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A NONLINEAR EVOLUTION PROBLEM ARISING IN THE PHYSICS OF IONIZED GASES

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by

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ABSTRACT

We consider a Coulomb gas in a special experimental situation: the pre-breakdown gas discharge between two electrodes. The equation for the negative charge density can be formulated as a nonlinear parabolic equation degenerate at the origin. We prove the existence and uniqueness of the solution as well as the asymptotic stability of its unique steady state. Also some results are given about the rate of convergence.

KEY WORDS & PHRASES: *nonlinear parabolic equation degenerate at the origin in one space dimension; pre-breakdown discharge in an ionized gas between two electrodes*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we study the nonlinear evolution problem,

$$P \begin{cases} u_t = \varepsilon x u_{xx} + (g(x)-u)u_x & \text{on } D = (0,\infty) \times (0,T) \\ u(0,t) = 0 & \text{for } t \in [0,T] \\ u(x,0) = \psi(x) & \text{for } x \in (0,\infty). \end{cases}$$

where ε is a positive constant, g is a given function which satisfies the hypothesis H_g: $g \in C^2([0,\infty))$; $g(0) = 0$; $g'(x) > 0$ and $g''(x) < 0$ for all $x \geq 0$ and the initial function ψ satisfies the hypothesis H_ψ:

(i) ψ is continuous, with piecewise continuous derivative on $[0,\infty)$;

(ii) $\psi(0) = 0$ and $\psi(\infty) = K \in (0,g(\infty))$;

(iii) there exists a constant $M_\psi \geq g'(0)$ such that $0 \leq \psi'(x) \leq M_\psi$ at all points x where ψ' is defined.

In section 2, we briefly describe how the problem arises in physics and give the derivation of the equations.

In section 3, we present maximum principles for certain linear and non-linear problems related to P; the uniqueness of the solution of P follows directly from those principles.

In section 4, we prove that P has a classical solution which satisfies furthermore the condition

$$(*) \qquad u(\infty,t) = K \qquad \text{for } t \in [0,T],\ T < \infty.$$

The methods used here are inspired by those of VAN DUYN [7],[8] and GILDING & PELETIER [13]. We also consider the limit case $\varepsilon \downarrow 0$ and prove that u tends to the generalized solution of the corresponding hyperbolic problem.

We then investigate the behaviour of u as $t \to \infty$ and prove that it converges towards the unique solution $\emptyset$ of the problem P_0 defined as follows

$$P_0 \begin{cases} \varepsilon \times \emptyset'' + (g(x)-\emptyset)\emptyset' = 0 \\ \emptyset(0) = 0 \qquad \emptyset(\infty) = \lambda_0 =: \min(\max(g(\infty)-\varepsilon,0),K) \end{cases}$$

Qualitative properties of $\emptyset$ have been extensively studied by DIEKMANN, HILHORST & PELETIER [6]. Here we analyse its stability. In section 5, following a method of ARONSON & WEINBERGER [2] based on the knowledge about lower and upper solutions for the steady state problem P_0, we prove that $\emptyset$ is asymptotically stable.

In section 6 we investigate the rate of convergence of u towards its steady state. The function $\emptyset$ turns out to be exponentially stable when the function g grows fast enough to infinity as $x \to \infty$; the proof, based on constructing upper and lower solutions for the function $u-\emptyset$, follows the same lines as that of FIFE & PELETIER [10]. We also consider the case when g increases less fast and show that provided that $\varepsilon < g(\infty) - K$ and that $\emptyset$ converges algebraically fast to K as $x \to \infty$, the function $u-\emptyset$ decays algebraically fast; this is done by obtaining first that property for a weighted integral of $u-\emptyset$ according to a method of IL'IN & OLEINIK [14] and VAN DUYN & PELETIER [9]. Finally we consider the corresponding hyperbolic problem and obtain a similar result of algebraic convergence.

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2. PHYSICAL DERIVATION OF THE EQUATIONS

The physical context of the present problem has been described in some detail by DIEKMANN, HILHORST & PELETIER [6]. Here we shall summarize it again and explain how one can obtain the time evolution problem P.

One considers an ionized gas between two electrodes in which the ions and electrons are present with densities $n_i(\vec{r})$ and $n_e(\vec{r},t)$ respectively, where $\vec{r} = (x_1,x_2,x_3)$. The ions are heavy and slow and the density $n_i(\vec{r})$ may therefore be regarded as fixed. The electrons are highly mobile. The problem is then to find $n_e(\vec{r},t)$ for given $n_i(\vec{r})$ and in particular to find out whether given an initial electron distribution the electrons stabilize and if so to evaluate the time needed for such a stabilization.

A special situation of practical interest is a so-called pre-breakdown discharge which spreads out in filamentary form (cf. MARODE [17] and MARODE, BASTIEN & BAKKER [18]). In this situation there is cylindrical symmetry about the x_3-axis and the particle densities depend on $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ only. We thus have effectively a two-dimensional Coulomb gas with circular symmetry. The starting equations are

(i) Coulomb's law for the electric field E,

$$(2.1) \qquad \frac{1}{r}\frac{\partial}{\partial r} r\, E = -C_d(n_e - n_i)$$

where C_d is a fixed constant;

(ii) a constitutive equation for the electric current j,

$$(2.2) \qquad j = n_e \mu E + kT\frac{\partial n_e}{\partial r},$$

in which the first term represents Ohm's law and the second term is due to thermal diffusion, μ being the mobility, k Boltzmann's constant and T the temperature; and

(iii) the continuity equation for the electron density,

$$(2.3) \qquad \frac{\partial n_e}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} rj.$$

If we set

$$u(x,t) = \int_0^{\sqrt{x}} n_e(r,t)r\, dr$$

and

$$g(x) = \int_0^{\sqrt{x}} n_i(r)r\, dr,$$

we obtain after redefining the constants the equation

$$(2.4) \qquad u_t = \varepsilon x u_{xx} + (g(x)-u)u_x$$

where $\varepsilon = 2kT/(\mu C_d)$ and the boundary condition

(2.5) $\quad u(0,t) = 0.$

Furthermore one makes the hypothesis that the total charge is positive and fixed, that is

$$\int_0^\infty (n_i(r) - n_e(r,t))r\,dr = N > 0$$

from which we deduce the boundary condition at infinity,

(2.6) $\quad u(\infty,t) = K := g(\infty) - N.$

Clearly $K \in (0,g(\infty))$.

Equations (2.4) and (2.5) together with the initial condition

(2.7) $\quad u(x,0) = \psi(x)$

constitute the mathematical formulation of the problem which we propose to study in this paper. Furthermore the condition (2.6) will turn out te be satisfied at all finite times t and also, for low enough values of the small parameter ε, at the time $t = \infty$. This latter property expresses the fact that all the electrons stay attached to the ions at low enough temperature; we shall also see that if the temperature rises above a critical value, then some of the electrons escape to infinity and if it rises even further above a second critical value, then all the electrons escape to infinity.

3. MAXIMUM PRINCIPLES FOR SOME DEGENERATE PARABOLIC OPERATORS - UNIQUENESS THEOREM

In this section we prove maximum principles for some linear and non-linear operators which have a degeneracy at the origin; these principles hold for functions $u \in C^{2,1}(D) \cap C(\overline{D})$ where $C^{2,1}(D)$ is the set of continuous functions on D with two continuous x-derivatives and one continuous t-derivative. It will follow easily from those maximum principles that P can have at most one solution $u \in C^{2,1}(D) \cap C(\bar{D})$ such that u_x is bounded in $\bar{D}$.

We begin by defining a linear operator L as follows

$$(3.1) \qquad Lu = \varepsilon x u_{xx} + b(x,t)u_x + c(x,t)u - u_t$$

where the functions b and c are continuous on D and such that the quantities $b/(1+x)$ and c are bounded on $\bar{D}$. First we consider the bounded domain $D_R :=$ $(0,R) \times (0,T)$, where R is a positive constant. In the same way as for a uniformly parabolic operator one can prove the following maximum principle which holds in fact for a much wider class of degenerate parabolic operators (see for example IPPOLITO [15] or COSNER [4])

THEOREM 3.1. *Suppose* $c \le 0$. *Let* $u \in C^{2,1}(D_R) \cap C(\bar{D}_R)$ *satisfy* $Lu \ge 0$ *on* $(0,R) \times (0,T]$. *Then if* u *has a positive maximum in* $\bar{D}_R$, *that maximum is attained on* $((0,R) \times \{0\}) \cup (\{0,R\} \times [0,T])$.

Next following a method due to ARONSON & WEINBERGER [2] we derive a comparison theorem for a class of nonlinear evolution problems.

THEOREM 3.2. *Let* u *and* $v \in C^{2,1}(D_R) \cap C(\bar{D}_R)$ *and suppose that either* u_x *or* v_x *is bounded on* $\bar{D}_R$. *Let* u *and* v *satisfy*

$$Lv - vv_x \ge Lu - uu_x \qquad \text{on } (0,R) \times (0,T]$$

and let

$$0 \le v \le u \le K \qquad \text{on } (0,R) \times \{0\} \text{ and } \{0,R\} \times [0,T].$$

Then $v \le u$ *in* $(0,R) \times (0,T]$.

PROOF. Let

$$w = (v-u)e^{-\alpha t}$$

where

$$\alpha = \max_{(x,t)\,\in\,\overline{D}} (c(x,t) - u_x(x,t))$$

(in the case where u_x is bounded). Then w satisfies

$$\varepsilon x w_{xx} + (b(x,t) - v)w_x + (c(x,t) - u_x - \alpha)w - w_t \geq 0$$

and

$$w \leq 0 \text{ on } (0,R) \times \{0\} \text{ and } \{0,R\} \times [0,T].$$

Thus we deduce from Theorem 3.1 that

$$w \leq 0 \text{ in } (0,R) \times (0,T]$$

which completes the proof of theorem 3.2. □

Now let us consider the unbounded domain D. To begin with we present a Phragmèn-Lindelöf principle which is a special case of a theorem due to COSNER [4].

<u>THEOREM 3.3</u>. *Suppose that* $b/(1+x)$ *and* c *are continuous and bounded in* $\overline{D}$. *Let* $u \in C^{2,1}(D) \cap C(\overline{D})$ *satisfy* $Lu \geq 0$ *on* $(0,\infty) \times (0,T]$ *and the growth condition*

$$\liminf_{\mathcal{R} \to \infty} e^{-B\mathcal{R}} [\max_{0\leq t\leq T} u(\mathcal{R},t)] \leq 0 \tag{3.2}$$

for some positive constant B. *If* $u \leq 0$ *for* $t = 0$ *and on* $\{0\} \times [0,T]$, *then* $u \leq 0$ *in* $(0,\infty) \times (0,T]$.

Making use of Theorem 3.3 one can prove a comparison theorem on the unbounded domain D.

<u>THEOREM 3.4</u>. *Let* u *and* $v \in C^{2,1}(D) \cap C(\overline{D})$ *be such that either* u_x *and* v *or* u *and* v_x *are bounded on* $\overline{D}$ *and that*

$$|u(x,t)|,\ |v(x,t)| \leq C\,e^{B_1 x}$$

for some positive constants C and B_1 *and uniformly in* $t \in [0,T]$. *Suppose that*

$$Lv - vv_x \geq Lu - uu_x \qquad \text{on } (0,\infty) \times (0,T]$$

and that

$$0 \leq v \leq u \leq K \qquad \text{on } (0,\infty) \times \{0\} \text{ and } \{0\} \times [0,T].$$

Then $v \leq u$ *in* $(0,\infty) \times (0,T]$.

Finally let us come to the question of uniqueness of the solution of problem P.

DEFINITION. We shall say that u is a classical solution of Problem P if it is such that (i) $u \in C^{2,1}(D) \cap C(\overline{D})$, (ii) u and u_x are bounded in $\overline{D}$, (iii) u satisfies the equation in D, (iv) u satisfies the initial and boundary conditions.

THEOREM 3.5. *Problem* P *can have at most one solution.*

PROOF. Apply Theorem 3.4 twice to deduce that if u and v are two such solutions then their difference $w = u-v$ satisfies $w \geq 0$ and $w \leq 0$ and thus $w \equiv 0$. □

4. EXISTENCE AND REGULARITY OF THE SOLUTION

In order to be able to prove the existence of a solution of the nonlinear degenerate parabolic problem P, we consider certain related nonlinear uniformly parabolic problems on bounded domains and observe that they have a unique solution; we then deduce that P has a generalized solution, in a certain sense. It finally turns out that this solution is in fact a classical solution of P and thus the unique solution of P and that it also satisfies condition (*). Finally we consider its limiting behaviour as $\varepsilon \downarrow 0$.

4.1. Existence

Let us first introduce some notation. Let $D_n := (0,n) \times (0,T)$. We denote by $C_{2+\alpha}([0,n])$ the space of functions v which are twice differentiable and such that v" is Hölder continuous on $[0,n]$ with exponent α. We also use the spaces $\bar{C}_\alpha(D_n)$, $\overline{C_{2+\alpha}}(D_n)$ and $C_{2+\alpha}(D_n)$, defined in FRIEDMAN [11] p. 62 and 63.

Consider the problem

$$P_n \quad \begin{cases} u_t = \varepsilon(x+1/n)u_{xx} + (g(x)-u)u_x & \text{in } D_n \\ u(0,t) = 0 \qquad u(n,t) = K & t \in [0,T] \\ u(x,0) = \psi_n(x) & x \in (0,n). \end{cases}$$

with $n \geq g^{-1}(K)$ and where ψ_n is such that

(i) $\psi_n \in C^\infty([0,\infty])$,
(ii) ψ_n satisfies H_ψ,
(iii) $\psi''_n(0) = 0$ and $\psi_n(x) = K$ for $x \in [n-1,\infty)$.

In what follows we shall denote by H_n properties (i) - (iii). The following theorem holds:

<u>THEOREM 4.1</u>. *There exists a unique solution* $u_n \in \bar{C}_{2+\alpha}(D_n)$ *of* P_n *for any* $\alpha \in (0,1)$; *furthermore* u_n *satisfies the inequalities*

$$(4.1) \qquad 0 \leq u_n(x,t) \leq \min(M_{\psi_n} x, K)$$

$$(4.2) \qquad 0 \leq u_{nx}(x,t) \leq M_{\psi_n}$$

for all $(x,t) \in \bar{D}_n$.

<u>PROOF</u>. The existence and uniqueness of $u_n \in \overline{C_{2+\alpha}}(D_n)$ is a consequence of Theorem 5.2 of LADYŽENSKAJA ([16] p. 564-565). The inequalities in (4.1) can be deduced by means of a comparison theorem analogous to theorem 3.2. From the linear theory (FRIEDMAN [11] p. 72) we deduce that the function $w := u_{nx} \in C_{2+\alpha}(D_n)$; thus $w \in C^{2,1}(D_n) \cap C(\bar{D}_n)$. Furthermore w satisfies

$$(4.3) \qquad \begin{cases} w_t = \varepsilon(x+1/n)w_{xx} + (g(x)-u_n+\varepsilon)w_x + (g'(x)-w)w \\ 0 \leq w(0,t) \leq M_{\psi_n} \qquad\qquad 0 \leq w(n,t) \leq M_{\psi_n} \\ w(x,0) = \psi'_n(x). \end{cases}$$

The bounds on the function $w(n,t)$ follow from the fact that the function $\max(0,M_{\psi_n}(x-n)+K)$ is a lower solution of the boundary value problem

$$\varepsilon(x+1/n)\phi'' + (g(x)-\phi)\phi' = 0$$

$$\phi(0) = 0 \qquad\qquad \phi(n) = K$$

and consequently a lower bound for u_n. Clearly the set

$$\{w \in C([0,n]) \text{ such that } 0 \le w(x) \le M_{\psi_n}\}$$

is invariant with respect to the problem (4.3) and thus the inequalities (4.2) are satisfied.

Next we deduce from theorem 4.1 the existence of solution of P. We begin by approximating the initial function ψ by a sequence of smooth functions $\{\psi_n\}$.

<u>LEMMA 4.2</u>. *Let the function* ψ *satisfy* H_ψ. *Then there exists a sequence* $\{\psi_n\}$ *which satisfies the properties* H_n *given at the beginning of this section with* $M_{\psi_n} = M_\psi$ *for all* n, *such that* $\psi_n \to \psi$ *as* $n \to \infty$, *uniformly on* $[0,\infty)$.

<u>PROOF</u>. Let $n_0 \ge g^{-1}(K)$ be such that for all $n \ge n_0$ the point x_{1n} defined by $M_\psi(x_{1n}-1/n) = \psi(x_{1n})$ is such that $1/n < x_{1n} \le n-2$ and that the point x_{2n} defined by $x_{2n} = n - 2 + (K-\psi(n-2))/M_\psi$ satisfies $n-2 < x_{2n} < n-1$. Also define

$$\psi_n^*(x) = \begin{cases} 0 & -\infty < x \le 1/n \\ M_\psi(x-1/n) & 1/n < x \le x_{1n} \\ \psi(x) & x_{1n} < x \le n-2 \\ M_\psi(x-n+2) + \psi(n-2) & n-2 < x \le x_{2n} \\ K & x_{2n} < x < +\infty \end{cases}$$

Note that for all x

$$|\psi_n^*(x) - \psi(x)| \le \max(M_\psi/n,\ K - \psi(n-2)).$$

Next introduce the function

$$\rho(x) = \begin{cases} 0 & \text{if } |x| \ge 1 \\ C\exp(1/(|x|^2-1)) & \text{if } |x| < 1 \end{cases}$$

where the constant C is such that $\int_{\mathbb{R}} \rho dx = 1$, and let

$$\rho_\delta(x) = \rho(x/\delta)/\delta.$$

Finally define

$$\psi_n(x) = \int_{\mathbb{R}} \rho_{\delta_n}(x-y)\psi_n^*(y)dy \qquad x \in [0,n]$$

with $\delta_n = \min(1/n, x_{1n}-1/n, n-2-x_{1n}, x_{2n}-n+2, n-1-x_{2n})/10$. We now show that ψ_n has the desired properties. Firstly $\psi_n \in C^\infty([0,n])$. The uniform convergence of $\{\psi_n\}$ to ψ follows from the continuity of ψ_n^*, uniformly in n and in x and the uniform convergence of ψ_n^* to ψ as $n \to \infty$. Finally properties (ii) and (iii) of H_n can be deduced for ψ_n from the fact that ψ also satisfies them. □

Next we prove the following theorem

THEOREM 4.3. *P has a unique classical solution. Furthermore this solution also satisfies condition* (*):

$$\lim_{x\to\infty} u(x,t) = K \quad \textit{for each } t \in (0,T].$$

PROOF. We rewrite the parabolic equation of Problem P_n as

$$(4.4) \qquad u_t = \varepsilon(x+1/n)u_{xx} + c(x,t)u_x,$$

where

$$c(x,t) = g(x) - u_n(x,t).$$

From Theorem 4.1 we know that for all $(x',t),(x'',t) \in \overline{D}_n$ and for all $n \ge n_0$

$$(4.5) \qquad |u_n(x',t) - u_n(x'',t)| \le M_\psi|x'-x''|.$$

Now fix $I \geq n_0$; (4.4) and (4.5) enable us to apply a theorem of GILDING [12] about the Hölder continuity of solutions of parabolic equations and we obtain

$$|u_n(x,t') - u_n(x,t")| \leq C\ |t'-t"|^{\frac{1}{2}}$$

for all $n \geq I$ and for all $(x,t'),(x,t") \in \bar{D}_I$, with $|t'-t"| \leq 1$. Here the constant C depends on I but not on n. The set $\{u_n(x,t)\}_{n=I}^{\infty}$ is bounded and equicontinuous in D_I and thus there exists a continuous function $u_I(x,t)$ and a convergent subsequence $\{u_{n_k}(x,t)\}$ with $n_k \geq I$ such that $u_{n_k}(x,t) \to u_I(x,t)$ as $n_k \to \infty$, uniformly on $\bar{D}_I$. Then, by a diagonal process, it follows that there exists a function $u(x,t)$ defined on $\bar{D}$ and a convergent subsequence, denoted by $\{u_j(x,t)\}$ such that $u_j(x,t) \to u(x,t)$ as $j \to \infty$, pointwise on $\bar{D}$. Since this convergence is uniform on any bounded subset of $\bar{D}$, the limit function u is continuous on $\bar{D}$.

It remains to show that u is a solution of P; to that purpose we shall proceed in two steps: firstly we show that u is a generalized solution of P in a certain sense and then we conclude that it is in fact a classical solution. We shall say that u is a generalized solution of P if it has the following properties:

(i) u is continuous and uniformly bounded in $\bar{D}$;

(ii) $u(0,t) = 0$ for all $t \in [0,T]$;

(iii) u has a bounded generalized derivative with respect to x in D;

(iv) u satisfies the identity

$$(4.6)\qquad \iint_D [u\phi_t - \varepsilon(xu_x-u)\phi_x - (g-u/2)u\phi_x - ug'\phi]dxdt + \int_0^\infty \psi(x)\phi(x,0)\,dx = 0$$

for all $\phi \in C^1(\bar{D})$ which vanish for $x = 0$, large x and $t = T$.

Let us check that u satisfies those properties.

(i) We already know that u is continuous on $\bar{D}$ and furthermore, since $u(x,t) = \lim_{j\to\infty} u_j(x,t)$, we have that $0 \leq u \leq K$.

(ii) This property follows from a similar boundary condition in P_n.

(iii) Let ϕ be an admissible test function and let $L \geq n_0$ be such that supp $\phi \subset D_L$. Since $|u_{jx}|$ is uniformly bounded with respect to $j \geq L$ for all $(x,t) \in D_L$, it follows that there exists a subsequence $\{(u_{j_k})_x\}$ and a bounded function $p \in L^2(D_L)$ such that

$$(u_{j_k})_x \rightharpoonup p \quad \text{in } L^2(D_L) \quad \text{as } j_k \to \infty .$$

Now let $\zeta \in C_0^1(\overline{D}_L)$. Then

$$(4.7) \qquad ((u_{j_k})_x, \zeta) \to (p, \zeta) \quad \text{as } j_k \to \infty$$

where $(.,.)$ denotes the inner product in $L^2(D_L)$. But since $u_{j_k} \to u$ as $j_k \to \infty$, uniformly on $\overline{D}_L$, we have

$$(4.8) \qquad (u_{j_k}, \zeta_x) \to (u, \zeta_x) \quad \text{as } j_k \to \infty.$$

Hence combining (4.7) and (4.8) we find that p is the generalized derivative of u.

(iv) Since u_{j_k} is a classical solution of P_n it follows that

$$(4.9) \qquad \iint\limits_{D_L} [u_{j_k}\phi_t - \varepsilon((x+1/j_k)(u_{j_k})_x - u_{j_k})\phi_x - (g-u_{j_k}/2)u_{j_k}\phi_x - u_{j_k}g'\phi]dxdt + \int_0^L \psi_{j_k}(x)\phi(x,0)dx = 0$$

The sequences $\{u_{j_k}\}$ and $\{u_{j_k}^2\}$ converge to u and u^2, respectively, strongly in $L^2(D_L)$ as $j_k \to \infty$. Furthermore since $(u_{j_k})_x$ is uniformly bounded we have

$$\iint\limits_{D_L} \frac{1}{j_k} (u_{j_k})_x \phi_x \, dxdt \to 0 \quad \text{as } j_k \to \infty.$$

Thus letting $j_k \to \infty$ we obtain (4.6). Because ϕ has been chosen arbitrarily, we may conclude that u is indeed a generalized solution of P.

It remains to show that u is a classical solution of P. One can do it by using a classical bootstrap argument (see for example GILDING & PELETIER [13]) to show that for whatever $\eta, L > 0$ there exists $\alpha(\eta,L) \in (0,1)$ such that

$$(4.10) \qquad u \in \overline{C_{2+\alpha}}((\eta,L) \times (\eta,T))$$

where α and $\|u\|_{\overline{C_{2+\alpha}}}$ may be estimated independently of T. In particular

$$u \in C^{2,1}(D) \cap C(\overline{D}).$$

Since furthermore u and u_x are uniformly bounded, u is a classical solution of Problem P and by theorem 3.5 it is the unique solution of P.

Finally let us analyze the behaviour of u for large x; since we have $0 \le u \le K$ and $u_x \ge 0$, $u(\infty,t) = \lim_{x\to\infty} u(x,t)$ is well defined for all $t \in [0,T]$ and such that $0 \le u(\infty,t) \le K$. Next we show that $u(\infty,t) \equiv K$ by constructing a time dependent lower solution for P. Consider the problem

$$(4.11) \qquad \begin{cases} u_t = \varepsilon x u_{xx} + (K-u)u_x \\ u(x_0,t) = 0 & x_0 \ge g^{-1}(K) \\ u(x,0) = \psi(x) \end{cases}$$

Since $u_x \ge 0$ we have that

$$\begin{aligned} \varepsilon x u_{xx} + (g(x)-u)u_x - u_t &= \varepsilon x u_{xx} + (K-u)u_x - u_t + (g(x)-K)u_x \\ &\ge \varepsilon x u_{xx} + (K-u)u_x - u_t \quad \text{for all } x \ge g^{-1}(K). \end{aligned}$$

Thus a lower solution $\hat{u}$ of (4.11) with $\hat{u}_x \ge 0$ is also a lower solution of P on $[x_0,\infty) \times [0,T]$. We search such functions $\hat{u}_k$ which satisfy furthermore

$$\hat{u}_k(\infty,t) = K - k \quad \text{for all } t \in [0,T] \text{ and with } k \in (0,K).$$

Writing

$$\hat{v} = K - \hat{u}$$

this comes down to finding an upper solution $\hat{v}_k$ of

$$\begin{cases} v_t = \varepsilon x v_{xx} + v v_x \\ v(x_0,t) = K \qquad v(\infty,t) = 0 \end{cases}$$

Next we look for such a function $\hat{v}_k$, also requiring that

$$\hat{v}_k(x,t) = \hat{f}_k(x/(t+1)).$$

Setting

$$\eta = x/(t+1)$$

one can easily derive that $\hat{f}_k$ should be an upper solution for the boundary value problem

$$\pi \begin{cases} \varepsilon\eta f'' + (f+\eta)f' = 0 \\ f(x_0) = K \quad f(\infty) = 0. \end{cases}$$

Let $x_0 > \max(\varepsilon, g^{-1}(K))$ and take

$$\hat{f}_k(\eta) = k + (K-k)(\eta/x_0)^{1-x_0/\varepsilon}.$$

One can check that indeed $\hat{f}_k$ is an upper solution for problem π and consequently that $\hat{u}_k(x,t) = K - \hat{f}_k(x/(t+1))$ is a lower solution for Problem P on the sector $\{t \geq 0,\ x \geq x_0(t+1)\}$ provided that x_0 is large enough. Since k can be chosen arbitrarily in $(0,K)$ it follows that $u(\infty,t) = K$ for all $t < \infty$. □

4.2. <u>The limiting behaviour as $\varepsilon \downarrow 0$.</u>

In this section we study the limiting behaviour of the solution u of P as $\varepsilon \downarrow 0$. To begin with we consider the following hyperbolic problem

$$H \begin{cases} u_t = (g(x)-u)u_x & \text{in } D \\ u(x,0) = \psi(x) & \text{for all } x \in (0,\infty) \end{cases}$$

and make some heuristic considerations about the solution $\bar{u}$ of Problem H; they are due to WILDERS [23]. One possible configuration of g and ψ is drawn in Figure 1; the corresponding characteristics are represented in Figure 2.

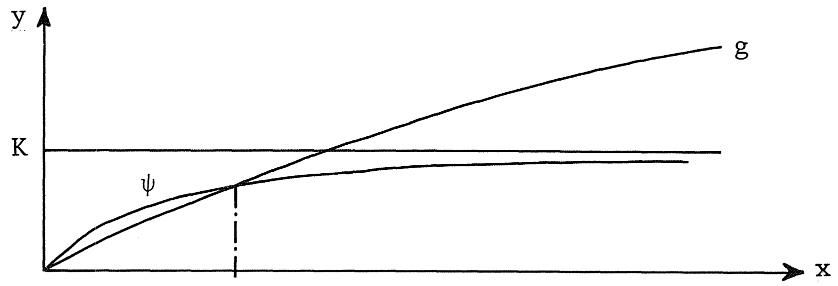

Fig. 1

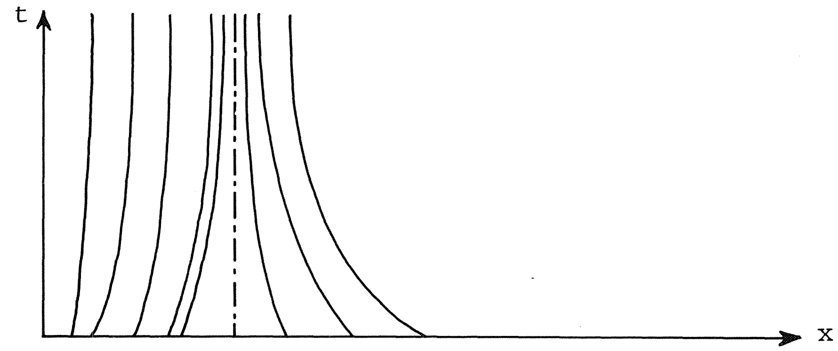

Fig. 2

Their equations are

$$\frac{dx}{dt} = -(g(x)-\psi(x(0))).$$

Along those characteristics $\bar{u}$ is constant, i.e. $\bar{u} = \psi(x(0))$. Also since $\psi(0) = 0$ it follows that the line $x = 0$ is the characteristic passing through the point $(0,0)$ and consequently that $\bar{u}$ automatically satisfies a boundary condition of the form $\bar{u}(0,t) = 0$. Next we deduce from the fact that ψ is nondecreasing that two characteristics do not intersect. Suppose that there exist two characteristics, issuing from the points $x = a$ and $x = b$ $(a < b)$ on the initial line, intersecting each other at the point $(x,t) = (x^*,t^*)$. Then if they would intersect transversally, we would have $-(g(x^*)-\psi(a)) > -(g(x^*)-\psi(b))$ and hence $\psi(a) > \psi(b)$ which is impossible. Now if the characteristics would be tangent to each other at the point (x^*,t^*) we would have $-(g(x^*)-\psi(a)) = -(g(x^*)-\psi(b))$ and consequently $\psi(a) = \psi(b)$; both characteristics would then be described by the same differential equation $\frac{dx}{dt} = -(g(x)-\psi(a))$ which, by the standard uniqueness theorem for ordinary differential equations, implies $a = b$. Finally we conclude that since the initial condition ψ is continuous and nondecreasing, no shock wave can occur and $\overline{u}(\cdot,t)$ is continuous at all times.

In [19] OLEINIK proved existence and uniqueness of the generalized solution of Cauchy problems and boundary value problems related to Problem H; but since the boundary line $x = 0$ is a characteristic for H (which is reflected in the relation $g(0)-\overline{u}(0,0) = 0$), Problem H does not satisfy all the assumptions made in [19]. This leads us to give here a proof of the existence of a solution of Problem H, by showing that the solution u of Problem P tends to a limit as $\varepsilon \downarrow 0$; the uniqueness is a consequence of [19]. Following lemmas 18 and 19 from [19] we say that $\overline{u}$ is a generalized solution of H if it satisfies

(i) $\overline{u}$ is bounded and measurable in $\overline{D}$.

(ii) $\dfrac{\overline{u}(x_1,t) - \overline{u}(x_2,t)}{x_1-x_2} \leq M_\psi$ for all points $(x_1,t),(x_2,t) \in \overline{D}$

(iii) $\overline{u}$ satisfies the identity

$$\iint_D [\overline{u}\,\phi_t - (g-\overline{u}/2)\overline{u}\phi_x - \overline{u}g'\phi]dxdt + \int_0^\infty \psi(x)\phi(x,0)dx = 0 \tag{4.12}$$

for all $\phi \in C^1(\overline{D})$ which vanish for large x and $t = T$.

Next we shall prove the theorem

<u>THEOREM 4.4</u>. *The solution* $u(x,t)$ *of* P *tends uniformly on all compact subdomains of* D *to a limit* $\overline{u}$ *as* $\varepsilon \downarrow 0$, *where* $\overline{u}$ *is the unique generalized solution of* H. *The function* $\overline{u}$ *is furthermore continuous, nondecreasing in* x *at all times* $t \in [0,T]$ *and satisfies the boundary conditions* $\overline{u}(0,t) = 0$ *and* $\overline{u}(\infty,t) = K$.

Before proving theorem 4.4 let us introduce a class of upper and lower solutions for Problem P which depend neither on ε nor on time. They will turn out to be very useful both to prove that $\overline{u}(\infty,t) = K$ in theorem 4.4 and to study the asymptotic behaviour of u as $t \to \infty$ in the next sections. Next we define

$$s^+(x) := \min\,(M_\psi x, K)$$

and

$$s^-(x,\lambda,x_1,\nu) := \max(0,\lambda(1-(x/x_1)^{-\nu}))$$

where the constants $\lambda \in [0,K]$, $\nu > 0$ and $x_1 > 0$ are chosen in the following manner:

(a) if $\varepsilon \le g(\infty)$, we choose $x_1 > 0$ so that

$$g(x_1) > \varepsilon,$$

then $\lambda > 0$ so that

$$\lambda < g(x_1) - \varepsilon$$

and finally $\nu > 0$ so that

$$\nu \le \varepsilon^{-1}(g(x_1) - \lambda) - 1 \tag{4.13}$$

(b) if $\varepsilon \ge g(\infty)$, we set $\lambda = 0$, which amounts to setting $s^- \equiv 0$. It is easily seen that s^- satisfies the inequality

$$\hat{\varepsilon}x(s^-)'' + (g-s^-)(s^-)' \ge 0 \quad \text{for all } x \in [0,\infty)\backslash\{x_1\},\ \hat{\varepsilon} \in (0,\varepsilon).$$

Thus if $\varepsilon < g(\infty)$, given any $\hat{\lambda} < \lambda_0 = \min(g(\infty) - \varepsilon, K)$, one can find $\hat{x}_1$ and $\hat{\nu}$ satisfying (4.13) and such that $s^-(.,\hat{\lambda},\hat{\hat{x}}_1,\hat{\nu}) \le \psi$. Applying the comparison theorem 3.4 we deduce that $s^-(.,\hat{\lambda},\hat{x}_1,\hat{\nu}) \le u$ (and thus that $\lambda_0 \le u(\infty,t)$ for all $t \le \infty$). Similarly one can check that $u \le s^+$.

PROOF of Theorem 4.4. The uniqueness of the solution of Problem H can be proven along the same lines as in the proof of Theorem 1 and Lemma 21 of [19]. Next we show its existence. Fix $I \ge 1$. Since u and u_x are bounded uniformly in ε we deduce from GILDING [12] that u is equicontinuous on $\overline{D}_I$; thus there exists a subsequence $\{u_{\varepsilon_n}\}_{n=I}^{\infty}$ of u and a function $\overline{u}_I \in C(\overline{D}_I)$, such that $u_{\varepsilon_n} \to \overline{u}_I$ as $\varepsilon_n \downarrow 0$ uniformly in $\overline{D}_I$ and such that for all $\lambda < K$, one can find x_1 and ν satisfying (4.13) and $s^-(.,\lambda,x_1,\nu) \le \overline{u}_I(.,t) \le s^+(.)$. Then by a diagonal process, it follows that there exists a bounded continuous function $\overline{u}$ and a converging subsequence denoted by $\{u_{\varepsilon_k}\}$ such that $u_{\varepsilon_k} \to \overline{u}$ as $\varepsilon_k \downarrow 0$, pointwise on D and uniformly on all compact subsets of D. Since $0 \le (u_{\varepsilon_k})_x \le M_\psi$, $\overline{u}$ is nondecreasing in the x-direction and satisfies (ii); $u_{\varepsilon_k}(0) = 0$ implies the same property for $\overline{u}$. The boundary condition $\overline{u}(\infty,t) = K$ follows from the inequalities $s^-(.,\lambda,x_1,\nu) \le \overline{u}(.,t) \le s^+(.)$ for all $\lambda < K$.

It remains to show that $\overline{u}$ is a generalized solution of H. Let $\phi \in C^1(\overline{D})$ vanish for large x and $t = T$ and let $L \ge 1$ be such that ϕ vanishes in the neighbourhood of $x = L$ and for $x > L$. Because the functions u_{ε_k} are classical solutions of P, we have

$$\iint_{D_L} [u_{\varepsilon_k}\phi_t - \varepsilon_k(xu_{\varepsilon_k x} - u_{\varepsilon_k})\phi_x - (g - u_{\varepsilon_k}/2)u_{\varepsilon_k}\phi_x - u_{\varepsilon_k}g'\phi]dxdt$$
$$+ \int_0^L \psi(x)\phi(x,0)dx = 0$$

Now letting $\varepsilon_k \downarrow 0$, we deduce that $\overline{u}$ satisfies (4.12); because ϕ has been chosen arbitrarily we conclude that $\overline{u}$ is indeed the generalized solution of H and that $\{u_\varepsilon\}$ converges to $\overline{u}$ as $\varepsilon \downarrow 0$. □

5. ASYMPTOTIC STABILITY OF THE STEADY STATE

Adapting a method due to ARONSON & WEINBERGER [2] we investigate the stability of the solution $\emptyset$ of Problem P_0. To that purpose we consider the solution u of the corresponding evolution problem P; since its dependence on ψ plays a central rôle in what follows, we denote this solution by $u(x,t,\psi)$. We show that for all the functions ψ satisfying the hypothesis H_ψ given in the introduction we have that

$$u(x,t,\psi) \to \emptyset(x) \qquad \text{as } t \to \infty.$$

To begin with we prove two auxiliary lemmas.

<u>LEMMA 5.1.</u>

(i) *Let* $\varepsilon < g(\infty)$ *and* $\hat{\lambda},\hat{x}_1,\hat{\nu}$ *satisfy* (4.13). *The function* $u(x,t,s^-(.,\hat{\lambda},\hat{x}_1,\hat{\nu}))$ *is nondecreasing in time and such that*

$$\lim_{t\to\infty} u(x,t,s^-(.,\hat{\lambda},\hat{x}_1,\hat{\nu})) = \phi_{\hat{\lambda}}(x) \tag{5.1}$$

where $\phi_{\hat\lambda}$ *is the unique solution of*

$$(5.2)\qquad \begin{cases} \varepsilon x\phi'' + (g(x)-\phi)\phi' = 0 \\ \phi(0) = 0 \qquad \phi(\infty) = \hat\lambda \end{cases}$$

(ii) *The function* $u(x,t,s^+)$ *is nonincreasing in time. Furthermore*

$$(5.3)\qquad \lim_{t\to\infty} u(x,t,s^+) = \emptyset.$$

PROOF. First note that it follows from the proofs in section 4 that Problem P with initial value $s^-(x,\hat\lambda,\hat x_1,\hat\nu)$ has a unique classical solution $u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu))$ with $u(\infty,t) = \hat\lambda$ for all $t \le \infty$. Applying repeatedly theorem 3.4 one can show that $u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu))$ is nondecreasing in time and that $u(x,t,s^+)$ is nonincreasing in time; it also follows from theorem 3.4 that

$$u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu)) \le \phi_{\hat\lambda}(x)$$

and that

$$u(x,t,s^+) \ge \emptyset(x).$$

Now for each x, $u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu))$ is nondecreasing in t and bounded from above. Therefore it has a limit $\tau^-(x)$ as $t \to \infty$ and one can use standard arguments (see for example ARONSON & WEINBERGER [2]) to show that $\tau^- \in C_{2+\alpha}((0,\infty)) \cap C([0,\infty))$ and satisfies the differential equation in (5.2) and the boundary conditions $\tau^-(0) = 0$ and $\tau(\infty) = \hat\lambda$. Finally since $\phi_{\hat\lambda}$ is the unique solution of Problem (5.2) we have that $\tau^- = \phi_{\hat\lambda}$. Similarly one can show that $u(x,t,s^+)$ converges to a function $\tau^+ \in C_{2+\alpha}((0,\infty)) \cap C([0,\infty))$ which satisfies the steady state equation, the boundary condition $\tau^+(0) = 0$ and the condition $\emptyset(\infty) \le \tau^+(\infty) \le K$. The fact that $\tau^+(\infty) = \emptyset(\infty)$ follows from [6, Lemma 5.1]. Consequently $\tau^+ = \emptyset$.

LEMMA 5.2. $\phi_{\hat\lambda}$ *is an increasing and continuous function of* $\hat\lambda$. *More precisely if* $\hat\lambda_1 \ge \hat\lambda_2$ *we have*

$$0 \le \phi_{\hat\lambda_1} - \phi_{\hat\lambda_2} \le \hat\lambda_1 - \hat\lambda_2.$$

PROOF. *Let* $m = \phi_{\hat\lambda_1} - \phi_{\hat\lambda_2}$. It satisfies the differential equation

$$\varepsilon x m'' + (g-\phi_{\hat\lambda_1})m' - \phi'_{\hat\lambda_2} m = 0$$

and the boundary conditions $m(0) = 0$ and $m(\infty) = \hat\lambda_1 - \hat\lambda_2 \geq 0$. Suppose that m attains a negative minimum at a certain point $\xi \in (0,\infty)$; then $m(\xi) < 0$, $m'(\xi) = 0$ and $m''(\xi) \geq 0$ which is in contradiction with $\varepsilon\xi m''(\xi) = \phi'_{\hat\lambda_2}(\xi)m(\xi)$. Thus $m \geq 0$. In the same way one can show that m cannot attain a positive maximum which implies $m \leq \hat\lambda_1 - \hat\lambda_2$.

Finally we are in a position to prove the following theorem

THEOREM 5.3. *Let* $\emptyset(x)$ *be the solution of* Problem P_0. *Suppose* ψ *satisfies the hypothesis* H_ψ, *then for each* $x \geq 0$

$$\lim_{t\to\infty} u(x,t,\psi) = \emptyset(x).$$

If $\varepsilon \leq g(\infty) - K$, *the convergence is uniform on* $[0,\infty)$; if $\varepsilon > g(\infty) - K$, *it is uniform on all compact intervals of* $[0,\infty)$.

PROOF. Since the functions u and u_x are bounded uniformly in t, we apply the Arzela-Ascoli theorem and a diagonal process to deduce that there exists a function $\tau \in C([0,\infty))$ and a sequence $\{u(t_n)\}$ with $u(t_n) = u(.,t_n,\psi)$ such that $u(t_n) \to \tau$ as $t_n \to \infty$, uniformly on all compact subsets of $[0,\infty)$. Let $\varepsilon < g(\infty)$; then for each $\hat\lambda < \lambda_0 = \min(g(\infty)-\varepsilon,K)$ one can find $\hat\nu$ and $\hat x_1$ satisfying (4.13) and such that $s^-(.,\hat\lambda,\hat x_1,\hat\nu) \leq \psi$. Applying Theorem 3.4 we obtain

$$u(x,t,s^-(.,\hat\lambda,\hat x_1,\hat\nu)) \leq u(x,t,\psi) \leq u(x,t,s^+). \tag{5.4}$$

Letting $t \to \infty$ in (5.4) and applying lemma 5.1 we obtain

$$\phi_{\hat\lambda} \leq \tau \leq \emptyset \qquad \text{for all } \hat\lambda < \lambda_0.$$

Next we deduce from lemma 5.2 that

$$\emptyset - \tau \leq \lambda_0 - \hat\lambda \qquad \text{for all } \hat\lambda < \lambda_0$$

and thus that $\tau = \emptyset$. If $\varepsilon \geq g(\infty)$ then the inequalities

$$0 \leq u(x,t,\psi) \leq u(x,t,s^{+})$$

imply

$$0 \leq \tau \leq \emptyset = 0.$$

Thus also in this case we have that $\tau = \emptyset$. Finally we conclude that as $t \to \infty$, $u(.,t,\psi)$ converges to $\emptyset$, uniformly on all compact intervals of $[0,\infty)$. This convergence result can be made slightly stronger in the case that $\varepsilon \leq g(\infty) - K$: since then $\emptyset(\infty) = K$ and since u is nondecreasing in x one can apply Lemma 2.4 of DIEKMANN [5] to deduce that the convergence is uniform on $[0,\infty)$. □

6. RATE OF CONVERGENCE OF THE SOLUTION TOWARDS THE STEADY STATE

In this section we analyse the rate of convergence of the solution u of P towards its steady state $\emptyset$. The results which we are able to derive depend strongly on the behaviour of g as $x \to \infty$. If g tends to infinity fast enough, we can prove exponential convergence with a certain weighted norm. In the more general case when $\varepsilon < g(\infty) - K$ we find that the solution converges algebraically fast towards its steady state on all finite x-intervals. No results are available in the case $\varepsilon \geq g(\infty) - K$, which coincides with the physical situation when some (or all the) electrons escape to infinity.

We write

$$u(x,t,\psi) = \emptyset(x) + v(x,t).$$

Then v satisfies the problem

$$(6.1) \qquad \begin{cases} v_t = \varepsilon x v_{xx} + (g-\emptyset)v_x - \emptyset' v - v v_x \\ v(0,t) = 0 \\ v(x,0) = \psi(x) - \emptyset(x). \end{cases}$$

Now let us make the change of function

$$v(x,t) = \exp\left(-\int_0^x \frac{g(\zeta)-\emptyset(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\tilde{v}(x,t).$$

Problem (6.1) becomes

$$(6.2)\qquad \begin{cases} \tilde{v}_t = \varepsilon x\tilde{v}_{xx} - q(x)\tilde{v} + h(x,\tilde{v},\tilde{v}_x) \\ \tilde{v}(0,t) = 0 \\ \tilde{v}(x,0) = \exp\left(\int_0^x \frac{g(\zeta)-\emptyset(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)(\psi(x)-\emptyset(x)) \end{cases}$$

where

$$q(x) = \frac{(g(x)-\emptyset(x))^2}{4\varepsilon x} + \frac{g'(x)+\emptyset'(x)}{2} - \frac{g(x)-\emptyset(x)}{2x}$$

and

$$h(x,\tilde{v},\tilde{v}_x) = -\exp\left(-\int_0^x \frac{g(\zeta)-\emptyset(\zeta)}{2\varepsilon\zeta}\,d\zeta\right)\tilde{v}\left(\tilde{v}_x - \frac{g(x)-\emptyset(x)}{2\varepsilon x}\,\tilde{v}\right).$$

In particular, there exists $M > 0$ such that

$$|h(x,\tilde{v},\tilde{v}_x)| \le M(\|\tilde{v}\|^2 + \|\tilde{v}_x\|^2) \qquad 0 < x < \infty.$$

where the notation $\|\cdot\|$ indicates the sup-norm.

In what follows we shall distinguish two cases: (i) the case when $\liminf_{x\to\infty} q(x) = \delta > 0$: this is so if $g(x) \ge C_0\sqrt{x}$ for all $x \ge x_2$ for some positive constants C_0 and x_2; (ii) the case when $\liminf_{x\to\infty} q(x) = 0$.

6.1. Case when g tends to infinity at least at fast as $\sqrt{x}$ for $x \to \infty$

The theorem we give next is very similar in its form and in its proof to a theorem of FIFE & PELETIER [10].

THEOREM 6.1. *Suppose that there exist constants* $x_2, C_0 > 0$ *such that*

$$(6.3)\qquad g(x) \ge C_0\sqrt{x} \qquad \textit{for all } x \ge x_2,$$

then there exist positive constants δ, μ, C *such that if*

$$\left\| \exp\left(\int_0^{\cdot} \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta} \, d\zeta \right) (\psi - \phi) \right\| \le \delta$$

then

$$\left\| \exp\left(\int_0^{\cdot} \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta} \, d\zeta \right) (u(.,t,\psi) - \phi) \right\| \le Ce^{-\mu t} \qquad t \ge 0$$

where the notation $\|.\|$ *indicates the sup-norm.*

<u>PROOF</u>. To begin with we note that with the hypothesis of Theorem 6.1 we have that $v(\infty,t) = 0$ (since $\varepsilon < g(\infty) - K$) or equivalently

$$\lim_{x \to \infty} \exp\left(- \int_0^x \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta} \, d\zeta \right) \tilde{v}(x,t) = 0.$$

Next let us consider the boundary value problem

$$(6.4) \qquad \begin{aligned} &\varepsilon x w'' - (q(x) + \lambda) w = -\theta(\phi'(R) + \lambda) \min(\tilde{\phi}(x), (x/R)^{-\nu_0} \tilde{\phi}(R)) \\ &w(0) = 0 \end{aligned}$$

where

$$\tilde{\phi}(x) = \exp\left(\int_0^x \frac{g(\zeta) - \phi(\zeta)}{2\varepsilon\zeta} \, d\zeta \right) \phi(x).$$

The right hand side of the differential equation in (6.4) has been chosen in a special manner so that one can exhibit upper and lower solutions for a problem closely related to (6.4); more precisely we shall prove in the appendix that this problem has at least one solution $w \in C^2([0,\infty))$ with w, w' and w'' bounded such that

$$0 < w(x) \le \min(\tilde{\phi}(x), (x/R)^{-\nu_0} \tilde{\phi}(R))$$

for all constants $\nu_0 > 1$ provided that the constants $\theta \in (0,1)$, $R > 0$ and $\lambda < 0$ satisfy certain conditions. We adjust θ such that $\|w\| + \|w'\| \le 1$.

We are now in a position to prove theorem 6.1. Let

$$z(x,t) = \beta(w(x) + \gamma)e^{-\mu t},$$

in which β, γ and μ are positive constants still to be determined, and let

$$\mathcal{M}z = \varepsilon x z_{xx} - q(x)z + h(x,z,z_x) - z_t.$$

(i) The function q is positive for x near zero and, because of condition (6.3), also for large x; thus there exists $\overline{q}_0 > 0$ and $\zeta_1, \zeta_2 \in (0,\infty)$ such that $\overline{q}_0 = \min\{q(x) : x \in [0,\zeta_1] \cup [\zeta_2,\infty)\}$ is positive; therefore

$$\mathcal{M}z \le \beta e^{-\mu t}((\lambda+\mu)w + \gamma(-\overline{q}_0+\mu) + M\beta(1+\gamma)^2)$$

Choose

$$0 < \mu < \min(-\lambda, \overline{q}_0),$$

assume that γ is known (we shall specify it later) and choose

$$\beta = \frac{\gamma(\overline{q}_0-\mu)}{M(1+\gamma)^2}.$$

Then $\mathcal{M}z \le 0$ for all $x \in [0,\zeta_1] \cup [\zeta_2,\infty)$ and $t \ge 0$.

(ii) Let $\zeta_1 \le x \le \zeta_2$; since $w(x) > 0$ on $(0,\infty)$ and since w is continuous we have

$$m = \min\{w(x) : \zeta_1 \le x \le \zeta_2\} > 0.$$

Therefore

$$\mathcal{M}z \le \beta e^{-\mu t}((\lambda+\mu)m + \gamma(-\overline{q}+\mu) + M\beta(1+\gamma)^2)$$

where $\overline{q}$ is an arbitrary constant such that

$$\overline{q} < \min\{q(x) : x \in [0,\infty)\}.$$

Hence

$$\mathcal{M}z \le \beta e^{-\mu t}((\lambda+\mu)m + \gamma(-\overline{q}+\overline{q}_0)).$$

Therefore if we choose

$$\gamma = -\frac{\lambda+\mu}{-\overline{q}+\overline{q}_0}\, m$$

we have

$$\mathcal{M}z \le 0 \qquad \text{for } \zeta_1 \le x \le \zeta_2 \text{ and } t \ge 0.$$

Thus for the above choice of β, γ and μ the function z is an upper solution of the equation $M\tilde{v} = 0$. Let

$$\sup_{[0,\infty)} \tilde{v}(x,0) \leq \delta$$

where $\delta = \beta\gamma$. Then

$$\tilde{v}(x,0) \leq z(x,0) \qquad \text{for all } x \in [0,\infty)$$

and hence by theorem 3.4

$$\tilde{v}(x,t) \leq z(x,t) \qquad \text{for all } x \in [0,\infty),\ t \geq 0.$$

In a similar manner one can show that if

$$\inf_{[0,\infty)} \tilde{v}(x,0) \geq -\delta$$

then

$$\tilde{v}(x,t) \geq -z(x,t) \qquad \text{for all } x \in [0,\infty),\ t \geq 0.$$

Hence if

$$\|\tilde{v}(.,0)\| \leq \delta$$

then

$$\|\tilde{v}(.,t)\| \leq Ce^{-\mu t}$$

where we define

$$C = \beta(1+\gamma) = (1+1/\gamma)\delta. \quad \square$$

6.2. Algebraic decay rate in the case that $\varepsilon < g(\infty) - K$

Provided that $\varepsilon < g(\infty) - K$ and that the initial function ψ converges algebraically fast to K as $x \to \infty$, we prove that the solution u of P converges algebraically fast to the steady state solution $\emptyset$ for all finite values of x. To that purpose we show that a certain weighted space integral of the function $|u-\emptyset|^p$, for some integer $p \geq 1$, decays algebraically in time; a similar proof, with exponent $p = 1$, has been given for example by van DUYN & PELETIER [9].

THEOREM 6.2. *Provided that* $\varepsilon < g(\infty) - K$ *and that* $\psi \geq s^-(.,K,\overline{x}_1,\overline{\nu})$ *for some* $\overline{x}_1,\overline{\nu}$ *satisfying* (4.13) *with* $\lambda = K$, *we have that*

$$(6.5)\qquad \int_0^\infty (g'(x) + (p-1)\ \emptyset'(x))\,|u(x,t,\psi) - \emptyset(x)|^p\,dx \leq \left[\int_0^\infty ((s^+-\emptyset)^p + (\emptyset - s^-)^p)\,dx\right]/t$$

for all $t > 0$ *and* $p = [1/\overline{\nu}] + 1$.

PROOF. Since $|v(x,t)|^p \leq (s^+(x) - s^-(x,K,\overline{x}_1,\overline{\nu}))^p$, it follows that $\int_0^\infty (v(x,t))^p dx$ is defined for all $t \geq 0$. If $p \geq 2$ let us multiply the differential equation in (6.1) by v^{p-1} and integrate with respect to x; we obtain

$$\frac{d}{dt}\int_0^\infty \frac{v^p}{p}\,dx = \left[\varepsilon x v_x\, v^{p-1}\right]_0^\infty - \left[\varepsilon\,\frac{v^p}{p}\right]_0^\infty - \varepsilon(p-1)\int_0^\infty x v^{p-2}(v_x)^2\,dx + \left[g\,\frac{v^p}{p}\right]_0^\infty - \int_0^\infty (g' + \emptyset'(p-1))\frac{v^p}{p}\,dx - \left[\emptyset\,\frac{v^p}{p} + \frac{v^{p+1}}{p+1}\right]_0^\infty.$$

Since v tends to zero at least as fast as $x^{-\overline{\nu}}$ as $x \to \infty$, the equation above can be written in the simpler form

$$(6.6)\qquad \frac{d}{dt}\int_0^\infty \frac{v^p}{p}\,dx = \left[\varepsilon x v_x v^{p-1}\right]_0^\infty - \varepsilon(p-1)\int_0^\infty x v^{p-2}(v_x)^2 dx - \int_0^\infty (g' + \emptyset'(p-1))\frac{v^p}{p}\,dx.$$

Now let us define the functions v^+ and v^- as the solutions of (6.1) with initial values $v^+(x,0) = s^+(x) - \emptyset(x)$ and $v^-(x,0) = s^-(x,K,\overline{x}_1,\overline{\nu}) - \emptyset(x)$ respectively. By Theorem 3.4 we know that $v^+ \geq 0$ and $v^- \leq 0$. Furthermore it follows from Lemma 5.1 that v^+ is nonincreasing in time and v^- nondecreasing. Of course both v^+ and v^- satisfy (6.6) and in order to simplify this expression we use the following lemma which we shall prove later.

LEMMA 6.3. *Let* $\varepsilon < g(\infty) - K$. *Then* $\lim_{x\to\infty} x\emptyset'(x) = 0$. *If furthermore* $\psi \geq s^-(.,K,\overline{x}_1,\overline{\nu})$ *for some* $\overline{x}_1,\overline{\nu}$ *satisfying* (4.13) *with* $\lambda = K$ (*we suppose furthermore that* $\overline{\nu} > 1$ *if* $\varepsilon < (g(\infty) - K)/2$) *and* $\psi \in C_{1,\alpha}([x_3,\infty))$ *for* *some* $\alpha, x_3 > 0$, *then* $\lim_{x\to\infty} xu_x(x,t) = 0$ *for all* $t \in (0,\infty)$.

From Lemma 6.3 and formula (6.6) we deduce that v^+ satisfies

$$\frac{d}{dt}\int_0^\infty \frac{(v^+)^p}{p}\,dx = -\varepsilon(p-1)\int_0^\infty x(v^+)^{p-2}\,(v_x^+)^2 dx - \int_0^\infty (g' + \emptyset'(p-1))\frac{(v^+)^p}{p}dx.$$

If $p = 1$ similar calculations yield

$$\frac{d}{dt}\int_0^\infty v^+\,dx = -\int_0^\infty g'v^+\,dx.$$

Since $0 < g'(x) < g'(0)$ and $0 < \emptyset'(x) < \emptyset'(0)$ we have for all $p \geq 1$

$$\int_0^\infty (v^+(x,t))^p\,dx \geq \frac{1}{g'(0) + (p-1)\emptyset'(0)}\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(v^+(x,t))^p\,dx$$

and thus

$$\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(v^+(x,t))^p\,dx \leq (g'(0)+(p-1)\emptyset'(0))\int_0^\infty (v^+(x,0))^p dx$$
$$- (g'(0) + (p-1)\emptyset'(0))\int_0^t d\tau\int_0^\infty (g'(x)+(p-1)\emptyset'(x))(v^+(x,\tau))^p dx.$$

In what follows we apply the following lemma that we shall prove later.

<u>LEMMA 6.4</u>. *Let* $y \in C([0,\infty))$ *with* $y' \in L^1((0,\infty))$ *and* $y' \leq 0$ *such that*

(6.7) $\qquad 0 \leq y(t) \leq N - M\int_0^t y(\tau)d\tau$

for some constants $N \geq 0$, $M > 0$. *Then*

(6.8) $\qquad y(t) \leq N/(Mt)$.

Since the function $\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(v^+(x,t))^p\,dx$ is continuous and nonincreasing (because v^+ is nonincreasing), we deduce from Lemma 6.4 that

$$\int_0^\infty (g'(x) + (p-1)\emptyset'(x))(v^+(x,t))^p dx \le \left(\int_0^\infty (v^+(x,0))^p dx \right) / t.$$

Similarly one can show that

$$\int_0^\infty (g'(x) + (p-1)\emptyset'(x)(-v^-(x,t))^p dx \le \left(\int_0^\infty (-v^-(x,0))^p dx \right) / t.$$

Formula (6.5) is then deduced from the fact that

$$|v(x,t)|^p \le \max((v^+(x,t))^p, (-v^-(x,t))^p) \le (v^+(x,t))^p + (-v^-(x,t))^p. \quad \square$$

PROOF of Lemma 6.3. We first show that $\lim_{x\to\infty} x\emptyset'(x) = 0$. Since

$$\varepsilon x\emptyset'(x) = \varepsilon\emptyset(x) - \int_0^x (g(\zeta) - \emptyset(\zeta))\emptyset'(\zeta)d\zeta \le \varepsilon K,$$

we have

$$0 \le x\emptyset'(x) \le K.$$

Furthermore

$$(x\emptyset')' = x\emptyset'' + \emptyset' = -\frac{g-\emptyset-\varepsilon}{\varepsilon}\emptyset' \le 0 \qquad \text{for } x \text{ large enough.}$$

Since the function $x\emptyset'$ is bounded and decreasing for large x, we deduce that there exists $E \in [0,K]$ such that

$$\lim_{x\to\infty} x\emptyset'(x) = E$$

which implies

$$\emptyset(x) \sim E \ln x + C \qquad \text{as } x \to \infty.$$

Since

$$\lim_{x\to\infty} \emptyset(x) = K$$

we deduce that $E = 0$.

Next we show that $\lim_{x\to\infty} xu_x = 0$ by making use of Bernstein's argument, in a similar way as in ARONSON [1] and PELETIER & SERRIN [21].

Let

$$R_n = (n/2,\ 3n/2) \times (0,T],\ n > 3x_3$$

and let

$$\phi(r) = Nr(4-r)/3$$

where $N = \sup_{R_n} u - \inf_{R_n} u$. The function ϕ increases from 0 to N as r increases from 0 to 1. Note that $\phi'(r) = 2N(2-r)/3 > 0$ and $\phi''(r) = -2N/3 < 0$ and define a new function w such that

$$u = \inf_{R_n} u + \phi(w).$$

Then w satisfies the differential equation

$$w_t = \varepsilon x w_{xx} + \varepsilon x \frac{\phi''(w)}{\phi'(w)} (w_x)^2 + (g - \phi(w) - \inf_{R_n} u) w_x.$$

Set $p = w_x$ and differentiate the last equation with respect to x; we get

$$p_t = \varepsilon x p_{xx} + \varepsilon p_x + \varepsilon \frac{\phi''}{\phi'} p^2 + 2\varepsilon x \frac{\phi''}{\phi'} p p_x + \varepsilon x \left(\frac{\phi''}{\phi'}\right)' p^3$$

$$+ (g - \phi - \inf_{R_n} u) p_x + (g' - \phi' p) p$$

and thus

$$\tfrac{1}{2}(p^2)_t - \varepsilon x p p_{xx} = \varepsilon x \left(\frac{\phi''}{\phi'}\right)' p^4 + \varepsilon \left(\frac{\phi''}{\phi'} - \phi'\right) p^3 \tag{6.9}$$

$$+ 2\varepsilon x \frac{\phi''}{\phi'} p^2 p_x + (g - \phi - \inf_{R_n} u + \varepsilon) p p_x + g' p^2.$$

Let $R_n^* = (3n/4,\ 5n/4) \times (0,T]$ and let $\zeta = 1 - 4(x-n)^2/n^2$. Set $z = \zeta^2 p^2$.

(i) If z attains its maximum value at the lower boundary of R_n we have

$$\sup_{R_n^*} z \le z(\tilde{x}, 0) \qquad \text{where } \tilde{x} \in [n/2, 3n/2].$$

Hence

$$\sup_{R_n^*} \zeta |w_x| \leq \zeta(\tilde{x}) |w_x(\tilde{x},0)|.$$

Since $\zeta \geq 3/4$ in $(3n/4, 5n/4)$ and since $u_x = \phi'(w)w_x$ we find

$$\sup_{R_n^*} |u_x| \leq \frac{4}{3} \frac{\sup \phi'}{\inf \phi'} |\psi'(\tilde{x})| \leq 8\, M_\psi/3.$$

(ii) If z attains its maximum value at an interior point $(\tilde{x},\tilde{t})$ of R_n we have at that point

$$(6.10) \qquad \begin{cases} z_x = 2\zeta\zeta' p^2 + 2\zeta^2 p p_x = 0 \\ \varepsilon x z_{xx} - z_t \leq 0. \end{cases}$$

The last inequality can be cast in the more explicit form

$$\zeta^2(\tfrac{1}{2}(p^2)_t - \varepsilon x p p_{xx}) \geq \varepsilon x(\zeta'^2 p^2 + \zeta\zeta'' p^2 + 4\zeta\zeta' p p_x + \zeta^2 p_x^2).$$

Using (6.9), (6.10) and the inequality

$$|4\zeta\zeta' p p_x| \leq \zeta^2 p_x^2 + 4\zeta'^2 p^2$$

we obtain

$$-\zeta^2 \varepsilon \left(\frac{\phi''}{\phi'}\right)' p^4 \leq \left(-2\varepsilon\zeta\zeta' \frac{\phi''}{\phi'} + \varepsilon \frac{\zeta^2}{x} \frac{\phi''}{\phi'} - \frac{\zeta^2}{x} \phi'\right) p^3$$

$$+ \left(\zeta^2 \frac{g'}{x} + 3\varepsilon\zeta'^2 - \varepsilon\zeta\zeta'' - \frac{g - \phi - \inf_{R_n} u + \varepsilon}{x} \zeta\zeta'\right) p^2.$$

Since $\left(\frac{\phi''}{\phi'}\right)' \leq -1/4$, this implies

$$2\zeta^2 p^4 \leq C_1 p^2 + \zeta\, C_2\, |p|^3$$

where the C_i's are positive and depend only on N and n. Since

$$\zeta\, C_2\, |p|^3 \le \zeta^2 p^4 + \frac{C_2^{\,2}}{4}\, p^2$$

it follows that

$$z(x,t) \le \max_{\overline{R}_n} (z(x,t)) \le C_1 + \frac{C_2^{\,2}}{4} \equiv C_3.$$

Therefore

$$\max_{R_n^*} |w_x| \le 4\, C_3^{\frac{1}{2}}/3.$$

Finally, $u_x = \phi'(w)w_x$ and $\phi' \le 4\, N/3$ imply that

$$\max_{R_n^*} |u_x| \le 16N\, C_3^{\frac{1}{2}}/9.$$

Note that $N \le \sup_{R_n}(K-s^-(x,K,\overline{x}_1,\overline{\nu}))$ (which behaves as $x^{-\overline{\nu}}$ where $\overline{\nu} > 0$ is furthermore such that $\overline{\nu} > 1$ if $\varepsilon < (g(\infty)-K)/2)$.
Thus

$$(6.11) \qquad \max_{R_n^*} |u_x| \le 16\, C_3^{\frac{1}{2}} \sup_{R_n} (K-s^-(x,K,\overline{x}_1,\overline{\nu}))/9.$$

If $\varepsilon < (g(\infty)-K)/2$, C_3 is bounded uniformly in n and we deduce that xu_x tends to zero as $x \to \infty$. If on the other hand $(g(\infty)-K)/2 \le \varepsilon < g(\infty)-K$, then we only have that $\overline{\nu} > 0$ in (6.11) and $\sup_{R_n} (K-s^-(x,K,\overline{x}_1,\overline{\nu}))$ tends to zero as $x \to \infty$ but then $C_3^{\frac{1}{2}}$ tends to zero as $1/x$ when $x \to \infty$ which also yields the result. □

<u>PROOF of Lemma 6.4</u>. Integrating by parts we get

$$\int_0^t y(\tau)d\tau = ty(t) - \int_0^t \tau y'(\tau)d\tau \ge ty(t).$$

Also we deduce from (6.7) that

$$\int_0^t y(\tau)d\tau \le N/M$$

and thus (6.8) follows. □

Next we deduce from theorem 6.2 that there is also pointwise convergence. More precisely we prove the following theorem.

THEOREM 6.5. *Provided that* $\varepsilon < g(\infty)-K$ *and that* $\psi \geq s^{-}(.,K,\bar{x}_1,\bar{\nu})$ *for some* $\bar{x}_1,\bar{\nu}$ *satisfying* (4.13) *with* $\lambda = K$, *we have that*

$$\| (g'(.) + (p-1)\emptyset'(.))^{1/p} (u(.,t,\psi)-\emptyset)\| \leq C_\varepsilon / t^{\frac{1}{2}p} \qquad \text{for all } t > 0 \tag{6.12}$$

and $p = [1/\bar{\nu}] + 1$, *where*

$$C_\varepsilon = \left[2\left((K^{p-1} p^2 + K^p \frac{p-1}{\varepsilon})(g'(0))^2 + K^p \sup_{x\in[0,\infty)} |g''(x)|\right) \int_0^\infty ((s^+ - \emptyset)^p + (\emptyset - s^-)^p)\,dx\right]^{\frac{1}{2}p}. \tag{6.13}$$

In particular, if $\varepsilon < (g(\infty) - K)/2$ and $\bar{\nu} > 1$, then $p = 1$ and formulas (6.12) and (6.13) simplify as follows

$$\| g'(.)(u(.,t,\psi) - \emptyset)\| \leq C/\sqrt{t} \qquad \text{for all } t > 0 \tag{6.14}$$

where

$$C = \left[2\left((g'(0))^2 + K \sup_{x\in[0,\infty)} |g''(x)|\right) \int_0^\infty (s^+(x) - s^-(x,K,\bar{x}_1,\bar{\nu}))\,dx\right]^{\frac{1}{2}}.$$

PROOF. To prove Theorem 6.5 we need the following auxiliary lemma:

LEMMA 6.6. *Let* ϕ *be defined for* $0 \leq x < \infty$ *and satisfy the conditions*

(i) $\phi(x) \geq 0$ *and* $\phi(0) = 0$;

(ii) ϕ *is Lipschitz continuous with constant* ℓ;

(iii) $\int_0^\infty \phi(x)\,dx \leq N$,

then

$$\sup_{0\leq x<\infty} |\phi(x)| \leq \sqrt{2N\ell}.$$

We omit here the demonstration of this lemma since the main ideas of the proof are given in the proof of Lemma 3 of PELETIER [20].

Now let us apply Lemma 6.6 to the function $(g' + (p-1)\phi')|u-\phi|^p$; it is nonnegative, equal to zero at the origin and its derivative is continuous by parts and bounded by

$$\{(K^{p-1}p^2 + K^p \frac{p-1}{\varepsilon})(g'(0))^2 + K^p \sup_{x\in[0,\infty)} |g''(x)|\}$$

at all points where it is defined. Finally the bound on its integral is given in theorem 6.2. Inequality (6.12) follows. □

6.3. Asymptotic behaviour of the solution $\overline{u}$ of the hyperbolic problem H as $t \to \infty$

THEOREM 6.7. *Let* ψ *satisfy* H_ψ *and be such that* $\psi \geq s^-(.,K,\overline{x}_1,\overline{\nu})$ *for some* $\overline{x}_1 > 0$, $\overline{\nu} > 1$ *satisfying* (4.13) *with* $\lambda = K$ *and define* $\overline{\phi}(x) = \min(g(x),K)$. *Then*

$$\| g'(.)(\overline{u}(.,t,\psi) - \overline{\phi})\| \leq C/\sqrt{t} \qquad \text{for all } t > 0$$

where C *is the constant defined in Theorem* 6.5.

PROOF. Let $\varepsilon \in (0,(g(\infty) - K)/2) \downarrow 0$ in inequality (6.14), note that the constant C does not depend in ε and use the fact that ϕ converges to $\overline{\phi}$ uniformly on $[0,\infty)$ as $\varepsilon \downarrow 0$ (see [6]). □

APPENDIX

In what follows we shall prove the following theorem:

THEOREM A1. *There exists* $\theta \in (0,1)$, $R > 0$ *and* $\lambda < 0$ *such that the Cauchy Dirichlet problem* (6.4) *has at least one solution* $w \in C^2([0,\infty))$ *with* w,w',w'' *bounded and*

$$0 < w(x) \leq \min(\tilde{\phi}(x),\ (x/R)^{-\nu_0}\ \tilde{\phi}(R)) \qquad \text{for all } x \in (0,\infty).$$

PROOF. Let $n \geq 1$ and consider the boundary value problem

$$\text{(A1)} \qquad \varepsilon(x+\tfrac{1}{n})\, w'' - (q_n(x)+\lambda)w = -\theta(\emptyset'(R)+\lambda)\min(\tilde{\emptyset}_n(x), (x/R)^{-\nu_0}\,\tilde{\emptyset}_n(R))$$

$$\text{(A2)} \qquad w(0) = 0$$

where

$$\tilde{\emptyset}_n(x) = \exp\left(\int_0^x \frac{g(\zeta)-\emptyset(\zeta)}{2\varepsilon(\zeta+1/n)}\, d\zeta\right)\emptyset(x)$$

and

$$q_n(x) = \frac{(g(x)-\emptyset(x))^2}{4\varepsilon(x+1/n)} + \frac{g'(x)+\emptyset'(x)}{2} - \frac{g(x)-\emptyset(x)}{2(x+1/n)}$$

$\nu_0 > 1$ is arbitrary and where the constants $\theta \in (0,1)$, $R > 0$ and $\lambda \in (-\emptyset'(R), 0)$ satisfy some additional conditions which will be given later. Obviously zero is a lower solution for the differential equation in (A1). We shall now construct an upper solution. Firstly we deduce from the asymptotic behaviour of g that there exists $R_1 \geq 1$ and $q_0 > 0$ such that $q_n(x) \geq 2q_0$ for $x \geq R_1$. Also if $\lambda > \max(-q_0, -\emptyset'(R))$ and $\theta < (q_0+\lambda)/(\emptyset'(R)+\lambda)$, then the function $(x/R)^{-\nu_0}\,\tilde{\emptyset}_n(R)$ is an upper solution of the differential equation (A1) for $x \geq R := \max(R_1, 2\varepsilon\nu_0(\nu_0+1)/q_0)$. Next we note that $\tilde{\emptyset}_n$ is an upper solution of (A1) on $[0,R]$ and thus that $\min(\tilde{\emptyset}_n(x), (x/R)^{-\nu_0}\,\tilde{\emptyset}_n(R))$ is an upper solution of (A1) on $[0,\infty)$. Finally we conclude that there exists at least one solution $w_n \in C^2([0,\infty))$ of (A1), (A2) [3, Theorem 1.7.1], such that

$$0 \leq w_n(x) \leq \min(\tilde{\emptyset}_n(x),\ (x/R)^{-\nu_0}\,\tilde{\emptyset}_n(R))$$

which, since $\tilde{\emptyset}_n \leq \tilde{\emptyset}$, implies that

$$\text{(A3)} \qquad 0 \leq w_n(x) \leq \min(\tilde{\emptyset}(x),\ (x/R)^{-\nu_0}\,\tilde{\emptyset}(R)).$$

Furthermore the inequalities (A3) and

$$\text{(A4)} \qquad |q_n(x)| \leq \frac{(g-\emptyset)^2}{4\varepsilon x} + \frac{g'+\emptyset'}{2}$$

yield, together with (A1),

$$|w_n''(x)| \le C \qquad \text{for all } x \in [0,\infty)$$

where $C > 0$ is independent of n. Now let us integrate (A1); we get

$$\text{(A5)}\quad w_n'(x) = w_n'(0) + \int_0^x \frac{(q_n(\zeta)+\lambda)w_n(\zeta)-\theta(\emptyset'(R)+\lambda)\min(\tilde{\emptyset}_n(\zeta),(\zeta/R)^{-\nu_0}\,\tilde{\emptyset}_n(R)\,d\zeta}{\varepsilon(\zeta+1/n)}$$

and using again (A3) and (A4) we obtain

$$|w_n'(x)| \le C \qquad \text{for all } x \in [0,\infty).$$

Using the Arzela-Ascoli theorem and a diagonal process, we deduce that there exists a function $w \in C^1([0,\infty))$ and a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to w$ as $n_k \to \infty$, uniformly in $C^1([0,\infty))$ on all compact subsets of $[0,\infty)$. Also setting $n = n_k$ in (A5) and letting $n_k \to \infty$, we deduce that w satisfies the differential equation

$$\text{(A6)}\qquad \varepsilon x w'' - (q(x)+\lambda)w = -\theta(\emptyset'(R)+\lambda)\min(\tilde{\emptyset}(x),(x/R)^{-\nu_0}\,\tilde{\emptyset}(R))$$

and the boundary condition

$$w(0) = 0.$$

It follows from (A6) that $w \in C^2((0,\infty))$ and since

$$\lim_{x\to\infty} w''(x) = [(\emptyset'(0) + \lambda)w'(0) - \theta(\emptyset'(R)+\lambda)\;\tilde{\emptyset}'(0)]/\varepsilon$$

we deduce that in fact $w \in C^2([0,\infty))$. Finally the strict inequality $w > 0$ is proven by means of a maximum principle argument. □

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