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A Note on Negative Customers, GI/G/1 Workload, and Risk Processes

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Abstract

Recently the workload distribution in the M/G/1 queue with work removal has been analysed, and has been shown to exhibit a generalized Pollaczek-Khintchine form. The latter result is explained in this note by transforming the model into a standard GI/G/1 queue. Some extensions are also discussed, as well as a connection with a 'dual' risk process.

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1 Introduction

The following model has been studied in [2, 5]. Customers arrive at a single-server queue according to a Poisson process with rate λ^+. Their service requirements $\{B_n\}$ are i.i.d. with distribution $B(\cdot)$, finite mean β and Laplace-Stieltjes Transform (LST) $\beta(s)$. We shall refer to these customers as ordinary or positive customers. In addition to the ordinary customers, and independent of them, negative customers arrive at the queue according to a Poisson process with rate λ^-. These negative customers reduce the amount of work in the queue according to a distribution $C(\cdot)$, with mean γ and LST $\gamma(s)$. These reduction amounts are denoted by

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$\{C_n\}$ and are assumed i.i.d. Assume that $\lambda^+\beta < 1 + \lambda^-\gamma$; this is the *stability condition* for this M/G/1 generalization [2, 5]. One can even allow the case in which negative customers always remove *all* the work present; this is the so-called disaster model [2, 6].
Let V denote the steady-state workload in the system. In [2] its LST $\phi(s) = \mathrm{E}[exp(-sV)]$ was obtained by solving a Wiener-Hopf problem; in [5] this transform was shown to be of the Pollaczek-Khintchine form:

$$\phi(s) \;=\; \frac{1-\nu}{1-\nu\eta(s)}, \qquad Re\ s \geq 0, \tag{1}$$

with $\eta(s)$ the LST of the distribution of a non-negative random variable and $0 < \nu < 1$. Note that for the ordinary M/G/1 queue without negative arrivals, (1) holds with $\nu = \lambda^+\beta$ and $\eta(s) = (1-\beta(s))/\beta s$.
In [2] a transformation is presented to show that the workload distribution equals the waiting time distribution in an ordinary GI/G/1 queue with *positive* customers only - thus yielding access to the rich literature on the GI/G/1 queue. The purpose of the present note is twofold: to show how that transformation directly leads to (1), and to establish a connection between this transformation and a dual risk process, in which - in addition to claims and a unit rate premium - there are also lump additions.

2 A transformation

Consider the M/G/1 queue with additional removal of work, as described in Section 1, with workload $V(t)$ at time t, steady-state workload V, and V_a denoting steady-state workload as found by arriving ordinary customers. Let t_n denote the arrival epoch of the nth ordinary customer, with $t_0 \stackrel{\text{def}}{=} 0$. During the interarrival time $\tau_n \stackrel{\text{def}}{=} t_{n+1} - t_n$, K_n negative arrivals occur, removing the i.i.d. amounts of work $C_1^n, \ldots, C_{K_n}^n$. It was observed in [2] that

$$V(t_{n+1}) = (V(t_n) + B_n - \tau_n^*)^+, \quad n = 0, 1, \ldots, \tag{2}$$

where $a^+ \stackrel{\text{def}}{=} \max(a, 0)$ and

$$\tau_n^* \stackrel{\text{def}}{=} \tau_n + \sum_{j=1}^{K_n} C_j^n, \quad n = 0, 1, \ldots. \tag{3}$$

Clearly, (2) is the recurrence relation for the waiting time in a GI/G/1 queue with service times B_n but *longer interarrival times* τ_n^*. Thus we conclude that V_a has the same distribution as the steady state waiting time in this GI/G/1 queue.
A Pollaczek-Khintchine form then follows from random walk theory. Consider the random walk

$$R_n = B_1 - \tau_1^* + \cdots + B_n - \tau_n^*, \ n \geq 1, \ R_0 = 0,$$

with i.i.d. increments $B_i - \tau_i^*$. The steady-state waiting time distribution in the FIFO GI/G/1 queue is that of the maximum $M = \max\{R_n : n \geq 0\}$. It follows from random walk theory (see for example Ch. 9 in [12]) that the LST of M is of the form (1) where $\nu = P(R_n > 0,$ for some $n \geq 0)$ is the probability of at least one strictly ascending ladder height, and $\eta(s)$

the LST of such a ladder height distribution (conditional on it occurring).
Since V_a has the same distribution as M, while V has the same distribution as V_a by PASTA, we conclude that the desired Pollaczek-Khintchine formula (1) also holds for V. In this context $\nu = P(V > 0)$.

Remark 2.1
The transformation idea of [2], lengthening of interarrival times to take work removal into account, gives direct access to the literature on the GI/G/1 queue, see e.g. Cohen [3]. If $B(\cdot)$ is general but $\gamma(s)$ is rational, the denominator being a polynomial of degree m (denote this by K_m), then τ_n^* has a K_{m+1} distribution. One can now apply known results for the $K_{m+1}/G/1$ queue, cf. [3], Section II.5.11.
If $B(t) = 1 - exp(-t/\beta)$ then an even better tractable model results (even if the ordinary customers arrive according to a renewal process, and work removals follow an arbitrary distribution): the transformation yields a GI/M/1 queue and hence V has an exponential distribution with a point mass at the origin.
The transformation holds more generally in a G/G/1 queue with a stationary sequence $\{(B_n, \tau_n)\}$ of service and interarrival times for positive customers. The transformed sequence becomes the new stationary sequence $\{(B_n, \tau_n^*)\}$ as defined by (3) (the new sequence remains stationary because the negative arrivals are Poisson and independent of all else). In the GI/G/1 case where the interarrival times are i.i.d., the LST $\tau(s)$ of interarrival time is transformed into $\tau(s + \lambda^-(1 - \gamma(s)))$.
The transformation idea even applies to a GI/G/1 queue in which each arriving customer is with probability p a positive customer who requires some service, and with probability $1 - p$ a negative customer who removes some work; now $\tau(s)$ transforms into $p\tau(s)/(1 - (1 - p)\tau(s))$. Of course in these cases, PASTA may not hold and the distributions of V and V_a are typically different.

Remark 2.2
In [5] formula (1) is derived for the model of negative arrivals by utilizing the Preemptive LIFO discipline (PL) and extending a result of Fakinos [4] (see also [8]): Under PL $\eta(s)$ represents the LST of the distribution of the remaining service time of the customer found in service by an arrival, say $B^*(\cdot)$, and

$$P(Q = n,\ V_j \leq x_j,\ j = 1, \ldots, n) = (1 - \nu)\nu^n \prod_{j=1}^{n} B^*(x_j), \tag{4}$$

where Q is the number of customers found by an arrival and V_j the remaining service requirement of customer j, $j = 1, \ldots, n$.

Remark 2.3
It is known (cf. [3], p. 282) that the waiting time distribution in the GI/G/1 queue is infinitely divisible. Interestingly, this immediately follows from the form (1) and Theorem 12.2.3 in [7] (which states that such a form implies infinite divisibility).

3 Risk processes with lump additions

Consider an insurance business that starts off initially with $x \geq 0$ units of money and earns a rate 1 premium. Claims occur as a Poisson process at rate μ^- with interarrival times $\{\tau_n\}$, and claim sizes $\{B_n\}$ are non-negative and i.i.d. with distribution $B(\cdot)$. Furthermore, independently, lump sums of money are added according to a Poisson process at rate μ^+, and the lumps $\{C_n\}$ are non-negative and i.i.d. with distribution $C(\cdot)$. The total reserve at time t is given by the *risk process*

$$X_x(t) \stackrel{\text{def}}{=} x + t - \sum_{i=1}^{N^-(t)} B_i + \sum_{i=1}^{N^+(t)} C_i, \; t \geq 0, \tag{5}$$

where N^- and N^+ are the Poisson counting processes for claims and lumps respectively; $X_x(0) = x$. Of intrinsic interest is to compute the probability of *ruin*, $P(\tau(x) < \infty)$, where

$$\tau(x) \stackrel{\text{def}}{=} \inf\{t > 0 : X_x(t) < 0\}.$$

($\tau(x) \stackrel{\text{def}}{=} \infty$ if $X_x(t)$ never enters $(-\infty, 0)$.) When the model has no lump additions, then it is well known (see [9] for example) that

$$P(\tau(x) < \infty) = P(V > x), \tag{6}$$

where V is steady-state workload for an M/G/1 queue with arrival rate μ^- and service times $\{B_n\}$. It is intuitive that (6) should also hold for our risk model with lump additions, where V is the steady-state workload in the M/G/1 queue with negative customers in which $\lambda^+ = \mu^-$ and $\lambda^- = \mu^+$. We now show this, following the duality theory from [1].
Observing that ruin can only occur right after a claim epoch (denoted by t_n, $n \geq 1$, with $t_0 \stackrel{\text{def}}{=} 0$), we conclude that

$$\tau(x) = \min\{t_n > 0 : X_n < 0\},$$

where $X_n \stackrel{\text{def}}{=} X_x(t_n+)$, $n \geq 1$, and $X_0 \stackrel{\text{def}}{=} 0$. But X_n is a random walk starting at x with increments $\tau^*_{n-1} - B_n$, $n \geq 1$:

$$X_n = x + \tau^*_0 - B_1 + \cdots + \tau^*_{n-1} - B_n, \; n \geq 1, \; X_0 = x,$$

where τ^*_n is defined exactly as in (3), which in the current notation is given by

$$\tau^*_n = \tau_n + \sum_{j=N^+(t_n)+1}^{N^+(t_{n+1})} C_j, \tag{7}$$

and represents the cumulative earnings (interest plus lump sums) in the time interval $(t_n, t_{n+1}]$. Thus from Example 1 in [1] (and the fact that Poisson processes are time reversible) we conclude that $\{X_n\}$ is the dual of the *reflected random walk* given by

$$W_{n+1} = (W_n + B_n - \tau^*_n)^+, \tag{8}$$

and that (6) holds when $V = W$, the steady-state for this reflected random walk. But (8) is the same recursion as (2); so from Section 2, W can be identified with the stationary workload in the M/G/1 queue with negative customers in which $\lambda^+ = \mu^-$ and $\lambda^- = \mu^+$.

Remark 3.1
The lump additions can be replaced by any Levy process $\{A(t)\}$ with non-negative increments, in which case (7) is generalized to $\tau_n^* = \tau_n + A(t_{n+1}) - A(t_n)$; the duality with the reflected random walk remains valid.
The duality between $\{X_n\}$ and $\{W_n\}$ extends to the case when $\{(B_n, \tau_n)\}$ forms a stationary sequence, but then its time reversal must be used in recursion (8); see [1].

Remark 3.2
Using a continuous time analogue of [1] developed in [11] (or, since workload is a Markov process, by using Siegmund Duality [10]), it can be shown that the duality in (6) holds for all $t \geq 0$:

$$P(X_x(t) \leq y) = P(V_y(t) \geq x), \ t \geq 0, \tag{9}$$

where $V_y(t)$ denotes the workload at time t in the M/G/1 queue with negative customers in which $V(0) = y \geq 0$. This means that the continuous time $\{X(t)\}$ and $\{V(t)\}$ are duals of one another. This can be generalized further (using time reversal) to a time-stationary setting in which $\{(B_n, \tau_n)\}$ is defined from a time-stationary marked point process or $\{A(t)\}$ from Remark 3.1 is a process with non-negative stationary increments.

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