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LOCATION ESTIMATORS AND SPREAD

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Location estimators and spread*)

by

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ABSTRACT

In the location estimation problem translation equivariant estimators are considered. It is shown that under a mild regularity condition the distribution of such estimators is more spread out than a particular distribution which is defined in terms of the sample size and the density of the i.i.d. observations. Some consequences of this so-called spread-inequality are discussed, namely the Cramér-Rao inequality, an asymptotic inequality of Hájek and the efficiency of the maximum likelihood estimator in some nonregular cases.

KEY WORDS & PHRASES: *Location estimator, translation equivariance, spread, Cramér-Rao inequality, maximum likelihood estimation in nonregular cases.*

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1. INTRODUCTION AND MAIN RESULT

We shall consider one of the classical problems in statistical inference, to wit the estimation of a location parameter. Let $X_1,\dots,X_n$ be independent and identically distributed random variables with common density $f(.-\theta)$, $\theta \in \mathbb{R}$, with respect to Lebesgue measure on $(\mathbb{R},\mathcal{B})$. The location parameter θ is estimated by an estimator T_n which is a measurable function $t_n : \mathbb{R}^n \to \mathbb{R}$ of the random variables $X_1,\dots,X_n$, i.e. $T_n = t_n(X_1,\dots,X_n)$. We are interested in the distribution of T_n under $f(.-\theta)$.

Our estimation problem is invariant under translation. Hence it is natural to estimate the parameter θ with a translation equivariant estimator whenever we want to be impartial with respect to the possible values which the parameter can adopt. Therefore, we assume that T_n is translation equivariant, i.e. for all real a and Lebesgue almost all $x_1,\dots,x_n$

$$(1.1) \qquad t_n(x_1+a,\dots,x_n+a) = t_n(x_1,\dots,x_n) + a .$$

Because of the translation equivariance of T_n we have

$$(1.2) \qquad P_{f(.-\theta)}(T_n \le x) = P_f(T_n \le x-\theta),\ x \in \mathbb{R},\ \theta \in \mathbb{R} ,$$

and we see that it suffices to study the distribution of T_n under f, i.e. with $\theta = 0$.

Let a_n be positive. We denote the distribution function of a_nT_n under f by G_n,

$$(1.3) \qquad G_n(x) = P_f(a_nT_n \le x),\ x \in \mathbb{R} .$$

Furthermore we assume that the density f is absolutely continuous with an integrable Radon-Nikodym derivative f' and we define the distribution function K_n for some $w \in (0,1)$ by

$$(1.4) \qquad K_n^{-1}(u) = \int_w^u \frac{1}{\int_s^1 H_n^{-1}(t)dt}\, ds ,\quad 0 < u \le 1 ,$$

where

$$(1.5) \qquad H_n(x) = P_f\left(a_n^{-1} \sum_{i=1}^n \left[-\frac{f'}{f}(X_i)\right] \le x\right),\ x \in \mathbb{R} .$$

The distribution functions G_n and K_n are related by the fact that any two quantiles of G_n are further apart than the corresponding quantiles of K_n ; more precisely

<u>THEOREM 1.1</u>. *If the density* f *is absolutely continuous with respect to Lebesgue measure with Radon-Nikodym derivative* f' *satisfying*

$$\int |f'| < \infty \tag{1.6}$$

and if T_n *is translation equivariant (cf.* (1.1)*), then* G_n *and* K_n *are differentiable with derivatives* g_n *respectively* k_n *satisfying (cf.* (1.3), (1.4) *and* (1.5)*)*

$$g_n\left(G_n^{-1}(s)\right) \le k_n\left(K_n^{-1}(s)\right) = \int_s^1 H_n^{-1}(t)dt \ , \quad 0 < s < 1 \ . \tag{1.7}$$

This implies

$$G_n^{-1}(v) - G_n^{-1}(u) \ge K_n^{-1}(v) - K_n^{-1}(u) \ , \quad 0 \le u \le v \le 1 \ . \tag{1.8}$$

We say that G_n is more spread out than K_n . This concept of spread has been introduced by BICKEL and LEHMANN (1979). Note that the inequalities (1.7) and (1.8) are insensitive to translations and that hence the choice of $w \in (0,1)$ is immaterial. The important point in the spread-inequality (1.8) is that K_n is defined in terms of the sample size n and the density f of the observations. Hence K_n does not depend on T_n and consequently Theorem 1.1 gives a uniform upperbound to the accuracy of translation equivariant estimators T_n . Well-known upperbounds to the accuracy of estimators are provided by the Cramér-Rao inequality and by the asymptotic result of HÁJEK (1972). Restricted to the location estimation problem we are considering these results are implied by Theorem 1.1. Our spread-inequality also implies that the maximum likelihood estimator is asymptotically efficient in the nonregular location estimation problem of WOODROOFE (1972) and that its rate of convergence to its limit distribution is of the right order in the nonregular location estimation problem of WOODROOFE (1974). These consequences of Theorem 1.1 will be discussed in the next section. Section 3 consists of the proofs.

2. SOME CONSEQUENCES OF THE SPREAD-INEQUALITY

From the spread-inequality (1.8) nontrivial lower bounds may be obtained for the risk of translation equivariant location estimators, both for finite sample sizes and asymptotically. Such bounds are presented in the following theorem.

THEOREM 2.1. *Let* $\ell : \mathbb{R} \to \mathbb{R}$ *be a measurable function, which is nonincreasing on* $(-\infty,0]$ *and nondecreasing on* $[0,\infty)$. *Under the conditions of* Theorem 1.1 *we have*

$$(2.1) \qquad \inf_{a\in\mathbb{R}} E_f \ell(a_n T_n - a) \geq \inf_{a\in\mathbb{R}} \int_0^1 \ell\left(K_n^{-1}(u) - a\right) du \ .$$

Furthermore, if for some distribution function K *the sequence* $\{K_n\}$ *converges weakly to* K *as* n *tends to infinity and if at least one of the following conditions holds:*

(2.2) ℓ *is lower semicontinuous,*

(2.3) $\int_A dK = 0$ *for each countable set* $A \subset \mathbb{R}$,

then

$$(2.4) \qquad \liminf_{n\to\infty} \inf_{a\in\mathbb{R}} E_f \ell(a_n T_n - a) \geq \inf_{a\in\mathbb{R}} \int_0^1 \ell(K^{-1}(u) - a) du \ .$$

Finally, if $G_n \overset{w}{\to} G$ *and* $K_n \overset{w}{\to} K$ *as* $n \to \infty$ *for some distribution functions* G *and* K , *then*

$$(2.5) \qquad G^{-1}(v) - G^{-1}(u) \geq K^{-1}(v) - K^{-1}(u) \ , \quad 0 \leq u \leq v \leq 1 \ .$$

For quadratic loss functions inequality (2.1) of Theorem 2.1 implies an extension of the Cramér-Rao inequality.

COROLLARY 2.1. (Cramér-Rao inequality) *Under the conditions of* Theorem 1.1

(2.6) $\quad \mathrm{var}_f \; a_n T_n \geq \mathrm{var}_{k_n} X$.

If f *has finite Fisher information* $I(f) = \int (f'/f)^2 f$, *this implies*

(2.7) $\quad \mathrm{var}_f \; T_n \geq (nI(f))^{-1}$.

In the remainder of this section we will discuss three special cases of inequality (2.4) of Theorem 2.1. The first one is closely related to the result of HÁJEK (1972) and arises if f has finite Fisher information.

COROLLARY 2.2. (Hájek) *If* f *has finite Fisher information* $I(f)$ *then for* $a_n = (nI(f))^{\frac{1}{2}}$ *the sequence* $\{K_n\}$ *with* $w = \frac{1}{2}$ *converges weakly to the standard normal distribution function* Φ *as* n *tends to infinity and hence, under the conditions of* Theorem 1.1, *we have*

$$(2.8) \qquad \liminf_{n\to\infty} \inf_{a\in\mathbb{R}} E_f \ell((nI(f))^{\frac{1}{2}} T_n - a) \geq \inf_{a\in\mathbb{R}} \int_0^1 \ell(\Phi^{-1}(u) - a)du$$

for each measurable function $\ell : \mathbb{R} \to \mathbb{R}$ *which is nonincreasing on* $(-\infty,0]$ *and nondecreasing on* $[0,\infty)$.

Furthermore, we'll consider densities f of the following very special type. Let $c \in (0,\infty)$. If f satisfies (1.6), f vanishes on $(-\infty,0]$, $\lim_{x\downarrow 0} f'(x) = c$ and if $\int_\varepsilon^\infty (f'(x)/f(x))^2 f(x)dx < \infty$ for all $\varepsilon > 0$, then f will be said to belong to the class D(c) . The gamma and Weibull distributions with shape parameter 2 belong to this class. We note that $I(f) = \infty$ for all $f \in D(c)$. Nevertheless the following analogue of Corollary 2.2 holds.

COROLLARY 2.3. *If* $f \in D(c)$, *then for* $a_n = (\frac{1}{2} cn \log n)^{\frac{1}{2}}$ *the sequence* $\{K_n\}$ *with* $w = \frac{1}{2}$ *converges weakly to* Φ *as* n *tends to infinity and hence, if* T_n *is translation equivariant, we have*

$$(2.9) \qquad \liminf_{n\to\infty} \inf_{a\in\mathbb{R}} E_f \ell((\tfrac{1}{2} cn \log n)^{\frac{1}{2}} T_n - a) \geq \inf_{a\in\mathbb{R}} \int_0^1 \ell(\Phi^{-1}(u) - a)du$$

for each measurable function $\ell : \mathbb{R} \to \mathbb{R}$ *which is nonincreasing on* $(-\infty,0]$ *and nondecreasing on* $[0,\infty)$.

WOODROOFE (1972) has shown under some regularity conditions, that the asymptotic distribution of the maximum likelihood estimator for this case is standard normal if it is normed as in (2.9). Together with the convergence of K_n to Φ and (2.5) of Theorem 2.1 this implies that both the maximum likelihood estimator and the spread-inequality (2.5) are asymptotically efficient in this nonregular case.

Finally, we'll consider densities f which behave like $x^{\alpha-1}$, $1 < \alpha < 2$, near the origin.

<u>COROLLARY 2.4</u>. *Let* $\alpha \in (1,2)$, f *vanish on* $(-\infty,0]$, f *satisfy* (1.6) *and let* $f'(x) \sim \alpha(\alpha-1)x^{\alpha-2}L(x)$ *as* $x \downarrow 0$, *where* $L(x)$ *varies slowly as* $x \downarrow 0$. *Furthermore let* $\int_\varepsilon^\infty (f'(x)/f(x))^2 f(x)dx < \infty$ *for all* $\varepsilon > 0$. *If* $\{a_n\}$ *is such that*

$$(2.10) \qquad \lim_{n\to\infty} n\, a_n^{-\alpha} L\left(a_n^{-1}\right) = 1 ,$$

then $H_n \xrightarrow{w} H$ *as* $n \to \infty$, *where* H *is a stable distribution function with exponent* α *and cumulant generating function*

$$(2.11) \qquad \psi_H(t) = - d|t|^\alpha(1 + i \operatorname{sgn} t \operatorname{tg}(\tfrac{1}{2}\alpha\pi))$$

with

$$(2.12) \qquad d = (\alpha-1)^{\alpha-1}\Gamma(2-\alpha)[- \cos \tfrac{1}{2}\alpha\pi] .$$

Furthermore, $K_n \xrightarrow{w} K$ *as* $n \to \infty$, *where* K *is defined by*

$$(2.13) \qquad K^{-1}(u) = \int_w^u \frac{1}{\int_s^1 H^{-1}(t)dt}\, ds , \quad 0 < u \le 1 ,$$

and hence, if T_n *is translation equivariant,* (2.4) *holds with* a_n *and* K *as in* (2.10) *respectively* (2.13).

In WOODROOFE (1974) the asymptotic distribution $\hat{G}$ of the maximum likelihood estimator for this case has been derived under some regularity conditions. This has been done with the norming constants a_n as in (2.10). We infer that the rates of convergence of both the maximum likelihood estimator and the spread lower bound K_n are of the right order, i.e. that the a_n defined by (2.10) are suitable norming constants for the maximum likelihood estimator to attain a limit distribution without mass at infinity and for K_n to have a nondegenerate limit distribution. However, $\hat{G}$ and K are different as can be seen from the following lemma.

<u>LEMMA 2.1</u>. *Under the conditions of* Corollary 2.4 *we have for* $\delta \downarrow 0$

$$(2.14) \qquad K^{-1}(1-\delta) \sim [\Gamma(2-\alpha)]^{-\frac{1}{\alpha}} [-\log \delta]^{\frac{1}{\alpha}} ,$$

$$(2.15) \qquad \liminf_{\delta \downarrow 0} K^{-1}(\delta) > -\infty .$$

Under the conditions of Theorem 2.4 *of* WOODROOFE (1974)

$$(2.16) \qquad \limsup_{\delta \downarrow 0} \hat{G}(1-\delta)[-\log \delta]^{-\frac{1}{\alpha}} \le \left[1 + \frac{\alpha([2\alpha-3]^{+})^{2}}{8(\alpha-1)^{2}(2-\alpha)}\right]^{-\frac{1}{\alpha}} ,$$

$$(2.17) \qquad \hat{G}^{-1}(\delta) \sim -\rho^{-\frac{1}{\alpha}} [-\log \delta]^{\frac{1}{\alpha}},$$

where ρ *satisfies (*(3.2) of WOODROOFE (1974)*)*

$$\rho \ge \tfrac{1}{2}\Gamma(\alpha-1)\Gamma(2-\alpha) .$$

Since $\hat{G}$ and K are different, either the maximum likelihood estimator or the spread-inequality (1.8) respectively (2.5) or both are asymptotically nonoptimal for this case. In view of (2.5) it seems reasonable to compare $\hat{G}$ and K by the quantity

$$(2.18) \qquad Q(\hat{G},K) = \limsup_{\delta \downarrow 0} \frac{\hat{G}^{-1}(1-\delta) - \hat{G}^{-1}(\delta)}{K^{-1}(1-\delta) - K^{-1}(\delta)} .$$

From (2.5) and Lemma 2.1 we can only infer that

$$(2.19) \qquad 1 \le Q(\hat{G},K) \le \left[\frac{1}{\Gamma(2-\alpha)} + \frac{\alpha([2\alpha-3]^{+})^{2}}{8(\alpha-1)^{2}\Gamma(3-\alpha)}\right]^{-\frac{1}{\alpha}} + [\tfrac{1}{2}\Gamma(\alpha-1)]^{-\frac{1}{\alpha}} ,$$

which for $\alpha \downarrow 1$ tends to 1 and for $\alpha \uparrow 2$ tends to $2 + 2^{\frac{1}{2}}$. A simple and satisfactory estimator in this situation is the minimum of the observations. Normed by the a_n from (2.10) it has the limit distribution $\tilde{G}$ with

$$(2.20) \qquad \tilde{G}(x) = 1 - e^{-x^{\alpha}} , \quad x \ge 0 .$$

From Lemma 2.1 we see that

$$(2.21)\qquad Q(\tilde{G},K) = [\Gamma(2-\alpha)]^{\frac{1}{\alpha}} \quad ,$$

which for $\alpha \downarrow 1$ tends to 1 as it should.

The reader interested in the asymptotics of nonregular estimation problems should also consult LE CAM (1972), IBRAGIMOV and HAS'MINSKIĬ (1981) and AKAHIRA and TAKEUCHI (1981). Finite sample results on the tail behavior of the distributions of location estimators have been obtained by JUREČKOVÁ (1981a, 1981b).

As a curiosity we mention the following immediate consequence of (1.7)

$$(2.22)\qquad g_n(x) \le \tfrac{1}{2} \int_0^1 \left|H_n^{-1}(t)\right| dt \ , \quad x \in \mathbb{R} \ .$$

With $n = 1$, $a_n = 1$ and $T_1 = X_1$ this reduces to the simple inequality

$$(2.23)\qquad f(x) \le \tfrac{1}{2} \int |f'| \ , \quad x \in \mathbb{R} \ ,$$

which can easily be proved directly.

In the above we have discussed some consequences of the spread-inequality (1.8). Other consequences of it can be found in KLAASSEN (1981), which restricts attention to the case of symmetric densities with finite Fisher information.

3. PROOFS

<u>PROOF OF THEOREM 1.1.</u> First we note that for $\theta > 0$

$$\int_{-\infty}^{\infty} |\theta^{-1}(f(x+\theta) - f(x))|dx = \int_{-\infty}^{\infty} |\theta^{-1} \int_x^{x+\theta} f'(y)dy|dx$$

$$\le \int_{-\infty}^{\infty} \int_x^{x+\theta} \theta^{-1}|f'(y)|dydx = \int_{-\infty}^{\infty} \int_{y-\theta}^{y} \theta^{-1}|f'(y)|dxdy$$

$$= \int_{-\infty}^{\infty} |f'(x)|dx < \infty \ .$$

Clearly the same is true for $\theta < 0$. Since the set $\{x | f(x) = 0 \ , \ f'(x) \ne 0\}$ has Lebesgue measure zero, this yields for all $\theta \ne 0$ and for $j = 1,\dots,n$

(3.1)
$$\int_{\mathbb{R}^n}\!\!\cdots\!\int \left| \prod_{i=1}^{j-1} f(x_i+\theta) \prod_{i=j+1}^{n} f(x_i)\theta^{-1}(f(x_j+\theta) - f(x_j))\right| dx_1 \ldots dx_n$$
$$\leq \int_{\mathbb{R}^n}\!\!\cdots\!\int \left| \frac{f'}{f}(x_j) \prod_{i=1}^{n} f(x_i)\right| dx_1 \ldots dx_n < \infty .$$

By Vitali's theorem it follows that for $j = 1,\ldots,n$

$$\lim_{\theta\to 0} \int_{\mathbb{R}^n}\!\!\cdots\!\int \left| \prod_{i=1}^{j-1} f(x_i+\theta) \prod_{i=j+1}^{n} f(x_i)\theta^{-1}(f(x_j+\theta) - f(x_j)) - \frac{f'}{f}(x_j) \prod_{i=1}^{n} f(x_i)\right| dx_1 \ldots dx_n = 0 ,$$

and this implies that

(3.2)
$$\lim_{\theta\to 0} \int_{\mathbb{R}^n}\!\!\cdots\!\int \left| \theta^{-1} \left(\prod_{i=1}^{n} f(x_i+\theta) - \prod_{i=1}^{n} f(x_i)\right) - \left(\sum_{j=1}^{n} \frac{f'}{f}(x_j)\right) \prod_{i=1}^{n} f(x_i)\right| dx_1 \ldots dx_n = 0 .$$

Now by the translation equivariance of T_n

$$\theta^{-1}(G_n(y+\theta) - G_n(y))$$

(3.3)
$$= \int_{a_n t_n(x_1,\ldots,x_n) \leq y} \cdots\cdots\cdots \int \theta^{-1} \left\{\prod_{i=1}^{n} f(x_i+a_n^{-1}\theta) - \prod_{i=1}^{n} f(x_i)\right\} dx_1 \ldots dx_n$$
$$= \int_{a_n t_n(x_1,\ldots,x_n) > y} \cdots\cdots\cdots \int - \theta^{-1} \left\{\prod_{i=1}^{n} f(x_i+a_n^{-1}\theta) - \prod_{i=1}^{n} f(x_i)\right\} dx_1 \ldots dx_n ,$$

and it follows from (3.2) that G_n is differentiable with derivative g_n given by

$$(3.4) \qquad g_n(y) = \int_{a_n t_n(x_1,\ldots,x_n) > y} \cdots\cdots\cdots \int a_n^{-1} \sum_{j=1}^{n} \left[-\frac{f'}{f}(x_j) \right] \prod_{i=1}^{n} f(x_i)dx_1 \ldots dx_n \ .$$

With

$$(3.5) \qquad S_n = a_n^{-1} \sum_{j=1}^{n} \left[-\frac{f'}{f}(X_j) \right]$$

and $y = G_n^{-1}(s)$ formula (3.4) may be rewritten as (cf. (1.5))

$$(3.6) \qquad g_n\left(G_n^{-1}(s)\right) = \int_0^1 H_n^{-1}(t) E_f\left(1_{(G_n^{-1}(s),\infty)}(a_n T_n) \middle| S_n = H_n^{-1}(t) \right) dt \ .$$

Furthermore it is easy to verify that

$$(3.7) \qquad 1 - s = \int_0^1 E_f\left(1_{(G_n^{-1}(s),\infty)}(a_n T_n) \middle| S_n = H_n^{-1}(t) \right) dt \ .$$

Since the integrand in (3.7) takes on values in [0,1] and since H_n^{-1} is non-decreasing the Neyman-Pearson lemma (cf. Theorem 5(ii) with $m = 1$ of Chapter 3 of LEHMANN (1959)) applied to (3.6) and (3.7) yields

$$(3.8) \qquad g_n\left(G_n^{-1}(s)\right) \le \int_s^1 H_n^{-1}(t)dt \ , \quad 0 < s < 1 \ .$$

Because H_n is nondegenerate with mean 0, we have for all $s \in (0,1)$

$$0 < \int_s^1 H_n^{-1}(t)dt \le \tfrac{1}{2} \int_0^1 \left| H_n^{-1}(t) \right| dt \le \tfrac{1}{2} a_n^{-1} n \int |f'| < \infty \ .$$

Furthermore, $\int_s^1 H_n^{-1}(t)dt$ is concave and hence

$$\inf_{u \le s \le v} \int_s^1 H_n^{-1}(t)dt = \min\left\{ \int_u^1 H_n^{-1}(t)dt, \int_v^1 H_n^{-1}(t)dt \right\} \ , \quad 0 < u \le v < 1 \ .$$

Consequently K_n^{-1} is well defined by (1.4) and is differentiable with a positive and finite derivative on $(0,1)$. Hence K_n is differentiable with a positive and finite derivative k_n on $\left(K_n^{-1}(0+)\ ,\ K_n^{-1}(1)\right)$ and K_n satisfies

$$(3.9) \qquad x = \int_w^{K_n(x)} \frac{1}{\int_s^1 H_n^{-1}(t)dt}\, ds\ , \quad x \in \left(K_n^{-1}(0+), K_n^{-1}(1)\right)\ .$$

Differentiating (3.9) and combining the result with (3.8) we see that (1.7) holds. Combining (1.4) and (1.7) we obtain

$$\begin{aligned}
G_n^{-1}(v) - G_n^{-1}(u) &= \int_{-\infty}^{\infty} 1_{[G_n^{-1}(u),G_n^{-1}(v)]}(x)dx \\
&\geq \int_{-\infty}^{\infty} 1_{[G_n^{-1}(u),G_n^{-1}(v)]}(x)\ \frac{1}{g_n(x)}\ dG_n(x) \\
&= \int_0^1 1_{[G_n^{-1}(u),G_n^{-1}(v)]}(G_n^{-1}(s))\ \frac{1}{g_n(G_n^{-1}(s))}\ ds \\
&= \int_u^v \frac{1}{g_n(G_n^{-1}(s))}\ ds \geq \int_u^v \frac{1}{\int_s^1 H_n^{-1}(t)dt}\ ds \\
&= K_n^{-1}(v) - K_n^{-1}(u)\ , \quad 0 < u \leq v \leq 1\ .
\end{aligned}$$

Hereby (1.8) and the theorem have been proved. □

<u>PROOF OF THEOREM 2.1.</u> For every $a \in (G_n^{-1}(0),\ G_n^{-1}(1))$ there exist $\alpha,\ u \in [0,1]$ such that $a = \alpha G_n^{-1}(u) + (1-\alpha)G_n^{-1}(u+)$. Together with (1.8), the continuity of K_n^{-1} (cf. (1.4)) and the properties of ℓ this yields

$$\inf_{a\in\mathbb{R}} E_f \ell(a_n T_n - a) = \inf_{\alpha,u\in[0,1]} \int_0^1 \ell(\alpha[G_n^{-1}(v) - G_n^{-1}(u)]$$

$$+ (1-\alpha)[G_n^{-1}(v) - G_n^{-1}(u+)])dv$$

(3.10) $$\geq \inf_{\alpha,u\in[0,1]} \int_0^1 \ell(K_n^{-1}(v) - K_n^{-1}(u))dv$$

$$= \inf_{a\in\mathbb{R}} \int_0^1 \ell(K_n^{-1}(v) - a)dv ,$$

which implies (2.1). Let a_n , $n = 1,2,\ldots$, be such that

$$\inf_{a\in\mathbb{R}} \int_0^1 \ell(K_n^{-1}(u) - a)du \geq \int_0^1 \ell(K_n^{-1}(u) - a_n)du - \frac{1}{n} , \quad n = 1,2,\ldots .$$

Now we obtain from (2.1)

(3.11) $$\liminf_{n\to\infty} \inf_{a\in\mathbb{R}} E_f \ell(a_n T_n - a) \geq \liminf_{n\to\infty} \int_0^1 \ell(K_n^{-1}(u) - a_n)du .$$

Let $\{n_i\}$ be a sequence of positive integers and let $a_0 \in [-\infty,\infty]$ such that

(3.12) $$\liminf_{n\to\infty} \int_0^1 \ell(K_n^{-1}(u)-a_n)du = \lim_{i\to\infty} \int_0^1 \ell(K_{n_i}^{-1}(u) - a_{n_i})du .$$

and

(3.13) $$\lim_{i\to\infty} a_{n_i} = a_0 .$$

From Satz 2.11 of WITTING and NÖLLE (1970) it follows that

(3.14) $$\lim_{i\to\infty} K_{n_i}^{-1}(u) = K^{-1}(u)$$

for Lebesgue almost all $u \in (0,1)$. Under either of the conditions (2.2) and (2.3) we arrive by (3.13) and (3.14) at

$$\begin{aligned}\lim_{i\to\infty}\int_0^1 \ell\left(K_{n_i}^{-1}(u) - a_{n_i}\right)du &\ge \int_0^1 \liminf_{i\to\infty} \ell\left(K_{n_i}^{-1}(u) - a_{n_i}\right)du \\ &\ge \int_0^1 \ell(K^{-1}(u) - a_0)du \ge \inf_{a\in\mathbb{R}}\int_0^1 \ell(K^{-1}(u) - a)du \,. \end{aligned} \tag{3.15}$$

Combining (3.11), (3.12) and (3.15) we obtain (2.4). Since (3.14) holds also for G_n^{-1} and K_n^{-1} , the left continuity of G^{-1} and K^{-1} yields (2.5). □

<u>PROOF OF COROLLARY 2.1.</u> With $\ell(x) = x^2$ inequality (2.6) is a special case of (2.1). Let $u_0 \in (0,1)$ be such that H_n^{-1} is nonpositive on $(0,u_0)$ and non-negative on $(u_0,1)$. By Fubini's theorem we have

$$\begin{aligned}\int_0^1 K_n^{-1}(u)H_n^{-1}(u)du &= \int_0^1\int_{u_0}^{u} \frac{H_n^{-1}(u)}{\int_s^1 H_n^{-1}(t)dt}\,ds\,du \\ &= \int_0^{u_0} \frac{-\int_0^s H_n^{-1}(u)du}{\int_s^1 H_n^{-1}(t)dt}\,ds + \int_{u_0}^1 \frac{\int_s^1 H_n^{-1}(u)du}{\int_s^1 H_n^{-1}(t)dt}\,ds = 1\,. \end{aligned} \tag{3.16}$$

Consequently, if $I(f) < \infty$, the Cauchy-Schwarz inequality yields

$$\mathrm{var}_{k_n} X \ge \left(\int_0^1 [H_n^{-1}(u)]^2 du\right)^{-1} = a_n^2(n\ I(f))^{-1}$$

and hence (2.7). □

For the proofs of corollaries 2.2 through 2.4 the following lemma is useful.

<u>LEMMA 3.1.</u> *Let* H *be a nondegenerate distribution function such that* $H_n \overset{w}{\to} H$ *as* $n \to \infty$. *If* S_n , $n = 1,2,\ldots$ (cf. (3.5) and (1.5)) *are uniformly integrable, i.e.*

$$(3.17)\qquad \lim_{\beta\to\infty}\limsup_{n\to\infty} E_f\left\{|S_n|1_{[\beta,\infty)}(|S_n|)\right\} = 0 ,$$

then

$$(3.18)\qquad K_n \overset{w}{\to} K \quad \text{as} \quad n \to \infty ,$$

where the absolutely continuous distribution function K *is defined by (cf.* (1.4)*)*

$$(3.19)\qquad K^{-1}(u) = \int_w^u \frac{1}{\int_s^1 H^{-1}(t)dt}\, ds , \quad 0 < u \le 1 .$$

PROOF. As in (3.14) we have

$$(3.20)\qquad \lim_{n\to\infty} H_n^{-1}(t) = H^{-1}(t)$$

for Lebesgue almost all $t \in (0,1)$. If $H^{-1}(1)$ is finite then (3.20) and the dominated convergence theorem yield

$$(3.21)\qquad \lim_{\delta\downarrow 0}\limsup_{n\to\infty} \int_{1-\delta}^1 |H_n^{-1}(t)|dt = 0 .$$

If $H^{-1}(1)$ is infinite then we obtain by (3.17)

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \int_{1-\delta}^1 |H_n^{-1}(t)|dt \le \lim_{\delta\downarrow 0}\limsup_{n\to\infty} E_f\left\{|S_n|1_{[H_n^{-1}(1-\delta),\infty)}(S_n)\right\}$$

$$\le \lim_{\delta\downarrow 0}\limsup_{n\to\infty} E_f\left\{|S_n|1_{[H^{-1}(1-\delta)-1,\infty)}(S_n)\right\} = 0 .$$

Consequently we have for all $s \in (0,1)$

$$\limsup_{n\to\infty} \left|\int_s^1 H_n^{-1}(t)dt - \int_s^1 H^{-1}(t)dt\right|$$

$$\le \lim_{\delta\downarrow 0}\left\{\limsup_{n\to\infty}\left|\int_s^{1-\delta} H_n^{-1}(t)dt - \int_s^{1-\delta} H^{-1}(t)dt\right| + \limsup_{n\to\infty}\int_{1-\delta}^1 |H_n^{-1}(t)|dt + \int_{1-\delta}^1 |H^{-1}(t)|dt\right\}$$

$$\le 2 \lim_{\delta\downarrow 0}\limsup_{n\to\infty} \int_{1-\delta}^1 |H_n^{-1}(t)|dt = 0$$

or

$$(3.22)\qquad \lim_{n\to\infty}\int_s^1 H_n^{-1}(t)dt = \int_s^1 H^{-1}(t)dt \ .$$

In the same way we obtain

$$(3.23)\qquad \lim_{n\to\infty}\int_0^s H_n^{-1}(t)dt = \int_0^s H^{-1}(t)dt \ , \quad 0 < s < 1 \ .$$

From (3.22), (3.23) and formulas like (3.21) we see that

$$(3.24)\qquad \int_0^1 H^{-1}(t)dt = \lim_{n\to\infty} ES_n = 0$$

and

$$(3.25)\qquad \int_0^1 |H^{-1}(t)|dt < \infty \ .$$

Let $u \in [w,1)$ be fixed. Because $\int_s^1 H_n^{-1}(t)dt$ and $\int_s^1 H^{-1}(t)dt$ are concave and because H is nondegenerate, there exists in view of (3.22) and (3.24) an $\varepsilon > 0$ such that for all $s \in [w,u]$ and all sufficiently large n

$$\int_s^1 H_n^{-1}(t)dt \geq \varepsilon \ .$$

By the dominated convergence theorem and (3.22) this implies that (cf. (1.4))

$$\lim_{n\to\infty} K_n^{-1}(u) = K^{-1}(u) \ .$$

The same relation can be shown to hold for $u \in (0,w)$ too.
Consequently, for all bounded continuous functions $b : \mathbb{R} \to \mathbb{R}$

$$\lim_{n\to\infty}\int_0^1 b(K_n^{-1}(u))du = \int_0^1 b(K^{-1}(u))du$$

and hence $K_n \overset{w}{\to} K$ as $n \to \infty$. Finally we note that K is absolutely continuous because of (3.25). □

PROOF OF COROLLARY 2.2. Let $a_n = (nI(f))^{\frac{1}{2}}$. By the central limit theorem $H_n \overset{w}{\to} \Phi$. Since $ES_n^2 = 1$, (3.17) is satisfied and Lemma 3.1 with $w = \frac{1}{2}$ yields $K_n \overset{w}{\to} \Phi$. Applying (2.3) and (2.4) of Theorem 2.1 we obtain (2.8). □

PROOF OF COROLLARY 2.3. Let $a_n = (\frac{1}{2}cn\log n)^{\frac{1}{2}}$. From Lemma 3.2 of WOODROOFE (1972) it follows that $H_n \overset{w}{\to} \Phi$. Since for all $a, b \in \mathbb{R}$ and $\beta > 0$

$$|a+b|\,1_{[\beta,\infty)}(|a+b|) \le 2|a| + \beta^{-1}b^2 , \tag{3.26}$$

we have

$$\begin{aligned} E_f\left\{|S_n|\,1_{[\beta,\infty)}(|S_n|)\right\} &\le 2\, E_f\left|a_n^{-1}\sum_{i=1}^{n}\left[-\frac{f'}{f}(X_i)\,1_{(0,n^{-1}]}(X_i) + f(n^{-1})\right]\right| \\ &\quad + \beta^{-1}E_f\left\{a_n^{-1}\sum_{i=1}^{n}\left[-\frac{f'}{f}(X_i)\,1_{(n^{-1},\infty)}(X_i) - f(n^{-1})\right]\right\}^2 \\ &\le 2\, a_n^{-1}\, n\left\{\int_0^{n^{-1}} |f'(x)|dx + f(n^{-1})\right\} \\ &\quad + 2\,\beta^{-1}\, a_n^{-2}\, n\left\{\int_{n^{-1}}^{\infty}\left(\frac{f'}{f}(x)\right)^2 f(x)dx + f^2(n^{-1})\right\} . \end{aligned} \tag{3.27}$$

The properties of $f \in D(c)$ yield

$$\int_{n^{-1}}^{\infty}\left(\frac{f'}{f}(x)\right)^2 f(x)dx = O\left(\int_{n^{-1}}^{1} x^{-1}dx\right) = O(\log n) \tag{3.28}$$

and

$$\int_0^{n^{-1}} |f'(x)|dx = O(f(n^{-1})) = O(n^{-1}) . \tag{3.29}$$

Combining (3.27), (3.28) and (3.29) we see that

$$E_f\left\{|S_n|\,1_{[\beta,\infty)}(|S_n|)\right\} = O(a_n^{-1}) + \beta^{-1}O(1) \quad \text{as} \quad n \to \infty$$

and hence that (3.17) holds. Now Lemma 3.1 with $w = \frac{1}{2}$ yields $K_n \overset{w}{\to} \Phi$. By (2.3) and (2.4) of Theorem 2.1 we arrive at (2.9). □

PROOF OF COROLLARY 2.4. We define

$$\mu(x) = \int_{|f'(y)/f(y)|\le x} \left(\frac{f'}{f}(y)\right)^2 f(y)dy \ , \quad x > 0 \ .$$

Let $\varepsilon \in (0,1)$. There exists a $\delta > 0$ such that (cf. Lemma 4.1 of WOODROOFE (1974))

$$\left|\frac{f'}{f}(y) - \frac{\alpha-1}{y}\right| \le \frac{(\alpha-1)\varepsilon}{y} \ , \quad 0 < y \le \delta \ .$$

In view of the properties of f and Karamata's theorem (cf. Theorem VIII.9.1 of FELLER (1971)) this yields for $x \to \infty$

$$\mu(x) \le \int_\delta^\infty \left(\frac{f'}{f}(y)\right)^2 f(y)dy + \int_{(\alpha-1)(1-\varepsilon)x^{-1}}^{\delta} \alpha(\alpha-1)^2(1+\varepsilon)^2 y^{\alpha-3}L(y)dy$$

$$\sim \alpha(\alpha-1)^\alpha(2-\alpha)^{-1}(1+\varepsilon)^2(1-\varepsilon)^{\alpha-2}x^{2-\alpha}L(x^{-1})$$

and

$$\mu(x) \ge \int_{(a-1)(1+\varepsilon)x^{-1}}^{\delta} \alpha(\alpha-1)^2(1-\varepsilon)^2 y^{\alpha-3}L(y)dy$$

$$\sim \alpha(\alpha-1)^\alpha(2-\alpha)^{-1}(1-\varepsilon)^2(1+\varepsilon)^{\alpha-2}x^{2-\alpha}L(x^{-1}) \ .$$

Consequently $\mu(x)$ is regularly varying with exponent $2-\alpha$ and

$$\text{(3.30)} \qquad \lim_{n\to\infty} n\, a_n^{-2}\mu(a_n) = \alpha(\alpha-1)^\alpha(2-\alpha)^{-1} \ .$$

Since for ε and δ as above and $x \to \infty$

$$P_f\left(-\frac{f'}{f}(X_1)>x\right) \le x^{-2}E_f\left\{\left(\frac{f'}{f}(X_1)\right)^2 1_{(\delta,\infty)}(X_1)\right\} = O(x^{-2})$$

and

$$P_f\left(-\frac{f'}{f}(X_1)<-x\right) \ge P_f\left(0<X_1\le\delta, (\alpha-1)(1-\varepsilon)>x\, X_1\right)$$

$$= (\alpha-1)^\alpha(1-\varepsilon)^\alpha x^{-\alpha}L(x^{-1}) \ ,$$

we also have

$$(3.31)\qquad \lim_{x\to\infty} P_f\left(-\frac{f'}{f}(X_1)>x\right)\left[P_f\left(\left|\frac{f'}{f}(X_1)\right|>x\right)\right]^{-1} = 0 .$$

From (3.30), (3.31), the regular variation of μ and $\int f' = 0$ we obtain the weak convergence of H_n to H by Theorem XVII.5.3 of FELLER (1971).

Here we note that the + and - sign in (3.18) of section XVII.4 of FELLER (1971) should be interchanged. In view of this misprint the + sign in formula (2.4) of WOODROOFE (1974) should be replaced by a - sign and consequently the remark at the beginning of section 3 of WOODROOFE (1974) should be (in our notation): $\hat{G}(0) = 1 - \alpha^{-1}$, which for $\alpha \downarrow 1$ tends to 0. We proceed now with the proof of Corollary 2.4.

By (3.26) we have for all $\varepsilon > 0$

$$E_f\left\{|S_n| 1_{[\beta,\infty)}(|S_n|)\right\} \le 2\, E_f\left|a_n^{-1}\sum_{i=1}^n\left\{-\frac{f'}{f}(X_i)1_{(0,\varepsilon]}(X_i) + f(\varepsilon)\right\}\right|$$

$$(3.32)\qquad + \beta^{-1}E_f\left[a_n^{-1}\sum_{i=1}^n\left\{-\frac{f'}{f}(X_i)1_{(\varepsilon,\infty)}(X_i) - f(\varepsilon)\right\}\right]^2$$

$$\le 2\, a_n^{-1} n\left\{\int_0^\varepsilon |f'(x)|dx + f(\varepsilon)\right\} + 2\,\beta^{-1}a_n^{-2}\, n\left\{\int_\varepsilon^\infty\left(\frac{f'}{f}(x)\right)^2 f(x)dx + f^2(\varepsilon)\right\} .$$

Again by Karamata's theorem we obtain for $\varepsilon \downarrow 0$

$$(3.33)\qquad \int_\varepsilon^\infty \left(\frac{f'}{f}(x)\right)^2 f(x)dx = O\left(\int_\varepsilon^1 x^{-2}f(x)dx\right) = O(\varepsilon^{\alpha-2}L(\varepsilon))$$

and

$$(3.34)\qquad \int_0^\varepsilon |f'(x)|dx = O(f(\varepsilon)) = O(\varepsilon^{\alpha-1}L(\varepsilon)) .$$

Combining (3.32) through (3.34) and choosing $\varepsilon = a_n^{-1}\beta^{-1}$ we see that there exists a constant $c > 0$ such that for sufficiently large n and β

$$E_f\left\{|S_n| 1_{[\beta,\infty)}(|S_n|)\right\} \le c\, \beta^{1-\alpha}\, n\, a_n^{-\alpha}\, L(a_n^{-1}\beta^{-1}) .$$

In view of (2.10) this yields the validity of (3.17). Hence Lemma 3.1 implies that $K_n \overset{w}{\to} K$ and that K satisfies (2.3). Consequently (2.4) holds and the corollary has been proved. □

PROOF OF LEMMA 2.1. From entry VI of SKOROHOD (1961), (2.11) and (2.12) we see that the density h of H satisfies

$$(3.35) \qquad h(x) \sim p(x)\exp\left\{-c(\alpha)x^{\frac{\alpha}{\alpha-1}}\right\}, \quad x \to \infty,$$

where $p(x)$ is some polynomial in x and

$$c(\alpha) = [\alpha^{\alpha}\Gamma(2-\alpha)]^{-\frac{1}{\alpha-1}}.$$

This implies that for $u \uparrow 1$

$$(3.36) \qquad H^{-1}(u) \sim (c(\alpha))^{-\frac{\alpha-1}{\alpha}}[-\log(1-u)]^{1-\frac{1}{\alpha}}$$

and hence

$$(3.37) \qquad \int_u^1 H^{-1}(t)dt \sim (c(\alpha))^{-\frac{\alpha-1}{\alpha}}(1-u)[-\log(1-u)]^{1-\frac{1}{\alpha}}$$

and

$$(3.38) \qquad K^{-1}(u) \sim (c(\alpha))^{\frac{\alpha-1}{\alpha}}\alpha[-\log(1-u)]^{\frac{1}{\alpha}} = [\Gamma(2-\alpha)]^{-\frac{1}{\alpha}}[-\log(1-u)]^{\frac{1}{\alpha}},$$

which is (2.14).

By Theorem XVII.5.1 of FELLER (1971) we conclude from (2.11) and (2.12) that

$$(3.39) \qquad H^{-1}(u) \sim -(\alpha-1)u^{-\frac{1}{\alpha}}, \quad u \downarrow 0.$$

This implies that for all $\varepsilon > 0$ there exists a u_0 such that for $u < u_0$

$$(3.40) \qquad \begin{aligned} K^{-1}(u) - K^{-1}(u_0) &= \int_u^{u_0} \frac{1}{\int_0^s H^{-1}(t)dt}\, ds \ge -\int_u^{u_0} \alpha^{-1} s^{\frac{1}{\alpha}-1} ds(1+\varepsilon) \\ &\ge -(1+\varepsilon)u_0^{\frac{1}{\alpha}}, \end{aligned}$$

which implies (2.15).

From Theorem 2.3 and the formulas just before Lemma 3.1 on page 479 of WOODROOFE (1974) we see that for $t > 0$ and $\lambda > 0$

$$(3.41)\qquad E\, e^{-\lambda Z_t^*} = \exp\left\{\alpha t^{\alpha}\left\{\lambda + \int_1^{\infty}\left[e^{-\frac{\lambda(\alpha-1)}{y-1}} - 1 + \frac{\lambda(\alpha-1)}{y}\right]y^{\alpha-1}dy\right\}\right\} .$$

Since, trivially, for $\lambda \geq 0$

$$(3.42)\qquad P(Z_t^* < 0) \leq E\, e^{-\lambda Z_t^*} ,$$

we have for $t > 0$

$$(3.43)\qquad \begin{aligned} P(Z_t^* < 0) &\leq \inf_{\lambda\geq 0} \exp\left\{\alpha t^{\alpha}\left\{\lambda + \int_0^1 \left[e^{-\frac{\lambda(\alpha-1)z}{1-z}} - 1 + \lambda(\alpha-1)z\right]z^{-1-\alpha}\, dz\right\}\right\} \\ &= \inf_{\beta\geq 0} \exp\left\{\alpha t^{\alpha}\left\{\frac{\beta}{\alpha-1} + \int_0^1 \left[e^{-\frac{\beta z}{1-z}} - 1 + \beta z\right]z^{-1-\alpha}\, dz\right\}\right\} . \end{aligned}$$

Because for $z \in (0,1)$ and $\beta \geq 0$

$$\begin{aligned} e^{-\frac{\beta z}{1-z}} &\leq e^{-\beta z(1+z)} \leq 1 - \beta z - \beta z^2 + \tfrac{1}{2}\, \beta^2 z^2 (1+z)^2 \\ &\leq 1 - \beta z - \beta(1-2\beta)z^2 , \end{aligned}$$

we obtain by (3.43)

$$(3.44)\qquad P(Z_t^* < 0) \leq \exp\left\{ - \frac{\alpha([2\alpha - 3]^+)^2}{8(\alpha-1)^2(2-\alpha)}\, t^{\alpha}\right\} , \quad t > 0 .$$

Combining this inequality with Theorem 2.4 of WOODROOFE (1974) we arrive at (2.16). Finally we note that (2.17) is the content of Theorem 3.1 of WOODROOFE (1974). □

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