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Singularities of the generator of a Markov additive process with one-sided jumps

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ABSTRACT
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SINGULARITIES OF THE GENERATOR OF A MARKOV ADDITIVE PROCESS WITH ONE-SIDED JUMPS

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Abstract. We analyze the number of zeros of $\det(F(\alpha))$, where $F(\alpha)$ is the matrix cumulant generating function of a Markov Additive Process (MAP) with one-sided jumps. The focus is on the number of zeros in the right half of the complex plane, where $\det(F(\alpha))$ is well-defined. Moreover, we analyze the case of a killed MAP with state-dependent killing rates, and the limiting behavior of the zeros as all killing rates converge to 0. We argue that our results are particulary useful for the fluctuation theory of MAPs. For example, they lead, under mild assumptions, to a straightforward identification of the stationary distribution of a reflected MAP with one-sided jumps.

1. Introduction

In this paper we consider a Markov Additive Process (MAP) with one-sided jumps. Loosely speaking, such a process is actually a Markov modulated Lévy process with additional jumps at switching epochs; it is required that the process has either no negative jumps, or no positive jumps. Due to symmetry reasons we can restrict ourselves to the case of *no negative* jumps, which we will do throughout this work.

In order to describe our results, let us first formally introduce the model. Following [3] and [11], we consider a MAP $(X(t), J(t))$ specified by the characteristics: $q_{ij}, G_{ij}, a_i, \sigma_i, \nu_i(\mathrm{d}x)$, which we define in the following way. Let $J(t)$ be a right-continuous irreducible continuous-time Markov chain with state space $\{1, \ldots, N\}$ and intensity matrix $Q = (q_{ij})$. For each $i \in \{1, \ldots, N\}$, let $X_i(t)$ be a Lévy process with

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Laplace exponent

$$\phi_i(\alpha) := \log(\mathbb{E}e^{-\alpha X_i(1)})$$
$$(1) \qquad = a_i\alpha + \frac{1}{2}\sigma_i^2\alpha^2 + \int_0^\infty (-1 + e^{-\alpha x} + \alpha x \mathbb{1}_{\{x<1\}})\nu_i(\mathrm{d}x),$$

where $(a_i, \sigma_i, \nu_i(\mathrm{d}x))$ is a *Lévy triple*, that is, $a_i \in \mathbb{R}, \sigma_i \geq 0$ and $\nu_i(\mathrm{d}x)$ is a measure on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge x^2)\nu_i(\mathrm{d}x) < \infty$. Note that restricting the support of measure $\nu_i(\mathrm{d}x)$ to $(0,\infty)$ amounts to forbidding negative jumps. Let T_n (with $T_0 = 0$) be the n-th jump epoch of the Markov chain $J(t)$. Also, let (U_{ij}^n) be a sequence of i.i.d. non-negative random variables with distribution function $G_{ij}(\cdot)$, and corresponding Laplace-Stieltjes transform $\tilde{G}_{ij}(\alpha) := \mathbb{E}e^{-\alpha U_{ij}}$. This sequence describes the jumps of our MAP at the transitions of the Markov chain from state i to state j. Without loss of generality we set $U_{ii} \equiv 0$, and $U_{ij} \equiv 0$ whenever $q_{ij} = 0$. It is assumed that all the stochastic quantities considered above are independent.

We can now define our process $X(t)$, with $\mathbb{1}_{ij}^n(t)$ denoting the indicator function of the event $\{J(T_{n-1}) = i, J(T_n) = j, T_n \leq t\}$, and $\mathbb{1}_i^n(t)$ denoting the indicator function of $\{J(T_{n-1}) = i, T_{n-1} \leq t < T_n\}$, by

$$X(t) := X(0) + \sum_{n\geq 1} \sum_{i,j\in\{1,\ldots,N\}, i\neq j} (X_i(T_n) - X_i(T_{n-1}) + U_{ij}^n)\mathbb{1}_{ij}^n(t)$$
$$(2) \qquad + \sum_{n\geq 1} \sum_{i\in\{1,\ldots,N\}} (X_i(t) - X_i(T_{n-1}))\mathbb{1}_i^n(t).$$

It is observed that the former summation relates to the contributions of all states of the Markov chain that have been left before t, whereas the latter summation represents the contribution of the state the Markov chain is currently in at time t.

Letting $\tilde{G}(\alpha) := (\tilde{G}_{ij}(\alpha))$ and $A \circ B := (a_{ij}b_{ij})$, where A and B are two square matrices of the same dimensions, we define the *matrix cumulant generating function* or simply the *generator* of MAP $X(t)$ through

$$(3) \qquad F(\alpha) := Q \circ \tilde{G}(\alpha) + \mathrm{diag}(\phi_1(\alpha), \ldots, \phi_N(\alpha)).$$

It is noted that the generator of a MAP is the matrix-analogue of the Laplace exponent of a Lévy process, see [3] for related issues as well as a detailed probabilistic interpretation of $F(\alpha)$. It is easy to see that the absence of negative jumps implies that $F(\alpha)$ is finite for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$, where $\mathbb{C}^{\mathrm{Re}\geq 0} := \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) \geq 0\}$ (similarly we define $\mathbb{C}^{\mathrm{Re}>0}$ and $\mathbb{C}^{\mathrm{Re}<0}$).

The main contribution of this paper is an extensive analysis of the number of zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$. Our study was primarily motivated by [3], where a queueing model fed by a MAP with one-sided jumps was considered. There it is shown that the Laplace-Stieltjes transform of the stationary distribution of the buffer content has an appealing expression, but it involves an unknown vector of constants. We return to this model in Section 5, where we discuss the applicability of our results and indicate some important consequences for the fluctuation theory of MAPs. In particular, we shed some light on a relatively straightforward procedure that yields, under a mild assumption, the unknown vector of constants mentioned above using constraints induced by the zeros of $\det(F(\alpha))$.

A number of special cases of the present problem can be found in the literature, see e.g. [15], [17] and [12]. A common, rather restrictive, assumption in these and related papers is that the process $X(t)$ evolves *linearly* between jumps of the underlying Markov chain $J(t)$. In this respect the findings of our paper considerably generalize results from the existing literature.

Before we can state our main results, we introduce a number of useful notions. Firstly, Lévy processes whose paths are non-decreasing are called *subordinators*. The number of processes $X_i(t), i \in \{1, \ldots, N\}$ which are *not* subordinators plays a crucial role in our work. We denote this number by N^*. Secondly, Perron-Frobenius theory entails that there exists a unique eigenvalue $k(\alpha), \alpha \geq 0$ of $F(\alpha)$ with maximum real part. This eigenvalue is real and simple. Moreover, it is well known that $k(0) = 0$ and

$$(4) \qquad \lim_{t\to\infty} \frac{1}{t}\mathbb{E}(-X(t)|J(0) = i, X(0) = x) = k'(0^+) \text{ for any } i \text{ and } x,$$

where $k'(0^+)$ is the right-sided derivative of $k(\alpha)$ at 0. In this sense $k'(0^+)$ can be interpreted as the asymptotic drift of $-X(t)$. When $k'(0^+) > 0$, the process $X(t)$ drifts to $-\infty$; when $k'(0^+) < 0$, the process drifts to $+\infty$; and when $k'(0^+) = 0$ the process oscillates between $-\infty$ and $+\infty$, unless $X(t)$ is degenerate ($X(t) \equiv 0$). These results can be found in Chapter XI of [2].

We are now ready to state the main results. We start with the case where *killing* is present: we analyze the zeros of $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}))$, where $\boldsymbol{q}$ is a vector with non-negative entries, of which at least one is strictly positive. In this context, 'killing' means that we consider the MAP not on the full half-line $[0, \infty)$, but rather up to some random horizon; the value of this horizon has hazard rate (killing rate) q_i when the underlying Markov chain is in state i. It can then be seen that the matrix $F(\alpha) - \mathrm{diag}(\boldsymbol{q})$ is essentially the generator of the killed MAP. We emphasize that this 'killing principle' plays a crucial role in the fluctuation theory of Lévy processes (see Chapter 5 of [10] for example), which explains our interest in it.

Theorem 1. *If vector* $\boldsymbol{q} \geq \mathbf{0}$ *is not identically zero, then* $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}))$ *has no zeros on the imaginary axis and has exactly* N^* *zeros (counting multiplicities) in* $\mathbb{C}^{\mathrm{Re}>0}$.

Interestingly, in the situation *without* killing the statement becomes slightly less clean, as we see in Theorem 2. Note the important role played by the asymptotic drift: the result depends on whether the process tends to $-\infty$ or $+\infty$.

Theorem 2. *If* $N^* > 0$ *and* $k'(0^+)$ *is finite and non-zero, then* $\det(F(\alpha))$ *has a unique zero on the imaginary axis at* $\alpha = 0$ *and* $N^* - \mathbb{1}_{\{k'(0^+)>0\}}$ *zeros (counting multiplicities) in* $\mathbb{C}^{\mathrm{Re}>0}$.

We believe that the case when all the underlying Lévy processes $X_i(t)$ are subordinators, in other words, $N^* = 0$, is not of much interest. For completeness we make the following remark.

Remark 1.1. If $N^* = 0$ then either $X(t)$ is degenerate or $\det(F(\alpha))$ has no zeros in $\mathbb{C}^{\mathrm{Re}>0}$. In the latter case $\det(F(\alpha))$ has either a unique zero (at 0) or infinitely many distinct zeros on the imaginary axis.

Earlier we mentioned that, when considering MAPs with one-sided jumps, we can without loss of generality assume that there are no *negative* jumps. This claim is made precise in the following remark.

Remark 1.2. If $(X(t), J(t))$ is a MAP without positive jumps then $(-X(t), J(t))$ is a MAP without negative jumps. Let $F(\alpha)$ be the generator of the latter MAP. In the case of no positive jumps it is common to use $\phi_i(\alpha) := \log(\mathbb{E}e^{\alpha X_i(1)})$ and $\tilde{G}_{ij}(\alpha) := \mathbb{E}e^{\alpha U_{ij}}$ in the definition (3) of the generator. As a consequence $(X(t), J(t))$ has the same generator $F(\alpha)$. It is easy to see now that Theorem 1 and Theorem 2 also hold in the case of no positive jumps, but now N^* is defined as the number of processes which are not *downward* subordinators.

This paper is organized as follows. Section 2 presents two rather generally applicable results on the number of zeros of certain functions. In Section 3 we prove some analytic properties of the Laplace exponent of a Lévy process without negative jumps. Proofs of the main results, i.e., Theorem 1 and Theorem 2, are given in Section 4. This section also contains the analysis of the limiting behavior of the zeros when the killing rates $q_1, \ldots, q_N$ decrease to zero, see Theorem 10. The importance of the latter result lies in the fact that it does provide us with useful information on the number of zeros of $\det(F(\alpha))$ when $k'(0^+) = -\infty$ or $k'(0^+) = 0$, that is, when Theorem 2 does not apply. Finally, we give in Section 5 a brief outlook on envisaged applications of the main results.

Realizing that some parts of the paper are of a rather technical nature, we decided to include below the associated 'implication diagram', to enhance the paper's accessibility. Finally, we mention that Lemma 5

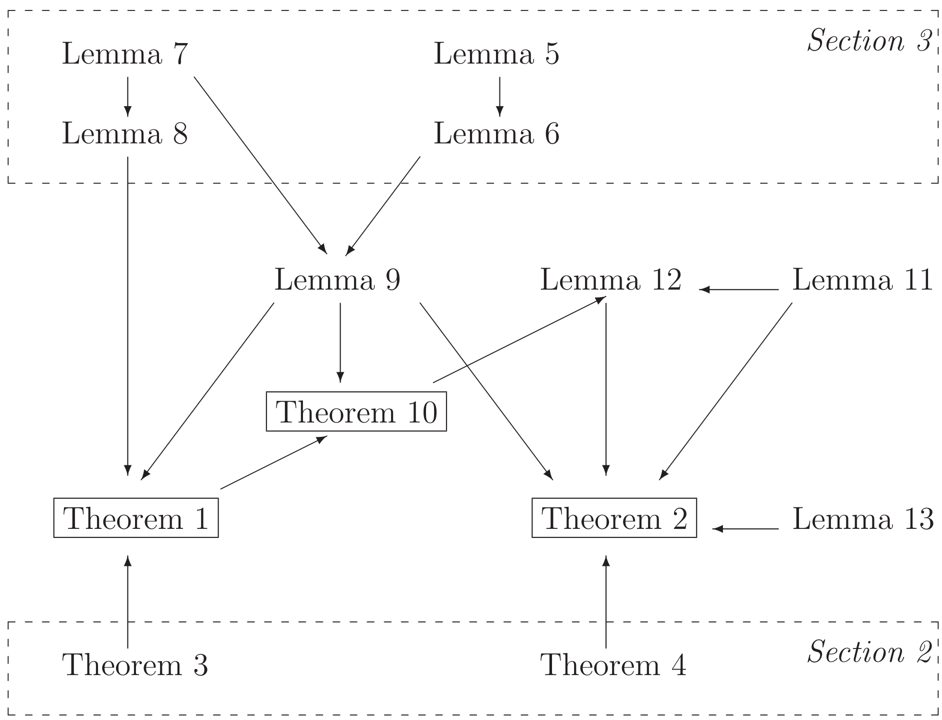

FIGURE 1. Implication diagram.

and the proofs of Lemma 6, Lemma 7 and Lemma 13 can be skipped during the first reading, because of the technicalities involved, as well as the fact that they do not provide much additional intuition.

2. ON THE NUMBER OF ZEROS OF CERTAIN FUNCTIONS

This section presents two general results on the number of zeros of certain functions (that is, functions satisfying a given set of assumptions) in a bounded domain. We would like to stress that we rely in this section on techniques that were developed earlier. To enhance the paper's transparency, we have isolated these results from the rest of the paper; for the sake of completeness their proofs are given in Appendix A.

In the following we assume that

$$\text{(5)} \qquad \begin{array}{l} D \subset \mathbb{C} \text{ is a bounded domain} \\ \text{which is the interior of a piecewise smooth simple loop } \gamma. \end{array}$$

One can find the basic notions of complex analysis in, e.g., [8]. We use $B(z, r)$ to denote an open ball of radius $r > 0$ centered at a point $z \in \mathbb{C}$.

The first theorem concerns the number of zeros of the determinant of a matrix-valued function in a bounded domain.

Theorem 3. *Let $M(z) = (m_{ij}(z))$ be a $n \times n$-matrix-valued function and $f(z) := \det(M(z))$. If*

A1 *$m_{ij}(z)$ are analytic on D and continuous on $D \cup \gamma$,*
A2 *$\forall i \in \{1, \ldots, n\}, z \in \gamma : |m_{ii}(z)| \geq \sum_{j \neq i} |m_{ij}(z)| \neq 0$,*
A3 *$f(z) \neq 0$ for $z \in \gamma$,*

then $f(z)$ and $\prod_{i=1}^{n} m_{ii}(z)$ have the same number of zeros in D.

Proof. See Appendix A. □

The main idea of the proof of Theorem 3 is taken from [7], where the authors use the following procedure. First they introduce an additional parameter t; the original function is retrieved by taking $t = 1$. For $t = 0$, however, the function has a nice form (that is, it nicely factorizes) making the analysis of the number of zeros easy. Then essentially continuity arguments are used to conclude that the number of zeros, as a function of the new parameter t, is constant. This basic idea used in a related context can be also found in [5] and [15].

It is noted that Theorem 3 does not allow $f(z)$ to be zero on the boundary of the domain. The analysis of the number of zeros becomes substantially harder if this assumption does not hold. In case of a simple zero on the boundary the following powerful result may be used.

Theorem 4 shows that if a function of interest and a given sequence of 'approximating' functions satisfy certain assumptions, then the functions in the tail of the sequence have the same number of zeros as the original function. This turns out to be useful in situations where the approximating functions have particular crucial properties (such as being analytic) which the original function does not necessarily have.

Theorem 4. *Let complex functions $f(z), f_n(z), n \in \mathbb{N}$ satisfy the following assumptions for some $z_0 \in \gamma$:*

A1 $f(z), f_n(z), n \in \mathbb{N}$ *are analytic on* D *and continuous on* $D \cup \gamma$,
A2 $f_n(z) \to f(z)$ *and* $f_n'(z) \to f'(z)$ *as* $n \to \infty$ *uniformly in* $z \in D$,
A3 $f(z_0) = f_1(z_0) = f_2(z_0) = \ldots = 0$ *and* $f(z) \neq 0, z \in \gamma \backslash \{z_0\}$,
A4 $\exists \epsilon > 0$, *such that* $f_n(z), n \in \mathbb{N}$ *are analytic on* $B(z_0, \epsilon)$,

$$f'(z_0) := \lim_{z \to z_0, z \in D} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, is non-zero and coincides with $\lim_{n \to \infty} f_n'(z_0)$.

Then for large enough n*, the functions* $f_n(z)$ *are non-zero on* $\gamma \backslash \{z_0\}$ *and have the same number of zeros in* D *as the function* $f(z)$.

Proof. The proof of this result relies on a technical argument borrowed from [1] and is given in Appendix A. □

3. Analytic Properties of the Laplace Exponent

In this section we discuss some analytic properties of the Laplace exponent of a Lévy process without negative jumps. These properties will turn out to be crucial in the analysis of the zeros of $\det(F(\alpha))$. Throughout this section we assume that $X(t)$ is a Lévy process without negative jumps, $(\alpha, \sigma, \nu(\mathrm{d}x))$ is the associated Lévy triple, and $\phi(\alpha)$ is the Laplace exponent of $X(t)$, cf. (1).

We start by recalling a number of well-known facts about Lévy processes, see [4] or [10] for a general reference. Firstly, it is well known that $\phi(\alpha)$ is finite on $\mathbb{C}^{\mathrm{Re} \geq 0}$. Due to dominated convergence, the derivative of $\phi(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}>0}$ can be computed by interchanging the differentiation and integration operators when using representation (1). It then follows easily that $\phi(\alpha)$ is analytic on $\mathbb{C}^{\mathrm{Re}>0}$. If it is additionally assumed that the jumps of $X(t)$ are bounded by a constant, then similar arguments show that $\phi(\alpha)$ is analytic on $\mathbb{C}$. Secondly, the following is well known:

$$\begin{array}{l} X(t) \text{ has paths of bounded variation iff} \\ \sigma = 0 \text{ and } \int_0^1 x\nu(\mathrm{d}x) < \infty. \end{array} \tag{6}$$

The Laplace exponent of such a process has a unique representation of the form

$$\phi(\alpha) = a'\alpha + \int_0^\infty (-1 + e^{-\alpha x})\nu(\mathrm{d}x), \tag{7}$$

where a' is usually referred to as the *drift term*. Note that any subordinator has paths of bounded variation, so it can be written in the form given in (7). We say that $X(t)$ is a *pure jump subordinator* if it is a subordinator with zero drift term. Finally, a compound Poisson process without drift is the same as a pure jump subordinator with finite Lévy measure. All the above facts can be found in [10].

In the rest of this section we restrict ourselves to $\mathbb{C}^{\mathrm{Re}\geq 0}$. The first lemma is similar to Proposition 2 on p. 16 of [4]. This lemma will only be used to prove Lemma 6.

Lemma 5. *It holds that*

$$\lim_{|\alpha|\to\infty,\alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} \alpha^{-2}\phi(\alpha) = \sigma^2/2. \tag{8}$$

Moreover, if $X(t)$ has paths of bounded variation, then

$$\lim_{|\alpha|\to\infty,\alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} \alpha^{-1}\phi(\alpha) = a', \tag{9}$$

where a' is the drift term as given in (7)*.*

Proof. First note that

$$|-1+e^{-y}+y| \leq 3|y|^2 \text{ for } y \in \mathbb{C}^{\mathrm{Re}\geq 0}.$$

This inequality holds, because if $|y| \geq 1$ then $|-1+e^{-y}+y| \leq 2+|y| \leq 3|y| \leq 3|y|^2$. On the other hand if $|y| < 1$ then using a power series expansion we have $|-1+e^{-y}+y| = |y^2/2! - y^3/3! + \ldots| \leq |y|^2(1/2! + |y|/3! + \ldots) \leq 3|y|^2$.

Now we see that $|\alpha|^{-2}|-1+e^{-\alpha x}+\alpha x| \leq 3x^2$ when $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}, \alpha \neq 0$ and $x > 0$. Since $\int_0^1 x^2\nu(\mathrm{d}x) < \infty$, dominated convergence gives

$$\lim_{|\alpha|\to\infty,\alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} \alpha^{-2}\int_0^1(-1+e^{-\alpha x}+\alpha x)\nu(\mathrm{d}x) = 0$$

and then (8) follows from (1). The second part can be proven in the same way by noting that $|-1+e^{-y}| \leq 2|y|$ for $y \in \mathbb{C}^{\mathrm{Re}\geq 0}$. $\square$

Lemma 6. *At least one of the following holds:*
(i) $\lim_{|\alpha|\to\infty,\alpha\in\mathbb{C}^{\mathrm{Re}\geq 0}} |\phi(\alpha)| = \infty$*, (ii)* $\mathrm{Re}(\phi(\alpha)) \leq 0$ *for all* $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$*.*

Proof. If the Gaussian component σ^2 (see (1)) is non-zero or $\phi(\alpha)$ can be written as in (7) with $a' \neq 0$, then the result follows trivially from Lemma 5. If, on the other hand, $\phi(\alpha) = \int_0^\infty(-1+e^{-\alpha x})\nu(\mathrm{d}x)$, then it is easy to see that $\mathrm{Re}(\phi(\alpha)) \leq 0$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. It follows from (6) that the only case left is the following:

$$\phi(\alpha) = a\alpha + \int_0^\infty(-1+e^{-\alpha x}+\alpha x\mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x),$$

where $\int_0^1 x\nu(\mathrm{d}x) = \infty$. We now show that in this case statement (i) holds. Note that $|\int_1^\infty(-1+e^{-\alpha x})\nu(\mathrm{d}x)|$ is bounded for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$, so we can truncate Lévy measure $\nu(\mathrm{d}x)$ to the interval $(0,1)$.

Step 1. We show that $\mathrm{Im}(\phi(u+\mathrm{i}v))/v \to \infty$ as $|v| \to \infty$ uniformly in $u \geq 0$. Note that

$$\mathrm{Im}(\phi(u+\mathrm{i}v)) = av + \int_0^1 (vx - e^{-ux}\sin(vx))\nu(\mathrm{d}x)$$

is an odd function in v, thus it is enough to consider the case when $v > 0$. Note also that $vx - e^{-ux}\sin(vx) \geq 0$ when $x > 0$. Thus we have for any $\epsilon > 0$

$$\frac{\mathrm{Im}(\phi(u+\mathrm{i}v))}{v} \geq a + \int_\epsilon^1 \left(x - \frac{e^{-ux}\sin(vx)}{v}\right)\nu(\mathrm{d}x)$$
$$\geq a + \int_\epsilon^1 x\nu(\mathrm{d}x) - \int_\epsilon^1 \frac{1}{v}\nu(\mathrm{d}x) \to a + \int_\epsilon^1 x\nu(\mathrm{d}x) \text{ as } v \to \infty.$$

Send ϵ to 0 and use $\int_0^1 x\nu(\mathrm{d}x) = \infty$ to complete the proof of the first step.

Step 2. We show that given any constants $M > 0$ and $V > 0$ one can choose a large $U > 0$, so that $\mathrm{Re}(\phi(u+\mathrm{i}v)) > M$ for all u and v such that $|v| \leq V$ and $u > U$. First recall that the process we consider has paths of unbounded variation and thus is not a subordinator. It is well known that in this case $\phi(u) \to \infty$ as $u \to \infty$. Next note that

$$\frac{\partial \mathrm{Re}(\phi(u+\mathrm{i}v))}{\partial v} = -\int_0^1 xe^{-ux}\sin(vx)\nu(\mathrm{d}x)$$

and

$$\left|\int_0^1 xe^{-ux}\sin(vx)\nu(\mathrm{d}x)\right| \leq V\int_0^1 x^2\nu(\mathrm{d}x) < \infty,$$

when $|v| \leq V$. So it is enough to choose U such that $\phi(u) > M + V^2\int_0^1 x^2\nu(\mathrm{d}x)$ for all $u > U$.

Now pick any $M > 0$. The result of Step 1 implies that there exists a large enough $V > 0$, so that $|\mathrm{Im}(\phi(u+\mathrm{i}v))| > M$ for all $u \geq 0$ and all v satisfying $|v| > V$. Combining this with the result of Step 2, we see that there exists $U > 0$, such that $|\phi(\alpha)| > M$ when $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ and $|\alpha| > U + V$, which implies (i). □

The above proof provides more information than stated in the lemma. Namely, we can add that the first statement is true at least for those $X(t)$ which are not pure jump subordinators. If $X(t)$ is a compound Poisson process without drift then $|\phi(r)|$ is bounded for all $r \in [0,\infty)$, and thus the first statement of the above lemma does not hold. We omit a full discussion of this issue, because such a result does not play a role in the following.

Lemma 7. *For $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}\backslash\mathbb{R}$ it holds that (i) $\phi(\alpha) \notin (0,\infty)$, (ii) $\phi(\alpha) \neq 0$ if $X(t)$ is not a compound Poisson process without drift.*

Proof. Let $u \geq 0, v \neq 0$ and assume that $\phi(u + \mathrm{i}v) \geq 0$, then

$$au + \frac{1}{2}\sigma^2(u^2 - v^2) + \int_0^\infty (-1 + e^{-ux}\cos(vx) + ux\mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x) \geq 0,$$

$$av + \sigma^2 uv + \int_0^\infty (-e^{-ux}\sin(vx) + vx\mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x) = 0.$$

Divide the second equation by v, multiply it by u and subtract it from the first inequality to obtain:

$$\frac{1}{2}\sigma^2(-u^2 - v^2) + \int_0^\infty \Big(\frac{u}{v}e^{-ux}\sin(vx) - 1 + e^{-ux}\cos(vx)\Big)\,\nu(\mathrm{d}x) \geq 0.$$

Now note that

$$\cos r + \frac{q}{r}\sin r \leq e^q \ \text{ when } \ q \geq 0, r \neq 0$$

with equality when $q = 0$ and $\cos r = 1$. This shows that the integrand is non-positive, which proves (i).

Finally, from the above we conclude that $\phi(u + \mathrm{i}v) = 0$ if and only if either (A) $X(t) \equiv 0$, or (B) $\sigma^2 = 0$, $u = 0$, and

$$\int_0^\infty (1 - \cos(vx))\nu(\mathrm{d}x) = 0, \quad av + \int_0^\infty (-\sin(vx) + vx\mathbb{1}_{\{x<1\}})\nu(\mathrm{d}x) = 0.$$

It can be further deduced that in the latter case $a = -\int_0^1 x\nu(\mathrm{d}x)$. Therefore we have that

$$\phi(\alpha) = \int_0^\infty (-1 + e^{-\alpha x})\nu(\mathrm{d}x)$$

with $\int_0^1 x\nu(\mathrm{d}x) < \infty$, which means that $X(t)$ is a compound Poisson process without drift. $\square$

Note that if $X(t)$ is not identically zero and for some $\alpha_0 \in \mathbb{C}^{\mathrm{Re}\geq 0}\backslash\mathbb{R}$ it holds that $\phi(\alpha_0) = 0$, then Lemma 7 *(ii)* implies that $X(t)$ is a compound Poisson process without drift. Moreover, the above proof shows that α_0 lies on the imaginary axis.

We finish this section with a simple lemma.

Lemma 8. *For any $c > 0$ function $\phi(\alpha) - c$ has no zeros in $\mathbb{C}^{\mathrm{Re}\geq 0}$ if $X(t)$ is a subordinator, and has a unique simple zero otherwise.*

Proof. Lemma 7 shows that $\phi(\alpha) \neq c$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}\backslash\mathbb{R}$. It remains to analyze the case when $\alpha \geq 0$. It is well known that $\phi(0) = 0$ and $\phi(\alpha), \alpha \geq 0$ is convex. The claim then follows from another well-known fact, viz. that $\phi(\alpha) \leq 0$ if $X(t)$ is a subordinator and $\lim_{\alpha\to\infty} \phi(\alpha) = \infty$ otherwise. □

From Lemma 8 we see that a special role is played by *subordinators*, which was to be expected in view of Theorem 2.

4. Proofs of The Main Results

The primary goal of this section is to prove our main results, viz. Theorems 1 and 2. Throughout this section we assume that $(X(t), J(t))$ is a MAP without negative jumps, and $F(\alpha)$ is the associated generator as defined in (3). In the following we extensively use a bounded domain D_R, defined through

$$D_R := \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) > 0, |\alpha| < R\}, \tag{10}$$

and its boundary γ_R. Note that this domain satisfies (5). Furthermore, recall that a square $n \times n$ matrix $M = (m_{ij})$ is called *non-strictly diagonally dominant* if $\forall i : |m_{ii}| \geq \sum_{j\neq i} |m_{ij}|$. If, moreover, M is irreducible and at least one of the above inequalities is strict then M is called *irreducibly diagonally dominant.* It is well-known that an irreducibly diagonally dominant matrix is non-singular, see for instance p. 226 of [13].

The following lemma is a key result on the way to prove the main theorems. It allows us to restrict our attention to a bounded domain D_R instead of considering the whole $\mathbb{C}^{\mathrm{Re}\geq 0}$. Note that this is an essential prerequisite required by Theorems 3 and 4. An important role in the lemma is played by a subset $\mathcal{S}$ of the set of functions $g : \mathbb{C}^{\mathrm{Re}\geq 0} \mapsto \mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$, where it is assumed that all the functions in $\mathcal{S}$ are bounded in absolute value by a common constant. Different choices of $\mathcal{S}$, suitable to the problem at hand, are made in the following.

Lemma 9. *For $R > 0$ large enough it holds that for all $g_1, \dots, g_N \in \mathcal{S}, \alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \backslash D_R$ the matrix $Q + \mathrm{diag}(\phi_1(\alpha) - g_1(\alpha), \dots, \phi_N(\alpha) - g_N(\alpha))$ is irreducibly diagonally dominant, if either $g_i(\alpha) \neq 0$ for some i, or $N^* > 0$ and $\alpha \neq 0$.*

Proof. Choose $i \in \{1, \dots, N\}$ and note that

$$e^{\mathrm{Re}(\phi_i(\mathrm{i}r))} = |e^{\phi_i(\mathrm{i}r)}| = |\mathbb{E}e^{-\mathrm{i}rX_i(1)}| \leq 1, r \in \mathbb{R}.$$

Therefore, $\mathrm{Re}(\phi_i(\alpha)) \leq 0$ for all α on the imaginary axis. This statement and Lemma 6 imply that there exists $R_i > 0$, such that, $\forall g_i \in \mathcal{S}, \alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \backslash D_{R_i}$ it holds that

$$|q_{ii} + \phi_i(\alpha) - g_i(\alpha)| > -q_{ii} \text{ or } \mathrm{Re}(\phi_i(\alpha)) \leq 0, \tag{11}$$

where the $q_{ii} = -\sum_{j\neq i} q_{ij} < 0$ $(i = 1, \ldots, N)$ are the diagonal elements of the intensity matrix Q. Note, we used that the functions in $\mathcal{S}$ are bounded in absolute value by a common constant. Now $\mathrm{Re}(\phi_i(\alpha)) \leq 0$ implies $|q_{ii} + \phi_i(\alpha) - g_i(\alpha)| \geq -q_{ii}$, because $\mathrm{Re}(g_i(\alpha)) \geq 0$. Hence for $R = \max\{R_1, \ldots, R_N\}$ our matrix is non-strictly diagonally dominant.

Assume for a moment that $\forall i : |q_{ii} + \phi_i(\alpha) - g_i(\alpha)| = -q_{ii}$. Then from (11) it follows that for all i we have $\mathrm{Re}(\phi_i(\alpha)) \leq 0$. Since by assumption $g_i(\alpha)$ takes values in $\mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$, we see that $g_i(\alpha) = 0$ and $\phi_i(\alpha) = 0$. To finish the proof, it is enough to show that $N^* > 0$ and $\alpha \neq 0$ imply that $\phi_i(\alpha) \neq 0$ for some i. Take i, such that $X_i(t)$ is not a subordinator, which is possible due to $N^* > 0$. Then $\phi_i(\alpha) \neq 0$ for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \backslash \mathbb{R}$ by Lemma 7 (ii). Considering $\phi_i(r), r \in \mathbb{R}^+$, we note that $\lim_{r\to\infty} \phi_i(r) = \infty$, thus $\phi_i(r)$ has no zeros larger than some constant C_i. Clearly, we were initially able to choose $R > C_i$. Hence $\phi_i(\alpha) \neq 0$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \backslash (D_R \cup \{0\})$, which concludes the proof. $\square$

Note that if matrix $Q + \mathrm{diag}(\phi_1(\alpha) - g_1(\alpha), \ldots, \phi_N(\alpha) - g_N(\alpha))$ is irreducibly diagonally dominant, then so is

$$Q \circ \tilde{G}(\alpha) + \mathrm{diag}(\phi_1(\alpha) - g_1(\alpha), \ldots, \phi_N(\alpha) - g_N(\alpha)),$$

because $0 < |\tilde{G}_{ij}(\alpha)| \leq 1$ and $\tilde{G}_{ii}(\alpha) = 1$ for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. Moreover, it is easy to see from the above proof that $\det(F(\alpha)) \equiv 0$ on $\mathbb{C}^{\mathrm{Re}>0}$ if and only if $\forall i, j : \phi_i(\alpha) \equiv 0$ and $\tilde{G}_{ij}(\alpha) \equiv 1$, which is the same as $X(t) \equiv 0$. Furthermore, it is a trivial consequence of the above lemma that $F(\alpha)$ is non-singular for all α on the imaginary axis except $\alpha = 0$, whenever $N^* > 0$. On the other hand, a simple non-degenerate example of $X(t)$ can be constructed with $N^* = 0$, such that $F(\alpha)$ is singular at infinitely many points on the imaginary axis (let $X_i(t), i \in \{1, \ldots, N\}$ be a Poisson processes and set $U_{ij} \equiv 0$).

4.1. **Killing is Present.** We are ready to prove our first main result, Theorem 1. The statement of the theorem is an immediate consequence of Lemma 9, Theorem 3 and Lemma 8.

Proof (of Theorem 1). Apply Lemma 9 with $\mathcal{S}$ containing constant functions equal to killing rates q_i to see that there exists $R > 0$, such that $F(\alpha) - \mathrm{diag}(\boldsymbol{q})$ is irreducibly diagonally dominant (and thus non-singular) for $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0} \backslash D_R$, because $\boldsymbol{q} \neq \boldsymbol{0}$. Now we can apply Theorem 3 to show that $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}))$ and $\prod_{i=1}^{N}(q_{ii} + \phi_i(\alpha) - q_i)$ have the same number of zeros in D_R. But the latter function has no zeros in $\mathbb{C}^{\mathrm{Re}\geq 0} \backslash D_R$ (use diagonal dominance) and has exactly N^* zeros in D_R according to the statement of Lemma 8. $\square$

Next we study the limiting behavior of the zeros of $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}))$ in $\mathbb{C}^{\mathrm{Re}>0}$ as all the killing rates converge to 0. This is an important step in the analysis of the case of no killing. To be precise, we let $\boldsymbol{q}_n, n \in \mathbb{N}$ be a sequence of nonnegative vectors with at least one positive component, such that $||\boldsymbol{q}_n|| \to 0$ as $n \to \infty$ with $||\cdot||$ denoting the Euclidean norm.

Theorem 10. *If $N^* > 0$ then the zeros of $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}_n))$ in $\mathbb{C}^{\mathrm{Re}>0}$ converge as $n \to \infty$ to some limit points $z_1, \ldots, z_{N^*} \in \mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$ (not necessarily distinct). The set*

$$Z := \bigcup_{i=1}^{N^*} \{z_i\} \cup \{0\}$$

is the set of all the distinct zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$, and the multiplicity of an element $z \in Z, z \neq 0$ is given by the number of zeros of $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}_n))$ converging to z.

Proof. Let Z_0 be the set of all the distinct zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$. Recall that $\det(F(\alpha))$ is *not* identically zero, because $N^* > 0$. Now Hurwitz's theorem (see p. 173 of [8]) shows that every zero of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ (analyticity region) of multiplicity m is a limit point of exactly m zeros of $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}_n))$. Recall also that $\det(F(\alpha))$ has a unique zero on the imaginary axis, which is at 0. Clearly, Z_0 can have at most finitely many elements, so it remains to show that for sufficiently large n_0 the zeros of $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}_n)), n > n_0$ are arbitrarily close to the elements of Z_0. Suppose this is not true. So we can pick a sequence of the zeros which are at least ϵ away from the elements of Z_0.

Apply Lemma 9 with $\mathcal{S}$ containing the components of all the vectors $\boldsymbol{q}_n, n \in \mathbb{N}$ (these are all bounded by some constant) to see that we can choose $R > 0$, such that the zeros of $\det(F(\alpha) - \mathrm{diag}(\boldsymbol{q}_n)), n \in \mathbb{N}$ in $\mathbb{C}^{\mathrm{Re}>0}$ are all in D_R. Thus, given the above sequence of zeros, we can choose a converging subsequence (D_R is bounded) with some limit z_0. Clearly, $\det(F(z_0)) = 0$ and $z_0 \in \mathbb{C}^{\mathrm{Re}\geq 0}$ which means that $z_0 \in Z_0$, and, thus, the above sequence can not exist. □

It is easy to see that the above proof also shows that if $N^* = 0$ then either $X(t)$ is degenerate or $F(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}>0}$ is non-singular. Considering the question about the number of zeros of $\det(F(\alpha))$, we note that there is essentially one thing left unknown: the number of zeros which converge to 0 as all the killing rates go to 0. We address this seemingly simple question in the sequel of this section.

4.2. **No Killing.** We now concentrate on the proof of Theorem 2. The statement of Theorem 2 shows that a critical role is played by the sign of the asymptotic drift. The next lemma presents a relation between the sign of the asymptotic drift and the sign of $\det(F(0^+))'$, the right-sided derivative of $\det(F(r)), r \geq 0$ at 0.

Lemma 11. *It holds that*

$$\operatorname{sign}(k'(0^+)) = (-1)^{N-1}\operatorname{sign}(\det(F(0^+))'). \tag{12}$$

Proof. Let $\lambda_1(\alpha), \ldots, \lambda_{N-1}(\alpha), \lambda_N(\alpha) = k(\alpha)$ be the eigenvalues of $F(\alpha)$, then $\det(F(\alpha)) = \prod_{i=1}^N \lambda_i(\alpha)$. So we have

$$\det(F(0^+))' = k'(0^+) \prod_{i=1}^{N-1} \lambda_i(0),$$

because $k(0) = 0$. Hence it is enough to show that $\prod_{i=1}^{N-1}(-\lambda_i(0)) > 0$.

Take any $i < N$ and set $\lambda = \lambda_i(0)$. If λ is real then it is negative, since $k(0) = 0$ is a simple eigenvalue with the maximal real part. If, however, λ has a non-zero imaginary part and is of multiplicity m, then there is an eigenvalue $\bar{\lambda}$ (complex conjugate of λ) of multiplicity m. The product of these $2m$ eigenvalues is a positive number. □

The next lemma specifies the number of zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ under the additional assumption of analyticity.

Lemma 12. *Let $N^* > 0$ and $k'(0^+) \neq 0$. If the function $\det(F(\alpha))$ is analytic in some open neighborhood of 0, then it has $N^* - \mathbb{1}_{\{k'(0^+)>0\}}$ zeros in $\mathbb{C}^{\mathrm{Re}>0}$.*

Proof. Consider the setting of Theorem 10. In view of this result, we only need to show the following: (A) if $k'(0^+) > 0$ then exactly one zero out of the N^* zeros of $\det(F(\alpha) - \operatorname{diag}(\boldsymbol{q}_n))$ in $\mathbb{C}^{\mathrm{Re}>0}$ converges to 0, and (B) if $k'(0^+) < 0$ then none of these zeros converges to 0. Using Lemma 11 we note that $k'(0^+) \neq 0$ implies $\det(F(0))' \neq 0$, so the multiplicity of the zero of $\det(F(\alpha))$ at 0 is 1. The assumption of analyticity in the neighborhood of 0 allows us to apply Hurwitz's theorem to show that there is exactly one zero of $\det(F(\alpha) - \operatorname{diag}(\boldsymbol{q}_n))$ converging to 0. Note that this zero either converges from $\mathbb{C}^{\mathrm{Re}>0}$ or from $\mathbb{C}^{\mathrm{Re}<0}$. So it remains to show that the first case corresponds to $k'(0^+) > 0$ and the second to $k'(0^+) < 0$. Before we proceed we note that $(-1)^N \det(F(0) - \operatorname{diag}(\boldsymbol{q}_n)) > 0$, which follows by an argument similar to the one appearing in the proof of Lemma 11.

We restrict ourselves to the domain of reals and assume without loss of generality that $k'(0^+) > 0$. So $(-1)^{N-1} \det(F(0))' > 0$ by

Lemma 11. Now for any small $\delta > 0$ we can pick $x \in (0, \delta)$, such that $(-1)^{N-1}\det(F(x)) > 0$. Hence for large enough n the inequality $(-1)^{N-1}\det(F(x) - \mathrm{diag}(\boldsymbol{q}_n)) > 0$ holds. This means that $\det(F(x) - \mathrm{diag}(\boldsymbol{q}_n))$ and $\det(F(0) - \mathrm{diag}(\boldsymbol{q}_n))$ have opposite signs, thus by continuity there exists $x_n \in (0, x)$, such that, $\det(F(x_n) - \mathrm{diag}(\boldsymbol{q}_n)) = 0$. This concludes the proof. □

It is noted that the second paragraph of the above proof uses an idea from Proposition 9 of [7].

Now we outline the proof of Theorem 2. We start by constructing a sequence of functions, which approximates $\det(F(\alpha))$. Then Lemma 9 is applied to bound the region of zeros of the above functions. Next, using Theorem 4, we relate the number of zeros of $\det(F(\alpha))$ to the number of zeros of an approximating function from the tail of the sequence. Finally, due to the enlarged region of analyticity of the approximating functions, Lemma 12 can be applied to obtain the latter number.

In order to implement the above ideas, we introduce a sequence of 'truncations' of $(X(t), J(t))$. For every $n \in \mathbb{N}$ define a MAP $(X^{[n]}(t), J(t))$ through

$$\nu_i^{[n]}(\mathrm{d}x) := \mathbb{1}_{\{x \le n\}}\nu_i(\mathrm{d}x) \text{ and } U_{ij}^{[n]} := U_{ij}\mathbb{1}_{\{U_{ij} \le n\}}, \tag{13}$$

where the other characteristics are kept unchanged. Using self-evident notation, we note that $\tilde{G}_{ij}^{[n]}(\alpha)$, $\phi_i^{[n]}(\alpha)$, and thus $\det(F^{[n]}(\alpha))$ are analytic on $\mathbb{C}$ (see the introduction to Section 3). Next we consider a sequence of functions $\det(F^{[n]}(\alpha))$ and prove some convergence results required by Theorem 4. In the following lemma we implicitly assume that the derivative of any function $f(\alpha)$ at a point α_0 on the imaginary axis is understood in the following sense: $\lim_{h\to 0, h\in\mathbb{C}^{\mathrm{Re}\ge 0}}(f(\alpha_0+h) - f(\alpha_0))/h$. It is noted that $f(\alpha)$ may be infinite for all $\alpha \in \mathbb{C}^{\mathrm{Re}<0}$, and yet $f'(\alpha_0)$ is well-defined and finite.

Lemma 13. *If $\mathbb{E}X_i(1)$ and $\mathbb{E}U_{ij}$ exist for all i and j, then for any $R > 0$ it holds that*

$$\det(F^{[n]}(\alpha)) \to \det(F(\alpha)) \textit{ and } \det(F^{[n]}(\alpha))' \to \det(F(\alpha))' \tag{14}$$

as $n \to \infty$ uniformly in $\alpha \in D_R$. Moreover,

$$\lim_{n\to\infty} \det(F^{[n]}(0))' = \det(F(0))' \in (-\infty, \infty). \tag{15}$$

Proof. The statements of the lemma follow immediately from the following two observations: (A) $\phi_i^{[n]}(\alpha), \tilde{G}_{ij}^{[n]}(\alpha)$ as well as their derivatives converge to the corresponding 'non-truncated' functions as $n \to \infty$ uniformly in $\alpha \in \mathbb{C}^{\mathrm{Re}\ge 0}$, and (B) $|\phi_i(\alpha)|, |\phi_i'(\alpha)|, |\tilde{G}_{ij}(\alpha)|$ and $|\tilde{G}_{ij}'(\alpha)|$ are bounded on D_R. Statement (B) follows from (A) and the fact that the corresponding truncated functions are bounded for every n, which is true, because D_R is bounded and functions $\phi_i^{[n]}(\alpha)$ and $\tilde{G}_{ij}^{[n]}(\alpha)$ are analytic on $\mathbb{C}$.

With regard to statement (A) we only show uniform convergence of the derivatives of $\phi_i^{[n]}(\alpha)$, because the other results are either trivial or follow by a similar argument. That is we show that $\Delta_n(\alpha) := |\partial\phi_i^{[n]}(\alpha)/\partial\alpha - \partial\phi_i(\alpha)/\partial\alpha| \to 0$ as $n \to \infty$ uniformly in $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$. Recall that $\mathbb{E}X_i(1) < \infty$ implies $\int_1^\infty x\nu_i(\mathrm{d}x) < \infty$. Now use dominated convergence to see that

$$\begin{aligned}\Delta_n(\alpha) &= \left|\frac{\partial \int_n^\infty (-1+e^{-\alpha x})\nu_i(\mathrm{d}x)}{\partial\alpha}\right| = \left|\int_n^\infty xe^{-\alpha x}\nu_i(\mathrm{d}x)\right| \\ &\leq \int_n^\infty x\nu_i(\mathrm{d}x),\end{aligned}$$

which goes to 0 as $n \to \infty$. □

It is not difficult to show using (4) that $k'(0^+) \in [-\infty, \infty)$ and, moreover,

(16) $\quad k'(0^+)$ is finite if and only if $\forall i,j : \mathbb{E}X_i(1)$ and $\mathbb{E}U_{ij}$ exist.

Hence the above lemma can be applied whenever $k'(0^+) \neq -\infty$.

We are now ready to prove Theorem 2. In this proof we use $X^{[\infty]}(t)$ to denote the process $X(t)$.

Proof (of Theorem 2). Note that for all $n \in \mathbb{N} \cup \{\infty\}$ it holds that

$$\phi_i^{[n]}(\alpha) = \phi_i^{[1]}(\alpha) - \left(-\int_1^n (-1+e^{-\alpha x})\nu_i(\mathrm{d}x)\right) = \phi_i^{[1]}(\alpha) - g_i^n(\alpha),$$

where $g_i^n(\alpha) := -\int_1^n (-1+e^{-\alpha x})\nu_i(\mathrm{d}x)$. It is an easy exercise to show that for all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}$ functions $g_i^n(\alpha)$ take values in $\mathbb{C}^{\mathrm{Re}>0} \cup \{0\}$ and are bounded in absolute value by a common constant (take, e.g., $2\max_i\{\nu_i(1,\infty)\}$). So we can apply Lemma 9 to the MAP $(X^{[1]}(t), J(t))$ with $\mathcal{S} = \{g_i^n : i \in \{1,\ldots,N\}, n \in \mathbb{N} \cup \{\infty\}\}$ to show that there exists $R > 0$, such that the matrices $Q + \mathrm{diag}(\phi_1^{[n]}(\alpha), \ldots, \phi_N^{[n]}(\alpha))$ are irreducibly diagonally dominant for all $n \in \mathbb{N} \cup \{\infty\}$ and all $\alpha \in \mathbb{C}^{\mathrm{Re}\geq 0}\backslash(D_R \cup \{0\})$. Hence the zeros of $\det(F^{[n]}(\alpha)), n \in \mathbb{N} \cup \{\infty\}$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$ are all in $D_R \cup \{0\}$. Now use (16) and Lemma 13 to see that Theorem 4 applies. So it remains to analyze the number of zeros of $\det(F^{[n]}(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$ for a large n.

First note that $X_i(t)$ is a subordinator if and only if $X_i^{[n]}(t)$ is a subordinator. Thus the number of non-subordinators corresponding to any truncated MAP is N^*. Secondly, Lemma 13 and Lemma 11 show that $k^{[n]\prime}(0^+)$ has the same sign as $k'(0^+)$ for n large enough. Now Lemma 12 completes the proof. □

5. Discussion

This research was motivated by open issues in the analysis of the steady-state buffer content of queues with MAP input. There one is interested in determining the stationary distribution of a *reflected* MAP $(Z(t), J(t))$, where $(X(t), J(t))$ is a MAP without negative jumps and $Z(t) := X(t) - \inf\{X(s) : 0 \leq s \leq t\}$. For stability one has to require that $(X(t), J(t))$ has a negative asymptotic drift, which is the same as $k'(0^+) > 0$, see (4). Otherwise, the limiting distribution of $Z(t)$ as $t \to \infty$ is degenerate at ∞.

Let (Z, J) be a random vector distributed as the stationary version of $(Z(t), J(t))$. It is shown in [3] that the Laplace-Stieltjes transform of (Z, J) can be expressed in terms of the generator $F(\alpha)$ and a generally unknown *row* vector $\boldsymbol{\ell}$. More precisely,

$$\mathbb{E}[e^{-\alpha Z}; J] := (\mathbb{E}e^{-\alpha Z}\mathbb{1}_{\{J=1\}}, \ldots, \mathbb{E}e^{-\alpha Z}\mathbb{1}_{\{J=N\}}) = \alpha\boldsymbol{\ell}F(\alpha)^{-1}. \tag{17}$$

The authors observe that the computation of vector $\boldsymbol{\ell}$ in general is a difficult problem. In the first part of this section we aim to shed some light on a relatively straightforward procedure which yields $\boldsymbol{\ell}$ using constraints induced by the zeros of $\det(F(\alpha))$.

The analysis of the steady-state buffer content $(X(t), J(t))$ is simplified if it is assumed that none of the processes $X_i(t)$ is a subordinator. In order to demonstrate the main ideas in a clear way, we assume in the rest of this section that $N^* = N$.

5.1. A Straightforward Approach.

According to Theorem 2 the function $\det(F(\alpha))$ has $N^* - 1 = N - 1$ zeros in $\mathbb{C}^{\mathrm{Re}>0}$ and a unique simple zero on the imaginary axis at 0 (recall that $k'(0^+) > 0$ is the stability condition). Suppose there are k distinct zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$. Denote them by $\alpha_1, \ldots, \alpha_k$ and let $m_1, \ldots, m_k$ be the corresponding multiplicities, so $\sum_{i=1}^{k} m_i = N - 1$. Let $\boldsymbol{v}_i^1, \ldots, \boldsymbol{v}_i^{n_i}$ be a basis of the (right) null space of $F(\alpha_i)$. It is known that the dimension of the null space of $F(\alpha_i)$ satisfies $1 \leq n_i \leq m_i$. The right inequality is shown by expressing the n-th derivative of $\det(F(\alpha))$ at α_i as a sum of terms containing the minors of matrix $F(\alpha_i)$ of dimensions not smaller than $N - n$, and noting that these minors are all zero for $n < n_i$.

We argue in Section 5.2 below that

(18) the vectors $\boldsymbol{v}_i^j, 1 \leq i \leq k, 1 \leq j \leq n_i$ are linearly independent.

Now we assume that

$$\forall i : n_i = m_i. \tag{19}$$

Note that this assumption holds, e.g., when all the zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ are distinct. It is easy to see that (17) implies $\boldsymbol{\ell}\boldsymbol{v}_i^j = 0$, so $\boldsymbol{\ell}$ is orthogonal to the $(N-1)$-dimensional linear space spanned by the vectors $\boldsymbol{v}_i^j$. Thus $\boldsymbol{\ell}$ can be identified up to a scalar. Finally, a limiting argument shows that $\boldsymbol{\ell}\boldsymbol{e} = k'(0^+)$, where $\boldsymbol{e}$ is a vector of ones, see [3]. Hence under Assumption (19) the vector $\boldsymbol{\ell}$ can be uniquely identified, and thus (17) fully determines the stationary distribution of the reflected MAP $(Z(t), J(t))$.

It is important to note that Assumption (19) does not hold in general. Thus our straightforward approach does not always lead to the unique identification of $\boldsymbol{\ell}$. We illustrate this with the following example. On the other hand, it can be shown that Assumption (19) always holds for some classes of MAPs, see [16] and [9] for examples.

Example 5.1. We specify the jump-free MAP $(X(t), J(t))$ as follows. Let

$$U_{ij} \equiv 0, \phi_1(\alpha) = \alpha, \phi_2(\alpha) = \alpha + \alpha^2, \phi_3(\alpha) = \frac{2}{5}\alpha, \text{ and}$$

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Note that the drifts of the underlying Lévy processes are $-1, -1, -2/5$ respectively. So the asymptotic drift of the MAP is negative. Moreover, none of the Lévy processes $X_i(t)$ is a subordinator. Clearly, $\det(F(\alpha))$ is a fourth order polynomial. The zeros are $-3/2, 0, 2, 2$. Thus there is one multiple zero $(\alpha_1 = 2, m_1 = 2)$ in $\mathbb{C}^{\mathrm{Re}>0}$. The null space of $F(2)$, however, has dimension $n_1 = 1$.

5.2. **More Related Results.** Define a hitting time of the level $-x$ for each $x \geq 0$ by

$$\tau_x^- := \inf\{t \geq 0 : X(t) \leq -x\}.$$

It is well known that $J(\tau_x^-)$ is a Markov chain. Let Λ be its intensity matrix and let $\boldsymbol{\pi}_\Lambda$ be the stationary distribution of Λ. This intensity matrix plays a crucial role in the analysis of fluctuations of one-sided MAPs, see e.g. [11]. It is, however, an open problem to establish an explicit expression for Λ.

In the following we state some results based on the main theorems of the present paper. These results are not trivial and will be part of a forthcoming paper. Firstly, there is a one-to-one correspondence between the eigenvalues of Λ and the zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}\geq 0}$. More precisely, $\alpha_0 \in \mathbb{C}^{\mathrm{Re}\geq 0}$ is a zero of $\det(F(\alpha))$ of multiplicity m if and only if $-\alpha_0$ is an eigenvalue of Λ of algebraic multiplicity m. Secondly, the null space of $F(\alpha_0)$ coincides with the eigenspace of Λ corresponding to the eigenvalue $-\alpha_0$. This is a full specification of the eigensystem of Λ. It is noted that the above correspondence was shown in [14] for the case of Markov-modulated rate models, that is, when $F(\alpha) = Q + R\alpha$ and $R := \mathrm{diag}(r_1, \dots, r_N)$.

It is important to note that a number of properties of the zeros follow immediately from the above specified correspondence. For example, the zeros of $\det(F(\alpha))$ in $\mathbb{C}^{\mathrm{Re}>0}$ are symmetric with respect to the real axis. It also follows that the null spaces of $F(\alpha)$ are orthogonal, which proves (18). Moreover, if assumption (19) holds then we obtain an explicit diagonal-form expression of Λ.

Furthermore, there is a very close relation between Λ and the unknown vector $\boldsymbol{\ell}$, namely

$$\boldsymbol{\ell} = k'(0^+)\boldsymbol{\pi}_\Lambda. \tag{20}$$

This formula follows by a limiting argument from the observation of [6] that $\boldsymbol{\ell}$ is a left null-eigenvector of Λ. Alternatively, one can derive it using the probabilistic interpretation of $\boldsymbol{\ell}$ given in [3] or using Wiener-Hopf factorization. Now one can note that $\boldsymbol{\ell}\boldsymbol{v}_i^j = (k'(0^+)\boldsymbol{\pi}_\Lambda)(-\Lambda\boldsymbol{v}_i^j/\alpha_i) = 0$. So, as it was argued in Section 5.1, $\boldsymbol{\ell}$ is orthogonal to the null spaces of $F(\alpha), \alpha \in \mathbb{C}^{\mathrm{Re}>0}$.

The above discussion was simplified assuming that $N^* = N$. We finish by giving some comments about the case $0 < N^* < N$. It is easy to see that in this case the states of $J(\tau_x^-)$ corresponding to the subordinators are unreachable, thus the matrix Λ is N^*-dimensional. Moreover, it is known that the components of $\boldsymbol{\ell}$ corresponding to the subordinators are equal 0, so we have exactly N^* unknowns. On the other hand, Theorem 2 states that there are exactly N^* zeros in $\mathbb{C}^{\mathrm{Re}\geq 0}$. This correspondence allows us to proceed in a similar way as in the case of $N^* = N$, albeit with some additional technicalities.

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Appendix A. Proofs of the Results from Section 2

Proof (of Theorem 3). Define $f(z,t) := \det(M_t(z))$ for $t \in [0,1]$, where $M_t(z)$ is a $n \times n$ matrix obtained from $M(z)$ by multiplying the off-diagonal elements by t. Note that $f(z,0) = \prod_{i=1}^n m_{ii}(z)$ and $f(z,1) = f(z)$. Moreover, $f(z,t) \neq 0$ for all $z \in \gamma$. To see this use assumption A3 when $t = 1$ and A2 when $t < 1$. In the second case $M_t(z), z \in \gamma$ is strictly diagonally dominant and thus non-singular, see p. 226 of [13]. Since $f(z,t)$ is a continuous function on $\overline{D} \times [0,1]$, one can choose $\delta > 0$, such that $f(z,t) \neq 0$ on $[0,1] \times E_\delta$, where $E_\delta := \{z \in D : y \in \gamma, |z-y| < \delta\}$ is a boundary strip of D. This is true, because otherwise there exists a converging sequence of the zeros with a limit (z^*,t^*), such that $z^* \in \gamma$ and $f(z^*,t^*) = 0$.

Let n_t denote the number of zeros (counting multiplicities) of the function $f_t(z) := f(z,t)$ in D. Take some piecewise-smooth simple loop $\gamma' \subset E_\delta$ (which is possible) and write using the argument principle

$$n_t = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma'} \frac{f_t'(z)}{f_t(z)} \mathrm{d}z.$$

Note that n_t is integer-valued and continuous, because $f_t'(z)/f_t(z)$ is continuous in t uniformly in $z \in \gamma'$. This means that n_t is constant. □

Proof (of Theorem 4). We start by noting that there exists $\delta > 0$, such that $f(z) \neq 0$ on $E_\delta := \{z \in D : y \in \gamma, |z-y| \leq \delta\}$, because otherwise there would exist a converging sequence (z_n) in D with a limiting point $z^* \in \gamma$, such that $f(z_n) = 0$ for all n. But then $f(z^*) = 0$ and $\lim_{n\to\infty}(f(z^*) - f(z_n))/(z^* - z_n) = 0$, which contradicts the assumptions. Now take a piecewise-smooth simple loop $\gamma' \subset E_\delta$ and write using the argument principle and the fact that ${f_n}'(z)/f_n(z)$ converges uniformly to $f'(z)/f(z)$ on γ':

$$k = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma'} \frac{f'(z)}{f(z)} \mathrm{d}z = \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \oint_{\gamma'} \frac{{f_n}'(z)}{f_n(z)} \mathrm{d}z,$$

where k is the number of zeros of $f(z)$ inside γ'. Thus for a sufficiently large n the numbers of zeros of $f(z)$ and $f_n(z)$ inside γ' are the same.

It remains to show that $f_n(z)$ has neither zeros in E_δ, nor in $\gamma \backslash \{z_0\}$, for sufficiently large n. Uniform convergence $f_n'(z) \to f'(z), z \in D$ and continuity of $f_n'(z)$ on $D \cup \{z_0\}$ imply that $f'(z)$ is continuous on $D \cup \{z_0\}$, where $f'(z_0)$ is defined in the statement of the theorem. Now it is easy to see that one can pick $\eta > 0$, such that for a sufficiently small $\epsilon > 0$ and large n the following holds:

$$|f_n'(z) - f'(z_0)| < \frac{1}{2}\eta < \eta < |f'(z_0)|, z \in D \cap B(z_0, \epsilon),$$

which implies

$$|f_n'(z) - f'(z_0)| \le \frac{1}{2}\eta < \eta < |f'(z_0)|, z \in \overline{D} \cap B(z_0, \epsilon).$$

Here we assume that ϵ is taken small enough, so that the $f_n(z)$ are analytic on $B(z_0, \epsilon)$. Note that for a sufficiently small $\epsilon > 0$ one can connect the points z_0 and $z \in \overline{D} \cap B(z_0, \epsilon)$ by a piecewise smooth path $\tilde{\gamma}$, so that $\tilde{\gamma} \subset \overline{D} \cap B(z_0, \epsilon)$ and the length of $\tilde{\gamma}$ is less than $2|z - z_0|$, because the contour of D is assumed to be piecewise smooth. Now

$$\begin{aligned} |f_n(z) - f'(z_0)(z - z_0)| &= \left| \int_{\tilde{\gamma}} ({f_n}'(s) - f'(z_0)) \mathrm{d}s \right| \\ &\le 2|z - z_0| \max_{s \in \tilde{\gamma}} |{f_n}'(s) - f'(z_0)| \le \eta |z - z_0| \end{aligned}$$

and

$$\begin{aligned} |f_n(z)| &\ge |f'(z_0)(z - z_0)| - |f_n(z) - f'(z_0)(z - z_0)| \\ &\ge (|f'(z_0)| - \eta)|z - z_0| > 0 \end{aligned}$$

for $z \in \overline{D} \cap B(z_0, \epsilon), z \neq z_0$ and sufficiently large n.

Finally, consider the set $E' := (\gamma \cup E_\delta) \backslash B(z_0, \epsilon)$. The set E' is compact and $f(z) \neq 0$ on E', thus $f_n(z) \neq 0$ on E' for sufficiently large n, which completes the proof. $\square$

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