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Associated distributions in the analysis of two-dimensional random walks

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During the last ten years much insight has been obtained in the analysis of two-dimensional random walks on the lattice in the first quadrant of $\mathbb{R}_2$, in particular if they are homogeneous. Slight relaxation of the homogeneity condition soon leads to intricate analysis. In the present study it is shown that by using so called associated distributions for the characterization of the transition structure of the random walk a quite general analysis becomes possible for a fairly large class of transition distributions. The present study introduces these associated distributions and exposes their use in the derivation of the bivariate generating function of the stationary distribution of the random walk.

Key Words & Phrases: Random walk, two-dimensional,, nonhomogeneous, hitting point distribution, associated distributions, stationary distribution.

1. Introduction

This study concerns the analysis of two-dimensional random walks with state space S the lattice in the first quadrant of $\mathbb{R}_2$, the coordinate axis forming the boundary B of S. The one-step displacement vectors at the interior points of S, of those at points of the x-axis and at points of the y-axis, except the origin, are identically distributed, respectively, and characterized by the bivariate generating functions $\phi_3(p_1,p_2),\phi_1(p_1,p_2)$ and $\phi_2(p_1,p_2)$, $|p_1|\leqslant 1, |p_2|\leqslant 1$; that at the origin by $\phi_0(p_1,p_2)$; the general structure of the random walk is described in section 2.

In section 3 we study for this random walk the distribution of the hitting point $\mathbf{k}(z_0)\equiv(\mathbf{k}_1(z_0),\mathbf{k}_2(z_0))$, $z_0 = (x_0,y_0)\in S\setminus B$; the hitting point being that point of B at the moment of the first entrance into B when starting at $z_0\in S\setminus B$. Such a hitting point distribution has a remarkable property, viz. every zero tuple $(\hat{p}_1,\hat{p}_2)$ with $|\hat{p}_1|\leqslant 1, |\hat{p}_2|\leqslant 1$ of

$$p_1p_2-\phi_3(p_1,p_2)$$

is also a zero tuple of

$$p_1^{x_0}p_2^{y_0}-\mathrm{E}\{p_1^{\mathbf{k}_1(z_0)}p_2^{\mathbf{k}_2(z_0)}\}.$$

This property leads to the introduction of a class of distributions which are called z_0-*associated* with the distribution of which $\phi_3(p_1,p_2)$ is the bivariate generating function. The concept of associated distributions has been introduced in [2], hitting point distribution have been studied in [6], [7], [8], [9], [10], for a related study see [11]. The use of associated distributions has led to interesting results in the analysis of N-dimensional random walks, see [7].

In section 4 the concept of z_0-associated distributions is introduced and discussed.

The analysis of two-dimensional random walks with state space S is rather complicated, see [1] and [2], in particular if $\phi_1(.,.)$ and $\phi_2(.,.)$ differ from $\phi_3(.,.)$, i.e. if the transition characteristics at interior points of S differ from those at points of its boundary, see also [5]. In sections 5 and 6 it is shown by taking: for $j = 1,2$,

$$\phi_j(p_1,p_2) = \sum_{h=1}^{\infty} c_{jh}\phi_{1h}(p_1,p_2),$$

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$$\sum_{h=1}^{\infty} c_{jh} = 1, \quad c_{jh} \geqslant 0, \quad h = 1,2,...,$$

where ϕ_{1h} is $(1,h)$-associated with ϕ_3, and ϕ_{2h} is $(h, 1)$-associated with ϕ_3, that the analysis of the functional equation for the stationary distribution of the random walk (if it exists) becomes very simple. The results so obtained provide explicit results for a large class of bivariate generating functions ϕ_1, ϕ_2 and ϕ_0. Generalisations of the ideas exposed in this section to higher dimensional random walks seem to be possible.

In section 7 we consider some points concerning the ergodicity properties of the random walk, in the final section 8 some specific examples are discussed. There remain still many points which need further investigation, but the results of the present study indicate clearly that the concept of associated distributions is very useful in the analysis of random walks.

1. Description of the random walk

In this study we consider the two-dimensional process

$$\mathbf{z}_n, \; n = 0,1,2,...,$$

with

$$\mathbf{z}_n \equiv (\mathbf{x}_n, \mathbf{y}_n) \in S := \{0,1,2,...\} \times \{0,1,2,...\}, \tag{2.1}$$

and of which the stochastic structure is described as follows.

Assumption 2.1.

i. for every fixed $k = 0,1,2,3$:
   $(\xi_n^{(k)}, \eta_n^{(k)})$, $n = 0,1,2,...$, is a sequence of i.i.d. stochastic vectors with $(\xi_n^{(k)}, \mathbf{y}_n^{(k)}) \in S$;
ii. the families $\{(\xi_n^{(k)}, \eta_n^{(k)}), k = 0,1,2,3\}$, are independent families.

Assumption 2.2.

For $n = 0,1,2,...$, and for $\mathbf{x}_n > 0, \mathbf{y}_n > 0$:

$$\mathbf{x}_{n+1} = [\mathbf{x}_n - 1]^+ + \xi_n^{(3)},$$

$$\mathbf{y}_{n+1} = [\mathbf{y}_n - 1]^+ + \eta_n^{(3)};$$

for $\mathbf{x}_n = 0, \mathbf{y}_n > 0$:

$$\mathbf{x}_{n+1} = \xi_n^{(2)},$$

$$\mathbf{y}_{n+1} = [y_n - 1]^+ + \eta_n^{(2)};$$

for $\mathbf{x}_n > 0, \mathbf{y}_n = 0$:

$$\mathbf{x}_{n+1} = [\mathbf{x}_n - 1]^+ + \xi_n^{(1)},$$

$$\mathbf{y}_{n+1} = \eta_n^{(1)};$$

for $\mathbf{x}_n = 0, \mathbf{y}_n = 0$:

$$\mathbf{x}_{n+1} = \xi_n^{(0)},$$

$$\mathbf{y}_{n+1} = \eta_n^{(0)}.$$

Further

$$\mathbf{z}_0 \equiv (x_0, y_0) \in S,$$

shall be the starting point of the process defined above.

The assumptions 2.1 and 2.2 imply that the sequence $\mathbf{z}_n, n = 0,1,2,...$, is a discrete time parameter

Markov chain with state space S the lattice points with integer valued coordinates in the first quadrant of $\mathbb{R}_2$.

We introduce the following notation

(ξ_k, η_k) will be a stochastic vector with the same state and distribution as $(\xi_n^{(k)}, \eta_n^{(k)})$, (2.2)

$$\phi_k(p_1,p_2) := \mathrm{E}\{p_1^{\xi_k} p_2^{\eta_k}\}, \ |p_1| \leqslant 1, |p_2| \leqslant 1. \tag{2.3}$$

Concerning the bivariate generating functions $\phi_k(.,.)$ several assumptions will be introduced in this study, in order to restrict the large variety of possible variants. Here we introduce

ASSUMPTION 2.3.

i. $\mathrm{E}\{\xi_3\}<1$, $\mathrm{E}\{\eta_3\}<1$; (2.4)

$0<\mathrm{E}\{\xi_k\}<\infty, 0<\mathrm{E}\{\eta_k\}<\infty$, $k = 0,1,2$;

ii. for $k = 0,1,2,3$,

$|\phi_k(p_1,p_2)| = 1$ for $|p_1| = 1$, $|p_2| = 1 \Rightarrow p_1 = 1, p_2 = 1$;

iii. for every $(i,j) \in S$ the coefficient of $p_1^i p_2^j$ in the series development of $[\phi_k(p_1,p_2)/p_1p_2]^n$, n a positive integer, is positive for n sufficiently large.

REMARK 2.1. The condition (2.4)i implies, cf. ass. 2.2, that the $\mathbf{z}_n$ process has in the interior of S a 'drift' towards the coordinate axes; the condition (2.4)ii shows that the random walk $\mathbf{z}_n$ is aperiodic, and the condition (2.4)iii implies that its state space is irreducible.

For the analysis of the random walk we introduce: for $|r|<1, |p_1| \leqslant 1, |p_2| \leqslant 1$,

$$\Phi_0(r,p_1,p_2) := \sum_{n=0}^{\infty} r^n \mathrm{E}\{p_1^{\mathbf{x}_n} p_2^{\mathbf{y}_n} \mid \mathbf{x}_0 = x_0, \mathbf{y}_0 = y_0\}. \tag{2.5}$$

From the structure of the random walk $\mathbf{z}_n, n = 0,1,2,...,$ described above it is readily derived that for $\Phi_0(r,p_1,p_2)$ holds: for $|r|<1, |p_1|<1, |p_2| \leqslant 1$,

$$\begin{aligned}[]
&[p_1p_2 - r\phi_3]\Phi_0(r,p_1,p_2) = \\
&p_1^{x_0+1} p_2^{y_0+1} + r[p_1p_2\phi_0 + \phi_3 - p_1\phi_2 - p_2\phi_1]\Phi_0(1,0,0) + \\
&r[p_1\phi_2 - \phi_3]\Phi_0(r,0,p_2) + r[p_2\phi_1 - \phi_3]\Phi_0(r,p_1,0),
\end{aligned} \tag{2.6}$$

here we used the abbreviation

$$\phi_k \equiv \phi_k(p_1,p_2), \quad k = 0,1,2,3. \tag{2.7}$$

In sections 5 and 6 we shall analyse this functional relation (2.6) for $\Phi_0(r,p_1,p_2)$; note that the sequence

$$\mathrm{E}\{p_1^{\mathbf{x}_n} p_2^{\mathbf{y}_n} \mid \mathbf{x}_0 = x_0, \mathbf{y}_0 = y_0\} \quad n = 0,1,2,...,$$

is for $|p_1| \leqslant 1, |p_2| \leqslant 1$, recursively and uniquely determined by the structure of the random walk $\mathbf{z}_n, n = 0,1,,2,...,$ and hence $\Phi_0(r,p_1,p_2)$ is uniquely determined by it. It further follows that for fixed r with $|r|<1$,

$\Phi(r,p_1,p_2)$ is for every fixed p_2 with $|p_2| \leqslant 1$ regular in p_1, for $|p_1|<1$ and continuous in p_1 for $|p_1| \leqslant 1$, and similarly with p_1 and p_2 interchanged. (2.8)

3. The hitting point distribution

In the analysis of the functional equation (2.6) we shall use the distribution of the hitting point $\mathbf{k}(z_0)$ of the boundary

$$B := \{0,,1,2,...\}\times\{0\}\cup\{0\}\times\{0,1,2,...\}, \tag{3.1}$$

of the state space S at the moment $\mathbf{n}(z_0)$ of the first entrance into B when starting at $z_0 \in S$. Put for $z_0 \in S$,

$$\mathbf{n}(z_0) := \inf_{n=0,1,2,...} \{n : \mathbf{z}_n \in B \mid z_0 \in S\}, \tag{3.2}$$

$$\mathbf{k}(z_0) \equiv (\mathbf{k}_1(z_0), \mathbf{k}_2(z_0)) := (\mathbf{x}_{\mathbf{n}(z_0)}, \mathbf{y}_{\mathbf{n}(z_0)}). \tag{3.3}$$

We first formulate and prove

LEMMA 3.1. *For* $|r|<1, |p_1|\leqslant 1, |p_2|\leqslant 1$,

$$[p_1p_2 - r\phi_3(p_1,p_2)]\sum_{n=0}^{\infty} r^n \mathrm{E}\{p_1^{\mathbf{x}_n}p_2^{\mathbf{y}_n}(\mathbf{n}(z_0)>n)\} = \tag{3.4}$$

$$p_1p_2[p_1^{x_0}p_2^{y_0} - \mathrm{E}\{r^{\mathbf{n}(z_0)}p_1^{\mathbf{k}_1(z_0)}p_2^{\mathbf{k}_2(z_0)}\}].^*$$

PROOF. Define the sequence $(\mathbf{u}_n,\mathbf{v}_n), n = 0,1,...,$ as follows:

for $(\mathbf{u}_n,\mathbf{v}_n)\in S\setminus B$, (3.5)

$$\mathbf{u}_{n+1} = \mathbf{u}_n - 1 + \xi_n^{(3)},$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n - 1 + \eta_n^{(3)};$$

$$(\mathbf{u}_{n+1},\mathbf{v}_{n+1}) = (\mathbf{u}_n,\mathbf{v}_n) \quad \text{for} \quad (\mathbf{u}_n,\mathbf{v}_n)\in B,$$

$$(\mathbf{u}_0,\mathbf{v}_0) = (x_0,y_0).$$

Hence with

$$\mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}\} = \mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n} \mid \mathbf{u}_0 = x_0, \mathbf{v}_0 = y_0\}, \tag{3.6}$$

we have from (3.5) for $|p_1|\leqslant 1, |p_2|\leqslant 1$, $n = 0,1,2,...,$

$$\mathrm{E}_0\{p_1^{\mathbf{u}_{n+1}}p_2^{\mathbf{v}_{n+1}}\} = \mathrm{E}_0\{p_1^{\mathbf{u}_{n+1}}p_2^{\mathbf{v}_{n+1}}[((\mathbf{u}_n,\mathbf{v}_n)\in B)+((\mathbf{u}_n,\mathbf{v}_n)\notin B)]\} = \tag{3.7}$$

$$\frac{\phi_3(p_1,p_2)}{p_1p_2}\mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}((\mathbf{u}_n,\mathbf{v}_n)\notin B)\} + \mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}((\mathbf{u}_n,\mathbf{v}_n)\in B)\}.$$

Consequently for $|r|<1$, $|p_1|\leqslant 1, |p_2|\leqslant 1$,

$$\sum_{n=0}^{\infty} r^n \mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}\} = p_1^{x_0}p_2^{y_0} + r\sum_{n=0}^{\infty} r^n \mathrm{E}\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}((\mathbf{u}_n,\mathbf{v}_n)\in B)\} + \tag{3.8}$$

$$r\frac{\phi_3(p_1,p_2)}{p_1p_2}\sum_{n=0}^{\infty} r^n \mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}((\mathbf{u}_n,\mathbf{v}_x)\notin B)\}.$$

By writing

$$\sum_{n=0}^{\infty} r^n \mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}\} = \sum_{n=0}^{\infty} r^n \mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}[((\mathbf{u}_n,\mathbf{v}_n \notin B)) + ((\mathbf{u}_n,\mathbf{v}_n)\in B)]\}, \tag{3.9}$$

and by noting, cf. (3.2) and (3.3), that for $n = 0,1,2,...,$

* (A) stands for the indicator function 1_A of the event A.

$$((\mathbf{u}_n,\mathbf{v}_n)\in B) = (\mathbf{n}(z_0)\leqslant n), \tag{3.10}$$
$$(\mathbf{u}_n,\mathbf{v}_n)\notin B) = (\mathbf{n}(z_0)>n),$$

so that

$$\sum_{n=0}^{\infty} r^n \mathrm{E}_0\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}((\mathbf{u}_n,\mathbf{v}_n)\in B)\} = \tag{3.11}$$
$$\frac{1}{1-r}E\{r^{\mathbf{n}(z_0)}p_1^{\mathbf{k}_1(z_0)}p_2^{\mathbf{k}_2(z_0)}\},$$

it results from (3.8),...,(3.11) that

$$[1-r\frac{\phi_3(p_1,p_2)}{p_1p_2}]\sum_{n=0}^{\infty} r^n \mathrm{E}\{p_1^{\mathbf{u}_n}p_2^{\mathbf{v}_n}(\mathbf{n}(z_0)>n)\} = \tag{3.12}$$
$$p_1^{x_0}p_2^{y_0} - \mathrm{E}\{r^{\mathbf{n}(z_0)}p_1^{\mathbf{k}_1(z_0)}p_2^{\mathbf{k}_2(z_0)}\}.$$

Because

$$\mathbf{x}_n = \mathbf{u}_n, \mathbf{y}_n = \mathbf{v}_n \quad \text{for } n<\mathbf{n}(z_0),$$

the proof of the lemma follows. □

THEOREM 3.1. *For* $\mathrm{E}\{\xi_3\}<1, \mathrm{E}\{\eta_3\}<1,\ z_0\in S$,

i. $\Pr\{\mathbf{n}(z_0)<\infty\} = 1,$

$\Pr\{\mathbf{k}_i(z_0)<\infty\} = 1,\ i = 1,2,$

ii. $\mathrm{E}\{\mathbf{n}(z_0)\} = \dfrac{x_0-\mathrm{E}\{\mathbf{k}_1(z_0)\}}{1-\mathrm{E}\{\xi_3\}} = \dfrac{y_0-\mathrm{E}\{\mathbf{k}_2(z_0)\}}{1-\mathrm{E}\{\eta_3\}} < \infty,$

iii. $\mathrm{E}\{\mathbf{n}(z_0)\}\geqslant \min(x_0,y_0).$

PROOF. It is well known, since $\mathrm{E}\{\xi_3\}<1$, that the function

$$p_1 - r\phi_3(p_1,1),\ |p_1|\leqslant 1, \tag{3.13}$$

has for fixed r with $|r|\leqslant 1$ a unique zero $m_{31}(r)$ and that this zero is a generating function of a stochastic variable $\mathbf{m}_{31}$ with state space $\{1,2,...,\}$,

$$m_{31}(r) := \mathrm{E}\{r^{\mathbf{m}_{31}}\},\quad |r|\leqslant 1, \tag{3.14}$$

and that since $\mathrm{E}\{\xi_3\}<1$,

$$\Pr\{\mathbf{m}_{31}<\infty\} = 1,\quad \mathrm{E}\{\mathbf{m}_{31}\} = \frac{1}{1-\mathrm{E}\{\xi_3\}}; \tag{3.15}$$

it is readily shown that $m_{31}(r)$ is univalent for $|r|\leqslant 1$.

Take in (3.4) $p_2 = 1$, it then follows, because

$$\sum_{n=0}^{\infty} r^n \mathrm{E}\{p_1^{\mathbf{x}_n}(\mathbf{n}(z_0)>n)\},\quad |p_1|\leqslant 1,\quad |r|<1,$$

is bounded, that for $|r|<1,\ |p_1|\leqslant 1$,

$$m_{31}^{x_0}(r) = \mathrm{E}\{r^{\mathbf{n}(z_0)}m_{31}^{\mathbf{k}_1(z_0)}(r)(\mathbf{k}_2(z_0)<\infty)\}. \tag{3.16}$$

Since the lefthand side of (3.16) tends to one for $r\uparrow 1$ we obtain from (3.16),

$$1 = \mathrm{E}\{(\mathbf{n}(z_0)<\infty(\mathbf{k}_1(z_0)<\infty)(\mathbf{k}_2(z_0)<\infty)\}, \tag{3.17}$$

and so the first statement follows.

By differentiating (3.16) with respect to r and letting $r \to 1$ we obtain by using (3.15),

$$\infty > \frac{x_0}{1-\mathrm{E}\{\xi_3\}} = \mathrm{E}\{\mathrm{n}(z_0)\} + \frac{\mathrm{E}\{\mathrm{k}_1(z_0)\}}{1-\mathrm{E}\{\xi_3\}},$$

and this result proves the first equality of the second statement, the second one follows by symmetry.

To prove the last statement note that for the random walk z_n with $\mathrm{z}_n \in S \setminus B$ holds that

$$\mathrm{x}_{n+1} - \mathrm{x}_n \geqslant -1, \quad \mathrm{y}_{n+1} - \mathrm{y}_n \geqslant -1,$$

so iii follows directly. □

COROLLARY 3.1. *For* $\mathrm{E}\{\xi_3\} < 1, \mathrm{E}\{\eta_3\} < 1$, $|p_1| \leqslant 1, |p_2| \leqslant 1$,

$$[p_1 p_2 - \phi_3(p_1, p_2)] \sum_{n=0}^{\infty} \mathrm{E}\{p_1^{\mathrm{x}_n} p_2^{\mathrm{y}_n} (\mathrm{n}(z_0) > n)\} = \qquad (3.18)$$

$$p_1 p_2 [p_1^{x_0} p_2^{y_0} - \mathrm{E}\{p_1^{\mathrm{k}_1(z_0)} p_2^{\mathrm{k}_2(z_0)}\}].$$

PROOF. Because $\mathrm{E}\{\mathrm{n}(z_0)\} < \infty$ we have for $|r| \leqslant 1, |p_1| \leqslant 1, |p_2| \leqslant 1$,

$$\left| \sum_{n=0}^{\infty} r^n \mathrm{E}\{p_1^{\mathrm{x}_n} p_2^{\mathrm{y}_n} (\mathrm{n}(z_0) > n)\} \right| \leqslant \sum_{n=0}^{\infty} \mathrm{E}\{(\mathrm{n}(\mathrm{z}_0) > n)\} = \mathrm{E}\{\mathrm{n}(z_0)\} < \infty,$$

and the relation (3.18) follows by taking $r = 1$ in (3.4). □

Obviously

$$\mathrm{E}\{p_1^{\mathrm{k}_1(z_0)} p_2^{\mathrm{k}_2(z_0)}\}, \quad |p_1| \leqslant 1, \quad |p_2| \leqslant 1,$$

represents the bivariate generating function of the hitting point $\mathrm{k}(z_0)$ of the boundary B. Next we derive the hitting point identity.

Denote by $(\hat{p}_1, \hat{p}_2)$ a zero tuple of the kernel

$$Z_3(p_1, p_2) := p_1 p_2 - \phi_3(p_1, p_2), \quad |p_1| \leqslant 1, |p_2| \leqslant 1, \qquad (3.19)$$

i.e.

$$\hat{p}_1 \hat{p}_2 - \phi_3(\hat{p}_1, \hat{p}_2) = 0. \qquad (3.20)$$

For a discussion of the existence of zero tuples, cf. e.q. [1], [2]; note also that by applying Rouché's theorem it is readily seen, since $\mathrm{E}\{\xi_3\} < 1$, that for fixed p_2 with $|p_2| = 1$, $Z_3(p_1, p_2)$ has a unique zero in $|p_1| \leqslant 1$.

THEOREM 3.2. *If* $\mathrm{E}\{\xi_3\} < 1$, $\mathrm{E}\{\eta_3\} < 1$ *then every zero tuple* $(\hat{p}_1, \hat{p}_2)$ *of* $Z_3(p_1, p_2), |p_1| \leqslant 1, |p_2| \leqslant 1$, *is also a zero tuple of*

$$p_1^{x_0} p_2^{y_0} - \mathrm{E}\{p_1^{\mathrm{k}_1(x_0)} p_2^{\mathrm{k}_2(y_0)}\}, |p_1| \leqslant 1, |p_2| \leqslant 1, \ z_0 \in S,$$

i.e. for every zero tuple of $Z_3(p_1, p_2)$:

$$\hat{p}_1^{x_0} \hat{p}_2^{y_0} = \mathrm{E}\{\hat{p}_1^{\mathrm{k}_1(z_0)} \hat{p}_2^{\mathrm{k}_2(z_0)}\}. \qquad (3.21)$$

PROOF. The proof follows directly from (3.18) and (3.20) since for $|p_1| \leqslant 1, |p_2| \leqslant 1$,

$$\left| \sum_{n=0}^{\infty} \mathrm{E}\{p_1^{\mathrm{x}_n} p_2^{\mathrm{y}_n} (\mathrm{n}(z_0) > n)\} \right| < \infty.$$

REMARK 3.1. The relation (3.21) is called the *hitting point identity* at the node z_0.

4. Associated distributions

Let $(\mathbf{u},\mathbf{v})$ and $(\boldsymbol{\mu},\boldsymbol{\nu})$ be both stochastic vectors with state space S and: for $|p_1|\leqslant 1, |p_2|\leqslant 1$,

$$w(p_1,p_2) := \mathrm{E}\{p_1^{\mathbf{u}}p_2^{\mathbf{v}}\},$$

$$\omega(p_1,p_2) := \mathrm{E}\{p_1^{\boldsymbol{\mu}}p_2^{\boldsymbol{\nu}}\}.$$

Definition 4.1. If every zero tuple $(\hat{p}_1,\hat{p}_2)$ of

$$p_1p_2-w(p_1,p_2), |p_1|\leqslant 1, |p_2|\leqslant 1,$$

is a zero tuple of

$$p_1^{x_0}p_2^{y_0}-\omega(p_1,p_2), |p_1|\leqslant 1, |p_2|\leqslant 1, \; z_0\equiv(x_0,y_0)\in S,$$

then (the distribution of) $(\boldsymbol{\mu},\boldsymbol{\nu})$ is said to be z_0*-associated* with (the distribution of) $(\boldsymbol{\mu},\boldsymbol{\nu})$.

Obviously theorem 3.2 implies that the *hitting point* $\mathbf{k}(z_0)\equiv(\mathbf{k}_1(z_0),\mathbf{k}_2(z_0))$ introduced in the preceding section is z_0-associated with (ξ_3,η_3).

Theorem 4.1. *If* $(\boldsymbol{\mu}_k,\boldsymbol{\nu}_k)$ *are for every* $k = 1,2,...;$ z_0*-associated with* $(\mathbf{u},\mathbf{v})$ *then the stochastic vector* $(\boldsymbol{\mu},\boldsymbol{\nu})$ *with state space S and bivariate generating function*

$$\mathrm{E}\{p_1^{\boldsymbol{\mu}}p_2^{\boldsymbol{\nu}}\} := \sum_{k=1}^{\infty} c_k\mathrm{E}\{p_1^{\boldsymbol{\mu}_k}p_2^{\boldsymbol{\nu}_k}\}, \; |p_1|\leqslant 1, |p_2|\leqslant 1,- \tag{4.2}$$

where

$$\sum_{k=1}^{N} c_k = 1, \quad c_k\geqslant 0, \; k = 1,2,..., \tag{4.3}$$

is z_0*-associated with* $(\mathbf{u},\mathbf{v})$.

Proof. From: for $|p_1|\leqslant 1, |p_2|\leqslant 1$,

$$p_1^{x_0}p_2^{y_0}-\mathrm{E}\{p_1^{\boldsymbol{\mu}}p_2^{\boldsymbol{\nu}}\} = \sum_{k=1}^{\infty} c_k[p_1^{x_0}p_2^{y_0}-\mathrm{E}\{p_1^{\boldsymbol{\mu}_k}p_2^{\boldsymbol{\nu}_k}\}],$$

the proof follows immediately. □

Theorem 4.2. *Let* $(\boldsymbol{\mu},\boldsymbol{\nu})$ *be* z_0*-associated with* $(\mathbf{u},\mathbf{v})$ *and*

$$\mathrm{E}\{\mathbf{u}\}<1, \quad \mathrm{E}\{\mathbf{v}\}<1,$$

$$\mathrm{E}\{\boldsymbol{\mu}\}<\infty, \quad \mathrm{E}\{\boldsymbol{\nu}\}<\infty,$$

then with $z_0\equiv(x_0,y_0)$,

$$\frac{x_0-\mathrm{E}\{\boldsymbol{\mu}\}}{y_0-\mathrm{E}\{\boldsymbol{\nu}\}} = \frac{1-\mathrm{E}\{\mathbf{u}\}}{1-\mathrm{E}\{\mathbf{v}\}} = -\frac{d\mathrm{P}_2(p_1)}{dp_1}/_{p_1=1},$$

with $(\hat{p}_1,\hat{p}_2)$ *a zero tuple of*

$$p_1p_2-\mathrm{E}\{p_1^{\mathbf{u}}p_2^{\mathbf{v}}\}, \; |p_1|\leqslant 1, |p_2|\leqslant 1, \tag{4.4}$$

and

$$\hat{p}_2 := P_2(\hat{p}_1), \quad |p_1| = 1.$$

Proof. Because $\mathrm{E}\{\mathbf{u}\}<1$ it is wellknown that (4.4) for $|p_1| = 1$ has a unique zero $p_2 = P_2(p_1)$ in $|p_2|\leqslant 1$. It follows readily that: for $|p_1|\leqslant 1, |p_2|\leqslant 1$,

$$-\frac{dP_2(p_1)}{dp_1}[p_1-\mathrm{E}\{\mathbf{v}p_1^{\mathbf{u}}p_2^{\mathbf{v}-1}\}] = p_2-\mathrm{E}\{\mathbf{u}p_1^{\mathbf{u}-1}p_2^{\mathbf{v}}\}. \tag{4.5}$$

Since E{v}<1 so that $P_2(1) = 1$, and E{u}<1, the second equality in the statement of the theorem follows directly. Further, because (μ,ν) is z_0-associated with (u,v) we have

$$p_1^{x_0}p_2^{y_0}-\mathrm{E}\{p_1^{\mu}p_2^{\nu}\} = 0 \quad \text{for } p_2 = P_2(p_1),\ |p_1|\leqslant 1. \tag{4.6}$$

Hence

$$-\frac{dP_2(p_1)}{dp_1}[y_0p_1^{x_0}p_2^{y_0-1}-\mathrm{E}\{\nu p_1^{\mu}p_2^{\nu-1}\}] = x_0p_1^{x_0-1}p_2^{y_0}-\mathrm{E}\{\mu p_1^{\mu-1}p_2^{\nu}\}, \tag{4.7}$$

and by taking $p_1 = 1$ the proof of the theorem follows.

REMARK 4.1. From theorem 3.2 it is readily seen that the hitting point $\mathbf{k}(z_0)$ is z_0-associated with (ξ_3,η_3). For the construction of other associated distributions see also [7].

5. ON THE STATIONARY DISTRIBUTION OF $\mathbf{z}_n$

In this section it will be assumed that the $\mathbf{z}_n$-process described in section 2 possesses a stationary distribution. Let (x,y) be a stochastic vector with state space S and with distribution this stationary distribution. Put

$$\Phi(p_1,p_2) := \mathrm{E}\{p_1^{\mathbf{x}}p_2^{\mathbf{y}}\},\quad |p_1|\leqslant 1, |p_2|\leqslant 1, \tag{5.1}$$

so that

i. $\Phi(1,1) = 1,\quad 0<\Phi(0,0)<1,$ (5.2)

ii. $\Phi(p_1,p_2)$ is for every fixed p_2 with $|p_2|\leqslant 1$, regular in p_1 for $|p_1|<1$, continuous in p_1 for $|p_1|\leqslant 1$, and similarly with p_1 and p_2 interchanged.

iii. $[p_1p_2-\phi_3]\Phi(p_1,p_2) =$
$[p_1p_2\phi_0+\phi_3-p_1\phi_2-p_2\phi_1]\Phi(0,0)+$
$[p_1\phi_2-\phi_3]\Phi(0,p_2)+[p_2\phi_1-\phi_3]\Phi(p_1,0),\quad |p_1|\leqslant 1, |p_2|\leqslant 1.$

Note that

$$\Phi(p_1,p_2) = \lim_{r\uparrow 1}(1-r)\Phi(r,p_1,p_2), \tag{5.3}$$

so that (5.2)iii. follows directly from (2.6).

The determination of $\Phi(p_1,p_2)$ concerns the construction of a solution of the functional equation (5.2) which satisfies the conditions (5.2)i and ii. A technique for the construction of such a solution has been discussed in [1] and [2]. For general bivariate generating functions ϕ_1 and ϕ_2 such a construction may be very complicated, see [5].

In this study we shall investigate the problem for a special class of bivariate generating functions ϕ_1 and ϕ_2.

This class is described as follows. Let $\phi_{jh}(p_1,p_2)$, $h = 1,2,...;\ j = 1,2$, be bivariate generating functions of distributions with support contained in S and such that

$$\phi_{1h}(p_1,p_2),\ h = 1,2,..., \tag{5.4}$$

are $(1,h)$ associated with ϕ_3,

$$\phi_{2h}(p_1,p_2),\ h = 1,2,...,$$

are $(h,1)$ associated with ϕ_3; so that for every $h = 1,2,...$, and every zero tuple $\hat{p}_1,\hat{p}_2)$ of

$$Z_3(p_1,p_2) := p_1p_2-\phi_3(p_1,p_2),\ |p_1|\leqslant 1, |p_2|\leqslant 1, \tag{5.5}$$

we have

$$\phi_{1h}(\hat{p}_1,\hat{p}_2) = \hat{p}_1\hat{p}_2^h, \tag{5.6}$$
$$\phi_{2h}(\hat{p}_1,\hat{p}_2) = \hat{p}_1^h\hat{p}_2.$$

Further let

$$c_{jh},\ h = 1,2,...;\ j = 1,2, \tag{5.7}$$

be nonnegative constants such that for $j = 1,2$,

$$\sum_{h=1}^{\infty} c_{jh} = 1,\quad 1 \leqslant \sum_{h=1}^{\infty} hc_{jh} < \infty. \tag{5.8}$$

We next define the bivariate generating functions ϕ_1 and ϕ_2 by: for $|p_1| \leqslant 1,\ |p_2| \leqslant 1$,

$$\phi_1(p_1,p_2) = \sum_{h=1}^{\infty} c_{1h}\phi_{1h}(p_1,p_2), \tag{5.9}$$
$$\phi_2(p_1,p_2) = \sum_{h=1}^{\infty} c_{2h}\phi_{2h}(p_1,p_2).$$

Consequently, we have from (5.6) and (5.9) that for every zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z_3(p_1,p_2)$,

$$\hat{\phi}_1 \equiv \phi_1(\hat{p}_1,\hat{p}_2) = \hat{p}_1 \sum_{h=1}^{\infty} c_{1h}\hat{p}_2^h, \tag{5.10}$$
$$\hat{\phi}_2 \equiv \phi_2(\hat{p}_1,\hat{p}_2) = \hat{p}_2 \sum_{h=1}^{\infty} c_{2h}\hat{p}_1^h.$$

So far for the description of the class of functions ϕ_1 and ϕ_2 for which we shall investigate the functional equation (5.2)iii, obviously, this a quite general class.

Because

$$|\Phi(p_1,p_2)| \leqslant 1 \text{ for } |p_1| \leqslant 1, |p_2| \leqslant 1,$$

it follows that every zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z_3(p_1,p_2)$ should also be a zero tuple of the righthand side of (5.2)iii. Consequently, we have by using (5.10): for every zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z_3(p_1,p_2)$,

$$[\hat{p}_1\hat{p}_2\hat{\phi}_0 + \hat{p}_1\hat{p}_2 - \hat{p}_1\hat{p}_2 \sum_{h=1}^{\infty} c_{2h}\hat{p}_1^h - \hat{p}_1\hat{p}_2 \sum_{h=1}^{\infty} c_{1h}\hat{p}_2^h]\Phi(0,0) + \tag{5.11}$$
$$\hat{p}_1\hat{p}_2[-1 + \sum_{h=1}^{\infty} c_{2h}\hat{p}_1^h]\Phi(0,\hat{p}_2) + \hat{p}_1\hat{p}_2[-1 + \sum_{h=1}^{\infty} c_{1h}\hat{p}_2^h]\Phi(\hat{p}_1,0) = 0.$$

Before proceeding with the analysis of (5.11) we introduce an assumption concerning the function $\phi_0(p_1,p_2)$.

ASSUMPTION 5.1.
For the function $\phi_0(p_1,p_2)$ we take

$$\phi_0(p_1,p_2) = \sum_{h=1}^{\infty} \sum_{k=1}^{\infty} c_{1h}c_{2k}p_1^k p_2^h,\ |p_1| \leqslant 1, |p_2| \leqslant 1, \tag{5.12}$$

with

$$\text{g.c.d.}\{c_{jh},\ h = 1,2,...\} = 1,\ j = 1,2. \tag{5.13}$$

REMARK 5.1. The assumption 5.1 is not very essential for the determination of $\Phi(p_1,p_2)$; it will, however, simplify the analysis, see also remark 6.4. The condition (5.13) has been introduced in order that

$\phi_0(p_1,p_2)$ as given by (5.12) satisfies (2.4)ii. Also this condition is not very essential.

Due to (5.12) the relation (5.11) may be rewritten as,
for every zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z_3(p_1,p_2)$,

$$\hat{p}_1\hat{p}_2[(1-\sum_{h=1}^{\infty}c_{2h}\hat{p}_1^h)(1-\sum_{h=1}^{\infty}c_{1h}\hat{p}_2^h)]\times \tag{5.14}$$

$$[\Phi(0,0)-\frac{1}{1-\sum_{h=1}^{\infty}c_{2h}\hat{p}_1^h}\Phi(\hat{p}_1,0)-\frac{1}{1-\sum_{h=1}^{\infty}c_{1h}\hat{p}_2^h}\Phi(0,\hat{p}_2)] = 0.$$

REMARK 5.2. The condition (2.4)ii togather with (5.8) implies that: for $(\hat{p}_1,\hat{p}_2)$ a zero tuple of $Z_3(p_1,p_2)$ and $(\hat{p}_1,\hat{p}_2)\neq(1,1)$,

$$|\sum_{h=1}^{\infty}c_{2h}\hat{p}_1^h|<1, \quad |\sum_{h=1}^{\infty}c_{1h}\hat{p}_2^h|<1. \tag{5.15}$$

REMARK 5.3. The condition (5.11) becomes for $(\hat{p}_1,\hat{p}_2) = (1,1)$ an identity. However, if we take in (5.2)iii $p_2 = 1$ and divide both sides by $p_1-\phi_3(p, 1)$, and then let $p_1\uparrow 1$ the resulting righthand side should be equal to one, according to (5.2)i.

By noting that for a zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z_3(\overline{p}_1,p_2)$ holds, since $E\{\xi_3\}<1$, and $E\{\eta_3\}<1$, cf. ass. 2.3, that

$$\hat{p}_1 = 1 \Leftrightarrow \hat{p}_2 = 1, \tag{5.16}$$

it follows from (5.14) and (5.15) that: for every zero tuple $(\hat{p}_1,\hat{p}_2)$ of $Z_3(p_1,p_2)$,

$$\hat{p}_1\hat{p}_2[\frac{1-\hat{p}_1}{1-\sum_{h=1}^{\infty}c_{2h}\hat{p}_1^h}\Phi(\hat{p}_1,0)+\frac{1-\hat{p}_1}{1-\sum_{h=1}^{\infty}c_{1h}\hat{p}_2^h}\Phi(0,\hat{p}_2)-(1-\hat{p}_1)\Phi(0,0)] = 0. \tag{5.17}$$

In the next section we shall analyse the relation (5.17) under certain assumptions to be satisfied by $\phi_3(p_1,p_2)$.

6. ON THE SOLUTION OF THE FUNCTIONAL EQUATION

In this section we shall present the solution of the functional equation (5.17) under certain assumptions concerning $\phi_3(p_1,p_2)$, $|p_1|\leq 1, |p_2|\leq 1$.

ASSUMPTION 6.1.

i. $\phi_3(0,0)>0$; (6.1)

ii. there exist in the complex p_i-plane a simple, smooth contour S_i, $i = 1,2$, with

$$S_i\subset\{p_i = |p_i|\leq 1\}, \tag{6.2}$$

$$p_i = 1\in S_i, \tag{6.3}$$

$$p_i = 0\in S_i^+, \text{ the interior of } S_i, \tag{6.4}$$

and a one-to-one mapping $\hat{p}_1 = P_1(\hat{p}_2)$: $S_2\rightarrow S_1$ such that $(\hat{p}_1,\hat{p}_2)$ is for every $\hat{p}_2\in S_2$ a zero tuple of

$$Z_3(p_1,p_2) = p_1p_2-\phi_3(p_1,p_2),$$

$\hat{p}_2 = P_2(\hat{p}_1)$: $S_1\rightarrow S_2$ shall denote the inverse of $P_1(p_2)$.

REMARK 6.1. For $|s| = 1$ it is readily verified by using Rouché's theorem and assumption (2.3)i and (2.3)ii that the function

$$g^2 - \phi(gs, gs^{-1}), \quad |g| \leqslant 1,$$

has exactly two zeros; both are continuous in s on $|s| = 1$, and one, say $g(s)$, is equal to one for $s = 1$. Obviously, for every s with $|s| = 1$,

$$(\hat{p}_1, \hat{p}_2) := (g(s)s, g(s)s^{-1}), \tag{6.5}$$

is a zero tuple of $Z_3(p_1,p_2)$ for which (6.3) applies and also (6.4) as a consequence of (6.1). For the alternative of (6.1) see the discussion in [1]. Note that if also

$$\{p_1 : p_1 = g(s)s, |s| = 1\} \text{ and } \{p_2 : p_2 = g(s)s^{-1}, |s| = 1\},$$

are both simple contours then the existence of S_1, S_2 and $P_1(.), P_2(.)$ follows. Actually, if

$$\phi_3(p_1,p_2) = \phi_3(p_2,p_1) \text{ for all } |p_1| \leqslant 1, |p_2| \leqslant 1, \tag{6.6}$$

i.e. if ξ_3 and η_3 are exchangeable variables, then $g(s)$ is positive for all s with $|s| = 1$ and the existence of $S_i, i = 1,2$ satisfying the conditions of assumption 6.1ii is readily proved, see [1] for further details.

Another case for which the existence of a simple, smooth $S_i, i = 1,2$ is readily proved occurs if

$$\phi_3(p_1,p_2) = \phi_3(p_1,1)\phi_3(1,p_2), \quad |p_1| \leqslant 1, |p_2| \leqslant 1, \tag{6.7}$$

i.e. if ξ_3 and η_3 are independent variables. For this case define $\mathbf{m}_{31}$ as in (3.14), and $\mathbf{m}_{32}$, analogously for $\phi_3(1,p_2)$. It is then readily shown that for every z with $|z| = 1$,

$$(\hat{p}_1, \hat{p}_2) := (\mathrm{E}\{z^{\mathbf{m}_{31}}\}, \mathrm{E}\{z^{-\mathbf{m}_{32}}\}), \tag{6.8}$$

is a zero tuple of $Z_3(p_1,p_2)$ and that by taking

$$S_1 = \{p_1 : p_1 = \mathrm{E}\{z^{\mathbf{m}_{31}}\}, |z| = 1\}, \tag{6.9}$$

$$S_2 = \{p_2 : p_2 = \mathrm{E}\{z^{-\mathbf{m}_{32}}\}, |z| = 1\},$$

the conditions of assumption 6.1.ii are fulfilled.

REMARK 6.2. In [1] it has been shown that the assumptions 2.3 and 6.1 guarantee that there exist:
i. in the complex z-plane a simple, smooth contour L with

ii. $z = 1 \in L$,

$z = 0 \in L^+$ with L^+ the interior of L,

$|z| = \infty \in L^-$ with L^- the exterior of L;

iii. a function $p_1(z)$ regular for $z \in L^+$, continuous and univalent for $z \in L^+ \cup L$, with $z = 0$ a simple zero of $p_1(z)$, with $p_1(z) = 1$ and $p_1(z): L \to S_1$;
iv. a function $p_2(z)$ regular for $z \in L^-$, continuous for $z \in L \cup L^-$ with $z = 0$ a simple zero of $p_2(\frac{1}{z})$, with $p_2(1) = 1$ and $p_2(z): L \to S_2$;
and such that

v. $$(\hat{p}_1, \hat{p}_2) = (p_1,(z), p_2(z)) \tag{6.11}$$

is for every $z \in L$ a zero tuple of $Z_3(p_1,p_2)$.

REMARK 6.3. For the explicit determination of $L, p_1(z), z \in L^+ \cup L$ and $p_2(z), z \in L \cup L^-$, with S_1 and S_2 satisfying the conditions 6.1 see [1], chapter II.2, where it is also shown that $L, p_1(.)$ and $p_2(.)$ are uniquely determined. We further note that if ξ_3 and η_3 are exchangeable variables then L is the unit circle $|z| = 1$. This also holds if ξ_3 and η_3 are independent and $p_1(z)$ and $p_2(z)$ are represented by, cf. (6.8),

$$p_1(z) = \mathrm{E}\{z^{\mathbf{m}_{31}}\}, \; |z| \leqslant 1, \qquad (6.12)$$

$$p_2(z) = \mathrm{E}\{z^{-\mathbf{m}_{32}}\}, \; |z| \geqslant 1.$$

Next we return to the functional equation (5.17) and consider this equation for the zero tuple $(\hat{p}_1,\hat{p}_2)$ characterized in remark 6.2.v; so we replace in (5.17) $\hat{p}_i$ by $\hat{p}_i(z), z \in L, i = 1,2$. By noting that, cf. (6.2), (6.3) and remark 6.2,

$$|p_1(z)| < 1, \; z \in L, \; z \neq 1, \qquad (6.13)$$

$$p_1(z) = 1, \; z = 1 \in L,$$

and that S_1, S_2 and L are all smooth, also at, $p_1 = 1$, $p_2 = 1$, $z = 1$, it follows that (5.17) for $(\hat{p}_1,\hat{p}_2) \in S_1 \times S_2$ is equivalent with: for $z \in L$,

$$\frac{1-z}{1-\sum_{h=1}^{\infty} c_{2h} p_1^h(z)} \Phi(p_1(z),0) + \frac{1-z}{1-\sum_{h=1}^{\infty} c_{1h} p_2^h(z)} \Phi(0,p_2(z)) = (1-z)\Phi(0,0). \qquad (6.14)$$

Put, cf. (5.8),

$$\Pi_1 := \frac{dp_1(z)}{dz}\Big|_{z=1}, \; \Pi_2 := -\frac{dp_2(z)}{dz}\Big|_{z=1}, \qquad (6.15)$$

$$\mu_0 := \sum_{h=1}^{\infty} hc_{2h} < \infty, \; \nu_0 := \sum_{h=1}^{\infty} hc_{1h} < \infty,$$

and note that: for $z \in L$, $z \neq 1$,

$$\Big|\sum_{h=1}^{\infty} c_{2h} p_1^h(z)\Big| < 1, \; \Big|\sum_{h=1}^{\infty} c_{1h} p_2^h(z)\Big| < 1. \qquad (6.16)$$

Consequently it follows from (5.2)ii and the properties of the functions $p_i(z), i = 1,2$, cf. remark 6.2, that the first term in (6.14) is regular for $z \in L^+$, continuous for $z \in L^+ \cup L$, and similarly for the second term with L^+ replaced by L^-. Hence by rewriting (6.14) as

$$\frac{1-z}{1-\sum_{h=1}^{\infty} c_{2h} p_1^h(z)} \Phi(p_1(z),0) - (1-z)\Phi(0,0) = \qquad (6.17)$$

$$-\frac{1-z}{1-\sum_{h=1}^{\infty} c_{1h} p_2^h(z)} \Phi(0,p_2(z)),$$

it is seen that the term in the righthand side for $z \in L \cup L^-$ is the analytic continuation of the left-hand side of (5.17) for $z \in L^+ \cup L$. Hence since $p_2(z) \to 0$ for $|z| \to \infty$ it follows from Liouville's theorem that

$$\frac{1-z}{1-\sum_{h=1}^{\infty} c_{2h} p_1^h(z)} \Phi(p_1(z),0) - (1-z)\Phi(0,0) = C_1(1-z) + C_2, \; z \in L^+ \cup L, \qquad (6.18)$$

$$-\frac{1-z}{1-\sum_{h=1}^{\infty} c_{1h} p_2^h(z)} \Phi(0,p_2(z)) = C_1(1-z) + C_2, \; z \in L \cup L^-, \qquad (6.19)$$

with C_1 and C_2 constants. Taking $z = 0$ in (6.18) yields, because $p_1(0) = 0$, cf. remark 6.2.iii,

$$C_1 + C_2 = 0. \tag{6.20}$$

Dividing (6.19) by z and letting $|z|\to\infty$ yields

$$C_1 = -\Phi(0,0);$$

and so we obtain

$$\Phi(p_1(z),0) = \frac{1-\sum_{h=1}^{\infty} c_{2h}p_1^h(z)}{1-z}\Phi(0,0), \quad z\in L^+\cup L, \tag{6.21}$$

$$\Phi(0,p_2(z)) = \frac{1-\sum_{h=1}^{\infty} c_{1h}p_2^h(z)}{1-\frac{1}{z}}\Phi(0,0), \quad z\in L\cup L^-;$$

in particular for $z=1$, cf. (6.15),

$$\Phi(1,0) = \mu_0\Pi_1\Phi(0,0), \tag{6.22}$$

$$\Phi(0,1) = \nu_0\Pi_2\Phi(0,0).$$

The functions $p_1(z), z\in L^+\cup L$ and $p_2(z)$, $z\in L\cup L^-$ have each a unique inverse, since they are univalent, denote their inverses by,

$$z = p_{10}(p_1)\colon S_1^+\cup S_1 \to L^+\cup L, \tag{6.23}$$

$$z = p_{20}(p_2)\colon S_2^+\cup S_2 \to L\cup L^-.$$

Then (6.21) may be rewritten as

$$\Phi(p_1,0) = \frac{1-\sum_{h=1}^{\infty} c_{2h}p_1^h}{1-p_{10}(p_1)}\Phi(0,0), \quad p_1\in S_1^+\cup S_1, \tag{6.24}$$

$$\Phi(0,p_2) = \frac{1-\sum_{h=1}^{\infty} c_{1h}p_2^h}{1-p_{20}^{-1}(p_2)}\Phi(0,0), \quad p_2\in S_2^+\cup S_2. \tag{6.25}$$

Note that

$$[0,1]\subset S_i^+\cup S_i, \quad i = 1,2. \tag{6.26}$$

To determine $\Phi(0,0)$ we apply the condition (5.2)i. Take $p_2 = 1$ in (5.2)iii and divide both sides by p_1-1 and let $p_1\uparrow 1$, then with (6.15) and

$$\mu_k := \mathrm{E}\{\xi_k\}>0 \quad \nu_k := \mathrm{E}\{\eta_k\}>0, \quad k = 1,2,3, \tag{6.27}$$

we obtain

$$1-\mu_3 = [\mu_0+\mu_3-\mu_1-\mu_2]\Phi(0,0)+ \tag{6.28}$$

$$+[1+\mu_2-\mu_3]\Phi(0,1)+(\mu_1-\mu_3]\Phi(1,0),$$

and similarly,

$$1-\nu_3 = [\nu_0+\nu_3-\nu_2-\nu_1]\Phi(0,0)+ \tag{6.29}$$

$$[\nu_2-\nu_3]\Phi(0,1)+[1+\nu_1-\nu_3]\Phi(1,0).$$

From the derivations above, see also remark 5.3, it may be seen that (6.20) and (6.29) are linearly dependent. For a direct proof see the next section (7.4),...,(7.6).

Once $\Phi(0,0)$ has been determined, $\Phi(p_1,0)$ and $\Phi(0,p_2)$ follow from (6.24) and (6.25), and then $\Phi(p_1,p_2)$ can be found from (5.2)iii for $p_i \in S_i^+ \cup S_i$, $i = 1,2$. So if we know that the random walk $\mathbf{z}_n$, $n = 0,1,...,$ of section 5 possesses a stationairy distribution then we have constructed its bivariate generating function (under the assumptions made) since the construction described above leads to a unique solution. In the next section we shall consider the question of the existence of the stationary distribution. We conclude this section by considering the solution obtained above for the case with

$$c_{11} = c_{21} = 1, \tag{6.30}$$

$$c_{jh} = 0, \ h = 2,3,....; \ j = 1,2,$$

$$\phi_1(p_1,p_2) = \phi_2(p_1,p_2), \ |p_1| \leq 1, |p_2| \leq 1.$$

Hence it follows from (6.21), (6.22), (6.28), (cf. also (7.3)ii),

$$\Phi(p_1,0) = \frac{1-p_1}{1-p_{10}(p_1)}\Phi(0,0), \ p_1 \in S_1^+ \cup S_1, \tag{6.31}$$

$$\Phi(0,p_2) = \frac{1-p_2}{1-p_{20}^{-1}(p_2)}\Phi(0,0), \ p_2 \in S_2^+ \cup S_2,$$

$$\Phi(1,0) = \Pi_1 \Phi(0,0), \tag{6.32}$$

$$\Phi(0,1) = \Pi_2 \Phi(0,0),$$

$$\Phi^{-1}(0,0) = 1+\frac{\Pi_2}{1-\mu_3} = 1+\frac{\Pi_1}{1-\nu_3} > 1;$$

and from (5.2)iii, for $p_i \in S_i^+ \cup S_i$, $i = 1,2$,

$$(p_1p_2-\phi_3)\Phi(p_1,p_2) = [p_1^2p_2^2+(1-p_1-p_2)\phi_3(p_1,p_2)]\Phi(0,0)$$

$$+(p_1-1)\phi_3(p_1,p_2)\Phi(0,p_2)+(p_2-1)\phi_3(p_1,p_2)\Phi(p,0).$$

Obviously, if a stationary distribution exists then necessarily from (6.32),

$$\Pi_1 > 1, \ \Pi_2 > 1, \tag{6.33}$$

and $(1-p_1)/(1-p_{10}(p_1))$ should have for $p_1 \in [0,1) \subset S_1^+ \cup S_1$ a series expansion in powers of p_1 with nonnegative coefficients, analogously for $(1-p_2)/(1-p_{20}^{-1}(p_2))$. This observation leads directly to the conclusion that the series expansion of $\Phi(p_1,0)/\Phi(0,0)$ in (6.14) has also nonnegative coefficients, because of (5.7) and (5.8).

REMARK 6.4. The choice of $\phi_0(p_1,p_2)$ as it is made in (5.12) leads to a simple coefficient of $\Phi(0,0)$ in (5.17), cf. also (6.14). It is, however, not necessary to specify $\phi_0(p_1,p_2)$; the resulting functional equation can again be easily solved for general ϕ_0 by using the technique for the nonhomogeneous Riemann boundary value problem, see [1].

7. ON THE ERGODICITY CONDITIONS

Conditions for ergodicity of an aperiodic Markov chain with an irreducible state space as described in section 2 have been investigated by several authors of which we mention here Fayolle [3], Malyshev [4], Nauta [5], Vaninskii and Lazareva [12]. In [4] necessary and sufficient conditions have been formulated for the case that the displacement vectors at points of the interior of the state space S and at points of the boundary of S have finite support. The conditions involve only the first moments. In Fayolle's study the requirement of finite support is replaced by condition on finiteness of second moments. Nauta's results involve only first moments conditions, however, in his research the bivariate generating functions belong to a special class, although it is a quite general class; in [12] conditions for ergodicity and also for nonrecurrence are derived involving only the first moments without restrictions on the supports (see also a future study by the present author).

For the random walk described in section 2 the necessary and sufficient condition read:

$$(1-\nu_3)(1-\mu_1)>-(1-\mu_3)\nu_1, \tag{7.1}$$
$$(1-\mu_3)(1-\nu_3)>-(1-\nu_3)\mu_2,$$

for the case (cf. assumption 2.3 and (6.127)) with

$$\mu_3<1, \;\; \nu_0<1. \tag{7.2}$$

Before discussing the conditions (7.1) we first proceed with some analysis of the solution constructed in the preceding sections.
From, cf. assumption 6.1 and remark 6.2,

$$p_1(z)p_2(z)-\phi_3(p_1(z),p_2(z)) = 0 \;\text{ for }\; z\in L,$$

it follows by using (5.8), (5.10), (6.15) and (6.27) that

$$\text{i.}\;\; \mu_0\geqslant 1, \;\; \nu_0\geqslant 1, \tag{7.3}$$

$$\text{ii.}\;\; \frac{\Pi_1}{\Pi_2} = -\frac{d\hat{p}_1}{d\hat{p}_2}|_{z=1} = -\{\frac{dp_1(z)}{dz}/\frac{dp_2(z)}{dz}\}|_{z=1}$$

$$= \frac{1-\nu_3}{1-\mu_3} = \frac{h-\nu_{1h}}{1-\mu_{1h}} = \frac{1-\nu_{2k}}{k-\mu_{2k}}, \; k,h\in\{1,2,...,\},$$

$$= \frac{\nu_0-\nu_1}{1-\mu_1} = \frac{1-\nu_2}{\mu_0-\mu_2}>0,$$

with for $j = 1,2$; $h = 1,2,...,$

$$\mu_{jh} := \frac{d}{dp_1}\phi_{jh}(p_1,1)|_{p_1=1}>0, \;\nu_{jh} := \frac{d}{dp_2}\phi_{jh}(1,p_2)|_{p_2=1}>0. \tag{7.4}$$

By using the relations (6.22), the relations (6.20) and (6.29) may be rewritten as:

$$\Phi^{-1}(0,0) = \Pi_2\frac{\nu_0}{1-\mu_3}+\frac{\mu_0-\mu_3}{1-\mu_3}+\frac{\mu_2-\mu_3}{1-\mu_3}\{-1+\Pi_2\nu_0\}+\frac{\mu_1-\mu_3}{1-\mu_3}\{-1+\Pi_1\mu_0\}, \tag{7.5}$$

$$\Phi^{-1}(0,0) = \Phi_1\frac{\mu_0}{1-\nu_3}+\frac{\nu_0-\nu_3}{1-\nu_3}+\frac{\nu_1-\nu_3}{1-\nu_3}\{-1+\Pi_1\mu_0\}+\frac{\nu_2-\nu_3}{1-\nu_3}\{-1+\Pi_2\nu_0\}.$$

By subtracting the relation in (7.5) and by noting that (7.3) implies

$$\frac{1-\mu_1}{1-\mu_3} = \frac{1-\nu_1}{1-\nu_3}-\frac{1-\nu_0}{1-\nu_3}, \quad \frac{1-\nu_2}{1-\nu_3} = \frac{1-\mu_2}{1-\mu_3}-\frac{1-\mu_0}{1-\mu_3}, \tag{7.6}$$

it is readily seen that the relations (7.5) are linearly dependent. Note that the first relation of (7.5) may be also written as:

$$\Phi^{-1}(0,0) = \Pi_2\nu_0+\frac{\mu_0}{1-\mu_3}+\frac{\mu_2}{1-\mu_3}\{\Pi_2\nu_0-1\}+\frac{\mu_1-\mu_3}{1-\mu_3}\{\Pi_1\mu_0-1\}, \tag{7.7}$$

$$= \Pi_1\mu_0+\frac{\nu_0}{1-\nu_3}+\frac{\nu_1}{1-\nu_3}\{\Pi_1\mu_0-1\}+\frac{\nu_2-\nu_3}{1-\mu_3}\{\Pi_2\nu_0-1\}.$$

Next we confront the conditions (7.1) with the relations (7.3). It is seen by using (7.3) that the conditions of (7.1) become

$$-\nu_1<\frac{1-\nu_3}{1-\mu_3}(1-\mu_1) = \frac{\nu_0-\nu_1}{1-\mu_1}(1-\mu_1) = \nu_0-\nu_1, \tag{7.8}$$

$$-\mu_2 < \frac{1-\mu_3}{1-\nu_3}(1-\nu_2) = \frac{\mu_0-\mu_2}{1-\nu_2}(1-\nu_2) = \mu_0-\mu_2,$$

and consequently, the conditions (7.1) are always satisfied for the case that the bivariate generating functions $\phi_1(p_1,p_2),\phi_2(p_1,p_2)$ and $\phi_0(p_1,p_2)$ are as given by (5.9) and (5.12).

This is a rather remarkable conclusion, the more so since by using the relations (7.3) it does not seem possible to show directly that the righthand side of (7.7) is larger than one. In this respect it is noted that the relations (6.33) should also apply for the general case, i.e. not only for the case described by (6.30) since Π_1 and Π_2 do only depend on the character of $\phi_3(p_1,p_2)$ (and possibly on the assumption 6.1.ii). It is further noted that Π_1 and Π_2 seem to depend only on the first moments of ξ_3 and η_3, i.e. μ_3 and ν_3, so the question whether the righthand side of (7.7) is larger than one, is a question which involves only the first moments $\mu_k,\nu_k,k = 1,2,3$, and this indicates that next to the first moment conditions (7.1) no other conditions are needed for the present random walk in formulating necessary and sufficient conditions.

The difficulty in showing that the righthand of (7.7) is larger than one is presumably due to a lack of information concerning relations between a bivariate generating function and its z_0-associated bivariate generating functions, see also the next section, remark 8.1.

8. Some special cases

In this section we shall consider some special cases. A first special case has already been introduced in section 6, see (6.30) and the following discussion there.

As a second special case we take, cf. section 2, for $|p_1|\leq 1, |p_2|\leq 1$,

$$\phi_0(p_1,p_2) = p_1,p_2, \tag{8.1}$$

$$\phi_k(p_1,p_2) = d_{k1},p_1p_2+d_{k2}\phi_3(p_1,p_2),\ k = 1,2, \tag{8.2}$$

with

$$d_{k1}\geq 0,\ d_{k2}\geq 0,\ d_{k1}+d_{k2} = 1.$$

Then from (5.2)iii we obtain (in the same way as in section 5) for $(\hat{p}_1,\hat{p}_2) = (p_1(z),p_2(z)),z\in L$, a zero tuple of $z(p_1,p_2)$:

$$\frac{1-z}{1-p_1(z)}\Phi(p_1(z),0)+\frac{1-z}{1-p_2(z)}\Phi(0,p_2(z)) = (1-z)\Phi(0,0). \tag{8.3}$$

The solution of (8.3) reads:

$$\Phi(p_1,0) = \frac{1-p_1}{1-p_{10}(p_1)}\Phi(0,0),\ p_1\in S_1^+\cup S_1, \tag{8.4}$$

$$\Phi(0,p_2) = \frac{1-p_2}{1-p_{20}^{-1}(p_2)}\Phi(0,0),\ p_2\in S_2^+\cup S_2,$$

$$\Phi(1,0) = \Pi_1\Phi(0,0),\ \Phi(0,1) = \Pi_2\Phi(0,0),$$

and

$$\Phi^{-1}(0,0) = \frac{1+\mu_3-\mu_1-\mu_2}{1-\mu_3}+\frac{1+\mu_2-\mu_3}{1-\mu_3}\Pi_2+\frac{\mu_1-\mu_3}{1-\mu_3}\Pi_1. \tag{8.5}$$

Since

$$\mu_1 = d_{11}+d_{12}\mu_3 \Rightarrow 1>\mu_1>\mu_3, \tag{8.6}$$

$$\mu_2 = d_{21}+d_{22}\mu_3 \Rightarrow 1>\mu_2>\mu_3,$$

it follows from (8.5) and (6.33) that

$$\Phi^{-1}(0,0) = 1-d_{11}-d_{21}+\{d_{21}+\frac{1}{1-\mu_3}\}\Pi_2+d_{11}\Pi_1 \tag{8.7}$$

$$= 1+\frac{1}{1-\mu_3}\Pi_2+(\Pi_2-1)d_{21}+(\Pi_1-1)d_{11}>1.$$

REMARK 8.1. The present example is of wider interest because of

$$1>\mu_k>\mu_3, \quad 1>\nu_k>\nu_3, \quad k = 1,2.$$

The present idea may also be used in connection with $\phi_{kh}(p_1,p_2)$, cf. (5.4). For instance we may consider instead of $\phi_{1h}(p_1p_2)$, cf. (5.4), the bivariate generating functions: for $h = 1,2,\ldots$, and $|p_2|\leqslant 1, |p_2|\leqslant 1$,

$$\psi_{1h}(p_1,p_2) := d_{1h}p_1p_2^h+d_{2h}\phi_{1h}(p_1,p_2), \tag{8.8}$$

$$d_{1h}\geqslant 0, \; d_{2h}\geqslant 0, \; d_{1h}+d_{2h} = 1,$$

which are also $(1,h)$ associated with $\phi_3(p_1,p_2)$.

As a third special case we take for $k = 1,2$,

$$c_{k1} = 1, \; c_{kh} = 0, \; k = 2,3,\ldots, \tag{8.9}$$

$$\phi_{11}(p_1,p_2) = \phi_2(p_1,p_2) = \mathrm{E}\{p_1^{\mathbf{k}_1(1,1)}p_2^{\mathbf{k}_2(1,1)}\}, \tag{8.10}$$

with

$$\mathbf{k}(1,1) = (\mathbf{k}_1(1,1),\mathbf{k}_2(1,1)),$$

defined as the hitting point in section 3.

Hence, cf. theorems 3.1 and 4.2 with $(x_0,y_0) = (1,1)$,

$$\mu_1 = \mu_2 = \mathrm{E}\{\mathbf{k}_1(1,1)\}<\mu_3<1, \tag{8.11}$$

$$\nu_1 = \nu_2 = \mathrm{E}\{\mathbf{k}_2(1,1)\}<\nu_3<1.$$

It follows from (7.5) since $\mu_0 = \nu_0 = 1$ that

$$\Phi^{-1}(0,0) = 1+\frac{1}{1-\mu_3}\Pi_2+\frac{\mu_1-\mu_3}{1-\mu_3}\{-2+\Pi_1+\Pi_2\}. \tag{8.12}$$

Again it is not possible to show simply that the righthand side of (8.12) is larger than one, more information concerning μ_1,Π_1 and Π_2 is needed: for such an analysis one should start from the results in [6] and [7].

REMARK 8.1. If for $|p_1|\leqslant 1, |p_2|\leqslant 1$,

$$\phi_3(p_1,p_2) = \phi_3(p_1,1)\phi_3(1,p_2),$$

and

$$\phi_3(p_1,p_2) = \phi_3(p_2,p_1),$$

then it is readily shown that, cf. (2.15) and remark 6.1,

$$\Pi_1 = \Pi_2 = \frac{1}{1-\mu_3},$$

so from (8.12),

$$\Phi^{-1}(0,0) = 1+\frac{1}{(1-\mu_3)^2}\{1+2\mu_3(\mu_1-\mu_3)\}>1,$$

since

$$\mu_1>0,\ 0<\mu_3<1\ \Rightarrow\ 1+2\mu_3(\mu_1-\mu_3)>0.$$

As a third special case we take: for $|p_1|\leqslant 1, |p_2|\leqslant 1$

$$\phi_{1h}(p_1,p_2) = p_1p_2^h,\ \ \phi_{2h}(p_1,p_2) = p_1^hp_2,\ \ h = 1,2,..., \tag{8.13}$$

so

$$\phi_1(p_1,p_2) = \sum_{h=1}^{\infty} c_{1h}p_1p_2^h, \tag{8.14}$$

$$\phi_2(p_1,p_2) = \sum_{h=2}^{\infty} c_{2h}p_1^hp_2,$$

$$\phi_0(p_1,p_2) = \sum_{k=1}^{\infty}\sum_{h=1}^{\infty} c_{1h}c_{2k}p_1^kp_2^h.$$

It then follows that

$$\mu_1 = 1,\ \ \nu_1 = \nu_0, \tag{8.15}$$

$$\mu_2 = \mu_0,\ \ \nu_2 = 1,$$

and the relation (7.5) becomes

$$\Phi^{-1}(0,0) = \Pi_2\nu_0\{1+\frac{\mu_0}{1-\mu_3}\}+\Pi_1\overline{\mu_0-1}>1.$$

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