A CONTINUED FRACTION TITBIT

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Abstract

In 1812 Gauss, in a letter to Laplace, proposed without proof a formula explaining the statistical regularity of continued fractions. There has since been speculation concerning the manner in which Gauss arrived at this formula. In this article we present a plausible explanation, which at the same time gives an elementary proof of the full ergodic nature of the underlying dynamical system. The method seems to be of interest for other number–theoretic expansions.

1. INTRODUCTION

In 1812 Gauss, in a letter to Laplace,[2] announced his discovery of the statistical regularity underlying the seemingly random sequence of integers obtained by expanding a positive real number in a continued fraction. Over one hundred years later, Kuzmin[5] published a proof of this regularity, now known as Kuzmin's theorem. The nature of Kuzmin's proof and subsequent investigations (Refs. 1, 4 and 6 among others) led to the now prevalent point of view that Gauss, whose letter and other works contained no indications of proof, was led to his discovery through his considerable mathematical intuition, as a magician pulls a rabbit out of a hat. The interest in and the perplexing nature of similar statistical regularities for other number–theoretical expansions (e.g. the Jacobi–Perron algorithm, a two-dimensional continued fraction which is known to be regular but for which a formula à la Gauss is still lacking (but see Ref. 3) gave rise to a number of attempts to *derive* Gauss' formula (e.g. Refs. 1 and 6) Up until now, these interesting derivations have not been of a simple nature. In the following, we present using modern terminology a simple derivation which at the same time provides a plausible explanation of Gauss' thoughts. Our ideas, which we suspect were also known to Gauss, provide an elegant proof not only of Kuzmin's theorem, but also of the full ergodic nature, i.e. weak Bernoullicity, of the continued fraction dynamical system, in a predominately geometric setting. Moreover, the simple calculations are accessible at the first year undergraduate level and provide interesting exercises for an elementary course. Finally, we hope that extensions will be useful in answering related open problems.

2. A GRECIAN VIEW OF CONTINUED FRACTIONS

In ancient times, geometers were very interested in proportions. Consider two rods A and B, rod A being longer than rod B. Now use rod B as a measuring stick in an attempt to explain how long rod A is, relative to the length of rod B. If rod A is at least n_1 times as long as rod B, but less than $n_1 + 1$ times as long as rod B, then we record the integer n_1 and make a new rod C by removing from rod A a piece which is n_1 times as long as rod B.

We are now left with two rods B and C, rod B being longer than rod C, and thus we can repeat the above procedure. This is simply the original version of the Euclidean algorithm; the (lengths of the) original rods A and B are said to be *commensurable* if at some finite stage k, the longer rod is exactly *explained* by the shorter one, and otherwise incommensurable. If

$$\alpha := \frac{\text{length } B}{\text{length } A}$$

and if $n_1, n_2, \ldots, n_k$ are the successive positive integers obtained from the commensurable rods A and B, then it is easy to see that

$$\alpha = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}}$$

by induction, since for the first step

$$\text{length } A = n_1 \text{length } B + \text{length } C$$

and hence

$$\frac{\text{length } B}{\text{length } A} = \cfrac{1}{n_1 + \cfrac{\text{length } C}{\text{length } B}}.$$

If rods A and B are incommensurable, then clearly

$$\alpha = \lim_{k\to\infty} \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}}$$

and we write

$$\alpha = [n_1, n_2, \ldots];$$

the sequence $n_1, n_2, \ldots$ is called the *continued fraction expansion* of α, and n_k is called the k^{th} *partial quotient.*

Remark

- It is interesting to notice that the above algorithm is one of the first instances of repeated scaling, and the division of the initial rod A into successive intervals at scale level k can be seen as an elementary fractal object. Thus a number α in the unit interval can itself be thought of as a fractal, and the choice of a random number as a random fractal!
- The continued fraction representation of α is known to be the "most efficient" approximation of α by rationals.
- If each $n_k = 1$ in the continued fraction expansion of α, then α is called the *golden ratio.* A rectangle whose sides are in the proportion 1: α, α being the golden ratio, was (and perhaps still is) regarded as æsthetically the most pleasing to the eye.

3. THE STATISTICS OF CONTINUED FRACTIONS

Suppose now that a number α is chosen at random from the unit interval, and then expanded in its continued fraction:

$$\alpha = [n_1, n_2, \ldots].$$

Of course, since α was chosen at random, we can have no certainty about the individual values n_k appearing in the expansion. However, there is an amazing *statistical* regularity, as revealed by Gauss in his letter to Laplace.[2] Suppose we fix a finite block of integers (e.g. the block 1, 15, 3), and ask for the *frequency* of appearance of the fixed block in the continued fraction. Then with probability one, this frequency exists, does not depend on the particular α chosen, and can be calculated by the following procedure:

- First, determine the interval $I \subseteq (0, 1)$ for which the numbers belonging to $I = (a, b)$ have a continued fraction which *begins* with the given block of integers.
- Then calculate
$$\frac{1}{\ln 2} \int_I \frac{dx}{1+x} = \log_2 \frac{1+b}{1+a}$$
this number is the correct frequency.

For example, suppose that the given block consists of the single integer 1. Then the corresponding interval I is $(1/2, 1)$, since n_1 is one if and only if the shorter rod is more than half as long as the longer rod. Thus the frequency of 1 in the continued fraction expansion is typically (i.e. with probability one) equal to

$$\frac{1}{\ln 2} \int_{1/2}^{1} \frac{dx}{1+x} = 2 - \log_2 3 \doteq 0.41504.$$

More generally, for a fixed positive integer k (as the block of length one), the corresponding interval is $(1/k+1, 1/k)$ and the frequency with which k appears is typically

$$\frac{1}{\ln 2} \int_{1/k+1}^{1/k} \frac{dx}{1+x} = \log_2 (n+1)^2/n(n+2).$$

For the block 1, 15, 3 the corresponding interval is $(61/65, 46/49)$ and the frequency is approximately 0.000234.In the following sections we try to explain these formulas.

4. THE SCALING TRANSFORMATION

Suppose that $\alpha \in (0, 1)$ and that we wish to compute α's continued fraction. This can be accomplished by a simple iterative procedure. First, determine n_1 and β such that

$$\alpha = \frac{1}{n_1 + \beta}$$

and

$$\beta \in (0, 1).$$

Next, determine n_2 and γ such that

$$\beta = \frac{1}{n_2 + \gamma}$$

and

$$\gamma \in (0, 1).$$

Continuing in this manner yields $\alpha = [n_1, n_2, \ldots]$. The transformation from α to β, β to γ, etc., is called the *scaling* or *continued fraction* transformation. More formally, we define a mapping

$$T : (0, 1) \longrightarrow [0, 1)$$

by setting

$$T(\alpha) = \left\{\frac{1}{\alpha}\right\}$$

where the curly braces denote the fractional part, and another mapping

$$n : (0, 1) \longrightarrow \{1, 2, 3, \ldots\}$$

by setting

$$n(\alpha) = \left[\frac{1}{\alpha}\right],$$

the square brackets denoting integral part. Then clearly

$$\alpha = \frac{1}{n(\alpha) + T(\alpha)},$$

so that if we define

$$n_k := n(T^{k-1}(\alpha)),$$

then $\alpha = [n_1, n_2, \ldots]$ is the continued fraction of α. If α is a rational number, then at some point we reach $T^k(\alpha) = 0$ and cannot continue, but since here we are interested in typical α, this is of little importance. In modern terminology, Gauss' result can be stated as follows:

The mapping $T : (0,1) \to [0,1)$ possesses the invariant probability measure

$$d\mu = \frac{1}{\ln 2} \frac{dx}{1+x};$$

the corresponding dynamical system is ergodic.

It is an easy exercise to show that μ is a T-invariant probability measure, once its formula is known. Up until now, the ergodicity proof has required detailed calculations. We now proceed by showing how to determine the formula for μ, and as a corollary we can obtain the ergodicity (and even a much stronger property, weak Bernoullicity) without calculation.

5. QUADRATIC FORMS

Gauss was seriously interested in quadratic forms, and his results remain central for their classification. Thus the following viewpoint would have been quite natural.

Let us consider a number $\alpha \in (0, 1)$ by supposing that it is the solution of a quadratic equation

$$ax^2 + bx - c = 0\,,$$

where a, b, and c are real numbers. If now

$$\alpha = \frac{1}{n+\beta}$$

with n a positive integer and $\beta \in (0,1)$, then clearly the quadratic equation

$$c(n+y)^2 - b(n+y) - a = 0$$

possesses the solution β. We expand the terms in this equation to determine the corresponding coefficients:

$$cy^2 + (2cn - b)y - (a + bn - cn^2) = 0$$

has solution β. Hence we may be able to replace α by a quadratic equation defining α, with coefficients $(a,\, b,\, c)$ as above; in that case, the scaling transformation of the previous section will become

$$(a,\, b,\, c,) \longrightarrow (c,\, 2nc - b,\, a + bn - cn^2)$$

for n properly chosen. In order to make sure that α is defined by the equation $ax^2+bx-c=0$, it is natural to require that the other root of this equation does not belong to $(0,\, 1)$; to have a definite geometric picture, let us suppose that $a > 0$ and that the other root is to the left of the origin, so that we have

$$a + b - c > 0$$

(the value at 1 is positive) and

$$c > 0$$

(the value at 0 is negative). Now we can discuss the choice of n, which we want to make so that the same geometric picture appears. That is, n should be chosen such that

$$a + bn - cn^2 > 0$$

and

$$c + 2cn - b - (a + bn - cn^2) = -(a + b(n+1) - c(n+1)^2) < 0\,,$$

which are the corresponding conditions on the transformed values. Hence n should be chosen maximal such that

$$a + bn - cn^2 > 0;$$

since c was assumed positive, and since (the case $n = 1$) $a + b - c > 0$, this determines n uniquely.

Let us also discuss the value of the other root $\tilde{\beta}$ of the transformed equation. If the other root $\tilde{\alpha}$ of the original equation is less than zero, then the relation

$$\tilde{\beta} = \frac{1}{\tilde{\alpha}} - n$$

shows that $\tilde{\beta} \leq -1$, since $n \geq 1$, which then persists in the following steps. If at the beginning we would have had $\tilde{\alpha} > 1$, then in the second step $\tilde{\beta}$ becomes negative and thus the other root is less than -1 from the third step on. Thus it is natural to assume from the beginning that $\tilde{\alpha} < -1$.

Now we are prepared to turn the above discussion into mathematical statements. Consider the quadratic equation

$$p(x) := ax^2 + bx - c = 0,$$

with coefficients a, b, c real and $a > 0$.

Lemma 5.1. *The equation $p(x) = 0$ has one root between 0 and 1 and another root less than -1 if and only if*

$$c > 0$$

and

$$|a - c| < b.$$

Proof. The equivalent conditions $p(1) > 0$, $p(0) < 0$, and $p(-1) < 0$, together with $a > 0$, translate into

$$\begin{aligned} a &> 0 \\ a + b - c &> 0 \\ c &> 0 \\ a - b - c &< 0, \end{aligned}$$

yielding the desired result. ∎

Definition 5.1. Let

$$\Gamma := \{(a, b, c) : a > 0, c > 0, |a - c| < b\}.$$

For $(a, b, c) \in \Gamma$ define

$$\tilde{T}(a, b, c) := (c, 2nc - b, a + bn - cn^2),$$

where n is chosen maximal such that $a + bn - cn^2 > 0$.

Lemma 5.2. *$\tilde{T}$ maps Γ to $\bar{\Gamma}$ (closure of Γ). Moreover, $\tilde{T}$ is locally linear, a.e. invertible, and preserves Lebesgue measure restricted to Γ.*

Proof. Clearly

$$\tilde{T}(a,\, b,\, c) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2n \\ 1 & n & -n^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

on the set of $(a,\, b,\, c)$ corresponding to n, and the determinant of the given matrix is 1. Moreover, the value of n can be determined from the image, since the other root of the transformed equation must lie between $-n-1$ and $-n$. Hence $\tilde{T}$ is a.e. invertible and must preserve Lebesgue measure. ■

6. SOME DIFFERENTIAL GEOMETRY IN THREE DIMENSIONS

Among other things, Gauss had a special interest for differential geometry. Hence it is plausible that the following arguments were available.

The region Γ defined above consists of a family of rays emanating from the origin; on each ray, the two solutions of $p(x) = 0$ remain the same, since the coefficients a, b, and c are just multiplied by a positive constant. We remove this ambiguity by noting that if we set

$$D(a,\, b,\, c) := b^2 + 4ac$$

(the discriminant of $p(x)$), then

$$D(\tilde{T}(a,\, b,\, c)) = D(a,\, b,\, c)\,,$$

since

$$(2nc - b)^2 + 4c(a + bn - cn^2) = b^2 + 4ac$$

regardless of the value of n. Hence $\tilde{T}$ leaves each of the surfaces $D =$ constant in Γ invariant, and we may reduce the number of variables to two by requiring, say, $D = 1$.

Definition 6.1. Let

$$\Delta := \{(a,\, c) : a > 0, c > 0, a + c < 1\}$$
$$\tilde{\Delta} := \{(a,\, \sqrt{1-4ac},\, c) : (a,\, c) \in \Delta\}$$
$$T(a,\, c) := (c,\, a + n\sqrt{1-4ac} - n^2 c)$$

with n maximal such that $a + n\sqrt{1-4ac} - n^2 c > 0$.

Lemma 6.1. *Each ray in Γ pierces $\tilde{\Delta}$. Moreover, $\tilde{T}\tilde{\Delta} = \tilde{\Delta}$ almost everywhere.*

Proof. Obvious. ■

Now we can use a bit of elementary differential geometry to calculate the density of an invariant measure on Δ with respect to the mapping T. The idea is as follows. The surface

$\tilde{\Delta}$ lies above the triangle Δ in the a-c-plane, and has the equation $b = \sqrt{1-4ac}$. The mapping T on Δ corresponds to the mapping $\tilde{T}$ on $\tilde{\Delta}$. The pieces of rays from the origin to $\tilde{\Delta}$ are mapped to each other by $\tilde{T}$, which preserves Lebesgue measure. Hence if we take a surface element $da\,dc$ in the triangle Δ, lift it to $\tilde{\Delta}$ via $b = \sqrt{1-4ac}$, and calculate the volume of the cone between the origin and the lifted surface element, then this volume will be proportional to the density of a T-invariant measure on Δ. The calculation is simple. Fix $(a, c) \in \Delta$. Then the corresponding point on $\tilde{\Delta}$ has coordinates

$$(a,\ \sqrt{1-4ac},\ c)\,,$$

and the lifted surface element is determined by the partial differentials in the a and c directions:

$$\left(da,\ \frac{-2c\,da}{\sqrt{1-4ac}},\ 0\right)$$

and

$$\left(0,\ \frac{-2a\,dc}{\sqrt{1-4ac}},\ dc\right).$$

Thus we define $\rho(a,\ c)da\,dc$ to be the determinant of the matrix

$$\begin{bmatrix} da & \dfrac{-2c\,da}{\sqrt{1-4ac}} & 0 \\ a & \sqrt{1-4ac} & c \\ 0 & \dfrac{-2a\,dc}{\sqrt{1-4ac}} & dc \end{bmatrix}$$

or

$$\rho(a,\ c) = \frac{1}{\sqrt{1-4ac}}\,.$$

We have thus proved

Lemma 6.2. *ρ is the density of a T-invariant measure on Δ.*

7. GEOMETRICAL INTERPRETATION OF (δ, T) AND THE DERIVATION OF GAUSS' FORMULA

We now return to our primary objective. Each point $(a,\ c) \in \Delta$ gives rise to the quadratic equation

$$ax^2 + \sqrt{1-4ac}x - c = 0\,,$$

which has two solutions, one lying in the unit interval and one to the left of -1. We now wish to determine the invariant measure for the root in the unit interval by integrating over the second root. So fix α in the unit interval and set

$$F(\alpha) := \{(a,\ c) \in \Delta : a\alpha^2 + \sqrt{1-4ac}\alpha - c = 0\}\,.$$

We have

$$a\alpha^2 + \sqrt{1-4ac}\alpha - c = 0$$

if and only if

$$a\alpha + \frac{c}{\alpha} = 1\,,$$

so that $F(\alpha)$ is simply the straight line segment between the points $(0, \alpha)$ and $(1/1+\alpha, \alpha/1+\alpha)$ on the boundary of Δ. Similarly, if we fix the other root $\tilde{\alpha} < -1$ and set $\tilde{\alpha} = -\frac{1}{\omega}$ with ω in the unit interval, and define

$$P(\omega) := \left\{(a,\, c) \in \Delta : a\left(-\frac{1}{\omega}\right)^2 + \sqrt{1-4ac}\left(-\frac{1}{\omega}\right) - c = 0\right\},$$

then this results in the linear equation

$$\frac{a}{\omega} + c\omega = 1\,,$$

so that $P(\omega)$ is simply the straight line segment between the points $(\omega, 0)$ and $(\omega/1+\omega, 1/1+\omega)$ on the boundary of Δ. Solving these equations for a and c in terms of α and ω results in

$$a = \frac{\omega}{1+\alpha\omega}$$
$$c = \frac{\alpha}{1+\alpha\omega}$$

so that

$$\rho(a,\, c)da\,dc = \frac{da\,dc}{\sqrt{1-4ac}} = \frac{d\alpha\,d\omega}{(1+\alpha\omega)^2}$$

after a simple calculation. Finally, integration over ω yields

$$d\alpha \int_0^1 \frac{d\omega}{(1+\alpha\omega)^2} = \frac{d\alpha}{1+\alpha},$$

which is the Gauss measure up to a normalization factor. Thus we have proved in a constructive manner

Theorem 7.1. *The measure*

$$d\mu(\alpha) = \frac{1}{\ln 2}\frac{d\alpha}{1+\alpha}$$

is an invariant probability measure under the scaling transformation T of section 4.

8. ERGODIC PROPERTIES

Here we only sketch the simple proofs of ergodic properties using the representation above, assuming that the reader is acquainted with the basic notions of ergodic theory.

Theorem 8.1. *T is ergodic on Δ with respect to the measure $da\,dc\sqrt{1-4ac}$.*

Proof. If A is a T-invariant set, then A belongs both to the remote future and to the remote past. It is easy to see that, taking the standard defining partition, the future σ-algebra $\mathcal{F}$ consists of unions of the sets $F(\alpha)$, while the past σ-algebra $\mathcal{P}$ consists of unions of the sets $P(\omega)$. But $A \in \mathcal{F} \cap \mathcal{P}$ implies immediately that either A or its complement must have zero measure, from the geometry and the fact that the density of the invariant measure is strictly positive on Δ. ■

Theorem 8.2. *T is a K-system.*

Proof. If A belongs to the Pinsker algebra, then it also belongs both to the future and the past, and the argument in the proof of the preceding theorem applies. ■

Theorem 8.3. *T is weak Bernoulli.*

Proof. This follows again from the geometric picture and the formula for the invariant measure — the conditional measures are equivalent. ■

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