary value problem for the contour L is to be solved:

To construct in the z-plane a simply connected Jordan contour L and a pair of mappings $p_1(z)$, $z \in L \bigcup L^+$, $p_2(z)$, $z \in L \bigcup L^-$, such that

$\Phi(p_1(z),0)$ *should be regular* for $z \in L^+$, *(9.15)*

continuous for $z \in L \bigcup L^+$,

$$\lim_{\substack{\zeta\to z\\ \zeta\in L^+}} \Phi(p_1(\zeta),0) = \Phi(p_1^+(z),0), \quad z \in L;$$

$\Phi(0,p_2(z))$ *should be regular* for $z \in L^-$,

continuous for $z \in L \bigcup L^-$,

$$\lim_{\substack{\zeta\to z\\ \zeta\in L^-}} \Phi(0,p_2(\zeta)) = \Phi(0,p_2^-(z)), \quad z \in L.$$

The boundary condition:
for $z \in L$,

$$\Phi(p_1^+(z),0) = \frac{r_1}{r_2}\Phi(0,p_2^-(z)) + \frac{1}{r_2}H(z), \tag{9.16}$$

where

$$H(z) = \frac{g(e^{i\lambda(z)}) - r_1 e^{i\lambda(z)} - r_2 e^{-i\lambda(z)}}{e^{i\lambda(z)} - e^{-i\lambda(z)}}, \quad z \in L. \tag{9.17}$$

The solution of the Riemann boundary value problem presented above is given by:

$$r_2\Phi(p_1(z),0) = \frac{\Pi_0}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta-z} + r_1\Pi_0, \quad z \in L^+, \tag{9.18}$$

$$r_1\Phi(0,p_2(z)) = \frac{\Pi_0}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta-z} + r_1\Pi_0, \quad z \in L^-,$$

where $H(\cdot)$ is given by (9.17) and (cf. (8.10))

$$\Pi_0 := \Phi(0,0) = 1-\lambda\beta. \tag{9.19}$$

The expressions, analogous to the ones in Section 7 will not be given here, because we can compute the moments of the queue-length distribution without them.

Chapter III

Averages

In this chapter we investigate the first moments of the actual waiting-time distribution, the first moments of the sojourn-time distribution, and, for each queue, the mean number of customers measured at an arbitrary epoch or at a departure instant. Throughout this chapter we suppose that the system is in equilibrium. Furthermore we assume general α_i, r_i and β_i, but of course $a = a_1 + a_2 < 1$.

1. Introduction

According to Little's formula we have,

$$E\{\mathbf{x}_i\} = \lambda_i E\{\mathbf{s}_i\}, \quad i = 1,2, \tag{1.1}$$

where, for $i = 1,2$,

$\mathbf{x}_i$: *number of customers of type i in the system, measured at an arbitrary epoch,* (1.2)

$\mathbf{s}_i$: *sojourn time of a type $-i$ customer,*

λ_i: *arrival intensity of type $-i$ customers.*

Because the input of queue i is Poissonian, it follows that the distribution of the number of type-i customers at an arbitrary epoch is equal to the distribution of the number of type-i customers immediately before an arrival epoch of a type-i customer: Because of the "memoryless" property of the negative-exponential distribution the duration of time separating an arbitrary instant and the preceding arrival epoch has the same (exponential) distribution as does the time separating successive arrival epochs. Since the state of any system at any epoch is entirely determined by the sequence of arrivals and service times prior to the epoch, it follows, that, when the input process is Poisson, no inference about the state of the system at any epoch can be drawn from knowledge of whether or not the epoch in question is an arrival epoch. (cf. Wolff [19])

In Appendix A, Section 2 it is proven by an "up-and-down-crossings" argument that the distribution of the number of type-i customers immediately before an arrival epoch of type-i customer equals the distribution of the number of type-i customers left behind in the system after the service completion of a type-i customer.

From the two arguments above it is seen that the distribution of the number of type-i customers in the system at an arbitrary epoch is equal to the distribution of the number of type-i customers in the system at a departure epoch of a type-i customer. Hence we have,

$$E\{\mathbf{x}_i\} = E\{\mathbf{z}_n^{(i)} \mid \mathbf{r}_n = i\}, \tag{1.3}$$

where, for $i = 1,2$,

$$\begin{aligned} \mathbf{x}_i &: \textit{ number of type} - i \textit{ customers in the system,} \\ &\quad \textit{measured at an arbitrary epoch,} \\ \mathbf{z}_n^{(i)} &: \textit{ number of type} - i \textit{ customers, left behind in the system} \\ &\quad \textit{after the service completion of the } n^{th} \textit{ customer,} \\ \mathbf{r}_n = i &: \text{ if } \textit{the } n^{th} \textit{ departing customer is of type i.} \end{aligned} \tag{1.4}$$

In the preceding chapter we have determined,

$$\Phi(p_1,p_2) = E\{p_1^{\mathbf{z}_n^{(1)}} p_2^{\mathbf{z}_n^{(2)}}\}, \tag{1.5}$$

we now have to establish a relation between $E\{\mathbf{z}_n^{(i)} \mid \mathbf{r}_n = i\}$ and $E\{\mathbf{z}_n^{(i)}\}$ for $i = 1,2$. This will be effectuated in Section 2. Expressions for the various first moments will be given in Section 3, while Section 4 and Section 5 are devoted to a numerical evaluation of these expressions.

2. The relation between $E\{\mathbf{z}_n^{(i)} \mid \mathbf{r}_n = i\}$ and $E\{\mathbf{z}_n^{(i)}\}$, $i = 1,2$.

As before, let $\mathbf{z}_n^{(i)}$, $i = 1,2$, denote the number of customers in queue i, left behind at the departure of the n^{th} customer from the system, and let $\mathbf{r}_n = i$ if the n^{th} customer is of type i, $i = 1,2$. We abbreviate the event $\{\mathbf{z}_n^{(1)} = i_1, \mathbf{z}_n^{(2)} = i_2, \mathbf{r}_n = i\}$ to $(i_1,i_2,i)_n$.
It easily follows, that, for $k = 1,2$:

$$(1)\quad Pr\{(i_1,i_2,1)_{n+1} \mid (0,0,k)_n\} = r_1 \int_0^\infty \frac{(\lambda_1 t)^{i_1}}{i_1!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{i_2}}{i_2!} e^{-\lambda_2 t}\, dB_1(t); \tag{2.1}$$

$$(2)\quad Pr\{(i_1,i_2,1)_{n+1} \mid (j_1,0,k)_n\} = \int_0^\infty \frac{(\lambda_1 t)^{i_1+1-j_1}}{(i_1+1-j_1)!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{i_2}}{i_2!} e^{-\lambda_2 t}\, dB_1(t),$$

for $j_1 = 1,\ldots,i_1+1$;

$$(3)\quad Pr\{(i_1,i_2,1)_{n+1} \mid (j_1,j_2,k)_n\} = \alpha_1 \int_0^\infty \frac{(\lambda_1 t)^{i_1+1-j_1}}{(i_1+1-j_1)!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{i_2-j_2}}{(i_2-j_2)!} e^{-\lambda_2 t}\, dB_1(t),$$

for $j_1 = 1,\ldots,i_1+1$, $j_2 = 1,\ldots,i_2$;

$$(4)\quad Pr\{(i_1,i_2,1)_{n+1} \mid (j_1,j_2,k)_n\} = 0 \text{ for } j_1,j_2 \textit{ other than in } (1),(2) \textit{ or } (3).$$

and analogously,

$$(5)\quad Pr\{(i_1,i_2,2)_{n+1} \mid (0,0,k)_n\} = r_2 \int_0^\infty \frac{(\lambda_1 t)^{i_1}}{i_1!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{i_2}}{i_2!} e^{-\lambda_2 t}\, dB_2(t);$$

$$(6)\quad Pr\{(i_1,i_2,2)_{n+1} \mid (0,j_2,k)_n\} = \int_0^\infty \frac{(\lambda_1 t)^{i_1}}{i_1!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{i_2+1-j_2}}{(i_2+1-j_2)!} e^{-\lambda_2 t}\, dB_2(t),$$

for $j_2 = 1,\ldots,i_2+1$;

$$(7)\quad Pr\{(i_1,i_2,2)_{n+1} \mid (j_1,j_2,k)_n\} = \alpha_2 \int_0^\infty \frac{(\lambda_1 t)^{i_1-j_1}}{(i_1-j_1)!} e^{-\lambda_1 t} \frac{(\lambda_2 t^{i_2+1-j_2}}{(i_2+1-j_2)} e^{-\lambda_2 t}\, dB_2(t),$$

for $j_1 = 1,...,i_1,\ j_2 = 1,...,i_2+1;$

(8) $Pr\{(i_1,i_2,2)_{n+1} \mid (j_1,j_2,k)_n\} = 0$, for j_1,j_2 *other than in* (5),(6) *or* (7).

Note that, since we assume the system to be stationary, $Pr\{(i_1,i_2,i)_{n+1} \mid (j_1,j_2,k)_n\}$ does not depend on the value of n.
Next define for $|p_1| \leqslant 1,\ |p_2| \leqslant 1$ and $i=1,2$:

$$\Phi^{(i)}(p_1,p_2) := E\{p_1^{\mathbf{z}_{n+1}^{(1)}} p_2^{\mathbf{z}_{n+1}^{(2)}} (\mathbf{r}_{n+1}=i)\} \tag{2.2}$$
$$= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} p_1^{i_1} p_2^{i_2} Pr\{(i_1,i_2,i)_{n+1}\}.$$

Writing out (2.2) yields:

$$\Phi^{(i)}(p_1,p_2) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} p_1^{i_1} p_2^{i_2} \sum_{j=1}^{2} Pr\{(i_1,i_2,i)_{n+1} \mid (0,0,j)_n\} Pr\{(0,0,j)_n\} \tag{2.3}$$
$$+ \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} p_1^{i_1} p_2^{i_2} \sum_{j_1=1}^{\infty} \sum_{j=1}^{2} Pr\{(i_1,i_2,i)_{n+1} \mid (j_1,0,j)_n\} Pr\{(j_1,0,j)_n\}$$
$$+ \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} p_1^{i_1} p_2^{i_2} \sum_{j_2=1}^{\infty} \sum_{j=1}^{2} Pr\{(i_1,i_2,i)_{n+1} \mid (0,j_2,j)_n\} Pr\{(0,j_2,j)_n\}$$
$$+ \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} p_1^{i_1} p_2^{i_2} \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \sum_{j=1}^{2} Pr\{(i_1,i_2,i)_{n+1} \mid (j_1,j_2,j)_n\} Pr\{(j_1,j_2,j)_n\}.$$

By substituting (2.1) into (2.3) we obtain, with x given by II.(2.10):
for $i=1,\ |p_1| \leqslant 1,\ |p_2| \leqslant 1$,

$$\Phi^{(1)}(p_1,p_2) = r_1 \beta_1(x) Pr\{\mathbf{z}_n^{(1)}=0,\ \mathbf{z}_n^{(2)}=0\} \tag{2.4}$$
$$+ \frac{\beta_1(x)}{p_1} \sum_{j_1=1}^{\infty} p_1^{j_1} Pr\{\mathbf{z}_n^{(1)}=j_1,\ \mathbf{z}_n^{(2)}=0\}$$
$$+ \frac{\alpha_1}{p_1} \beta_1(x) \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} p_1^{j_1} p_2^{j_2} Pr\{\mathbf{z}_n^{(1)}=j_1,\ \mathbf{z}_n^{(2)}=j_2\};$$

for $i=2,\ |p_1| \leqslant 1,\ |p_2| \leqslant 1$,

$$\Phi^{(2)}(p_1,p_2) = r_2 \beta_2(x) Pr\{\mathbf{z}_n^{(1)}=0,\ \mathbf{z}_n^{(2)}=0\}$$
$$+ \frac{\beta_2(x)}{p_2} \sum_{j_2=1}^{\infty} p_2^{j_2} Pr\{\mathbf{z}_n^{(1)}=0,\ \mathbf{z}_n^{(2)}=j_2\}$$
$$+ \frac{\alpha_2}{p_1} \beta_1(x) \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} p_1^{j_1} p_2^{j_2} Pr\{\mathbf{z}_n^{(1)}=j_1,\ \mathbf{z}_n^{(2)}=j_2\}.$$

Recall from Chapter II, Section 9, for $|p_1| \leqslant 1,\ |p_2| \leqslant 1$:

$$\Phi(p_1,p_2) = E\{p_1^{\mathbf{z}_n^{(1)}} p_2^{\mathbf{z}_n^{(2)}}\}. \tag{2.5}$$

It follows from (2.4) and (2.5):

$$\Phi^{(1)}(p_1,p_2) = \frac{\alpha_1}{p_1} \beta_1(x) \Phi(p_1,\ p_2) + \frac{\alpha_2}{p_1} \beta_1(x) \Phi(p_1,0) - \frac{\alpha_1}{p_1} \beta_1(x) \Phi(0,p_2) \tag{2.6}$$
$$+ (r_1 - \frac{\alpha_2}{p_1}) \beta_1(x) \Phi(0,0);$$

$$\Phi^{(2)}(p_1,p_2) = \frac{\alpha_2}{p_2}\beta_2(x)\Phi(p_1,p_2) + \frac{\alpha_1}{p_2}\beta_2(x)\Phi(0,p_2) - \frac{\alpha_2}{p_2}\beta_2(x)\Phi(p_1,0)$$
$$+ (r_2-\frac{\alpha_1}{p_2})\beta_2(x)\Phi(0,0).$$

REMARK 2.1
Taking $p_1 = p_2 = 1$ in one of the relations (2.6) yields the following interesting relation:

$$r_1 = \alpha_1 + \alpha_2 Pr\{\mathbf{z}_n^{(2)}=0\} - \alpha_1 Pr\{\mathbf{z}_n^{(1)}=0\} + (r_1-\alpha_2)Pr\{\mathbf{z}_n^{(1)}=0,\ \mathbf{z}_n^{(2)}=0\}. \tag{2.7}$$

We now determine $E\{\mathbf{z}_n^{(1)}(\mathbf{r}_n=1)\}$. First substitute $p_2=1$ in the first relation of (2.6):

$$\Phi^{(1)}(p_1,1) = \frac{\alpha_1}{p_1}\beta_1\{r_1\lambda(1-p_1)\}\Phi(p_1,1) + \frac{\alpha_2}{p_1}\beta_1\{r_1\lambda(1-p_1)\}\Phi(p_1,0) \tag{2.8}$$
$$- \frac{\alpha_1}{p_1}\beta_1\{r_1\lambda(1-p_1)\}\Phi(0,1) + (r_1-\frac{\alpha_2}{p_1})\beta_1\{r_1\lambda(1-p_1)\}\Phi(0,0).$$

By differentiating (2.8) once with respect to p_1, substituting $p_1 = 1$ and using (2.7) we obtain:

$$E\{\mathbf{z}_n^{(1)}(\mathbf{r}_n=1)\} = \frac{d}{dp_1}\Phi^{(1)}(p_1,1)\,|_{p_1=1} \tag{2.9}$$
$$= \alpha_1 E\{\mathbf{z}_n^{(1)}\} + \alpha_2 E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\} - r_1 r_2\lambda\beta_2.$$

In a similar way we obtain:

$$E\{\mathbf{z}_n^{(2)}(\mathbf{r}_n=2)\} = \alpha_2 + \alpha_1 E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\} - r_1 r_2\lambda\beta_1. \tag{2.10}$$

The quantity we are interested in, however, is

$$E\{\mathbf{z}_n^{(i)}\ |\mathbf{r}_n=i\},\ \ i=1,2. \tag{2.11}$$

Of course we have

$$E\{\mathbf{z}_n^{(i)}\ |\mathbf{r}_n=i\} = \frac{1}{r_i}E\{\mathbf{z}_n^{(i)}(\mathbf{r}_n=i)\}. \tag{2.12}$$

Hence, from (2.9), (2.10) and (2.12):

$$E\{\mathbf{z}_n^{(1)}\ |\mathbf{r}_n=1\} = \frac{\alpha_1}{r_1}E\{\mathbf{z}_n^{(1)}\} + \frac{\alpha_2}{r_1}E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\} - r_2\lambda\beta_2; \tag{2.13}$$
$$E\{\mathbf{z}_n^{(2)}\ |\mathbf{r}_n=2\} = \frac{\alpha_2}{r_2}E\{\mathbf{z}_n^{(2)}\} + \frac{\alpha_1}{r_2}E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\} - r_1\lambda\beta_1.$$

The expressions (2.13) supply us with a relation between $E\{\mathbf{z}_n^{(i)}\}$ and $E\{\mathbf{z}_n^{(i)}\ |\mathbf{r}_n=i\}$, $i=1,2$. Now from (1.1),(1.3) and (2.13) we find for the mean sojourn time $E\{\mathbf{s}_i\}$ of a type-i customer:

$$E\{\mathbf{s}_1\} = \frac{\alpha_1}{r_1^2\lambda}E\{\mathbf{z}_n^{(1)}\} + \frac{\alpha_2}{r_1^2\lambda}E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\} - \frac{r_2}{r_1}\beta_2; \tag{2.14}$$
$$E\{\mathbf{s}_2\} = \frac{\alpha_2}{r_2^2\lambda}E\{\mathbf{z}_n^{(2)}\} + \frac{\alpha_1}{r_2^2\lambda}E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\} - \frac{r_1}{r_2}\beta_1.$$

3. EXPLICIT EXPRESSIONS FOR THE VARIOUS FIRST MOMENTS.

In this section we derive an expression for $E\{\mathbf{z}_n^{(i)}\}$, the mean number of type-i customers in the system, measured at an arbitrary departure epoch. With the help of Little's formula and the relations in the preceding section expressions for the mean actual waiting and sojourn times are derived.

We start from the functional equation in the stationary case. This equation reads: (cf.II.(9.2)) for $|p_1|\leq 1$, $|p_2|\leq 1$:

$$\begin{aligned}&[p_1p_2-\alpha_1p_2\beta_1(x)-\alpha_2p_1\beta_2(x)]E\{p_1^{\mathbf{z}_n^{(1)}}p_2^{\mathbf{z}_n^{(2)}}\} = \\ &[p_2\beta_1(x)-p_1\beta_2(x)][\alpha_2E\{p_1^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} - \alpha_1E\{p_2^{\mathbf{z}_n^{(2)}}(\mathbf{z}_n^{(1)}=0)\}] + \\ &[p_1p_2\{r_1\beta_1(x)+r_2\beta_2(x)\} + \alpha_2p_2\beta_1(x) + \alpha_1p_1\beta_2(x)]\Pi_0,\end{aligned} \tag{3.1}$$

with x given by (II.(2.10)) and (cf. II.(9.19)):

$$\Pi_0 := \Phi(0,0) = 1-\lambda_1\beta_1-\lambda_2\beta_2. \tag{3.2}$$

Put,

$$\tilde{x} := \lambda_1(1-p_1). \tag{3.3}$$

By substituting $p_2=1$ into (3.1) we find:

$$\begin{aligned}&[p_1-\alpha_1\beta_1(\tilde{x})-\alpha_2p_1\beta_2(\tilde{x})]E\{p_1^{\mathbf{z}_n^{(1)}}\} = \\ &[\beta_1(\tilde{x})-p_1\beta_2(\tilde{x})][\alpha_2E\{p_1^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} - \alpha_1Pr\{\mathbf{z}_n^{(1)}=0\}] + \\ &[p_1\{r_1\beta_1(\tilde{x})+r_2\beta_2(\tilde{x})\} - \alpha_2\beta_1(\tilde{x}) - \alpha_1p_1\beta_2(\tilde{x})]\Pi_0.\end{aligned} \tag{3.4}$$

Or, equivalently,

$$\begin{aligned}E\{p_1^{\mathbf{z}_n^{(1)}}\} = &\{[\beta_1(\tilde{x})-p_1\beta_2(\tilde{x})][\alpha_2E\{p_1^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\}-\alpha_1Pr\{\mathbf{z}_n^{(1)}=0\}-\alpha_2\Pi_0] \\ &+p_1r_1[\beta_1(\tilde{x})-\beta_2(\tilde{x})]\}\cdot\{p_1-\alpha_1\beta_1(\tilde{x})-\alpha_2p_1\beta_2(\tilde{x})\}^{-1}.\end{aligned} \tag{3.5}$$

By differentiating (3.5) once with respect to p_1, substituting $p_1=1$, twice applying l'Hopital's rule and using relation (2.7) we obtain the following expression for the mean number of type-1 customers at an arbitrary epoch:

$$\begin{aligned}E\{\mathbf{z}_n^{(1)}\} = &\frac{[\lambda_1^2(\beta_1^{(2)}-\beta_2^{(2)})-2\lambda_1\beta_2][r_1(1-\Pi_0)-\alpha_1]}{2[\alpha_1-\lambda_1(\alpha_1\beta_1+\alpha_2\beta_2)]} \\ &+\frac{[\lambda_1(\beta_1-\beta_2)-1]\alpha_2E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)})\}}{\alpha_1-\lambda_1(\alpha_1\beta_1+\alpha_2\beta_2)} \\ &+\frac{2r_1\lambda_1(\beta_1-\beta_2)+r_1\lambda_1^2(\beta_1^{(2)}-\beta_2^{(2)})}{2[\alpha_1-\lambda_1(\alpha_1\beta_1+\alpha_2\beta_2)]} \\ &+\frac{[\lambda_1(\beta_1-\beta_2)-1][r_1(1-\Pi_0)-\alpha_1][\lambda_1^2(\alpha_1\beta_1^{(2)}+\alpha_2\beta_2^{(2)})+2\alpha_2\lambda_1\beta_2]}{2[\alpha_1-\lambda_1(\alpha_1\beta_1+\alpha_2\beta_2)]^2}.\end{aligned} \tag{3.6}$$

If we interchange the indices "1" and "2" in (3.6) we obtain the analogous expression for $E\{\mathbf{z}_n^{(2)}\}$.

REMARK 3.1

The only factor in (3.6) (and in the first relation of (2.14)) that is not yet known, is $E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}$. In the following section it will be seen how this factor can be determined.

With the help of the formulas (2.13) we can now determine $E\{\mathbf{x}_1\} = E\{\mathbf{z}_n^{(1)} \mid \mathbf{r}_n=1\}$ from (3.6). As indicated in Section 2 we then obtain, by applying Little's formula, an expression for the mean sojourn time $E\{\mathbf{s}_1\}$. Finally, by using the obvious relation:

$$E\{\mathbf{w}_i\} = E\{\mathbf{s}_i\} - \beta_i, \quad i=1,2, \tag{3.7}$$

we can find an expression for the mean actual waiting time of a type-1 customer. Similarly, the expressions for the first moments for type-2 customers are derived.

We shall consider the above mentioned expressions for some special cases.

CASE I: $r_1 = \alpha_1$ and $r_2 = \alpha_2$:

$$E\{\mathbf{z}_n^{(1)}\} = \frac{\lambda_1^2(\beta_1^{(2)}-\beta_2^{(2)})(1+2\Pi_0)}{2\Pi_0} - 2\lambda_1\beta_2 - \frac{r_2}{r_1}\frac{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{\Pi_0}$$
$$+ \frac{\lambda_1(\beta_1-\beta_2)[1+\frac{r_2}{r_1}E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}]}{\Pi_0} + \frac{\lambda_2\beta_2[\lambda_1(\beta_1-\beta_2)(1-\Pi_0)+\Pi_0]}{\Pi_0^2}$$
$$+ \frac{[\lambda_1(\lambda_1\beta_1^{(2)}+\lambda_2\beta_2^{(2)})][\lambda_1(\beta_1-\beta_2)(1-\Pi_0)+\Pi_0]}{2\Pi_0^2}. \tag{3.8}$$

CASE II: $\beta_1 = \beta_2$ and $\beta_1^{(2)} = \beta_2^{(2)}$:

$$E\{\mathbf{z}_n^{(1)}\} = \lambda_1\beta + \frac{\lambda_1^2\beta^{(2)}}{2(\alpha_1-\lambda_1\beta)} + \frac{\alpha_2\lambda_1\beta}{\alpha_1-\lambda_1\beta} - \alpha_2\frac{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{\alpha_1-\lambda_1\beta}, \tag{3.9}$$
$$E\{\mathbf{z}_n^{(2)}\} = \lambda_2\beta + \frac{\lambda_2^2\beta^{(2)}}{2(\alpha_2-\lambda_2\beta)} + \frac{\alpha_1\lambda_2\beta}{\alpha_1-\lambda_2\beta} - \alpha_1\frac{E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\}}{\alpha_2-\lambda_2\beta}.$$

From (2.14) we have for the mean sojourn times:

$$E\{\mathbf{s}_1\} = \frac{\alpha_1-r_2}{r_1}\beta + \frac{\alpha_1\lambda\beta^{(2)}}{2(\alpha_1-\lambda_1\beta)} + \frac{\alpha_1\alpha_2\beta}{r_1(\alpha_1-\lambda_1\beta)} - \frac{\alpha_2\beta E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{r_1(\alpha_1-\lambda_1\beta)}, \tag{3.10}$$
$$E\{\mathbf{s}_2\} = \frac{\alpha_2-r_1}{r_2}\beta + \frac{\alpha_2\lambda\beta^{(2)}}{2(\alpha_2-\lambda_2\beta)} + \frac{\alpha_1\alpha_2\beta}{r_2(\alpha_2-\lambda_2\beta)} - \frac{\alpha_1\beta E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\}}{r_2(\alpha_2-\lambda_2\beta)}.$$

REMARK 3.2
As pointed out before, we have for Case II that

$$E\{\mathbf{z}_n^{(1)}\} + E\{\mathbf{z}_n^{(2)}\} = E\{\mathbf{z}_n\}, \tag{3.11}$$

where $E\{\mathbf{z}_n\}$ denotes the mean number of customers, measured at an arbitrary departure epoch in an ordinary M/G/1 queueing system with arrival intensity λ, mean service time β and second moment $\beta^{(2)}$. From M/G/1 theory we have, cf. Cohen [5], that

$$E\{\mathbf{z}_n\} = \lambda\beta + \frac{\lambda^2\beta^{(2)}}{2(1-\lambda\beta)}. \tag{3.12}$$

From (3.9) and remark 3.2 a relation between $E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}$ and $E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\}$ follows:

$$\frac{(\alpha_2-\lambda_2\beta)}{\alpha_1}E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\} + \frac{(\alpha_1-\lambda_1\beta)}{\alpha_2}E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\} = \tag{3.13}$$
$$\frac{\lambda^2\beta^{(2)}(\alpha_1-\lambda_1\beta)(\alpha_2-\lambda_2\beta)}{2\alpha_1\alpha_2(1-\lambda\beta)} - \frac{\lambda_1^2\beta^{(2)}(\alpha_2-\lambda_2\beta)}{2\alpha_1\alpha_2} - \frac{\lambda_2^2\beta^{(2)}(\alpha_1-\lambda_1\beta)}{2\alpha_1\alpha_2} -$$

$$-\frac{\lambda_1\beta(\alpha_2-\lambda_2\beta)}{\alpha_1}-\frac{\lambda_2\beta(\alpha_1-\lambda_1\beta)}{\alpha_2}.$$

CASE III: $\alpha_1 = r_1$, $\alpha_2 = r_2$ and $B(\cdot)\equiv B_1(\cdot) = B_2(\cdot)$:

With the notation II.(1.2) and II.(1.3) we have:

$$E\{\mathbf{z}_n^{(1)}\} = r_1[a+\frac{\lambda^2\beta^{(2)}}{2(1-a)}] + \frac{a_2}{(1-a)} - \frac{r_2}{r_1}\frac{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{(1-a)}, \tag{3.14}$$

$$E\{z_n^{(2)}\} = r_2[a+\frac{\lambda^2\beta^{(2)}}{2(1-a)}] + \frac{a_1}{(1-a)} - \frac{r_1}{r_2}\frac{E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\}}{(1-a)}.$$

Relation (3.13) now becomes:

$$\frac{r_2}{r_1}E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\} + \frac{r_1}{r_2}E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\} = a. \tag{3.15}$$

For the mean number of customers, measured at an arbitrary epoch we have:

$$E\{\mathbf{x}_1\} = aE\{\mathbf{z}_n^{(1)}\} + (1-a)r_1[a+\frac{\lambda^2\beta^{(2)}}{2(1-a)}], \tag{3.16}$$

$$E\{\mathbf{x}_2\} = aE\{\mathbf{z}_n^{(2)}\} + (1-a)r_2[a+\frac{\lambda^2\beta^{(2)}}{2(1-a)}].$$

Applying Little's formula to (3.16) we obtain for the mean sojourn times:

$$E\{\mathbf{s}_1\} = \frac{\beta E\{\mathbf{z}_n^{(1)}\}}{r_1} + (1-a)E\{\mathbf{s}\}, \tag{3.17}$$

$$E\{\mathbf{s}_2\} = \frac{\beta E\{\mathbf{z}_n^{(2)}\}}{r_2} + (1-a)E\{\mathbf{s}\}.$$

Here $E\{\mathbf{s}\}$ denotes the mean sojourn time in an ordinary M/G/1 queueing system with arrival intensity λ and service-time distribution $B(\cdot)$.

REMARK 3.3
By noting (3.7) we can, of course, immediately derive the mean actual waiting times from the mean sojourn times. A more direct approach is also possible, following a method of Cohen [5]. In Appendix D this approach is effectuated, yielding the stationary distribution of the actual waiting times, and as a by-result thereof, the mean actual waiting times.

REMARK 3.4
For the cross moment $E\{\mathbf{z}_n^{(1)}\mathbf{z}_n^{(2)}\}$ we have:

$$E\{\mathbf{z}_n^{(1)}\mathbf{z}_n^{(2)}\} = \frac{1}{2}\lim_{p\to 1}\frac{d^2}{dp^2}\Phi(p,p) - \frac{1}{2}E\{\mathbf{z}_n^{(1)}[\mathbf{z}_n^{(1)}-1]\} - \frac{1}{2}E\{\mathbf{z}_n^{(2)}[\mathbf{z}_n^{(2)}-1]\}. \tag{3.18}$$

4. THE NUMERICAL EVALUATION OF CASE III.

In this section the restrictions II.(4.1) are assumed to hold. As already announced in the preceding section an expression for $E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}$ will be derived, which will enable us to (numerically) compute $E\{\mathbf{z}_n^{(i)}\}$, $i=1,2$. Hence, by the results of the preceding section, also the various first moments can be computed. Firstly, some additional notation will be given:
We parametrise the contour S_1 by the mapping $\phi(\cdot) : [0,2\pi] \to \mathbb{C}$

$$\phi(t) := g(e^{\frac{1}{2}it})e^{\frac{1}{2}it}, \quad t \in [0,2\pi]. \tag{4.1}$$

further we shall take:
for $t \in [0,2\pi]$,

$$\eta(t) := p_{10}^{+}(\phi(t)), \tag{4.2}$$

where $p_{10}^{+}(\cdot)$ denotes the inverse mapping of $p_1^{+}(\cdot)$, cf.II.(7.1). Finally, let,

$$F_2(\cdot) : S_1 \to S_2, \tag{4.3}$$

denote a mapping from the contour S_1 onto the contour S_2, such that:
for every $z \in L$,

$$(p_1^{+}(z),\ F_2(p_1^{+}(z))), \tag{4.4}$$

is a zero pair of the kernel $K(p_1,p_2)$ (cf. II.(9.4)), i.e.,

$$F_2(p_1^{+}(z)) = p_2^{-}(z). \tag{4.5}$$

The following is a list of the most relevant derivatives we shall need in the sequel.
Implicit differentiation of the equation:

$$g(s) = (r_1 s^{-1} + r_2 s)\beta\{\lambda(1-g(s)(r_1 s + r_2 s^{-1}))\}, \tag{4.6}$$

with respect to s, and substituting $s = 1$ yields:

$$g'(1) = r_2 - r_1, \tag{4.7}$$

$$g''(1) = 2r_1(1+4r_2\frac{a}{1-a}).$$

Note that,

$$g(1) = 1. \tag{4.8}$$

From (4.1),(4.7) and (4.8) it follows:

$$\phi(0) = 1, \tag{4.9}$$

$$\phi'(0) = r_2 i,$$

$$\phi''(0) = r_2(r_1 - r_2 - 2r_1\frac{1}{1-a}).$$

Because $(p_1^{+}(z),\ F_2(p_1^{+}(z)))$, $z \in L$, is a zero pair of the kernel we have,

$$p_1^{+}(z)F_2(p_1^{+}(z)) = (r_1 F_2(p_1^{+}(z)) + r_2 p_1^{+}(z))\beta\{\lambda(1-r_1 p_1^{+}(z) - r_2 F_2(p_1^{+}(z)))\}. \tag{4.10}$$

Hence implicit differentiation of the formula

$$p_1 F_2(p_1) = (r_1 F_2(p_1) + r_2 p_1)\beta\{\lambda(1-r_1 p_1 - r_2 F_2(p_1))\}, \tag{4.11}$$

with respect to p_1 and substituting $p_1 = 1$ yields:

$$F_2'(1) = -\frac{r_1}{r_2}, \tag{4.12}$$

$$F_2''(1) = \frac{2r_1}{r_2^2}\frac{1}{1-a},$$

$$F_2'''(1) = \frac{6r_1}{r_2^3}(r_1 - r_2 - 2r_1\frac{1}{1-a})\frac{1}{1-a}.$$

From (4.9) and (4.12) we find:

$$\frac{d}{dt}\{\frac{\phi(t)F_2(\phi(t)) - r_1\phi(t) - r_2F_2(\phi(t))}{\phi(t) - F_2(\phi(t))}\}|_{t=0} = r_1r_2i\frac{a}{1-a}; \tag{4.13}$$

$$\frac{d^2}{dt^2}\{\frac{\phi(t)F_2(\phi(t)) - r_1\phi(t) - r_2F_2(\phi(t))}{\phi(t) - F_2(\phi(t))}\}|_{t=0} = r_1r_2\{r_1 - r_2\}\frac{a}{1-a}.$$

At this point we are ready to give a numerical evaluation of the expressions for $E\{\mathbf{z}_n^{(i)}\}$, $i = 1,2$. We start from (3.14) and write it somewhat differently:

$$E\{\mathbf{z}_n^{(1)}\} = \frac{a}{1-a} + r_1\frac{\lambda^2\{\beta^{(2)} - 2\beta^2\}}{2(1-a)} - \frac{r_2}{r_1}\frac{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{1-a}, \tag{4.14}$$

$$E\{\mathbf{z}_n^{(2)}\} = \frac{a}{1-a} + r_2\frac{\lambda^2\{\beta^{(2)} - 2\beta^2\}}{2(1-a)} - \frac{r_1}{r_2}\frac{E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\}}{1-a}.$$

REMARK 4.1
It may be seen that if $B(t) = 1 - e^{-t/\beta}$ then (4.14) becomes:

$$E\{\mathbf{z}_n^{(1)}\} = \frac{a}{1-a} - \frac{r_2}{r_1}\frac{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{1-a}, \tag{4.15}$$

$$E\{\mathbf{z}_n^{(2)}\} = \frac{a}{1-a} - \frac{r_1}{r_2}\frac{E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\}}{1-a}.$$

From (4.14) it is seen, that, in order to determine $E\{\mathbf{z}_n^{(1)}\}$ numerically, we need to evaluate $E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}$.
Let, for $\zeta \in L$,

$$f(\zeta) := \frac{p_1^+(\zeta)F_2(p_1^+(\zeta)) - r_1p_1^+(\zeta) - r_2F_2(p_1^+(\zeta))}{p_1^+(\zeta) - F_2(p_1^+(\zeta))}. \tag{4.16}$$

From II.(9.18) we have:
for $z \in L^+$,

$$r_2E\{[p_1(z)]^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} = r_1\Pi_0 + \frac{\Pi_0}{2\pi i}\int_{\zeta\in L}\frac{f(\zeta)}{\zeta - z}\,d\zeta, \tag{4.17}$$

whereas for $z \in L$, using the Plemelj-Sokhotski formulas:

$$r_2E\{[p_1^+(z)]^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} = r_1\Pi_0 + \frac{1}{2}\Pi_0 f(z) + \frac{\Pi_0}{2\pi i}\int_{\zeta\in L}\frac{f(\zeta)}{\zeta - z}\,d\zeta. \tag{4.18}$$

It follows by differentiating (4.18) with respect to z and substituting $z = 1$ that,

$$r_2E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}(p_1^+)'(1) = \frac{1}{2}\Pi_0\frac{d}{dz}f(z)|_{z=1} + \Pi_0\frac{d}{dz}\{\frac{1}{2\pi i}\int_{\zeta\in L}\frac{f(\zeta)}{\zeta - z}\,d\zeta\}. \tag{4.19}$$

Note that, cf. (4.13),

$$\frac{d}{dz}f(z)|_{z=1} = \frac{1}{\eta'(0)}\frac{d}{dt}\{f(\eta(t))\}|_{t=0} = \tag{4.20}$$

$$= \frac{1}{\eta'(0)} \frac{d}{dt} \{ \frac{\phi(t)F_2(\phi(t)) - r_1\phi(t) - r_2F_2(\phi(t))}{\phi(t) - F_2(\phi(t))} \}|_{t=0}$$

$$= r_1 r_2 \frac{i}{\eta'(0)} \frac{a}{1-a},$$

and that, cf. (4.2)

$$(p_1^+)'(1) = \frac{\phi'(0)}{\eta'(0)} = r_2 \frac{i}{\eta'(0)}. \tag{4.21}$$

Hence we have:

$$\frac{r_2}{r_1} \frac{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{1-a} = \frac{a}{2(1-a)} + \frac{\eta'(0)}{r_1 r_2 i} \frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} \frac{f(\zeta)}{\zeta - z} d\zeta \}|_{z=1}. \tag{4.22}$$

In a completely similar manner we find:

$$\frac{r_1}{r_2} \frac{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{1-a} = \frac{a}{2(1-a)} - \frac{\eta'(0)}{r_1 r_2 i} \frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} \frac{f(\zeta)}{\zeta - z} d\zeta \}|_{z=1}. \tag{4.23}$$

Note that (4.23) can also be directly obtained from relation (3.15).
We now direct our attention towards the last term in the relations (4.22) and (4.23):

$$\frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} f(\zeta) \frac{d\zeta}{\zeta - z} \}|_{z=1}. \tag{4.24}$$

We first divide the integral into a continuous part and a singular part:

$$\frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} f(\zeta) \frac{d\zeta}{\zeta - z} \}|_{z=1} = \frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} \frac{f(\zeta) - f(1) - (\zeta - 1)f'(1)}{\zeta - z} d\zeta \}|_{z=1} \tag{4.25}$$

$$+ \frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} \frac{f(1) + (\zeta - 1)f'(1)}{\zeta - z} d\zeta \}|_{z=1}.$$

Because the integrand and its partial derivatives with respect to z of the first term in the right-hand side of (4.25) are continuous for $z \in L$ it follows, cf. Titchmarsh [18], that

$$\frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} \frac{f(\zeta) - f(1) - (\zeta - 1)f'(1)}{\zeta - z} d\zeta \}|_{z=1} = \frac{1}{2\pi i} \int_{\zeta \in L} \frac{f(\zeta) - f(1) - (\zeta - 1)f'(1)}{(\zeta - 1)^2} d\zeta. \tag{4.26}$$

Further we have

$$\frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} \frac{f(1) + (\zeta - 1)f'(1)}{\zeta - z} d\zeta \}|_{z=1} = \frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} f(1) \frac{d\zeta}{\zeta - z} \}|_{z=1} \tag{4.27}$$

$$+ \frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} f'(1) \frac{\zeta - 1}{\zeta - z} d\zeta \}|_{z=1}.$$

Of course we have, cf. Gakhov[11], p.29,...,31 :

$$\frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} f(1) \frac{d\zeta}{\zeta - z} \}|_{z=1} = 0, \tag{4.28}$$

and

$$\frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} f'(1) \frac{\zeta - 1}{\zeta - z} d\zeta \}|_{z=1} = f'(1) \frac{d}{dz} \{ \frac{1}{2\pi i} \int_{\zeta \in L} [1 + \frac{\zeta - 1}{\zeta - z}] d\zeta \}|_{z=1} = \tag{4.29}$$

$$= f'(1) \frac{1}{2\pi i} \int_{\zeta \in L} \frac{1}{(\zeta - 1)^2} d\zeta =$$

$$= \tfrac{1}{2}f'(1).$$

From (4.25),...,(4.29) it follows that

$$\frac{d}{dz}\{\frac{1}{2\pi i}\int_{\zeta\in L}\frac{f(\zeta)}{\zeta-z}\,d\zeta\}|_{z=1} = \frac{1}{2\pi i}\int_{\zeta\in L}\frac{f(\zeta)-f(1)-(\zeta-1)f'(1)}{(\zeta-1)^2}\,d\zeta + \tfrac{1}{2}f'(1). \tag{4.30}$$

Changing the variable of integration ζ into $\eta(t)$ (cf. (4.2)) yields:

$$\frac{d}{dz}\{\frac{1}{2\pi i}\int_{\zeta\in L}\frac{f(\zeta)}{\zeta-z}\,d\zeta\}|_{z=1} = \frac{1}{2\pi i}\int_0^{2\pi}[f(\eta(t))-f(1)-(\eta(t)-1)f'(1)]\frac{\eta'(t)}{(\eta(t)-1)^2}\,dt \tag{4.31}$$

$$+ \tfrac{1}{2}f'(1).$$

By applying l'Hopital's rule it is easily seen from (4.16) that

$$f(1) = r_2-r_1. \tag{4.32}$$

Hence, from (4.14), (4.19), (4.31) and (4.32):

$$E\{\mathbf{z}_n^{(1)}\} = r_1\frac{\lambda^2\{\beta^{(2)}-2\beta^2\}}{2(1-a)} + \frac{\eta'(0)}{2r_1r_2\pi}\int_0^{2\pi}[\frac{\phi(t)F_2(\phi(t))-r_1\phi(t)-r_2F_2(\phi(t))}{\phi(t)-F_2(\phi(t))} - \tag{4.33}$$

$$(r_2-r_1) + (\eta(t)-1)r_1r_2\frac{a}{1-a}\frac{1}{i\eta'(0)}]\frac{\eta'(t)}{(\eta(t)-1)^2}\,dt.$$

The analogous expression for $E\{\mathbf{z}_n^{(2)}\}$ reads:

$$E\{\mathbf{z}_n^{(2)}\} = \frac{a}{1-a} + r_2\frac{\lambda^2\{\beta^{(2)}-2\beta^2\}}{2(1-a)} - \tag{4.34}$$

$$\frac{\eta'(0)}{2r_1r_2\pi}\int_0^{2\pi}[\frac{\phi(t)F_2(\phi(t))-r_1\phi(t)-r_2F_2(\phi(t))}{\phi(t)-F_2(\phi(t))} - (r_2-r_1) -$$

$$(\eta(t)-1)r_1r_2\frac{a}{1-a}\frac{1}{i\eta'(0)}]\frac{\eta'(t)}{(\eta(t)-1)^2}\,dt.$$

To calculate the integrand of the integral occurring in (4.33), (4.34) in $t = 0$, expand the numerator and the denominator of the integrand into their Taylor series.
Denoting the integrand in $t = 0$ by I_0, it follows:

$$I_0 = \frac{r_1r_2}{2\eta'(0)}\frac{a}{(1-a)}[r_1-r_2+\frac{\eta''(0)}{i\eta'(0)}]. \tag{4.35}$$

5. Numerical evaluation of the case $r_1 \neq \alpha_1$, $r_2 \neq \alpha_2$ and $B_1(\cdot) \equiv B_2(\cdot)$.

In this section we take $r_1 \neq \alpha_1$, $r_2 \neq \alpha_2$ and hence we have no longer a proof that the contours S_1 and S_2 are simply connected. In fact it will appear that easily cases can be found where S_1 or S_2 is not simple. We therefore are forced to introduce the following assumption:

Assumption 5.1
For the considered values of the parameters the contours S_1 and S_2 are smooth and simple.

In exactly the same manner as in Section 4 we shall derive expressions which enable us to compute

$E\{\mathbf{z}_n^{(i)}\}$, $i=1,2$, and hence, with the formulas derived in Section 3, the first moments of the sojourn and the actual waiting times. We follow the notation from Section 4.
From (3.9) we have

$$E\{\mathbf{z}_n^{(1)}\} = \frac{a_1}{\alpha_1 - a_1} + r_1^2 \frac{\lambda^2\{\beta^{(2)} - 2\beta^2\}}{2(\alpha_1 - a_1)} - \frac{\alpha_2 E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}}{\alpha_1 - a_1}, \tag{5.1}$$

$$E\{\mathbf{z}_n^{(2)}\} = \frac{a_2}{\alpha_2 - a_2} + r_2^2 \frac{\lambda^2\{\beta^{(2)} - 2\beta^2\}}{2(\alpha_2 - a_2)} - \frac{\alpha_1 E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\}}{\alpha_2 - a_2}.$$

As before, it is seen that, in order to determine $E\{\mathbf{z}_n^{(1)}\}$ numerically, we need to evaluate $E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\}$.
As in Section II.6 it follows that for $z \in L$:

$$E\{[p_1^+(z)]^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} = \frac{\alpha_1}{\alpha_2} E\{[F_2(p_1^+(z))]^{\mathbf{z}_n^{(2)}}(\mathbf{z}_n^{(1)}=0)\} + H(z)\Pi_0, \tag{5.2}$$

where $H(z)$, $z \in L$, is given by:

$$H(z) = \frac{p_1^+(z)F_2(p_1^+(z)) - \alpha_2 F_2(p_1^+(z)) - \alpha_1 p_1^+(z)}{p_1^+(z) - F_2(p_1^+(z))}. \tag{5.3}$$

As in Section II.6 we obtain:

$$\alpha_2 E\{p_1(z)^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} = \frac{\Pi_0}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta - z} + \alpha_1\Pi_0, \quad z \in L^+, \tag{5.4}$$

$$\alpha_1 E\{p_2(z)^{(\mathbf{z}_n^{(2)}}(\mathbf{z}_n^{(1)}=0)\} = \frac{\Pi_0}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta - z} + \alpha_1\Pi_0, \quad z \in L^-.$$

Using the Plemelj-Sokhotski formulas, we obtain:
for $z \in L$,

$$\alpha_2 E\{[p_1^+(z)]^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)}=0)\} = \tfrac{1}{2}\Pi_0 H(z) + \frac{\Pi_0}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta - z} + \alpha_1\Pi_0, \tag{5.5}$$

$$\alpha_1 E\{F_2(p_1^+(z))^{\mathbf{z}_n^{(2)}}(\mathbf{z}_n^{(1)}=0)\} = -\tfrac{1}{2}\Pi_0 H(z) + \frac{\Pi_0}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta - z} + \alpha_1\Pi_0.$$

Differentiating (5.5) once with respect to z and substituting $z = 1$ yields:

$$\alpha_2 E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\} = \frac{\Pi_0}{p_1'(1)}\{\tfrac{1}{2}H'(1) + \frac{d}{dz}\{\frac{1}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta - z}\}|_{z=1}\}, \tag{5.6}$$

$$\alpha_1 E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\} = \frac{\Pi_0}{F_2'(1)p_1'(1)}\{-\tfrac{1}{2}H'(1) + \frac{d}{dz}\{\frac{1}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta - z}\}|_{z=1}\}.$$

As in Section 4 it is shown that

$$\frac{d}{dz}\{\frac{1}{2\pi i}\int_{\zeta\in L} H(\zeta)\frac{d\zeta}{\zeta - z}\}|_{z=1} = \frac{1}{2\pi i}\int_0^{2\pi}[H(\eta(t)) - H(1) - (\eta(t)-1)H'(1)]\times \tag{5.7}$$

$$\times\frac{\eta'(t)}{(\eta(t)-1)^2}\,dt + \tfrac{1}{2}H'(1).$$

We now direct our attention towards the computation of $H'(1)$.
In the notation of Section 4,

$$H'(1) = \frac{d}{dz}H(z)|_{z=1} \tag{5.8}$$

$$= \frac{d}{dz}\left[\frac{p_1^+(z)F_2(p_1^+(z)) - \alpha_2 F_2(p_1^+(z)) - \alpha_1 p_1^+(z)}{p_1^+(z) - F_2(p_1^+(z))}\right]\Big|_{z=1}$$

$$= \frac{1}{\eta'(0)}\frac{d}{dt}\left[\frac{\phi(t)F_2(\phi(t)) - \alpha_2 F_2(\phi(t)) - \alpha_1\phi(t)}{\phi(t) - F_2(\phi(t))}\right]\Big|_{t=0}$$

$$= \frac{\phi'(0)}{\eta'(0)}\frac{F_2''(1) + 2F_2'(1)[1 - F_2'(1)]}{2[1 - F_2'(1)]^2}.$$

For $F_2(\cdot)$ we have:

$$p_1 F_2(p_1) = [\alpha_1 F_2(p_1) + \alpha_2 p_1]\beta\{\lambda(1 - r_1 p_1 - r_2 F_2(p_1))\}. \tag{5.9}$$

Differentiating (5.9) with respect to p_1 and substituting $p_1 = 1$ yields:

$$F_2'(1) = \frac{a_1 - \alpha_1}{\alpha_2 - a_2}. \tag{5.10}$$

REMARK 5.1
From (5.10) it can be seen for which values of the parameters problems can be expected with the contours S_1 and S_2. For instance, if $F_2'(1) > 0$ then both contours start "in the same direction" as s traverses the unit circle. If $F_2'(1) = 0$ then one of the contours probably has a cusp, and if $F_2'(1) < 0$ then both contours start "in the opposite direction" as s traverses the unit circle. In Appendix B some graphs have been plotted of the contours S_1 and S_2, where all the forementioned possibilities are shown.

By differentiating (5.10) twice with respect to P_1 and substituting $P_1 = 1$ we obtain:

$$F_2''(1) = \frac{1}{\alpha_2 - a_2}\{2(\alpha_1 F_2'(1) + \alpha_2)a(r_1 + r_2 F_2'(1)) - 2F_2'(1) + \lambda^2\beta^{(2)}(r_1 + r_2 F_2'(1))^2\}. \tag{5.11}$$

The derivative of $g(s)$ in $s=1$ may be computed from

$$g(s) = (\alpha_1 s^{-1} + \alpha_2 s)\beta\{\lambda(1 - g(s)(r_1 s + r_2 s^{-1}))\}, \quad |s| = 1. \tag{5.12}$$

It follows that

$$g'(1) = \frac{\alpha_2 - a_2 - (\alpha_1 - a_1)}{1 - a}. \tag{5.13}$$

Hence, cf. (4.1) :

$$\phi'(0) = \frac{\alpha_2 - a_2}{1 - a}i. \tag{5.14}$$

We write (5.1) as

$$E\{\mathbf{z}_n^{(1)}\} = \frac{a_1}{\alpha_1 - a_1} + r_1^2\frac{\lambda^2\{\beta^{(2)} - 2\beta^2\}}{2(\alpha_1 - a_1)} + \frac{\Pi_0^2 i\eta'(0)}{\alpha_1 - a_1)(\alpha_2 - a_2)} \times \tag{5.15}$$

$$\times\left[H'(1) + \frac{1}{2\pi i}\int_0^{2\pi}[H(\eta(t)) - H(1) - (\eta(t) - 1)H'(1)]\frac{\eta'(t)}{(\eta(t) - 1)^2}\,dt\right],$$

$$E\{\mathbf{z}_n^{(2)}\} = \frac{a_2}{\alpha_2 - a_2} + r_2^2\frac{\lambda^2\{\beta^{(2)} - 2\beta^2\}}{2(\alpha_2 - a_2)} - \frac{\Pi_0^2 i\eta'(0)}{(\alpha_2 - a_2)^2} \times$$

$$\times\frac{1}{2\pi i}\int_0^{2\pi}[H(\eta(t)) - H(1) - (\eta(t) - 1)H'(1)]\frac{\eta'(t)}{(\eta(t) - 1)^2}\,dt.$$

The formulas (5.15) can, with the help of the expressions derived above, immediately be computed by the iterations program that will be described in Appendix D. For those cases where this program does not work we have to use simulation. A description of the simulation program is also given in Appendix D.
For various values of the parameters, the results are given in Appendix E.

Chapter IV

Approximation of a model with N queues and random allocation by the model with two queues

In this chapter we shall consider the question whether or not we can use the model with random allocation and two queues to obtain mean value approximations (in the steady state) for the analytically untractable model with $N > 2$ queues and random allocation.

Consider a multi-queue single-server system with N, $N > 2$, queues and random allocation as described in Chapter I. Suppose we want to approximate the mean actual waiting time of a type-1 customer. The idea is now:

APPROXIMATION IDEA 1.1:
Aggregate queues $2,...,N$ into one queue, say $\tilde{Q}_2$.

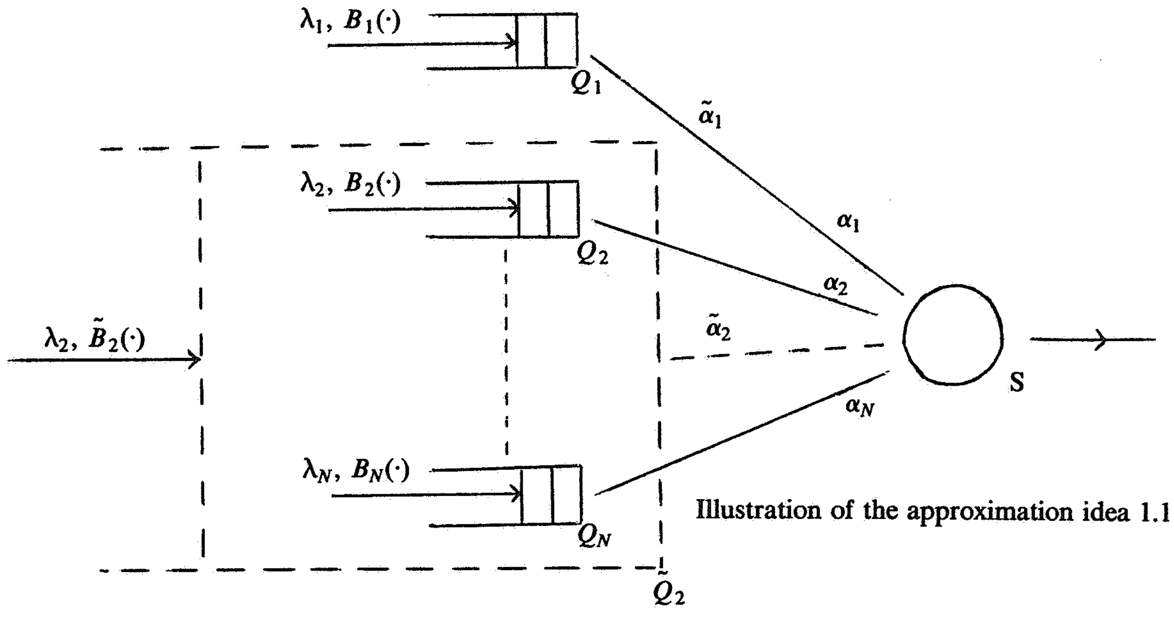


FIGURE 6

The arrival process at $\tilde{Q}_2$ is then the sum of N-1 independent Poisson processes with parameters

$\lambda_2, \ldots, \lambda_N$, hence is itself a Poisson process with parameter

$$\tilde{\lambda}_2 := \sum_{k=2}^{N} \lambda_k. \tag{1.1}$$

Since we are only interested in approximating mean values, the distribution of the required service time of a customer from $\tilde{Q}_2$ will be:

$$\tilde{B}_2(\cdot) := \sum_{k=2}^{N} r_k B_k(\cdot). \tag{1.2}$$

There now remains one problem: how to choose $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, the probabilities with which a customer is chosen from Q_1 and $\tilde{Q}_2$ respectively.

At first glance, several possibilities for the choice of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ seem reasonable. For example,

$$\tilde{\alpha}_1 \equiv \alpha_1, \tag{1.3}$$

$$\tilde{\alpha}_2 \equiv \sum_{k=2}^{N} \alpha_k.$$

Or, when the service-time distributions are all equal we could choose $\tilde{\alpha}_1$ equal to the fraction of the total traffic $a = \lambda\beta$, that is due to type-1 customers, i.e.,

$$\tilde{\alpha}_1 \equiv \frac{a_1}{a} = r_1, \tag{1.4}$$

$$\tilde{\alpha}_2 \equiv \frac{1}{a}\sum_{k=2}^{N} a_i = \sum_{k=2}^{N} r_k.$$

These seemingly reasonable choices will in most cases not give accurate results, due to a phenomenon that will be described below, by means of an example:

Consider a model with three queues and random allocation, where everything is symmetrical (figure 7):

$$r_1 = r_2 = r_3 = \frac{1}{3}, \tag{1.5}$$

$$\lambda = 1,$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3},$$

$$B(\cdot) \equiv B_1(\cdot) = B_2(\cdot) = B_3(\cdot),\ B(t) = 1 - e^{-t/0.8}.$$

Three queues, RA, symmetrical Approximation model

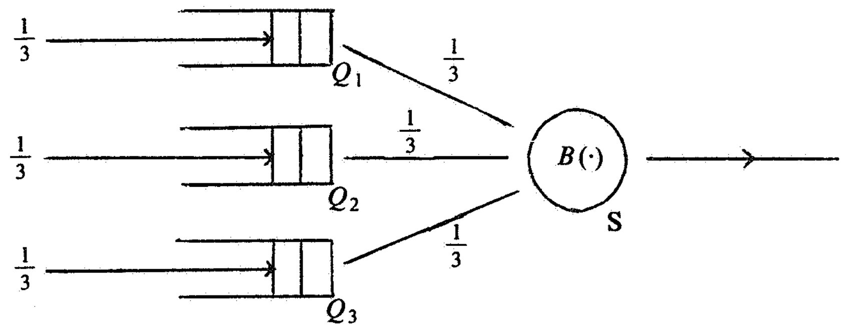


FIGURE 7

As can be seen from (1.5) we have taken the service-time distribution negative exponentially and we have chosen $\lambda = 1$ and $\beta = 0.8$, so that $a = \lambda\beta = 0.8 < 1$.
Now suppose we are interested in the mean waiting time of a type-1 customer. Then we replace the model above by the following model with two queues (as described in the preceding pages):
We take (cf. also figure 8):

$$\tilde{r}_2 := r_2 + r_3 = \frac{2}{3}, \tag{1.6}$$

$$\tilde{B}(t) := 1 - e^{-t/0.8},$$

$$\tilde{\alpha}_1 := \alpha_1 = r_1 = \frac{1}{3},\ \tilde{\alpha}_2 := \alpha_2 + \alpha_3 = r_2 + r_3 = \frac{2}{3}.$$

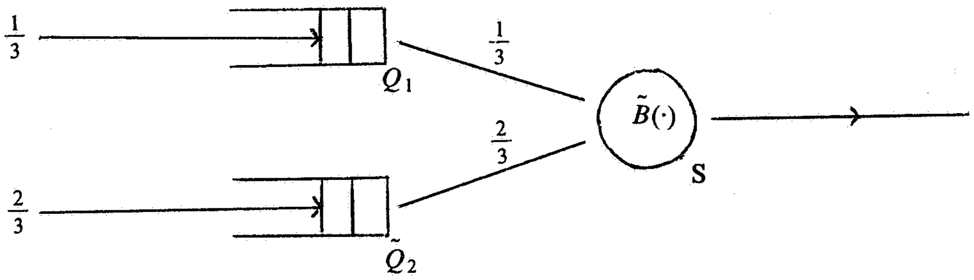


FIGURE 8

Denote by $E\{w_i\}$, $i=1,2,3$, the mean waiting time of a type-i customer in the model with three queues and by $E\{w\}$ the mean waiting time in an M/G/1 queueing model with arrival rate λ, mean service time β and second moment $\beta^{(2)}$. Then, owing to the symmetry of the model we have:

$$E\{w_1\} = \lambda_1 E\{w_1\} + \lambda_2 E\{w_2\} + \lambda_3 E\{w_3\} = E\{w\} = 3.2\,. \tag{1.7}$$

On the other hand we have for the model with two queues, if we denote by $E\{\tilde{w}_1\}$ the mean waiting time of a type-1 customer

$$E\{\tilde{w}_1\} = 3.74\,. \tag{1.8}$$

It may be clear that $E\{\tilde{w}_1\}$ - at least for this value of $\tilde{\alpha}_1$ - is not a very accurate approximation for $E\{w_1\}$. And things can get worse ! The phenomenon of overestimating the waiting time for type-1 customers can be explained as follows:

Suppose at a certain departure instant we have the following situation: Q_1 and Q_3 are not empty, whereas Q_2 is empty (see also figure 1.4). Then with probability 1/2 a customer is chosen from Q_1 and with probability 1/2 from Q_3. Consider now the approximation model at the same departure instant. There Q_1 is not empty and also $\tilde{Q}_2$ - the aggregation of Q_2 and Q_3 - is not empty, because Q_3 is not empty. Hence at this instant a customer is chosen from Q_1 with probability 1/3 (see figure 9). This leads to a certain underestimation of the probability with which customers from Q_1 are chosen and we can therefore expect to find an overestimation for the waiting time of a type-1 customer.

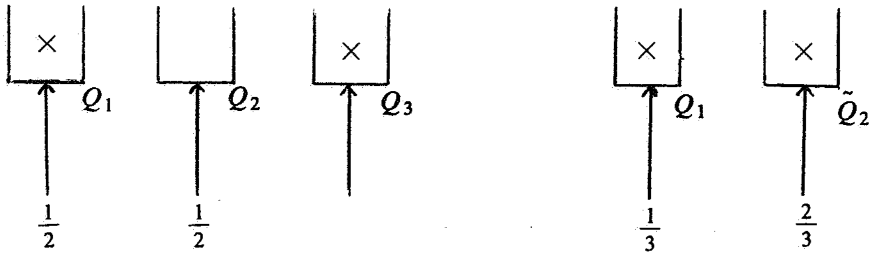


FIGURE 9

As is indicated in the discussion above we expect the probability that from $Q_2,...,Q_N$, k queues are empty, given that $\tilde{Q}_2$ is not empty, to appear in an expression for $\tilde{\alpha}_1, \tilde{\alpha}_2$. These considerations lead to the following:

APPROXIMATION IDEA 1.2 :

$$\tilde{\alpha}_1 \equiv \frac{\alpha_1}{\sum_{j=1}^{N} \alpha_j Pr\{\mathrm{x}_j>0 \mid \mathrm{x}_1>0, \sum_{j=2}^{N} \mathrm{x}_j>0\}} \tag{1.9}$$

$$= \frac{\alpha_1}{\alpha_1 + \sum_{j=2}^{N} \alpha_j Pr\{\mathrm{x}_j>0 \mid \mathrm{x}_1>0, \sum_{j=2}^{N} \mathrm{x}_j>0\}}.$$

For the definition of x_j, $j=1,...,N$, cf. III.(1.2).
Note that, for $k=2,...,N$,

$$Pr\{\mathrm{x}_k=0 \mid \mathrm{x}_1>0, \sum_{j=2}^{N} \mathrm{x}_j>0\} = \tag{1.10}$$

$$\sum_{m=2}^{\infty} \sum_{i=1}^{m-1} Pr\{\mathrm{x}_k=0, \mathrm{x}_1=i, \sum_{j=1}^{N} \mathrm{x}_j=m \mid \mathrm{x}_1>0, \sum_{j=2}^{N} \mathrm{x}_j>0\} =$$

$$\sum_{m=2}^{\infty} \sum_{i=1}^{m-1} Pr\{\mathrm{x}_k=0, \mathrm{x}_1=i \mid \mathrm{x}_1>0, \sum_{j=1}^{N} \mathrm{x}_j=m, \sum_{j=2}^{N} \mathrm{x}_j>0\} Pr\{\sum_{j=1}^{N} \mathrm{x}_j=m \mid \mathrm{x}_1>0, \sum_{j=2}^{N} \mathrm{x}_j>0\}.$$

Because we cannot compute the above probabilities exactly, we have to make two more approximations.

APPROXIMATION IDEA 1.3 :
The customers in the system are being served in arbitrary (FCFS) order.

By applying Approximation 1.3 to the first probability in the right-hand side of (1.10) it may be seen that this probability becomes equal to the probability that there are i type-1 and 0 type-k arrivals during the last m arrivals, given that the last m arrivals contain at least 1 type-2,...,N customer. This probability can be exactly determined.
Now only the second probability in the right-hand side of (1.10) remains to be determined. We make the following approximation:

APPROXIMATION 1.4 :

$$Pr\{\sum_{j=1}^{N}x_j=m \mid x_1>0, \sum_{j=2}^{N}x_j>0\} \approx Pr\{\sum_{j=1}^{N}x_j=m \mid \sum_{j=2}^{N}x_j>1\}. \tag{1.11}$$

The major disadvantage in this approach is, that we have little or none a priori knowledge of the influence of the various approximations upon the total: There simply are too many (4) needed to make one approximation. Specifically, we cannot tell a priori in which cases the approximation will be good or bad. A further drawback is, that the approximation is for the completely symmetrical case not exact, while we do know all averages exactly on basis of other arguments. A subject for further research in this area however, could be to investigate the sensitivity of the numerical results with respect to the several approximations.

Appendix A

Overview of definitions and theorems used in this report

SECTION 1

DEFINITION 1.1 (cf. Kuipers & Timman [12])
A function $f : \mathbb{C} \to \mathbb{C}$ is called *(complex) differentiable in the point* $z_0 \in \mathbb{C}$ if the limit

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists.

DEFINITION 1.2 (cf. Kuipers & Timman [12])
A function $f : \mathbb{C} \to \mathbb{C}$ is called *regular at the point* $z_0 \in \mathbb{C}$ if there exists a neighbourhood of z_0 in every point of which f is complex differentiable; f is called *regular in the domain D* if f is regular at every point of D.

DEFINITION 1.3 (cf. Muskhelishvili [15])
Let there be given on the arc L a function of position $\phi(t)$ (which is in general complex). The function $\phi(t)$ will be said to satisfy a *Holder condition on L* if for any two points t_1, t_2 of L

$$|\phi(t_2)-\phi(t_1)| \leqslant A|t_2-t_1|^{\mu},$$

where A and μ are positive constants. A is called the Holder constant and μ the Holder index.

DEFINITION 1.4 (cf. Gakhov [11])
Let there be given on the contour L a function $G(\cdot)$. The increment of the argument of $G(t)$, when t traverses L once in the positive direction, divided by 2π, is called the *index* of $G(\cdot)$ on L.

DEFINITION 1.5 (cf. Nehari [16])
A continuous function f is said to be *univalent in the domain D* if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in D$. A regular function that is univalent is called a *conformal mapping.*

DEFINITION 1.6 (cf. Markushevich [14])
A complex function $z = f(t)$ of a real variable which is defined and continuous on a closed interval $a \leqslant t \leqslant b$ is said to define a *(continuous) curve.*
If the same point z corresponds to more than one parameter value in the half-open interval $a \leqslant t < b$, we say that z is a *multiple point* of the curve $z = f(t)$, $a \leqslant t \leqslant b$. A curve with no

multiple points is called a *Jordan curve.*
A closed curve is called a contour.
A continuous curve L is said to be *smooth* if among its various parametric representations there is at least one representation

$$z = f(t) = \mu(t) + i\nu(t),$$

such that $f(t)$ has a continuous nonvanishing derivative $f'(t)$ at every point of the interval $[a,b]$.

THEOREM 1.1 (cf. Nehari [16]) *The corresponding boundaries theorem.*
Denote with L^+ the interior of a contour L. If L_1^+ and L_2^+ are two domains bounded by smooth contours then the conformal mapping $L_1^+ \to L_2^+$ is continuous in $L_1^+ \bigcup L_1$ and establishes a one-to-one correspondence between the points of L_1 and L_2.

THEOREM 1.2 (cf. Titchmarsh [18]) *Principle of corresponding boundaries.*
Let L_1^+ and L_2^+ be two domains bounded by piecewise smooth contours L_1 and L_2. If $f(z)$ is regular in L_1^+ and continuous in $L_1^+ \bigcup L_1$ and maps L_1 one-to-one onto L_2, then $f(z)$ is univalent in $L_1^+ \bigcup L_1$: if $f(z)$ preserves the positive directions on L_1 and L_2, then $f(z)$ maps L_1^+ conformally onto L_2^+, otherwise onto L_2^-.

Section 2

AD P.11

Let x *denote a nonnegative, discrete stochastic variable with probability distribution*

$$\{p_k,\ k=0,1,2,\ldots\}. \tag{2.1}$$

The generating function of this probability distribution,

$$\Psi(z) = \sum_{k=0}^{\infty} p_k z^k, \quad z \in \mathbb{C}, \tag{2.2}$$

is regular for $|z|<1$ *and continuous for* $|z|\leqslant 1$.

(Elementary) proof :

For $|z|\leqslant 1$:

$$\sum_{k=0}^{\infty} |p_k z^k| \leqslant \sum_{k=0}^{\infty} p_k = 1. \tag{2.3}$$

Hence the series (2.2) converges absolutely in the disk $\{z\colon |z|\leqslant 1\}$.
Note that the series

$$\phi(z) = \sum_{k=1}^{\infty} k p_k z^{k-1}, \tag{2.4}$$

converges absolutely for every z with $|z| < R$, $0 < R < 1$ fixed:

$$\sum_{k=1}^{\infty} |k p_k z^{k-1}| \leqslant \sum_{k=1}^{\infty} k|z|^{k-1}. \tag{2.5}$$

Let $|z| = R < 1$ and let h be a complex number such that

$$R+|h| < R_0 < 1. \tag{2.6}$$

Denote the absolute value of h by ϵ. We then have

$$\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| = \left|\frac{n(n-1)}{1\cdot 2} z^{n-2} h + \ldots + h^{n-1}\right| \tag{2.7}$$

$$\leqslant \frac{n(n-1)}{1\cdot 2} R^{n-2}\epsilon + \ldots + \epsilon^{n-1} = \frac{(R+\epsilon)^n - R^n}{\epsilon} - nR^{n-1}.$$

Hence,

$$\left|\frac{\Psi(z+h)-\Psi(z)}{h} - \phi(z)\right| = \left|\sum_{n=0}^{\infty} p_n\left[\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right]\right| \tag{2.8}$$

$$\leqslant \sum_{n=0}^{\infty}\left[\frac{(R+\epsilon)^n - R^n}{\epsilon} - nR^{n-1}\right] = \sum_{n=0}^{\infty} R_0^n\left[\frac{\left(\frac{R+\epsilon}{R_0}\right)^n - \left(\frac{R}{R_0}\right)^n}{\epsilon} - \frac{n}{R}\left(\frac{R}{R_0}\right)^n\right]$$

$$\leqslant \sum_{n=0}^{\infty}\left\{\left[\left(\frac{R+\epsilon}{R_0}\right)^n - \left(\frac{R}{R_0}\right)^n\right]\epsilon^{-1} - \frac{n}{R}\left(\frac{R}{R_0}\right)^n\right\} =$$

$$= \left[\frac{R_0}{R_0-R-\epsilon} - \frac{R_0}{R_0-R}\right]\epsilon^{-1} - \frac{R_0}{(R_0-R)^2} =$$

$$= \frac{\epsilon}{(R_0-R)^2(R_0-R-\epsilon)} \to 0 \quad (\epsilon = |h| \to 0).$$

It follows that $\Psi(z)$ is regular for $|z|<1$.
Next note that the series (2.2) converges uniformly for $z \in \{z: |z|\leq 1\}$. For every $\epsilon > 0$ there exists an N_0, independent of the choice of z in the considered area, such that

$$\left|p_{n+1}z^{n+1}+p_{n+2}z^{n+2}+...\right| \leq p_{n+1}+p_{n+2}+p_{n+3}+.... < \epsilon, \tag{2.9}$$

for all $n \geq N_0$.
Furthermore for every $n=0,1,2,...,$ the functions $p_n z^n$ are continuous. Recalling that the sum of a uniformly convergent series of continuous functions is a continuous function (cf. Titchmarsh [18]) we have proved the second statement.

AD P.13

A PROOF OF II.(3.10):

We have the following equation in g (cf. II.(3.9)):

$$g = r(\alpha_1 s^{-1}+\alpha_2 s)\beta\{\lambda(1-g(r_1 s+r_2 s^{-1}))\}. \tag{2.10}$$

Take,

$$h(r,s) = g(r_1 s+r_2 s^{-1}). \tag{2.11}$$

By substituting (2.11) into (2.10) we obtain:

$$h = r(\alpha_1 s^{-1}+\alpha_2 s)(r_1 s+r_2 s^{-1})\beta\{\lambda(1-h)\}. \tag{2.12}$$

Now applying Takacs' lemma (cf. Cohen [5]) to (2.12) we find if $a = \lambda\beta < 1$:

$$h(r,s) = \sum_{n=1}^{\infty} \frac{[r(\alpha_1 s^{-1}+\alpha_2 s)(r_1 s+r_2 s^{-1})]^n \lambda^{n-1}}{n!} \int_0^\infty e^{-\lambda t} t^{n-1}\, dB^{n*}(t) \tag{2.13}$$

$$= E\{r^{\mathbf{n}}(\alpha_1 s^{-1}+\alpha_2 s)^{\mathbf{n}}(r_1 s+r_2 s^{-1})^{\mathbf{n}}\},$$

with $\mathbf{n}$ as defined before.
From (2.11) and (2.13) it follows that:

$$g(r,s) = E\{r^{\mathbf{n}}(\alpha_1 s^{-1}+\alpha_2 s)^{\mathbf{n}}(r_1 s+r_2 s^{-1})^{\mathbf{n}-1}\}, \quad |s|=1. \tag{2.14}$$

AD P.43

Let, for $i = 1,2$,

$\mathbf{c}^{(i)}$: *duration of a busy cycle of queue i,* (2.15)

$\mathbf{n}^{(i)}$: *number of type* $-i$ *customers served during a busy period of queue i,*

$\mathbf{d}_m^{(i)}$: *moment of the* m^{th} *departure of a type* $-i$ *customer,*

$\mathbf{t}_m^{(i)}$: *moment of the* m^{th} *arrival of a type* $-i$ *customer,*

$\mathbf{x}_t^{(i)}$: *number of type* $-i$ *customers in the system at time t.*

THEOREM

For $h = 0,1,2,...$:

$$\lim_{m\to\infty} Pr\{\mathbf{z}_m^{(i)}=h \mid \mathbf{r}_m=i\} = \lim_{m\to\infty} Pr\{\mathbf{x}_{\mathbf{t}_m}^{(i)}=h \mid \mathbf{r}_m=i\}. \tag{2.16}$$

(i.e., the distribution of the number of type-i customers left behind in the system after the service completion of a type-i customer is equal to the distribution of the number of type-i customers immediately before an arrival epoch of a type-i customer.)

PROOF (cf. Cohen [4]):

If a change of $\mathbf{x}_t^{(i)}$ occurs it has the value 1 or -1, hence for every realisation of the $\mathbf{x}_t^{(i)}$-process during a busy cycle with every "upcrossing"

$$\mathbf{x}_{\mathbf{t}_m}^{(i)} = h, \quad \mathbf{x}_{\mathbf{t}_m}^{(i)} = h+1, \tag{2.17}$$

corresponds a "downcrossing"

$$\mathbf{x}_{\mathbf{d}_m}^{(i)} = h+1, \quad \mathbf{x}_{\mathbf{d}_m}^{(i)} = h, \tag{2.18}$$

and consequently,

$$E\{\sum_{m=1}^{\mathbf{n}^{(i)}}(\mathbf{x}_{\mathbf{t}_m}^{(i)}=h)\} = E\{\sum_{m=1}^{\mathbf{n}^{(i)}}(\mathbf{x}_{\mathbf{d}_m}^{(i)}=h)\}, \quad h = 0,1,2,\ldots . \tag{2.19}$$

The processes

$$\{\mathbf{x}_{\mathbf{d}_m}^{(i)}, \ m=1,2,...\} \ \text{and} \ \{\mathbf{x}_{\mathbf{t}_m}^{(i)}, \ m=1,2,...\}, \tag{2.20}$$

are regenerative with respect to the renewal sequence $\mathbf{n}_1, \mathbf{n}_2, \ldots$. Furthermore, since the interarrival times are negative exponentially distributed the busy cycle $\mathbf{c}$ has a nonlattice distribution (cf. Cohen [5], p.249 ff.). Hence it follows from (2.19) and the Key Renewal Theorem (cf. Cohen [5]) that for $h = 0,1,2,...$:

$$\begin{aligned} \lim_{m\to\infty} Pr\{\mathbf{x}_{\mathbf{d}_m}^{(i)}=h\} &= \frac{1}{E\{\mathbf{n}^{(i)}\}} E\{\sum_{m=1}^{\mathbf{n}^{(i)}}(\mathbf{x}_{\mathbf{d}_m}^{(i)}=h)\} \\ &= \frac{1}{E\{\mathbf{n}^{(i)}\}} E\{\sum_{m=1}^{\mathbf{n}^{(i)}}(\mathbf{x}_{\mathbf{t}_m}^{(i)}=h)\} \\ &= \lim_{m\to\infty} Pr\{\mathbf{x}_{\mathbf{t}_m}^{(i)}=h\}. \end{aligned} \tag{2.21}$$

Because $\mathbf{x}_{\mathbf{d}_m}^{(i)}$ is by definition equal to $\mathbf{z}_m^{(i)}$ the proof is completed.

Appendix B

Plots

In this section several graphs have been drawn of the S_1-contour, the S_2- contour and the L-contour. In all cases shown, the service times were taken negative exponential. In the cases where one of the contours S_1 or S_2 has loops or cusps a correct L-contour can not be constructed. Hence the plotted L-contour is in such cases simply the result of a few iterations of the start contour L_0.

As remarked on p.38 the expression

$$F_2'(1) = \frac{\alpha_1 - a_1}{a_2 - \alpha_2},$$

(cf. III.(5.10)) gives some information about the start direction of the S_1- and S_2-contours. Pages 52 - 56 provide an example of the various possibilities. We kept the parameters

$$r_1 = 0.75, \quad r_2 = 0.25; \qquad (1.1)$$
$$a_1 = 0.6, \quad a_2 = 0.2;$$

fixed and we varied α_1 (and hence α_2). Now 5 different possibilities were obtained:

(1) $\alpha_1 < 0.6$: $\frac{\alpha_1 - a_1}{a_2 - \alpha_2} > 0$

(2) $\alpha_1 = 0.6$: $\frac{\alpha_1 - a_1}{a_2 - \alpha_2} = 0$

(3) $0.6 < \alpha_1 < 0.8$: $\frac{\alpha_1 - a_1}{a_2 - \alpha_2} < 0$

(4) $\alpha_1 = 0.8$: $\frac{\alpha_1 - a_1}{a_2 - \alpha_2} = \infty$

(5) $\alpha_1 > 0.8$: $\frac{\alpha_1 - a_1}{a_2 - \alpha_2} > 0$

Note that only in Case (3), ($0.6 < \alpha_1 < 0.8$), we are able to compute the first moments using the iteration program.

On p.57 - 65 the total traffic, a, is kept fixed, and $r_1 = \alpha_1$ ($r_2 = \alpha_2$). The influence of varying $r_1 = \alpha_1$ (and hence $r_2 = \alpha_2$) is shown. Note that in the case $r_1 = \alpha_1 < 0.5$ the contours S_1 and S_2 are the same, but with the indices interchanged, as are the contours S_1 and S_2 for

$r_1' = \alpha_1' = r_1 + 0.5 = \alpha_1 + 0.5$.
Finally, on p.66 - 69, the total traffic, a, is again kept fixed and we let $r_1 \equiv \alpha_1 \uparrow 1$ (and hence $r_2 \equiv \alpha_2 \downarrow 0$). Note that on p.69 the S_1-contour has almost completely degenerated into a point and no correct L-contour has been computed.

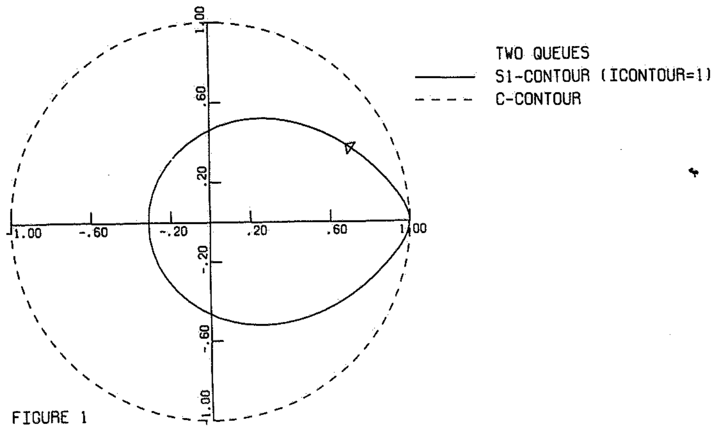


FIGURE 1

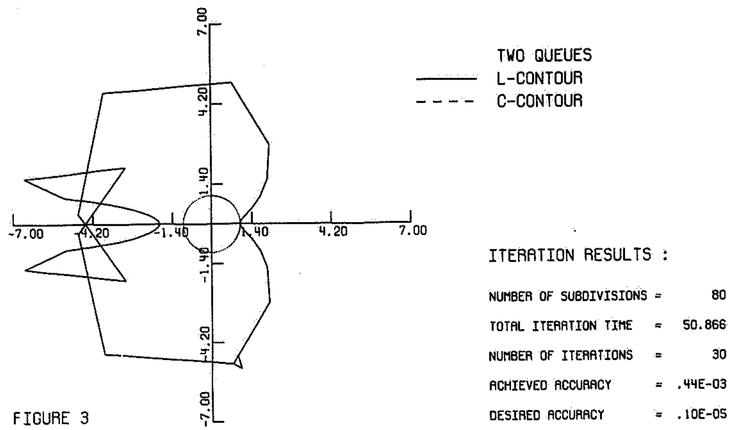


FIGURE 3

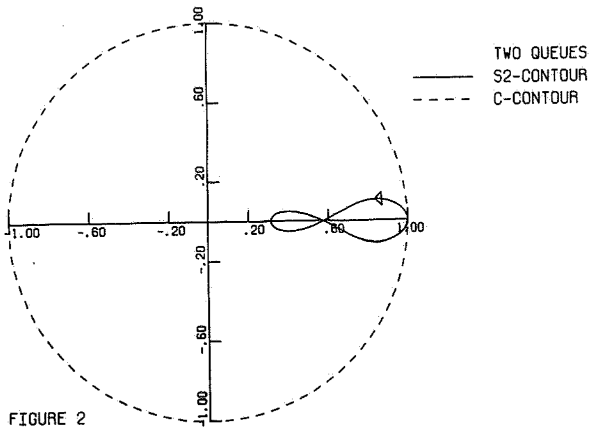


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .6000

A2 = .2000

ALFA1 = .2000

ALFA2 = .8000

R1 = .7500

R2 = .2500

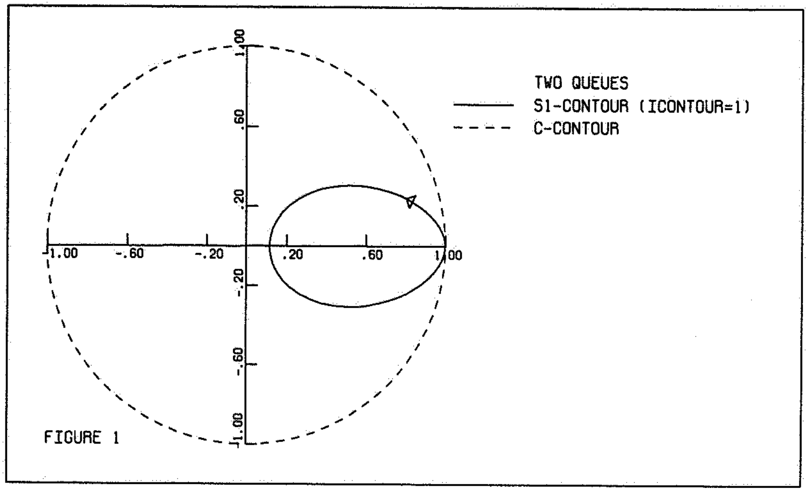


FIGURE 1

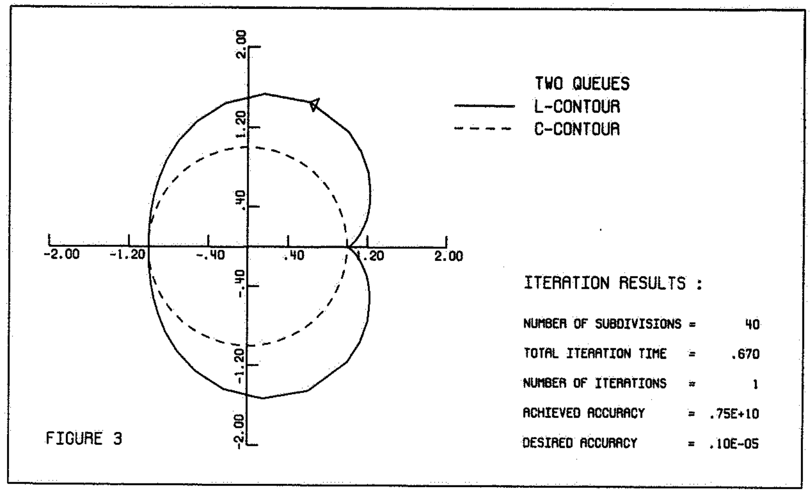


FIGURE 3

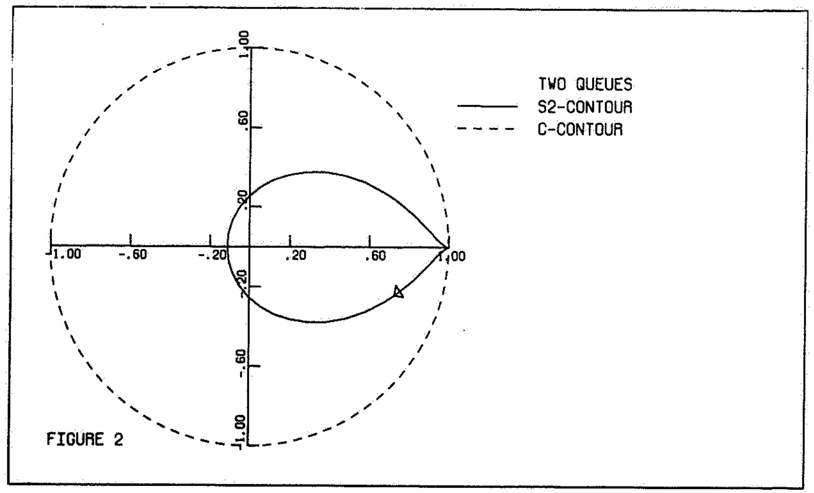


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .6000

A2 = .2000

ALFA1 = .6000

ALFA2 = .4000

R1 = .7500

R2 = .2500

$$F_2'(1) = \frac{\alpha_1 - a_1}{a_2 - \alpha_2} = 0$$

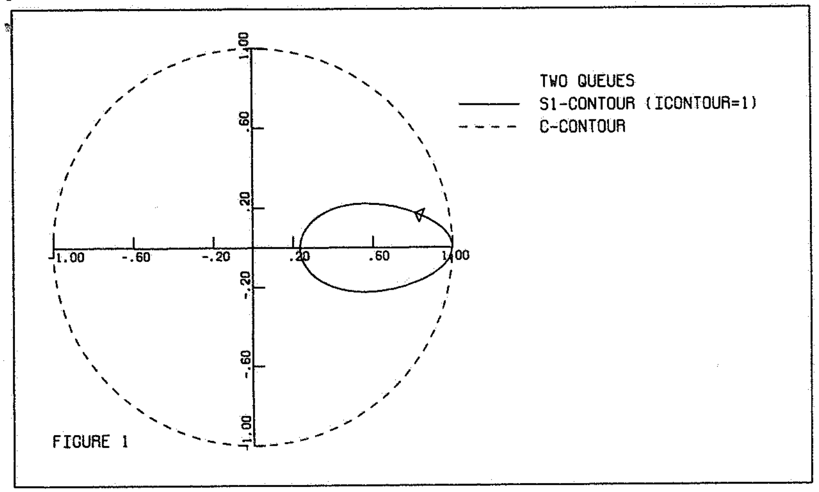


FIGURE 1

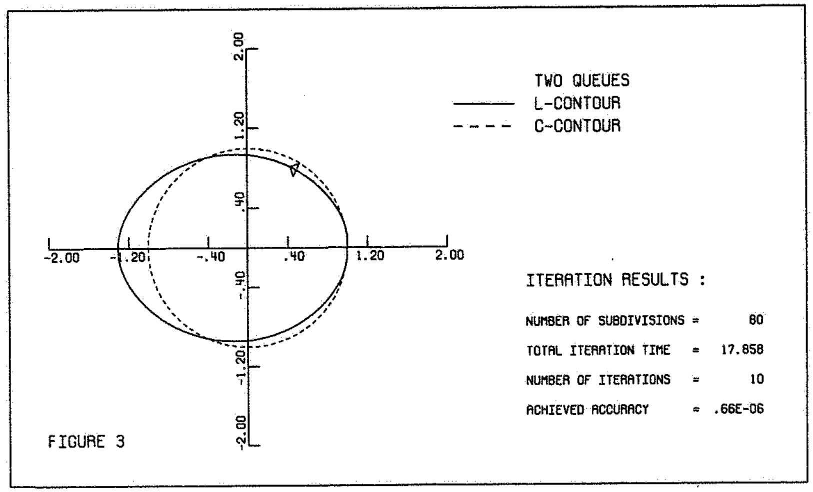


FIGURE 3

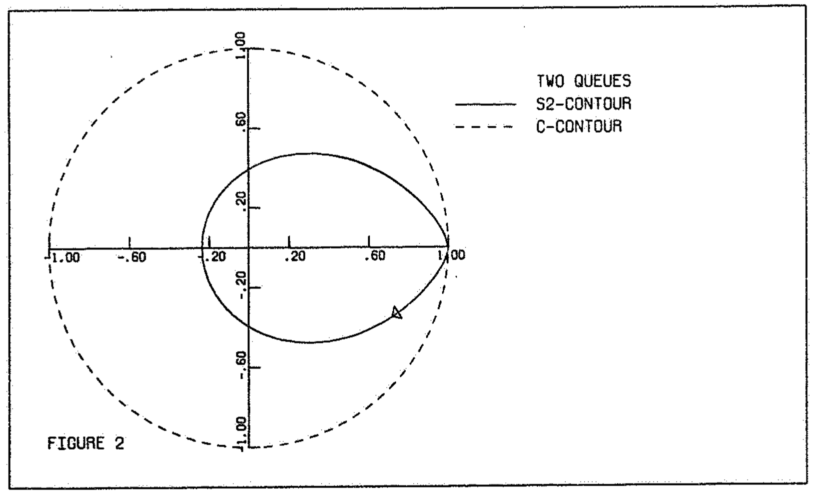


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .6000

A2 = .2000

ALFA1 = .7000

ALFA2 = .3000

R1 = .7500

R2 = .2500

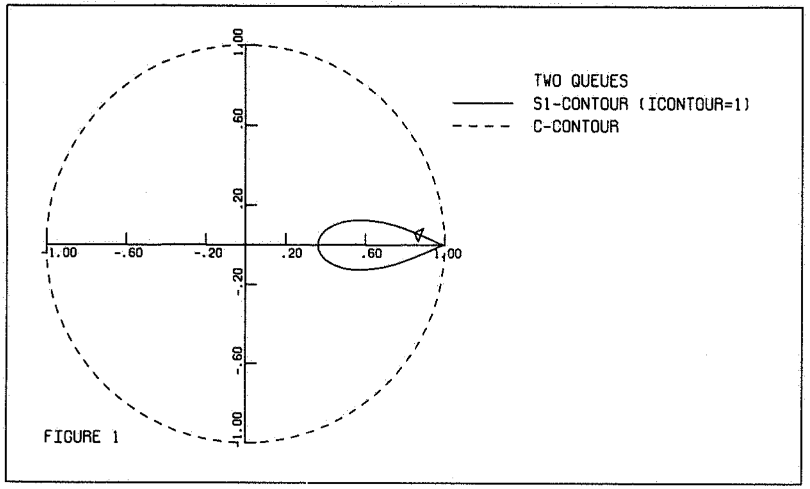


FIGURE 1

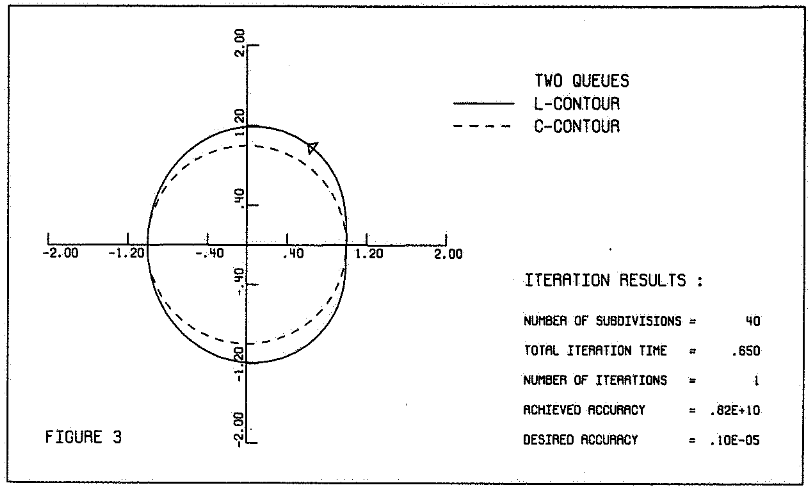


FIGURE 3

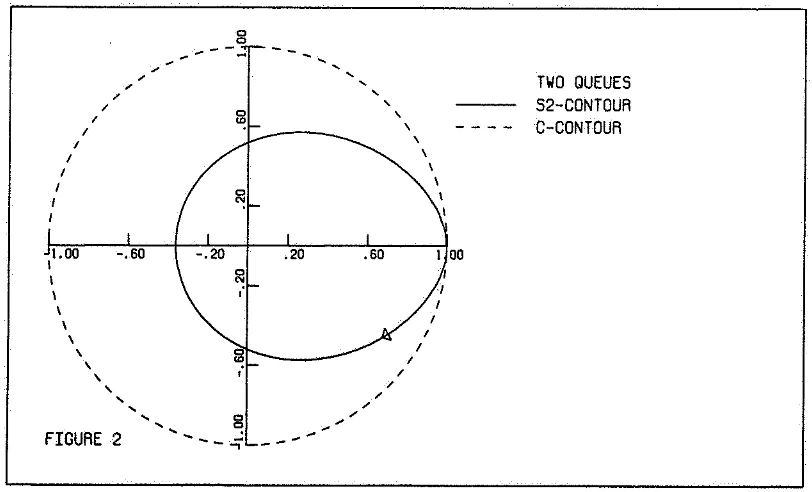


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1	=	.6000
A2	=	.2000
ALFA1	=	.8000
ALFA2	=	.2000
R1	=	.7500
R2	=	.2500

$$F_2'(1) = \frac{\alpha_1 - a_1}{a_2 - \alpha_2} = \infty$$

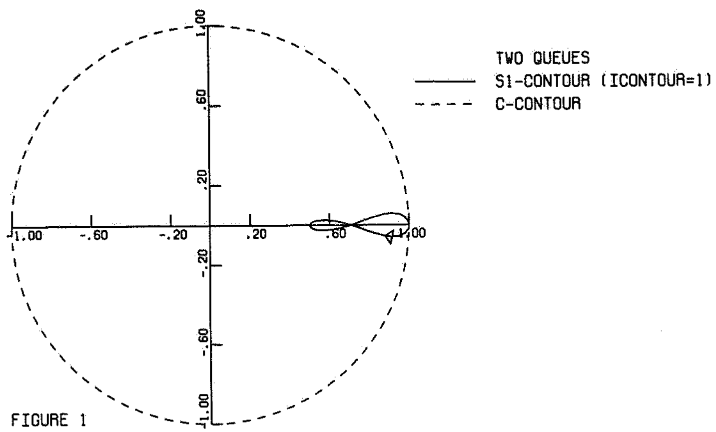


FIGURE 1

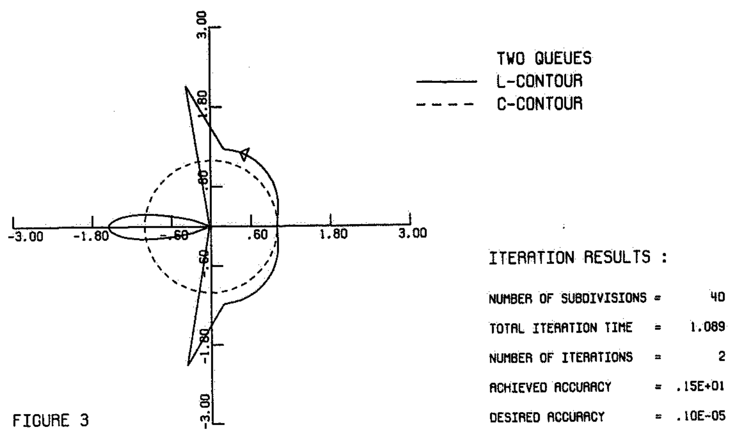


FIGURE 3

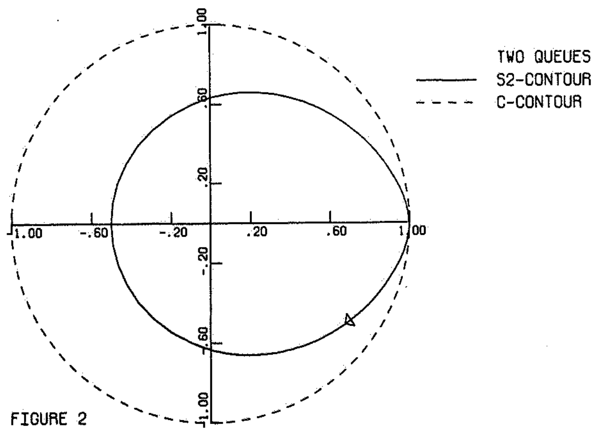


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .6000

A2 = .2000

ALFA1 = .9000

ALFA2 = .1000

R1 = .7500

R2 = .2500

$$F_2'(1) = \frac{\alpha_1 - a_1}{a_2 - \alpha_2} > 0$$

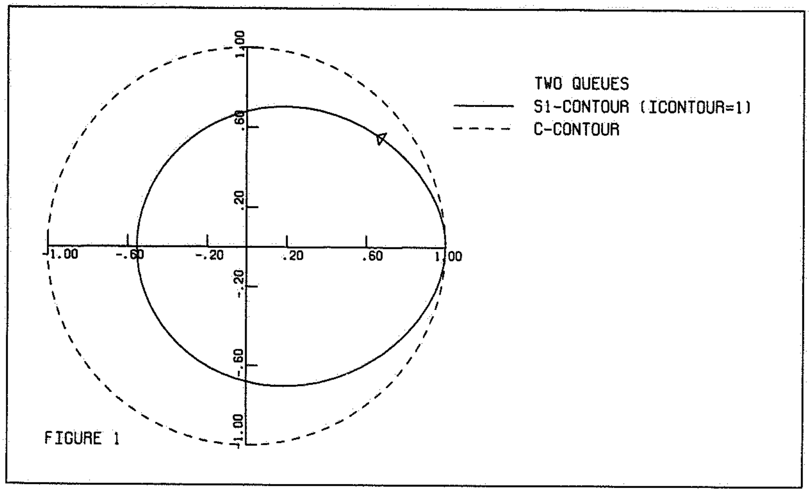


FIGURE 1

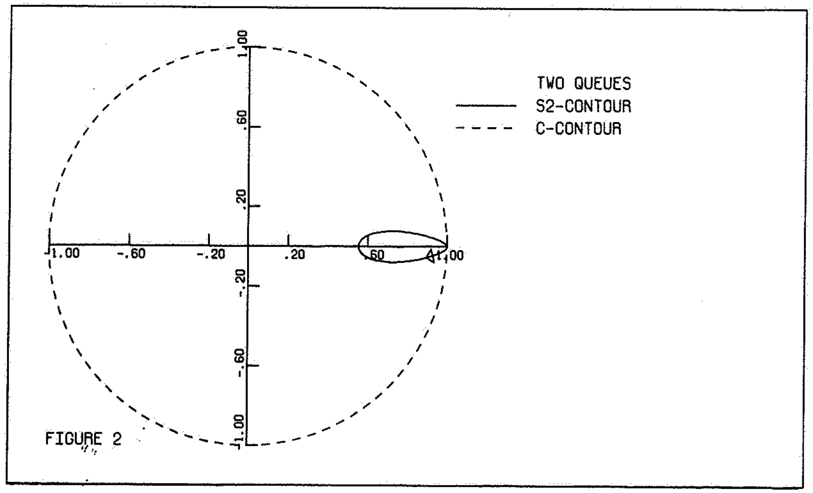


FIGURE 2

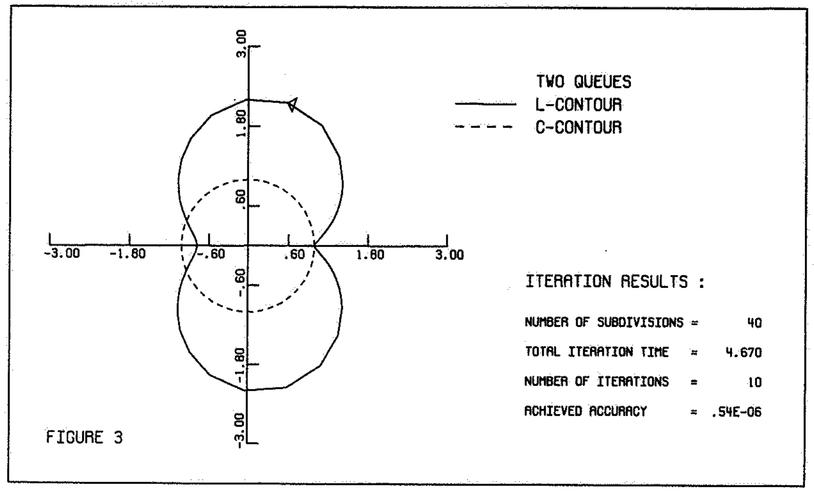


FIGURE 3

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .0800

A2 = .7200

ALFA1 = .1000

ALFA2 = .9000

R1 = .1000

R2 = .9000

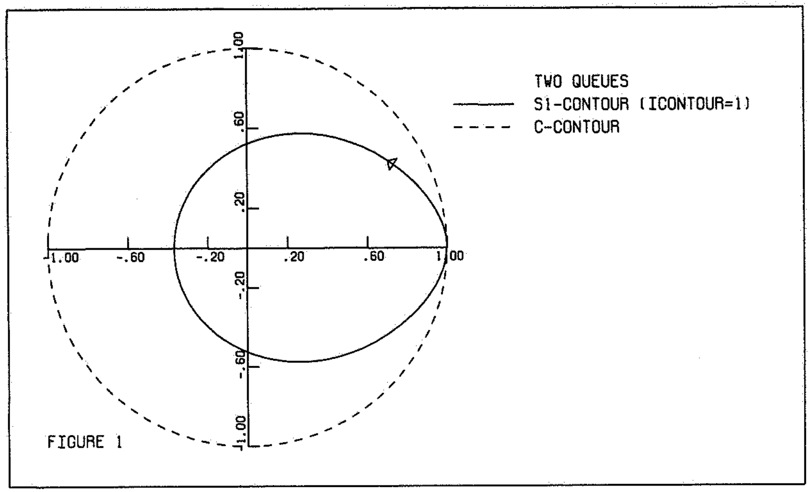


FIGURE 1

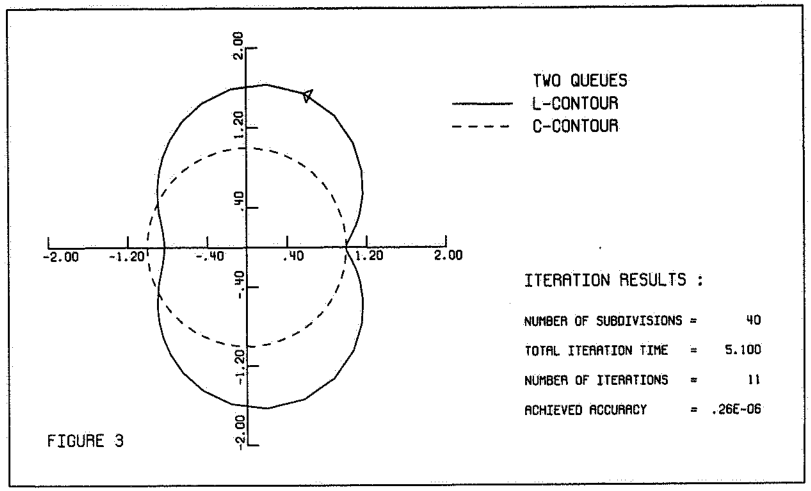


FIGURE 3

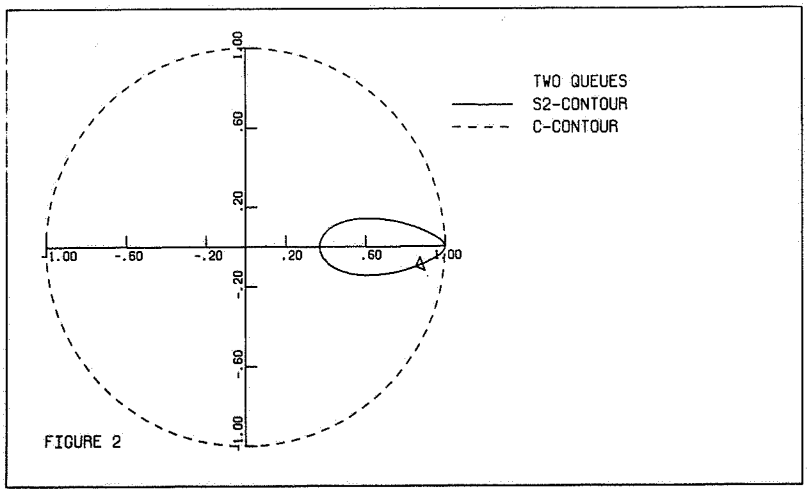


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .1600

A2 = .6400

ALFA1 = .2000

ALFA2 = .8000

R1 = .2000

R2 = .8000

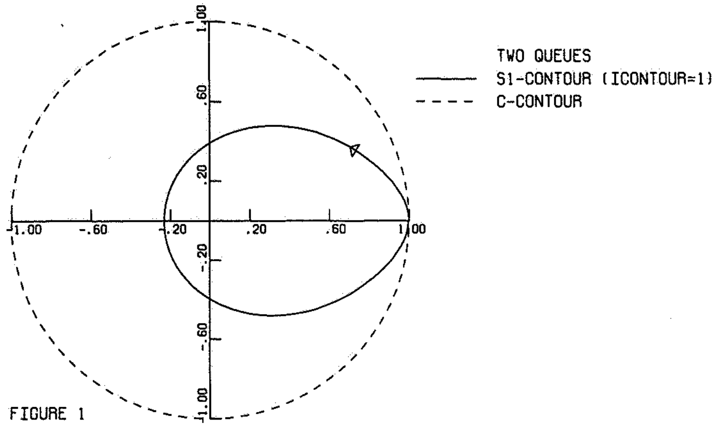


FIGURE 1

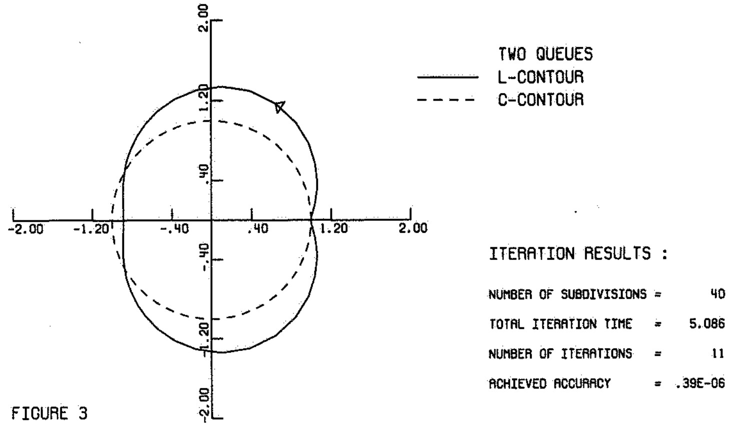


FIGURE 3

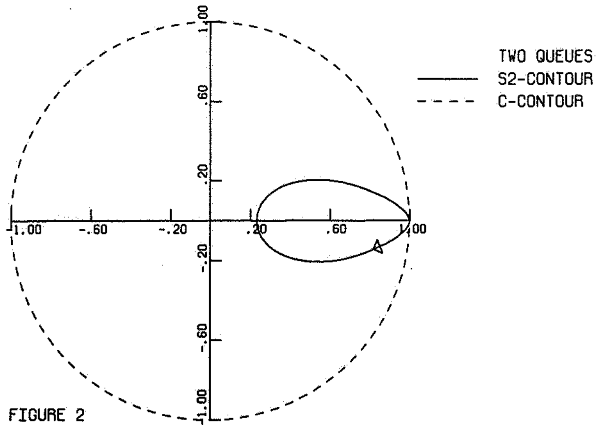


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .2400

A2 = .5600

ALFA1 = .3000

ALFA2 = .7000

R1 = .3000

R2 = .7000

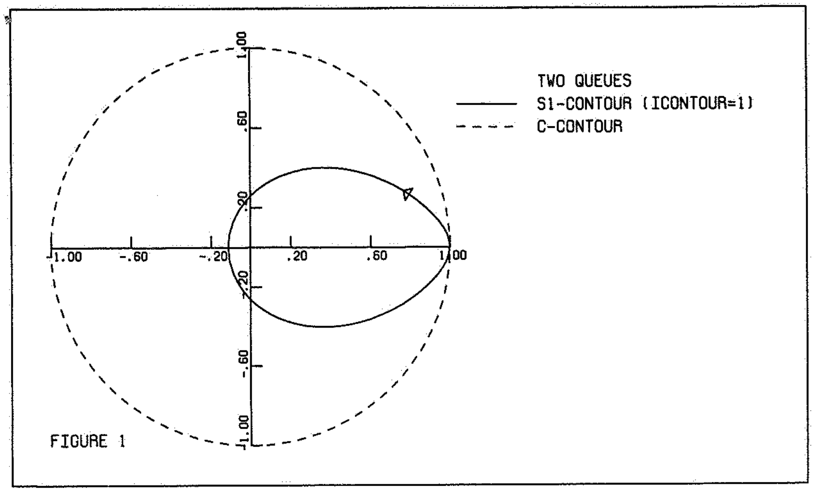


FIGURE 1

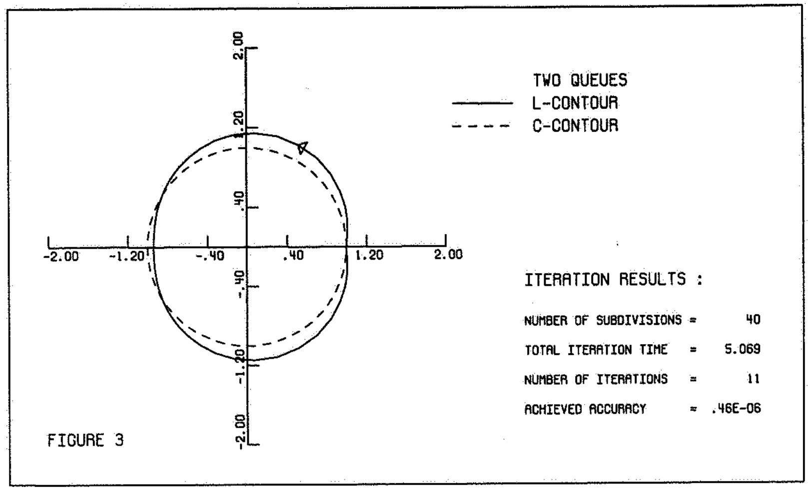


FIGURE 3

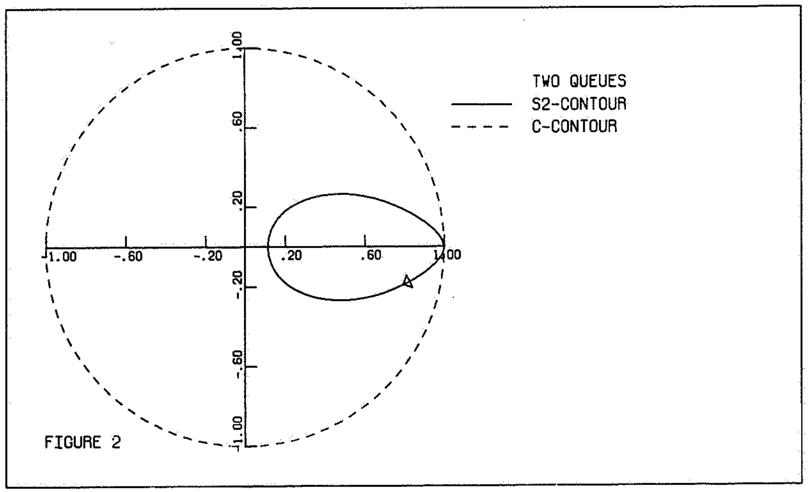


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .3200

A2 = .4800

ALFA1 = .4000

ALFA2 = .6000

R1 = .4000

R2 = .6000

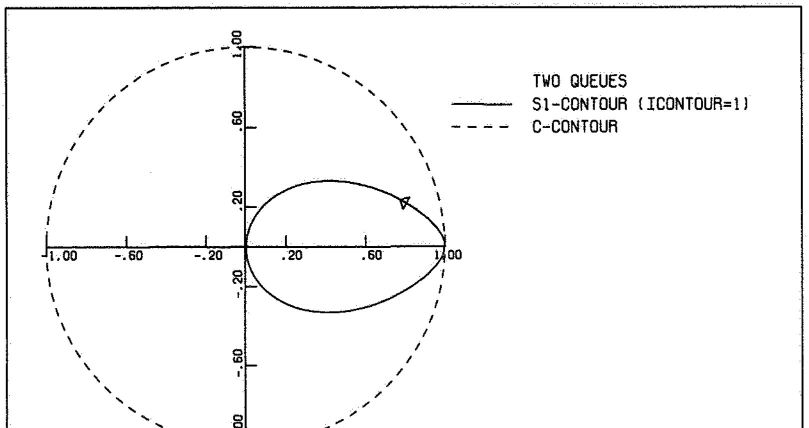


FIGURE 1

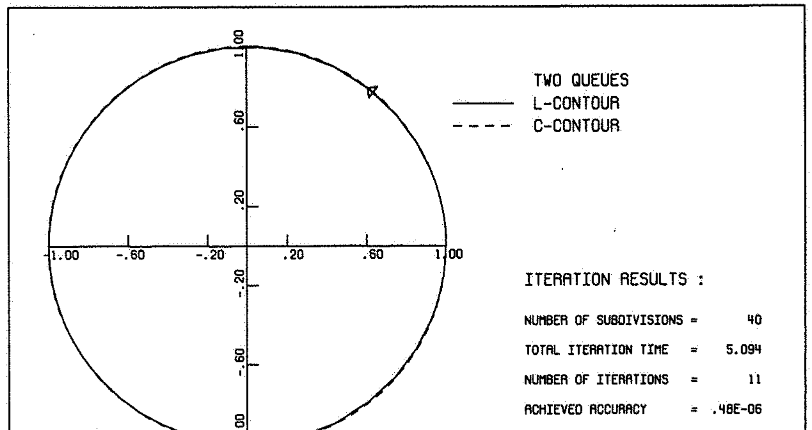


FIGURE 3

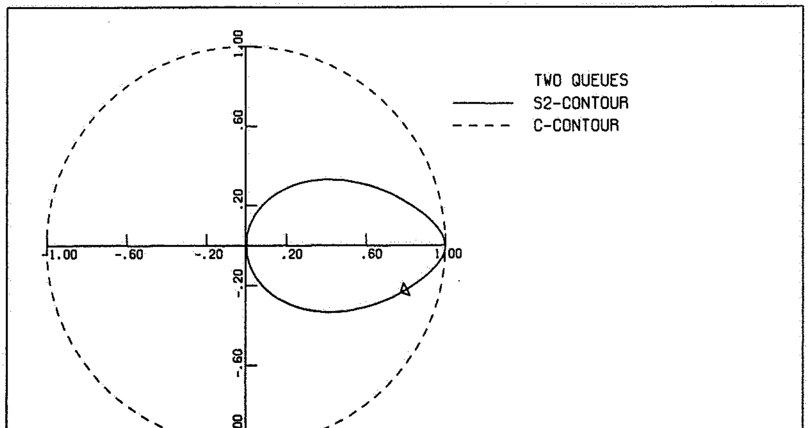


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .4000

A2 = .4000

ALFA1 = .5000

ALFA2 = .5000

R1 = .5000

R2 = .5000

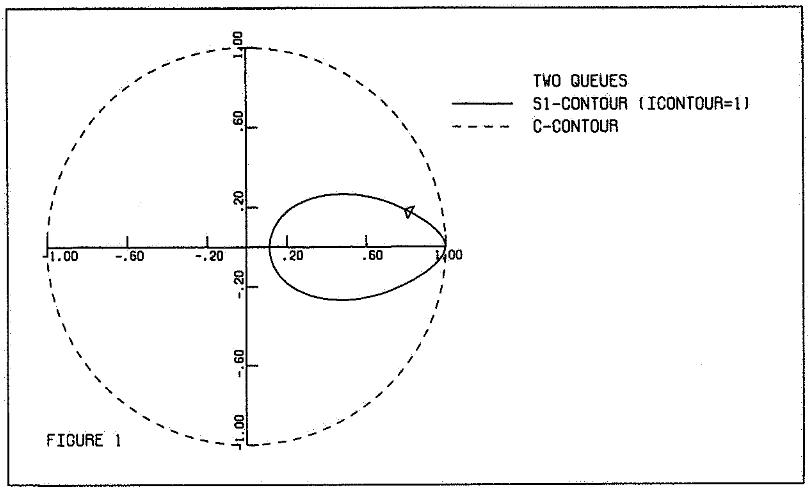


FIGURE 1

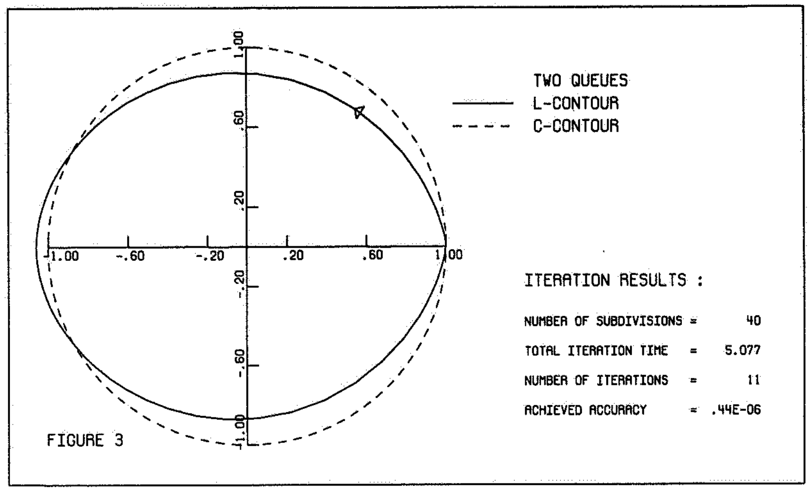


FIGURE 3

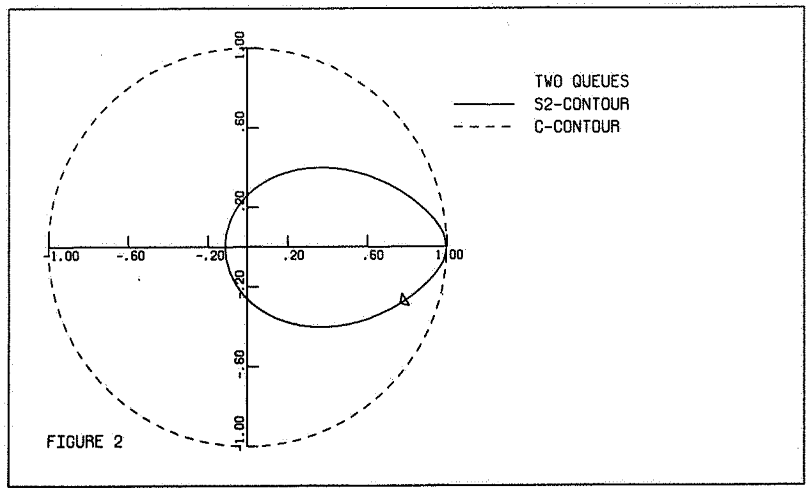


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .4800

A2 = .3200

ALFA1 = .6000

ALFA2 = .4000

R1 = .6000

R2 = .4000

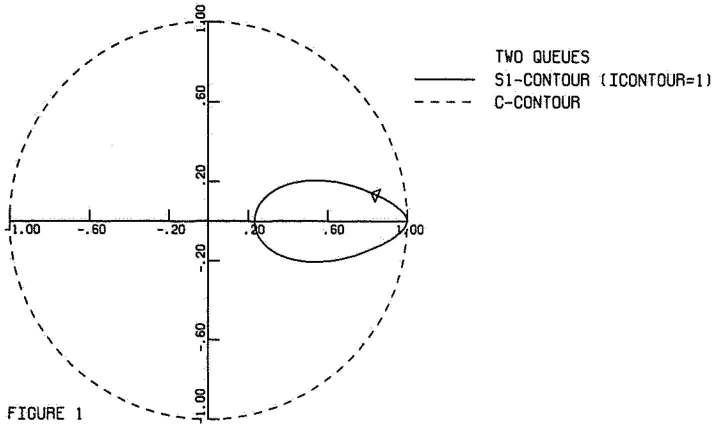
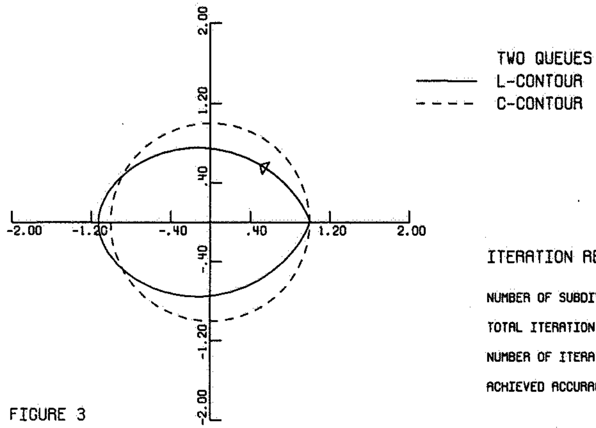


FIGURE 1



ITERATION RESULTS :

NUMBER OF SUBDIVISIONS = 40

TOTAL ITERATION TIME = 4.707

NUMBER OF ITERATIONS = 10

ACHIEVED ACCURACY = .93E-06

FIGURE 3

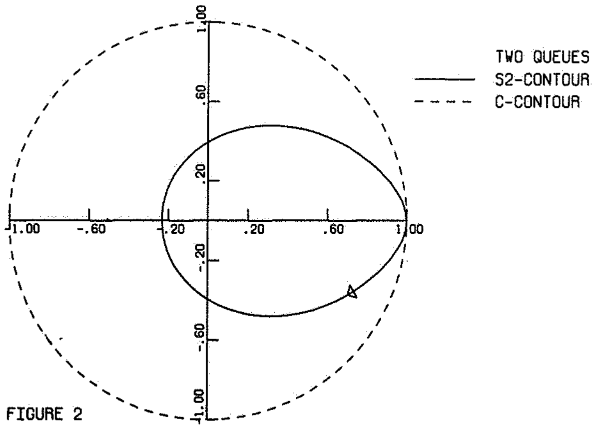


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .5600

A2 = .2400

ALFA1 = .7000

ALFA2 = .3000

R1 = .7000

R2 = .3000

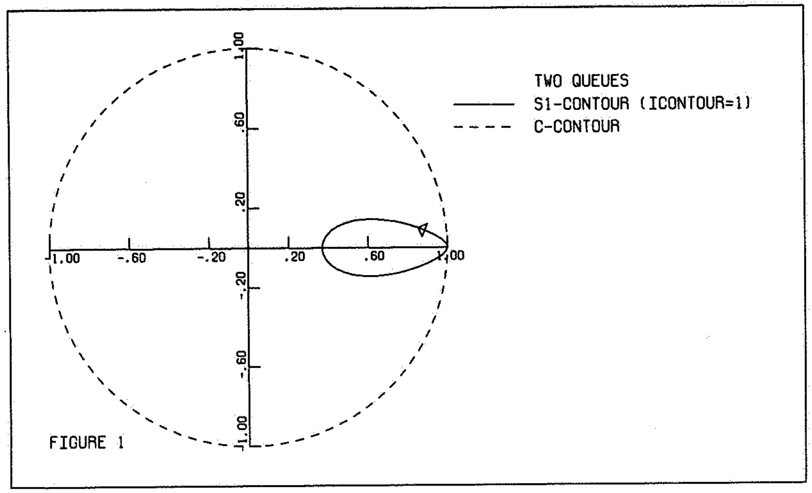


FIGURE 1

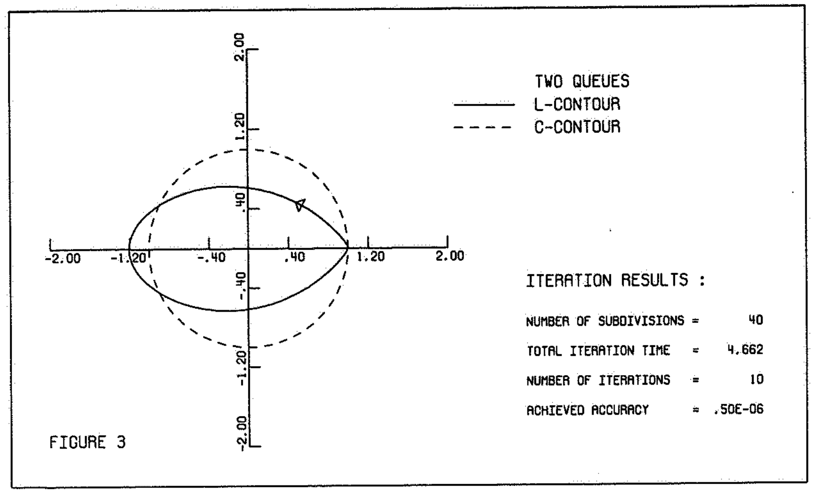


FIGURE 3

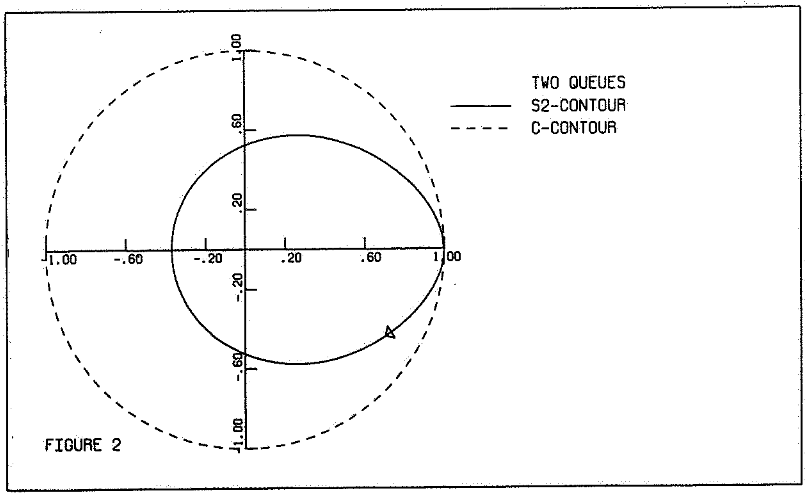


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1	=	.6400
A2	=	.1600
ALFA1	=	.8000
ALFA2	=	.2000
R1	=	.8000
R2	=	.2000

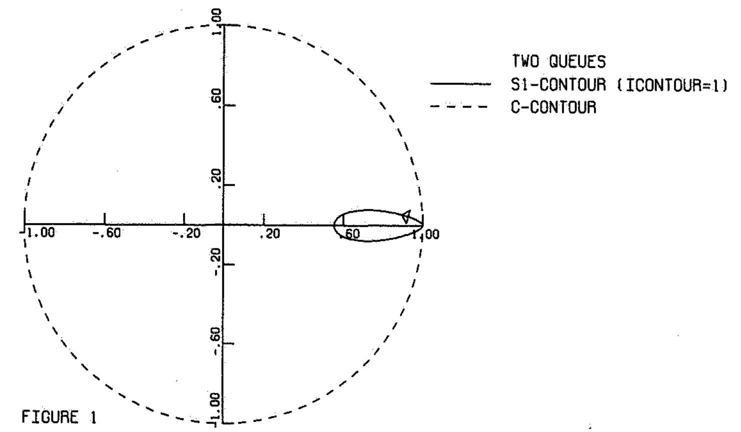


FIGURE 1

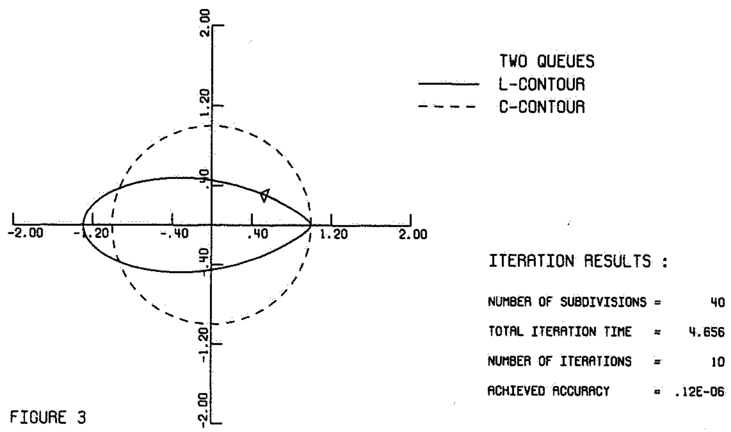


FIGURE 3

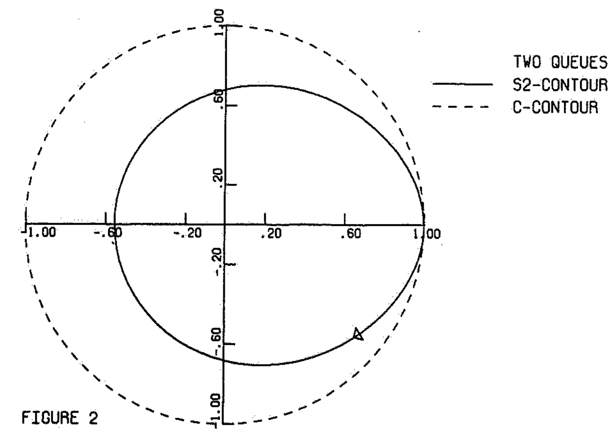


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .7200

A2 = .0800

ALFA1 = .9000

ALFA2 = .1000

R1 = .9000

R2 = .1000

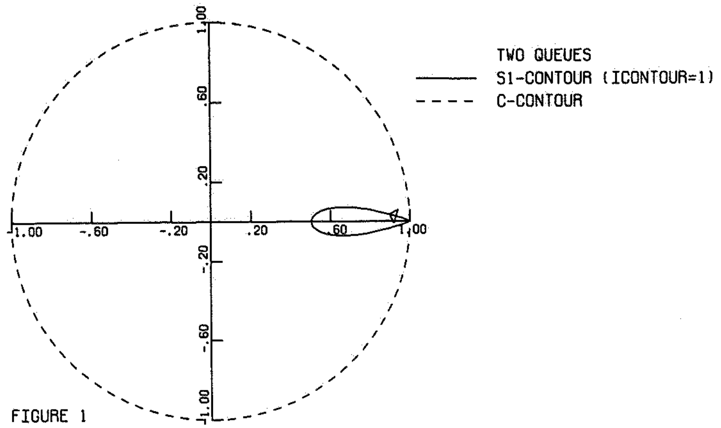


FIGURE 1

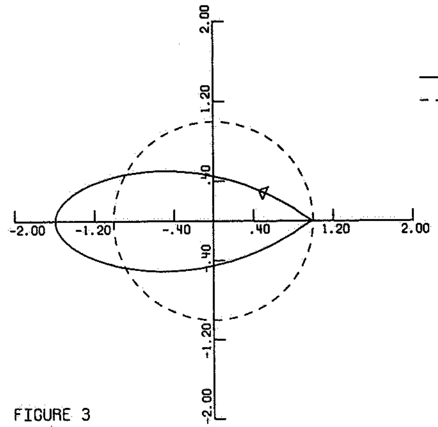


FIGURE 3

ITERATION RESULTS :

NUMBER OF SUBDIVISIONS = 40

TOTAL ITERATION TIME = 6.404

NUMBER OF ITERATIONS = 14

ACHIEVED ACCURACY = .87E-06

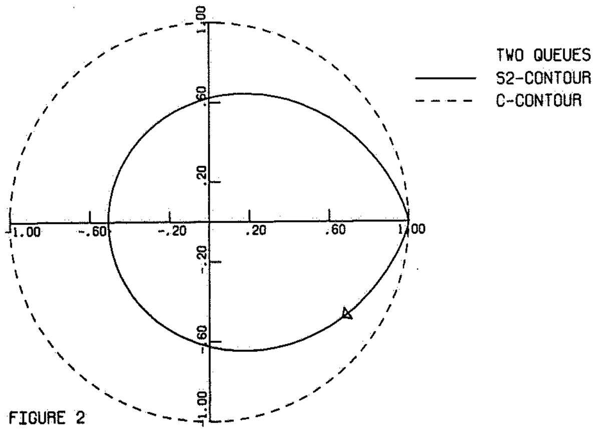


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .8820

A2 = .0980

ALFA1 = .9000

ALFA2 = .1000

R1 = .9000

R2 = .1000

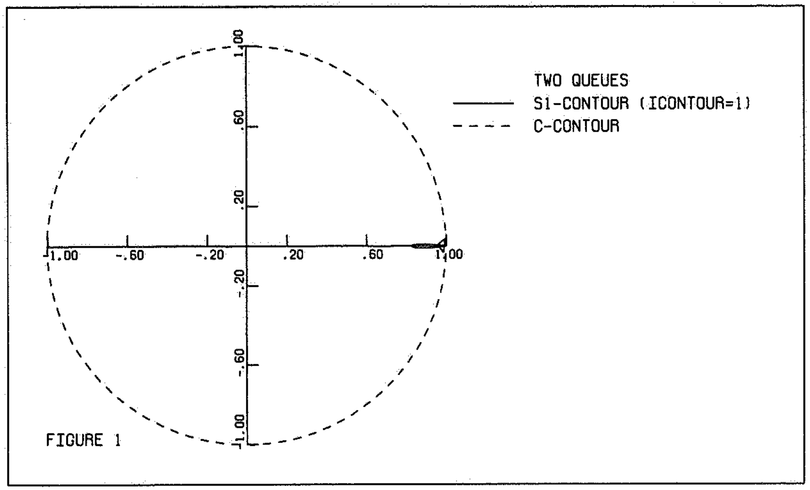


FIGURE 1

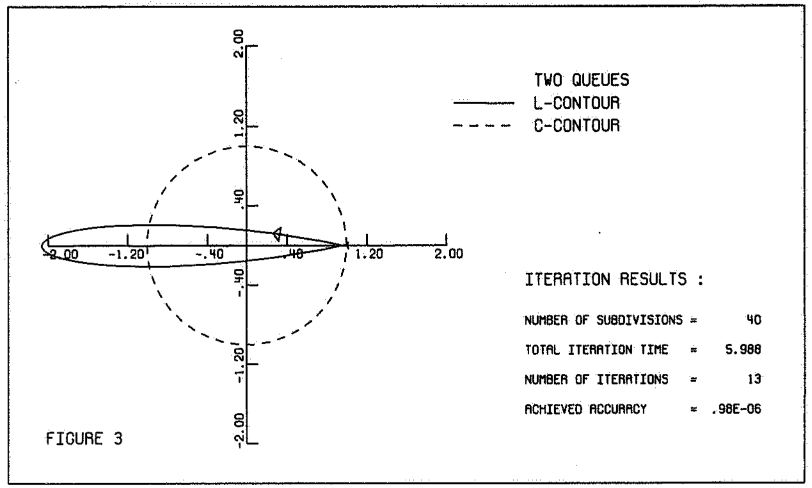


FIGURE 3

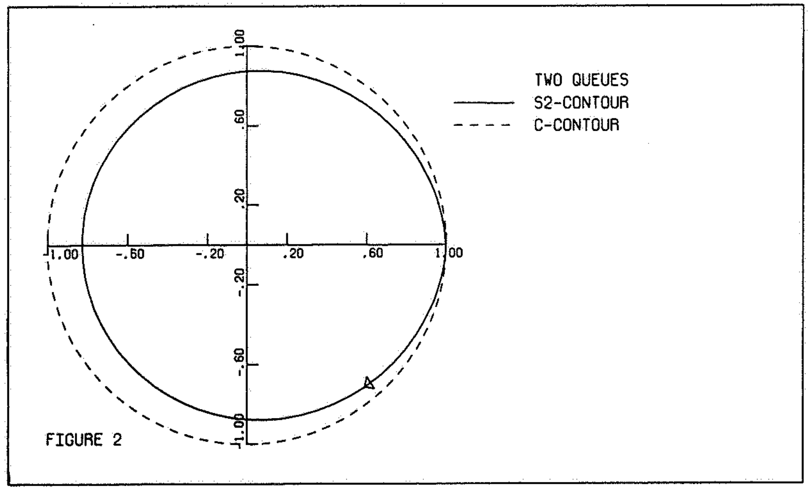


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .9702

A2 = .0098

ALFA1 = .9900

ALFA2 = .0100

R1 = .9900

R2 = .0100

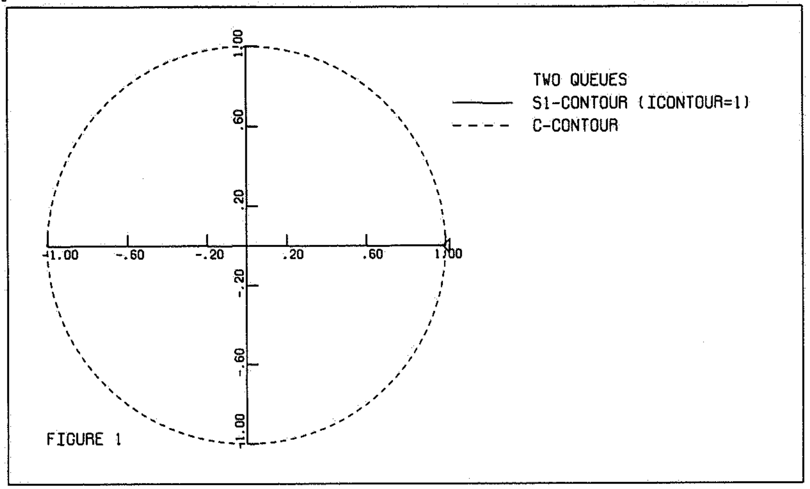


FIGURE 1

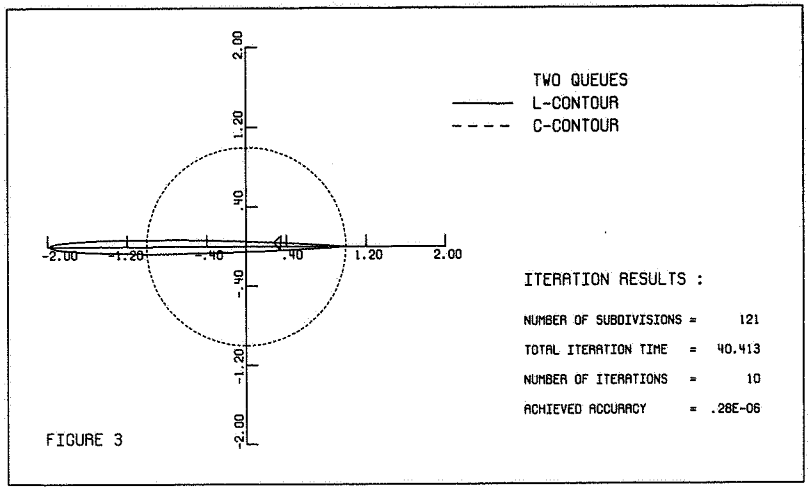


FIGURE 3

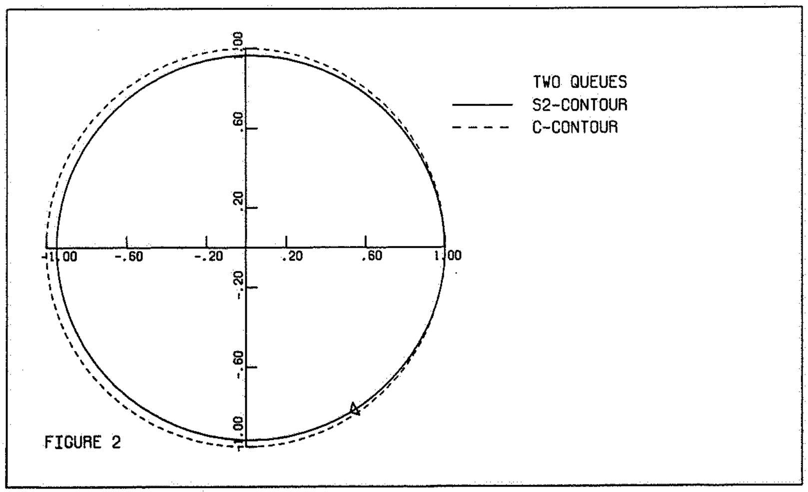


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .9790

A2 = .0010

ALFA1 = .9990

ALFA2 = .0010

R1 = .9990

R2 = .0010

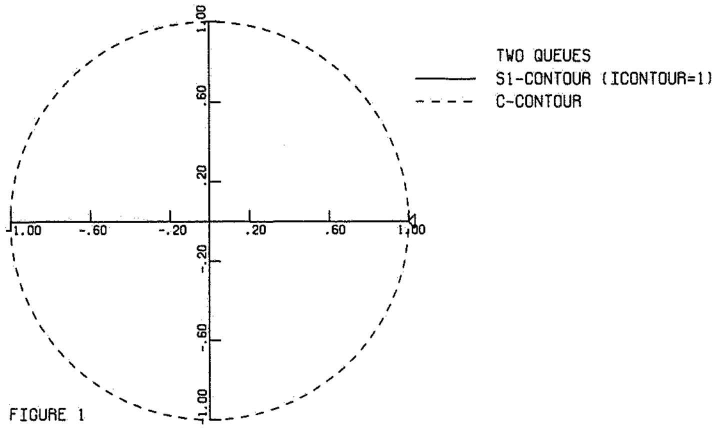


FIGURE 1

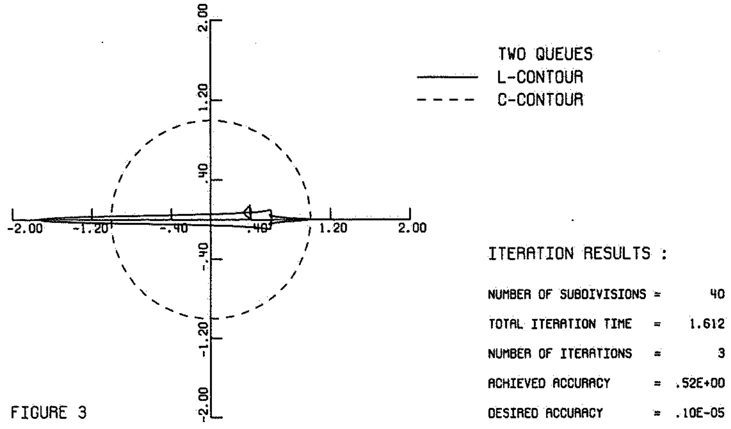


FIGURE 3

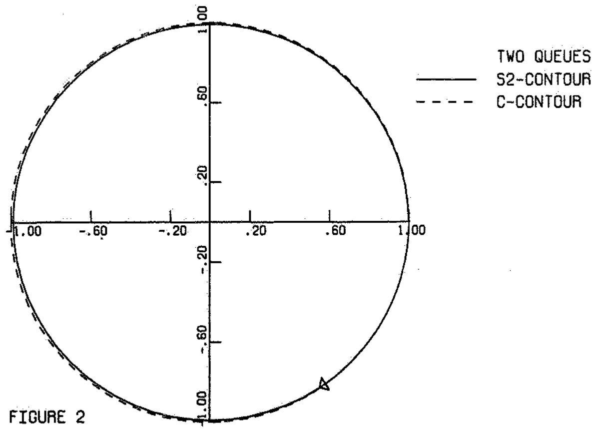


FIGURE 2

TWO QUEUES (IMODEL=4)

THE PARAMETERS ARE :

A1 = .9799

A2 = .0001

ALFA1 = .9999

ALFA2 = .0001

R1 = .9999

R2 = .0001

Appendix C

On a derivation of the stationary distribution of the actual waiting time of the n^{th} departing customer given this customer is of type i

In this section we consider the system as described in II.1. Let us assume the system is in equilibrium. Denote by

$$W_n^{(i)}(t) := Pr\{\mathbf{w}_n < t \mid \mathbf{r}_n = i\}, \tag{1.1}$$

the actual waiting-time distribution of the n^{th} departing customer, given that this customer is of type i.

Because the service discipline at each queue is FCFS, the number of type-i customers that is left behind by the n^{th} departing customer given this customer is of type i, is equal to the number of type-i customers that arrived during the waiting- and the service time of this customer. Hence,

$$Pr\{\mathbf{z}_n^{(i)} = j \mid \mathbf{r}_n = i\} = \int_{0-}^{\infty} \frac{(\lambda_i t)^j}{j!} e^{-\lambda_i t} \, d_t\{W_n^{(i)}(t) * B_i(t)\}, \tag{1.2}$$

because the service time and the waiting time of the n^{th} departing customer are independent and the arrival process is a Poisson process.

It follows that, for $|p| \leqslant 1$:

$$\int_{0-}^{\infty} e^{-\lambda_i(1-p)t} \, dW_n^{(i)}(t) = \frac{\sum_{j=0}^{\infty} p^j Pr\{\mathbf{z}_n^{(i)} = j \mid \mathbf{r}_n = i\}}{\beta_i\{\lambda_i(1-p)\}}. \tag{1.3}$$

Or, equivalently, cf. III.(2.2),

$$\int_{0-}^{\infty} e^{-\lambda_1(1-p)t} \, dW_n^{(1)}(t) = \frac{\Phi^{(1)}(p,1)}{r_1 \beta_1\{\lambda_1(1-p)\}}, \tag{1.4}$$

$$\int_{0-}^{\infty} e^{-\lambda_2(1-p)t} \, dW_n^{(2)}(t) = \frac{\Phi^{(2)}(1,p)}{r_2 \beta_2\{\lambda_2(1-p)\}}.$$

Denote the LST of $W_n^{(i)}(\cdot)$ by $\omega_n^{(i)}(\rho)$, Re $\rho \geqslant 0$. We have from III.(2.6) and (1.4):

$$\omega_n^{(1)}(\lambda_1(1-p)) = \frac{\alpha_1}{r_1 p}\left\{E\{p^{\mathbf{z}_n^{(1)}}\} - Pr\{\mathbf{z}_n^{(1)} = 0\}\right\} \tag{1.5}$$

$$+ \frac{\alpha_2}{r_1 p}\left\{E\{p^{\mathbf{z}_n^{(1)}}(\mathbf{z}_n^{(2)} = 0)\} - Pr\{\mathbf{z}_n^{(1)} = 0, \mathbf{z}_n^{(2)} = 0\}\right\} +$$

$$+ \; Pr\{\mathbf{z}_n^{(1)}=0,\; \mathbf{z}_n^{(2)}=0\},$$

$$\omega_n^{(2)}(\lambda_2(1-p)) = \frac{\alpha_2}{r_2 p}\left\{E\{p^{\mathbf{z}_n^{(2)}}\} - Pr\{\mathbf{z}_n^{(2)}=0\}\right\}$$

$$+ \; \frac{\alpha_1}{r_2 p}\left\{E\{p^{\mathbf{z}_n^{(2)}}(\mathbf{z}_n^{(1)}=0)\} - Pr\{\mathbf{z}_n^{(1)}=0,\; \mathbf{z}_n^{(2)}=0\}\right\}$$

$$+ \; Pr\{\mathbf{z}_n^{(1)}=0,\; \mathbf{z}_n^{(2)}=0\}.$$

From (1.5) the first moments are easily obtained:

$$E\{w_n^{(1)}\} = \frac{\alpha_1}{\lambda_1 r_1}\left\{E\{\mathbf{z}_n^{(1)}\} - Pr\{\mathbf{z}_n^{(1)}>0\}\right\} \tag{1.6}$$

$$+ \; \frac{\alpha_2}{\lambda_1 r_1}\left\{E\{\mathbf{z}_n^{(1)}(\mathbf{z}_n^{(2)}=0)\} - Pr\{\mathbf{z}_n^{(1)}>0,\; \mathbf{z}_n^{(2)}=0\}\right\},$$

$$E\{w_n^{(2)}\} = \frac{\alpha_2}{\lambda_2 r_2}\left\{E\{\mathbf{z}_n^{(2)}\} - Pr\{\mathbf{z}_n^{(2)}>0\}\right\}$$

$$+ \; \frac{\alpha_1}{\lambda_2 r_2}\left\{E\{\mathbf{z}_n^{(2)}(\mathbf{z}_n^{(1)}=0)\} - Pr\{\mathbf{z}_n^{(1)}=0,\; \mathbf{z}_n^{(2)}>0\}\right\}.$$

The expressions (1.6) could of course also be obtained by applying Little's formula to the first moments of the queue-length distribution.

Appendix D

Listing and description of the computer programs

Two computer programs have been used to help us in the investigation of the present model and to obtain some numerical results. In Section 1 a listing and description of the first program is given. It is a simulation program, written by the author of this report, to simulate the M/G/1 model with N queues and Random Allocation. In Section 2 a listing of the second program, written by drs. S.J. de Klein, in cooperation with the author of this report is presented. The program itself contains sufficient comments to provide in the documentation. It is written for general two dimensional random-walk cases, which include the present model.

1. The simulation program

The simulation program for the present model was written in Simula 67. For an introduction to this computer language cf. Zweerus-Vink [20].

On the next pages a listing of the simulation program is presented. The program is in principle identical to the program described in Zweerus-Vink [20] to simulate the waiting room of a doctor. There are some differences, due to the fact that, in the present model, there are N queues and a special service discipline. Most of the procedure- and class definitions in the listing speak for itself, but the purpose of some procedures and classes will be briefly outlined below.

The standard *integer procedure discrete (a,u)* has as input parameters a seed, *u*, and a cumulative row, *a*, of probabilities. The routine gives as output an integer in the range [lb.row, upb.row + 1]. To illustrate its use we give an example.
Suppose we have a biased die. The following table lists the probability of numbers to be thrown:

number	probability
1	0.1
2	0.1
3	0.2
4	0.1
5	0.2
6	0.3

Table 1.

To simulate the throwing of this particular die, we take

$$a := (0.1,0.2,0.4,0.5,0.7), \tag{1.1}$$

and then call the routine *discrete (a,seed).* The purpose of the procedure *cumulat,* line 49 - 57 may now be immediately clear.

The *real procedure stmean,* line 59 - 60, produces a positive, real number that is distributed according to the distribution function defined on line 60 (in this case negative exponential, but this may be adapted to arbitrary other distribution functions). The *array B* contains the means of the various service-time distributions, i.e., $B_j(\cdot)$ contains the mean service time of a type-j customer, j = 1,...,N.

The *integer procedure chooseclient* is called when the server has completed a service. Its purpose is to choose the next client to be served with the correct probabilities (as prescribed in Chapter I). This is realised as follows: When the routine is called, a *row [1 : N-1]* is made, with, for j = 1,...,N-1,

$$\begin{aligned} row(j) &= 0 \quad \text{if } \textit{queue j is empty}, \\ row(j) &= \alpha_j \quad \text{if } \textit{queue j is not empty}. \end{aligned} \tag{1.2}$$

The probabilities α_j, contained in the *row A,* times the indicator function of the event that queue j is not empty are summed for j = 1,...,N, and the result is put in the variable *sumr.* Then *row(j)* is divided by *sumr* for j = 1,...,N-1, *cumulat (row)* is called and then a call to *discrete (row,seed)* produces the number of the queue from which the next client should be chosen.
We illustrate this with an example.
Suppose the server has completed a service and we have the following situation:

$$\alpha_1 = 0.1,\ \alpha_2 = 0.3,\ \alpha_3 = 0.2,\ \alpha_4 = 0.4, \tag{1.3}$$

queue	status
1	not empty
2	empty
3	not empty
4	not empty

TABLE 2.

Firstly a *row[1,3]* is made as described above:

$$row := (0.1,0,0.2). \tag{1.4}$$

Next *sumr* is calculated:

$$sumr := 0.7, \tag{1.5}$$

and *row(j),* j = 1,...,N-1 is divided by *sumr:*

$$row := (\tfrac{1}{7},0,\tfrac{2}{7}). \tag{1.6}$$

Now a new client is chosen from queue j, j = 1,...,N, with the following probabilities (cf. Table 3):

queue	probability
1	$\frac{1}{7}$
2	0
3	$\frac{2}{7}$
4	$\frac{4}{7}$

TABLE 3.

It is easily checked that this is in accordance with the description of the model given in Chapter I.

The *process class experiment,* line 125 - 140, has been created to be able to implement the "single-run method" (cf. Lavenberg [13]) in a natural and efficient manner.

The *class statistics* computes for each experiment the mean, variance and standard deviation of the results. (line 142 - 162)

In the program, the waiting time and the sojourn time of a customer, and the maximal waiting time for each queue are recorded. After the listing, a list is made of the control statements, needed to run a job at the cyber 750 at SARA, and it is shown how a data file for this program should be organised.

LISTING OF THE SIMULATION PROGRAM

```
BEGIN
INTEGER N; N:=ININT;
SIMULATION BEGIN

  PROCEDURE READP(E,K); ARRAY E; INTEGER K;
   BEGIN
     INTEGER J;
     FOR J:=1 STEP 1 UNTIL K DO E(J):=INREAL;
     INIMAGE
   END;

  PROCEDURE P(X); REAL X;
   OUTFIX(X,4,15);

  PROCEDURE WRITEARRAY(E,T,K); VALUE T; ARRAY E; TEXT T;
                               INTEGER K;
   BEGIN
     INTEGER J;
     FOR J:=1 STEP 1 UNTIL K DO
       BEGIN
         OUTTEXT(T);OUTINT(J,2);OUTTEXT(":");
         P(E(J));OUTIMAGE;OUTIMAGE
       END
   END;

  PROCEDURE WRITEPARAMS;
   BEGIN
     OUTIMAGE;
     OUTTEXT(" LIST OF PARAMETERS USED DURING THIS SIMULATION");
     OUTIMAGE;OUTIMAGE;OUTIMAGE;
     WRITEARRAY(A,"  ALFA",N);
     WRITEARRAY(R,"     R",N-1);
     OUTTEXT("  LAMBDA:");P(LAMBDA);
     OUTIMAGE;OUTIMAGE;
     WRITEARRAY(B,"  BETA",N);
     OUTIMAGE;
     OUTTEXT("  SEED:             ");OUTINT(SEED,8);
     OUTIMAGE;
     OUTTEXT("  TRANSIENT PERIOD:");OUTINT(TRANSPER,8);
     OUTIMAGE;
     OUTTEXT("  RUN PERIOD:      ");OUTINT(RUNPER,8);
     OUTIMAGE;
     OUTTEXT("  NUMBER OF RUNS:  ");OUTINT(NRUNS,8);
     OUTIMAGE;OUTIMAGE;
     OUTTEXT("  TOTAL SIMULATION TIME:");
     OUTINT(TRANSPER+NRUNS*RUNPER,8);
     OUTIMAGE
   END;

  PROCEDURE CUMULAT(E); ARRAY E;
   BEGIN
     INTEGER I,J;
     FOR I:=1 STEP 1 UNTIL N-1 DO
       BEGIN
         FOR J:=I+1 STEP 1 UNTIL N-1 DO
           E(J):=E(J)+E(I);
       END
   END;
```

```

REAL PROCEDURE STMEAN(TYPE); INTEGER TYPE;
 STMEAN:=NEGEXP(1/B(TYPE),SEED);

BOOLEAN PROCEDURE NOTEMPTYQUEUES;
 BEGIN
   INTEGER J;
   NOTEMPTYQUEUES:=FALSE;
   FOR J:=1 STEP 1 UNTIL N DO
     IF NOT Q(J).EMPTY THEN NOTEMPTYQUEUES:=TRUE
 END;

INTEGER PROCEDURE CHOOSECLIENT;
BEGIN
  REAL ARRAY ROW(1:N-1);
  INTEGER J;
  REAL SUMR; SUMR:=0;
  FOR J:=1 STEP 1 UNTIL N-1 DO
    BEGIN
     IF Q(J).EMPTY THEN ROW(J):=0
                   ELSE BEGIN
                          ROW(J):=A(J);
                          SUMR:=SUMR+A(J)
                        END
    END;
  IF NOT Q(N).EMPTY THEN SUMR:=SUMR+A(N);
  FOR J:=1 STEP 1 UNTIL N-1 DO
    ROW(J):=ROW(J)/SUMR;
  CUMULAT(ROW);
  CHOOSECLIENT:=DISCRETE(ROW,SEED)
END;

PROCESS CLASS CLIENT(TYPE); INTEGER TYPE;
 BEGIN
   REAL TINSIDE,WTIME;
   INTO(Q(TYPE));
   TINSIDE:=TIME;
   IF RUN.S.IDLE THEN ACTIVATE RUN.S
                 ELSE PASSIVATE;
   WTIME:=TIME-TINSIDE
 END;

PROCESS CLASS SERVER;
 BEGIN
   REF (CLIENT) CL;
   INTEGER ARRAY NCL(1:N);
   REAL ARRAY SUMWTIME,MAXWTIME,SUMRESPT(1:N);
   WHILE TRUE DO
     BEGIN
       WHILE NOTEMPTYQUEUES DO
         BEGIN
           CL:-Q(CHOOSECLIENT).FIRST;
           CL.OUT;
           REACTIVATE CL;
           INSPECT CL DO BEGIN
           NCL(TYPE):=NCL(TYPE)+1;
           SUMWTIME(TYPE):=SUMWTIME(TYPE)+WTIME;
           IF WTIME > MAXWTIME(TYPE)
```

```
                THEN MAXWTIME(TYPE):=WTIME;
                HOLD(STMEAN(TYPE));
                SUMRESPT(TYPE):=SUMRESPT(TYPE)+TIME-TINSIDE
                END
              END;
              PASSIVATE
          END
      END;

    PROCESS CLASS EXPERIMENT(LENGTH); REAL LENGTH;
     BEGIN
       REAL START;
       INTEGER TYPE;
       REF (SERVER) S;
       S:-NEW SERVER;
       START:=TIME;
       WHILE TIME < START+LENGTH DO
         BEGIN
           HOLD(NEGEXP(LAMBDA,SEED));
           TYPE:=DISCRETE(R,SEED);
           ACTIVATE NEW CLIENT(TYPE)
         END;
       CANCEL(S);
       CANCEL(THIS EXPERIMENT)
     END;

    CLASS STATISTICS;
     BEGIN
       INTEGER J;
       REAL ARRAY SUM,SQSUM(1:N);
       INTEGER ARRAY K(1:N);
       PROCEDURE ADD(X,TYPE); REAL X; INTEGER TYPE;
        BEGIN
          SUM(TYPE):=SUM(TYPE)+X;
          SQSUM(TYPE):=SQSUM(TYPE)+X*X;
          K(TYPE):=K(TYPE)+1
        END;
       REAL PROCEDURE MEAN(TYPE); INTEGER TYPE;
        IF K(TYPE)>0 THEN MEAN:=SUM(TYPE)/K(TYPE)
                     ELSE MEAN:=0;
       REAL PROCEDURE VARIANCE(TYPE); INTEGER TYPE;
        IF K(TYPE)>1 THEN
        VARIANCE:=(SQSUM(TYPE)-SUM(TYPE)*SUM(TYPE)/K(TYPE))/(K(TYPE)-1);
        ELSE VARIANCE:=0;
       REAL PROCEDURE STDEV(TYPE); INTEGER TYPE;
        STDEV:=SQRT(VARIANCE(TYPE));
     END;

    PROCEDURE REPORT(E); REF (EXPERIMENT) E;
    BEGIN
      INTEGER J;
      FOR J:=1 STEP 1 UNTIL N DO
        BEGIN
          OUTTEXT("  QUEUE");OUTINT(J,2);OUTTEXT("   ");
          INSPECT E.S DO BEGIN
          OUTINT(NCL(J),10);TOTCL.ADD(NCL(J),J);
          P(SUMWTIME(J)/NCL(J));
          TOTWTIME.ADD(SUMWTIME(J)/NCL(J),J);
```

```
        P(SUMRESPT(J)/NCL(J));
        TOTRESPT.ADD(SUMRESPT(J)/NCL(J),J);
        P(MAXWTIME(J))
        END;OUTIMAGE
      END;
      OUTIMAGE;OUTIMAGE
  END;

  INTEGER SEED,TRANSPER,RUNPER,NRUNS,J;
  REAL ARRAY A,B(1:N),R(1:N-1);
  REF (HEAD) ARRAY Q(1:N);
  REAL LAMBDA;
  REF (STATISTICS) TOTCL,TOTWTIME,TOTRESPT;
  REF (EXPERIMENT) RUN;

  INIMAGE;
  READP(A,N);
  READP(R,N-1);
  READP(B,N);
  LAMBDA:=INREAL;
  INIMAGE;
  SEED:=ININT;
  INIMAGE;
  TRANSPER:=ININT;
  INIMAGE;
  RUNPER:=ININT;
  INIMAGE;
  NRUNS:=ININT;
  FOR J:=1 STEP 1 UNTIL N DO
    Q(J):-NEW HEAD;
  TOTCL:-NEW STATISTICS;
  TOTWTIME:-NEW STATISTICS;
  TOTRESPT:-NEW STATISTICS;
  WRITEPARAMS;
  FOR J:=1 STEP 1 UNTIL 40 DO OUTIMAGE;
  CUMULAT(R);

  RUN:-NEW EXPERIMENT(TRANSPER);
  ACTIVATE RUN;
  HOLD(TRANSPER);

  FOR J:=1 STEP 1 UNTIL NRUNS DO
    BEGIN
      OUTTEXT("  RUN NR.");OUTINT(J,2);
      OUTTEXT("       NUMBER     WAITINGTIME");
      OUTTEXT("      RESP.TIME     MAXWTIME");
      OUTIMAGE;OUTIMAGE;
      RUN:-NEW EXPERIMENT(RUNPER);
      ACTIVATE RUN;
      HOLD(RUNPER);
      REPORT(RUN)
    END;
```

```

    OUTIMAGE;OUTIMAGE;
    FOR J:=1 STEP 1 UNTIL N DO
      BEGIN
        OUTTEXT("                          NUMBER            ");
        OUTTEXT("WTIME           RESP.TIME");
        OUTIMAGE;OUTIMAGE;
        OUTTEXT("  THE RESULTS FOR QUEUE");OUTINT(J,2);
        OUTTEXT(":");OUTIMAGE;OUTIMAGE;
        OUTTEXT("  MEAN:     ");P(TOTCL.MEAN(J));
        P(TOTWTIME.MEAN(J));P(TOTRESPT.MEAN(J));
        OUTIMAGE;
        OUTTEXT("  VARIANCE:                     ");
        P(TOTWTIME.VARIANCE(J));P(TOTRESPT.VARIANCE(J));
        OUTIMAGE;
        OUTTEXT("  ST.DEV:                       ");
        P(TOTWTIME.STDEV(J));P(TOTRESPT.STDEV(J));
        OUTIMAGE;OUTIMAGE;OUTIMAGE;OUTIMAGE
      END;
      OUTTEXT("  NOTE THAT THE VARIANCE AND THE STANDARD ");
      OUTTEXT("DEVIATION");OUTIMAGE;OUTTEXT("  ABOVE ");
      OUTTEXT("ONLY APPLY TO THE SAMPLE RESULTS.");
      OUTIMAGE
  END
  END
```

CONTROL STATEMENTS

interactive:

```
etl,<nnn>
attach,simula
simula,i=<prog>,ql
lgo,i=<input>
```

batch:

```
<jobname>,LP=<nnn>.
attach,simula.
simula,i=input,ql.
lgo.
```

ORGANISATION OF THE DATAFILE:

```
<N>
<A(1)> <A(2)> <A(3)> .... <A(N)>
<R(1)> <R(2)> <R(3)> .... <R(N-1)>
<B(1)> <B(2)> <B(3)> .... <B(N)>
<lambda>
<seed>
<transper>
<runper>
<nruns>
```

SOME STATISTICAL ASPECTS OF THE SIMULATION.

As the listing on the previous pages shows, we have chosen for the single-run method. This was in fact a rather arbitrary choice, because in the considered cases, we could have chosen equally well for example the regenerative method (cf. Lavenberg [13]).
In all cases considered we have taken the transient period to be 500 seconds. We took $\lambda = 1$ and hence during the transient period approximately 500 customers arrived. Relatively little is known about relaxation times for queueing systems, especially for queueing networks (cf. Blanc & van Doorn [2]).
As recommended in Lavenberg [13] we take the number of subsequences equal to 10. The choice of the length of the subsequences presented more difficulties. If the length of the subsequences is not chosen large enough, the subsequence estimates are correlated enough to have a substantially narrowing effect on the confidence intervals. A length of 5000 seconds for each subsequence seemed in most cases large enough to overcome this difficulty. In cases with extremely high traffic however, 5000 seconds was not large enough to obtain a satisfactorily small confidence interval.

2. THE ITERATION PROGRAM

Because of its length the listing of the iteration program is not contained in this report. A copy of this listing is available on request.

Appendix E

Numerical results

In this section we shall present some numerical results. In the following tables we shall indicate by (I) that the results have been obtained using the iteration program and by (S) that the results have been obtained using simulation.

Firstly, we present some results for the case $r_1 = \alpha_1$, $r_2 = \alpha_2$ and the service-time distributions identical and negative exponential.

TABLE 1. (I)

$$r_1 = \alpha_1,\ r_2 = \alpha_2,\ \beta(\rho) = \frac{1}{1+\beta\rho},\ \text{Re}\ \rho \geqslant 0,\ \beta = 0.8,\ \lambda = 1,$$

\# *subdivisions* : 40

r_1	$E\{x_1\}$	$E\{x_2\}$	$E\{z_1(z_2=0)\}$
0.1	0.691	3.309	0.072
0.2	1.090	2.910	1.142
0.3	1.417	2.583	0.217
0.4	1.712	2.288	0.301
0.5	2.001	2.000	0.400
0.6	2.288	1.712	0.522
0.7	2.587	1.413	0.684
0.8	2.914	1.086	0.926
0.9	3.305	0.695	1.385

Note that if $r_1 = \alpha_1 < 0.5$ then $E\{x_1\}$ and $E\{x_2\}$ are the same, but with the indices interchanged, as $E\{x_1\}$ and $E\{x_2\}$ in the case $r_1' = \alpha_1' = r_1 + 0.5 = \alpha_1 + 0.5$. This is of course to be expected.

Next we investigate the influence of the number of subdivisions upon the accuracy of the results for a particular case.

TABLE 2 (I)

$$r_1 = \alpha_1 = 0.9,\ r_2 = \alpha_2 = 0.1,\ \beta(\rho) = \frac{1}{1+\beta\rho},\ \text{Re}\ \rho \geqslant 0,\ \beta = 0.8,\ \lambda = 1.$$

# subdivisions	$E\{x_1\}$	$E\{x_2\}$	$E\{z_1(z_2=0)\}$	CPU-time (sec.)
16	3.5271	0.4729	0.8839	1
40	3.3045	0.695		0.6461
80	3.3048	0.6952	1.3843	19
120	3.3061	0.6939	1.3811	40

We have simulated this case with a transient period of 500 seconds (customers) and 10 runs of 5000 seconds (customers) each and we obtained the following 95% confidence interval for $E\{x_1\}$ and $E\{x_2\}$:

$$\text{for } E\{x_1\}:\ [2.98,\ 3.42]; \qquad (1.1)$$

$$\text{for } E\{x_2\}:\ [0.619,\ 0.767].$$

This simulation run took about 50 seconds CPU time. It is seen that simulation is in this case much more costly and produces less accurate results then the iteration program. However if one of the contours S_1 and S_2 is not simple, we have to use simulation. Also the case that the service-time distributions are not identical is not included in the iteration program, so that also in this case we have to use simulation. If we want to use the iteration program to obtain results in the case $r_1 \neq \alpha_1$, $r_2 \neq \alpha_2$, and the service-time distributions identical, we should then first check the contours S_1 and S_2, to see if they are simple. A priori it can be said that one can never hope to obtain reasonable results if (cf. III.(5.10))

$$F_2'(1) = \frac{\alpha_1 - a_1}{a_2 - \alpha_2} \geqslant 0. \qquad (1.2)$$

Finally, as an example of where simulation can be used, we have investigated the influence of varying the β_i, $i=1,2$, with $\beta_i(\rho) = 1/(1+\beta_i\rho)$, Re $\rho \geqslant 0$, upon the sojourn times:

TABLE 3 (S)

$$\alpha_1 = r_2 = 0.1,\ \alpha_2 = r_1 + 0.9,\ \beta_i(\rho) = \frac{1}{1+\beta_i\rho},\ \text{Re}\ \rho \geqslant 0,\ \lambda = 1.$$

transient period : 500 *sec.* ; 10 *runs*, 5000 *sec. each*.

β_i, $i=1,2$	$E\{s_1\}$	$E\{s_2\}$
$\beta_1=0.8=\beta_2$	4.14	1.56
$\beta_1=0.1$; $\beta_2=0.9$	0.23	1.02
$\beta_1=0.9$; $\beta_2=0.1$	4.92	0.93

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