stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZN 62/75 AUGUST

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A NOTE ON EULER'S CONSTANT

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

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AMS(MOS) subject classification scheme (1970): 40A05, 10A40

A note on Euler's constant

by

J. van de Lune

0. INTRODUCTION

Writing

(1) $$H(n) = \sum_{k=1}^{n} \frac{1}{k},$$

Euler's constant γ is usually defined as the limit of the increasing sequence

(2) $$\{H(n) - \log(n+1)\}_{n=1}^{\infty},$$

or, equivalently, as the limit of the decreasing sequence

(3) $$\{H(n) - \log n\}_{n=1}^{\infty}.$$

Since

(4) $$H(n) - \log(n+1) = \sum_{k=1}^{n} \{\frac{1}{k} - \log(1 + \frac{1}{k})\}$$

it follows that

(5) $$\gamma = \sum_{k=1}^{\infty} \{\frac{1}{k} - \log(1 + \frac{1}{k})\}.$$

From (4) and (5) one may derive that

(6) $$\frac{1}{2n+1} - \frac{1}{6n^2} < \gamma - \{H(n) - \log(n+1)\} < \frac{1}{2n}, \quad (\forall n \in \mathbb{N}).$$

Since

(7) $$H(n) - \log n = H(n) - \log(n+1) + \log(1 + \frac{1}{n})$$

and

(8) $$\frac{1}{n} - \frac{1}{2n^2} < \log(1 + \frac{1}{n}) < \frac{1}{n}$$

it follows from (6) that

$$\frac{1}{2n} - \frac{1}{2n^2} < H(n) - \log n - \gamma < \frac{1}{2n} + \frac{2}{3n^2} . \tag{9}$$

From (6) and (9) it is clear that the sequences (2) and (3) converge rather slowly and that, from the numerical point of view, it would be better to consider γ as the common limit of, for example, the following two (monotonic) sequences

$$\{H(n) - \log(n+1) + \frac{1}{2n}\}_{n=1}^{\infty} \tag{10}$$

and

$$\{H(n) - \log n - \frac{1}{2n}\}_{n=1}^{\infty}, \tag{11}$$

where (10) is decreasing and (11) is increasing.

CESÀRO considered (cf. [2], p. 460) the sequence

$$\{H(n) - \frac{1}{2} \log n(n+1)\}_{n=1}^{\infty} \tag{12}$$

and showed that

$$0 < H(n) - \frac{1}{2} \log n(n+1) - \gamma < \frac{1}{6n(n+1)} . \tag{13}$$

It was shown by LODGE (cf. [2], p. 460) that a very good approximation of the n-th term of (12) is given by

$$\gamma + \frac{1}{6\{n(n+1) + \frac{1}{5}\}} \tag{14}$$

the error being of the order

$$n^{-6}. \tag{15}$$

In this note we will consider a number of variations on Cesàro's sequence (12). Some examples are:

(16) $$\{H(n) - \int_n^{n+1} \log x \, dx\}_{n=1}^{\infty}$$

which approximates γ from above, the error being less than $\frac{1}{12n^2}$;

(17) $$\{H(n) - \log(n+\tfrac{1}{2})\}_{n=1}^{\infty}$$

which tends decreasingly to γ, the error being less than $\frac{1}{24n^2}$;

(18) $$\{H(n) + \log(e^{\frac{1}{n+1}} - 1)\}_{n=1}^{\infty}$$

which tends increasingly to γ, the rapidity of convergence being about the same as that of (17). We will also determine all constants $c > -1$ for which

(19) $$\{H(n) - \log(n+c)\}_{n=1}^{\infty}$$

is monotonic. For more refined methods to compute γ numerically we refer to [1], [3], [4], [5] and [6].

We conclude this note by proving the remarkable identity

(20) $$1 - \gamma = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1} \left[\frac{\log n}{\log 2}\right]$$

where $[\cdot]$ denotes the greatest integer function.

1. The general term of Cesàro's sequence may be written as

(21) $$H(n) - \frac{\log n + \log(n+1)}{2}$$

in which the term $\frac{\log n + \log(n+1)}{2}$ may be considered as a trapezoidal approximation of $\int_n^{n+1} \log x \, dx$.

Because of the concavity of $\log x$ we have

(22) $$\frac{\log n + \log(n+1)}{2} < \int_n^{n+1} \log x \, dx.$$

Next we observe that

$$(23)\qquad H(n) - \frac{\log n + \log(n+1)}{2} > \gamma.$$

In order to see this it suffices to prove that (12) is decreasing in n. Hence, we want to show that

$$(24)\qquad H(n) - \frac{1}{2}\log n(n+1) > H(n+1) - \frac{1}{2}\log(n+1)(n+2)$$

or

$$(25)\qquad \log(n+2) - \log n > \frac{2}{n+1}$$

or

$$(26)\qquad \log\left(1+\frac{1}{n+1}\right) - \log\left(1-\frac{1}{n+1}\right) > \frac{2}{n+1}$$

which is true by the wellknown inequality

$$(27)\qquad \log(1+x) - \log(1-x) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} > 2x, \qquad (0<x<1).$$

After these observations it seems natural to investigate the behaviour of the sequence

$$(28)\qquad \{H(n) - \int_n^{n+1} \log x \, dx\}_{n=1}^{\infty}.$$

Since

$$(29)\qquad H(n) = \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n}\int_0^1 x^{k-1}\,dx =$$

$$= \int_0^1 \frac{1-x^n}{1-x}\,dx = \int_0^{\infty}\frac{1-e^{-nt}}{e^t-1}\,dt,$$

we define

$$(30)\qquad H(s) = \int_0^\infty \frac{1-e^{-st}}{e^t-1}\,dt, \qquad (s>-1),$$

and instead of (28) we will consider, more generally, the function

$$(31)\qquad \gamma(s) \overset{\text{def}}{=} H(s) - \int_s^{s+1} \log x\,dx, \qquad (s>0).$$

We first prove the following

PROPOSITION 1.1. $\gamma(s)$ *is decreasing on* $\mathbb{R}^+$.

PROOF. Since

$$(32)\qquad \log\alpha = \int_0^\infty \frac{e^{-t}-e^{-\alpha t}}{t}\,dt, \qquad (\alpha>0),$$

the derivative of $\gamma(s)$ may be written as

$$(33)\qquad \gamma'(s) = H'(s) - \log(s+1) + \log s =$$

$$= \int_0^\infty e^{-st}\frac{t}{e^t-1}\,dt - \int_0^\infty \frac{e^{-t}-e^{-(s+1)t}}{t}\,dt + \int_0^\infty \frac{e^{-t}-e^{-st}}{t}\,dt =$$

$$= \int_0^\infty e^{-st}\left\{\frac{t}{e^t-1} - \frac{1-e^{-t}}{t}\right\}dt, \qquad (s>0).$$

Now observe that for $t > 0$ we have

$$(34)\qquad t^2 < t^2 + 2\sum_{n=2}^\infty \frac{t^{2n}}{(2n)!} =$$

$$= t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots) + (-t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - + \ldots) =$$

$$= (e^t-1) + (e^{-t}-1) = (1-e^{-t})(e^t-1),$$

so that

(35) $$\frac{t}{e^t - 1} < \frac{1 - e^{-t}}{t}, \qquad (t>0)$$

From (33) and (35) it follows that

(36) $$\gamma'(s) < 0, \qquad (s>0).$$

proving the proposition. □

Next we have

PROPOSITION 1.2.

(36) $$\lim_{s\to\infty} \gamma(s) = \gamma.$$

PROOF. In view of proposition 1.1 it suffices to show that

(38) $$\lim_{n\to\infty} \gamma(n) = \gamma, \qquad (n\in\mathbb{N}).$$

Since we clearly have that

(39) $$H(n) - \log(n+1) < \gamma(n) < H(n) - \log n$$

the proposition follows. □

As to the rapidity of convergence we have

PROPOSITION 1.3.

(40) $$\gamma < \gamma(s) < \gamma + \frac{1}{12s^2}, \qquad (s>0).$$

PROOF. From propositions 1.1 and 1.2 it is clear that $\gamma < \gamma(s)$ for all $s > 0$. From (33) we infer that for $a,b > 0$ we have

$$(41)\qquad \gamma(b) - \gamma(a) = \int_a^b \gamma'(s)\,ds =$$

$$= \int_a^b \int_0^\infty e^{-st}\left\{\frac{t}{e^t - 1} - \frac{1 - e^{-t}}{t}\right\} dt\,ds =$$

$$= \int_0^\infty \left\{\frac{t}{e^t - 1} - \frac{1 - e^{-t}}{t}\right\} \int_a^b e^{-st}\,ds\,dt =$$

$$= \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \left\{\frac{t}{e^t - 1} - \frac{1 - e^{-t}}{t}\right\} dt.$$

Replacing a by s and letting $b \to \infty$ we obtain

$$(42)\qquad \gamma(s) = \gamma + \int_0^\infty \frac{e^{-st}}{t} \left\{\frac{1 - e^{-t}}{t} - \frac{t}{e^t - 1}\right\} dt, \qquad (s>0).$$

Now observe that

$$(43)\qquad (0 <)\ \frac{1 - e^{-t}}{t} - \frac{t}{e^t - 1} < \frac{t^2}{12}, \qquad (t>0).$$

In order to see this we may argue as follows:

If $n \geqq 3$ then

$$(44)\qquad \frac{24}{27} < \frac{3}{3} \cdot \frac{4}{3} \cdot \frac{5}{3} \cdot \frac{6}{3} \leqq \frac{n}{3} \cdot \frac{n+1}{3} \cdot \frac{n+2}{3} \cdots \frac{n+n}{3},$$

so that

$$(45)\qquad 24 \cdot 3^{n-2} < n(n+1)(n+2)\ldots(2n) = \frac{(2n)!}{(n-1)!}$$

or

$$(46)\qquad 24 \cdot \frac{3^{n-2}}{(2n)!} \leqq \frac{1}{(n-1)!}.$$

Hence, if $0 < t \leqq 3$ and $n \geqq$ then

(48) $$24 \cdot \frac{t^{2n}}{(2n)!} < \frac{t^{n+2}}{(n-1)!} .$$

Consequently we have

(49) $$24 \cdot \sum_{n=3}^{\infty} \frac{t^{2n}}{(2n)!} < \sum_{n=3}^{\infty} \frac{t^{n+2}}{(n-1)!} , \quad (0<t\leqq 3).$$

Since $24 \cdot \frac{t^{2.2}}{(2.2)!} = \frac{t^{2+2}}{(2-1)!}$, it follows that

(50) $$24 \cdot \sum_{n=2}^{\infty} \frac{t^{2n}}{(2n)!} < \sum_{n=2}^{\infty} \frac{t^{n+2}}{(n-1)!} , \quad (0<t\leqq 3),$$

from which it is easily seen that

(51) $$e^{t} + e^{-t} - 2 - t^{2} < \frac{t^{3}}{12} (e^{t}-1), \quad (0<t\leqq 3),$$

so that

(52) $$(1-e^{-t})(e^{t}-1) - t^{2} < \frac{t^{2}}{12} \cdot t(e^{t}-1),$$

or, equivalently,

(53) $$\frac{1-e^{-t}}{t} - \frac{t}{e^{t}-1} < \frac{t^{2}}{12} , \quad (0<t\leqq 3).$$

If $t > 3$ then certainly

(54) $$\frac{1}{t} < \frac{t^{2}}{12} ,$$

Since we obviously have that

(55) $$\frac{1-e^{-t}}{t} - \frac{t}{e^{t}-1} < \frac{1}{t} , \quad (t>0),$$

it follows that also

(56) $$\frac{1-e^{-t}}{t} - \frac{t}{e^{t}-1} < \frac{t^{2}}{12} , \quad (t>3).$$

Combining (53) and (56) it follows that (43) holds.

From (42) and (43) it is clear now that

$$(57)\qquad \gamma(s) = \gamma + \int_0^\infty \frac{e^{-st}}{t}\left\{\frac{1-e^{-t}}{t} - \frac{t}{e^t - 1} - \frac{t^2}{12}\right\} dt + \int_0^\infty \frac{e^{-st}}{t} \cdot \frac{t^2}{12}\, dt <$$

$$< \gamma + \frac{1}{12}\int_0^\infty e^{-st}\, t\, dt = \gamma + \frac{1}{12s^2}, \qquad (s>0),$$

completing the proof. □

REMARK. From (42) one may derive the following asymptotic expansion

$$\gamma(s) \sim \gamma + \sum_{n=2}^\infty \frac{1}{n}\left\{\frac{(-1)^n}{n+1} - B_n\right\} s^{-n} =$$

$$= \gamma + 1 + \frac{1}{2s} - (s+1)\log\left(1+\frac{1}{s}\right) - \sum_{n=2}^\infty \frac{B_n}{n} s^{-n}, \qquad (s\to\infty)$$

where the B_n are Bernoulli's numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^\infty \frac{B_n}{n!} t^n, \qquad (|t|<2\pi).$$

2. In this section we consider the sequence

$$(58)\qquad \{H(n) - \log(n+c)\}_{n=1}^\infty$$

where c is some constant in the open interval $(-1,\infty)$.

PROPOSITION 2.1. *If* $-1 < c \leq \frac{1}{2}$ *then the sequence* (58) *tends decreasingly to* γ.

Before proving this proposition we establish the following

LEMMA 2.1. *The function*

$$(59)\qquad f(x) = \frac{1}{e^x - 1} - \frac{1}{x}, \quad (x>0)$$

is increasing. Moreover,

(60) $$\lim_{x\downarrow 0} f(x) = -\frac{1}{2}\,.$$

PROOF. Since for $|x| < 2\pi$ we have

(61) $$f(x) = \frac{1}{x}\left\{\frac{x}{e^x - 1} - 1\right\} = \frac{1}{x}\left\{\left(1 - \frac{x}{2} + \frac{x^2}{12} - + \ldots\right) - 1\right\} = -\frac{1}{2} + \frac{x}{12} - + \ldots$$

it is clear that $\lim_{x\downarrow 0} f(x) = -\frac{1}{2}$.

In order to see that $f(x)$ is increasing on $\mathbb{R}^+$ we may argue as follows. Since

(62) $$f'(x) = \frac{-e^x}{(e^x-1)^2} + \frac{1}{x^2}$$

it suffices to show that

(63) $$(e^x-1)^2 > x^2 \cdot e^x, \qquad (x>0)$$

or, equivalently, that

(64) $$e^{2x} - 2e^x + 1 > x^2 e^x, \qquad (x>0).$$

Since

(65) $$e^{2x} - 2e^x + 1 = \sum_{n=2}^{\infty} \frac{2^n - 2}{n!} x^n,$$

and

(66) $$x^2 e^x = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}\,,$$

and

(67) $$\frac{2^n - 2}{n!} = \frac{1}{(n-2)!} \quad \text{for } n = 2 \text{ and } n = 3,$$

we are done if we can show that

(68) $$\frac{2^n - 2}{n!} > \frac{1}{(n-2)!} \qquad \text{for } n \geq 4,$$

or, equivalently, that

(69) $$2^n - 2 > n(n-1), \qquad (n \geq 4).$$

It is easily seen by induction that

(70) $$2^n \geq n^2, \qquad (n \geq 4),$$

so that $2^n - 2 \geq n^2 - 2 > n^2 - n = n(n-1)$, $(n \geq 4)$ completing the proof of the lemma. □

REMARK. Lemma 2.1 may also be proved by means of the identity

$$\sum_{k=1}^{\infty} \frac{2^{-k}}{e^{2^{-k}x} + 1} = \frac{1}{x} - \frac{1}{e^x - 1}, \qquad (x>0),$$

which may be deduced from

$$(1 - e^{-\frac{x}{2^n}}) \prod_{k=1}^{n} (1 + e^{-\frac{x}{2^k}}) = 1 - e^{-x},$$

by logarithmic differentiation and taking the limit for $n \to \infty$.

PROOF OF PROPOSITION 2.1. Fix any c such that $-1 < c \leq \frac{1}{2}$. In order to show that (58) is decreasing we have to prove that for all $n \in \mathbb{N}$

(71) $$H(n) - \log(n+c) > H(n+1) - \log(n+1+c),$$

or, equivalently, that

(72) $$\log(1 + \frac{1}{n+c}) > \frac{1}{n+1},$$

or

(73) $$c < \frac{1}{e^{\alpha} - 1} - \frac{1}{\alpha} + 1, \qquad (\alpha = \frac{1}{n+1}).$$

In view of lemma 2.1 and our assumption that $c \leqq \frac{1}{2}$ it follows that (73) is true indeed. Since it is obvious that (58) has the limit γ this completes the proof of the proposition.

PROPOSITION 2.2. *If* $c > \frac{1}{e^{\frac{1}{2}} - 1} - 1$ $(= 0.54149\ldots)$ *then the sequence* (58) *tends increasingly to* γ.

PROOF. Similarly as in the proof of proposition 2.1 it suffices to show that for all $n \in \mathbb{N}$ we have

(74) $$c > \frac{1}{e^{\alpha} - 1} - \frac{1}{\alpha} + 1, \qquad (\alpha = \frac{1}{n+1}).$$

Since $\alpha = \frac{1}{n+1} \leqq \frac{1}{2}$, (74) follows from lemma 2.1, completing the proof. □

PROPOSITION 2.3. *If* $\frac{1}{2} < c \leqq \frac{1}{e^{\frac{1}{2}} - 1} - 1$ *then the sequence*

(75) $$\{H(n) - \log(n+c)\}_{n=1}^{\infty}$$

is eventually increasing.

PROOF. Fix any $c > \frac{1}{2}$. Similarly as before we have

(76) $$H(n) - \log(n+c) < H(n+1) - \log(n+1+c)$$

if and only if

(77) $$c > \frac{1}{e^{\frac{1}{n+1}} - 1} - (n+1) + 1 = \frac{1}{e^{\alpha} - 1} - \frac{1}{\alpha} + 1, \qquad (\alpha = \frac{1}{n+1}).$$

If follows from lemma 2.1 and our assumption that $c > \frac{1}{2}$ that (77) holds if n is large enough. □

A somewhat closer examination of the above argument reveals that for all $n \in \mathbb{N}$

$$(78) \qquad H(n) - \log\left(1 + \frac{1}{e^{\frac{1}{n}} - 1}\right) < \gamma.$$

More precisely we have

PROPOSITION 2.4. *The sequence*

$$(79) \qquad \left\{H(n) - \log\left(1 + \frac{1}{e^{\frac{1}{n}} - 1}\right)\right\}_{n=1}^{\infty}$$

converges increasingly to γ.

PROOF. It is easy to see that

$$(80) \qquad \gamma = \lim_{n\to\infty} \left\{H(n) - \log\left(1 + \frac{1}{e^{\frac{1}{n}} - 1}\right)\right\}.$$

In order to see that (79) is increasing we may argue as follows.

In order to prove that

$$(81) \qquad H(n) - \log\left(1 + \frac{1}{e^{\frac{1}{n}} - 1}\right) < H(n+1) - \log\left(1 + \frac{1}{e^{\frac{1}{n+1}} - 1}\right)$$

we may just as well show that

$$(82) \qquad \log \frac{1 + \{e^{\frac{1}{n+1}} - 1\}^{-1}}{1 + \{e^{\frac{1}{n}} - 1\}^{-1}} < \frac{1}{n+1}$$

or

$$(83) \qquad \log \frac{e^{\frac{1}{n+1}}\{e^{\frac{1}{n+1}} - 1\}^{-1}}{e^{\frac{1}{n}}\{e^{\frac{1}{n}} - 1\}^{-1}} < \frac{1}{n+1}$$

or

$$(84) \qquad \log \frac{e^{\frac{1}{n}} - 1}{e^{\frac{1}{n+1}} - 1} < \frac{1}{n}$$

or

$$(85)\qquad \frac{e^{\frac{1}{n}} - 1}{e^{\frac{1}{n+1}} - 1} < e^{\frac{1}{n}}$$

or

$$(86)\qquad \{e^{\frac{1}{n+1}} - 1\}^{-1} < e^{\frac{1}{n}}\{e^{\frac{1}{n}} - 1\}^{-1} = 1 + \{e^{\frac{1}{n}} - 1\}^{-1}$$

or

$$(87)\qquad \{e^{\frac{1}{n+1}} - 1\}^{-1} - (n+1) < \{e^{\frac{1}{n}} - 1\}^{-1} - n$$

or

$$(88)\qquad \frac{1}{e^{\alpha} - 1} - \frac{1}{\alpha} < \frac{1}{e^{\beta} - 1} - \frac{1}{\beta}$$

where $\alpha = \frac{1}{n + 1} < \frac{1}{n} = \beta$. Hence, the proposition follows from lemma 2.1. □

PROPOSITION 2.5. *The sequence*

$$(89)\qquad \{H(n) + \log\,(e^{\frac{1}{n+1}} - 1)\}_{n=1}^{\infty}$$

tends increasingly to γ.

PROOF. Observe that

$$(90)\qquad H(n) + \log(e^{\frac{1}{n+1}} - 1) = H(n+1) - \frac{1}{n + 1} + \log(e^{\frac{1}{n+1}} - 1) =$$

$$= H(n+1) - \log \frac{e^{\frac{1}{n+1}}}{e^{\frac{1}{n+1}} - 1} =$$

$$= H(n+1) - \log\left(1 + \frac{1}{e^{\frac{1}{n+1}} - 1}\right)$$

so that our assertion follows from proposition 2.4. □

Concerning the case $c = \frac{1}{2}$ in proposition 2.1 we have

PROPOSITION 2.6.

(91) $$\gamma < H(n) - \log(n+\tfrac{1}{2}) < \gamma + \frac{1}{24n^2}\ .$$

PROOF. First observe that for $s > 0$

(92) $$H(s) - \log(s+\tfrac{1}{2}) =$$

$$= \int_0^\infty \frac{1-e^{-st}}{e^t-1}\,dt - \int_0^\infty \frac{e^{-t}-e^{-(s+\frac{1}{2})t}}{t}\,dt =$$

$$= \int_0^\infty \left(\frac{1}{e^t-1} - \frac{e^{-t}}{t}\right) dt + \int_0^\infty e^{-st}\left\{\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t-1}\right\} dt =$$

$$= \gamma + \int_0^\infty e^{-st}\left\{\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t-1}\right\} dt.$$

From proposition 2.1 or from the fact that

(93) $$\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t-1} > 0, \qquad (t>0)$$

(the proof of which is left to the reader) it is clear that $\gamma < H(s) - \log(s+\frac{1}{2})$, $(s>0)$.

Now observe that

(94) $$\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t-1} < \frac{t}{24}\,, \qquad (t>0).$$

In order to see this we may argue as follows:

First let $t \geqq 4$. Then

(95) $$\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t-1} < \frac{e^{-\frac{t}{2}}}{t} \leqq \frac{e^{-2}}{4} < \frac{1}{28} < \frac{t}{24}$$

so that (94) holds for $t \geqq 4$.

Now let $0 < t < 4$. If in addition $n \geqq 3$, then $0 < t < n+1$, so that $t^{n+1} < (n+1)^{n+1}$, from which it is easily seen that

$$(96) \qquad 24 \cdot t^{n-1} < 2^{2n}(n+1)(n+2)\dots(2n+1),$$

or

$$(97) \qquad \frac{t^{2n+1}}{2^{2n}(2n+1)!} < \frac{t^{n+2}}{24.n!} .$$

Since (97) also holds for $n = 2$ and $0 < t < 4$ we have

$$(98) \qquad 2 \sum_{n=2}^{\infty} \frac{(\frac{t}{2})^{2n+1}}{(2n+1)!} < \frac{1}{24} \sum_{n=2}^{\infty} \frac{t^{n+2}}{n!} ,$$

from which it follows that

$$(99) \qquad e^{\frac{t}{2}} - e^{-\frac{t}{2}} - t < \frac{t^2}{24}(e^t-1)$$

or

$$(100) \qquad (e^t-1)\, e^{-\frac{t}{2}} - t < \frac{t^2}{24}(e^t-1)$$

or

$$(101) \qquad \frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t - 1} < \frac{t}{24} , \qquad (0<t<4).$$

Combining (95) and (101) we find that (94) is true. Consequently we have

$$(102) \qquad H(s) - \log(s+\tfrac{1}{2}) = \gamma + \int_0^\infty e^{-st} \left\{\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t - 1} - \frac{t}{24}\right\} dt +$$

$$+ \int_0^\infty e^{-st} \frac{t}{24}\, dt < \gamma + \frac{1}{24}\int_0^\infty e^{-st}\, t\, dt = \gamma + \frac{1}{24.s^2}$$

completing the proof of the proposition. □

REMARK. Numerically it turns out that (for rather small n) the rapidity of convergence of (79) (or (89)) is about the same as that in (91).

3. In this section we prove

PROPOSITION 3.1. *The sequence*

$$\{H(n) + \log\log(1 + \frac{1}{n})\}_{n=1}^{\infty} \tag{103}$$

tends decreasingly to γ.

PROOF. Since $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$ and

$$H(n) + \log\log(1 + \frac{1}{n}) = H(n) - \log n + \log\log(1 + \frac{1}{n})^n \tag{104}$$

it is clear that (103) tends to γ. In order to show that (103) is decreasing we have to prove that

$$H(n) + \log\log(1 + \frac{1}{n}) > H(n+1) + \log\log(1 + \frac{1}{n+1}) \tag{105}$$

or

$$\log \frac{\log(1 + \frac{1}{n})}{\log(1 + \frac{1}{n+1})} > \frac{1}{n+1} \tag{106}$$

or

$$\frac{\log(1 + \frac{1}{n})}{\log(1 + \frac{1}{n+1})} > e^{\frac{1}{n+1}} \tag{107}$$

or

$$\frac{-\log(1 - \frac{1}{n+1})}{\log(1 + \frac{1}{n+1})} > e^{\frac{1}{n+1}} . \tag{108}$$

Hence it certainly suffices to show that

$$\frac{-\log(1-x)}{\log(1-x)} > e^{x}, \quad (0<x<1), \tag{109}$$

or

$$(110) \qquad -\log(1-x) > e^{x}\{-\log(1-\frac{x}{1+x})\}.$$

Since for $0 < t < 1$ we have

$$(111) \qquad -\log(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots < t + \frac{t^2}{2} + \frac{1}{3}\,\frac{t^3}{1-t}\,,$$

it suffices to show that

$$(112) \qquad x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots > e^{x}\left\{\frac{x}{1+x} + \frac{1}{2}\,\frac{x^2}{(1+x)^2} + \frac{1}{3}\,\frac{x^3}{(1+x)^3} \cdot \frac{1}{1-\frac{x}{1+x}}\right\} =$$

$$= e^{x}\left\{\frac{x}{1+x} + \frac{1}{2}\,\frac{x^2}{(1+x)^2} + \frac{1}{3}\,\frac{x^3}{(1+x)^2}\right\},$$

or

$$(113) \qquad (1+x)^2 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} > e^{x}.\{1 + \frac{3}{2}\,x + \frac{1}{3}\,x^2\},$$

or

$$(114) \qquad \sum_{n=3}^{\infty} \left(\frac{1}{n+1} + \frac{2}{n} + \frac{1}{n-1}\right) x^n > \sum_{n=3}^{\infty} \left(\frac{1}{n!} + \frac{3}{2.(n-1)!} + \frac{1}{3.(n-2)!}\right) x^n.$$

It is easily seen that

$$(115) \qquad \frac{1}{n+1} + \frac{2}{n} + \frac{1}{n-1} > \frac{1}{n!} + \frac{3}{2.(n-1)!} + \frac{1}{3.(n-2)!}\,,$$

for all $n \geqq 3$, so that (114) follows, completing the proof. □

<u>REMARK</u>. Since

$$(116) \qquad (1+\frac{1}{n})^{n+\frac{1}{2}} > e, \qquad (n \in \mathbb{N})$$

we have

$$(117) \qquad H(n) + \log\log(1+\frac{1}{n}) =$$

$$= H(n) - \log(n+\frac{1}{2}) + \log\log(1+\frac{1}{n})^{n+\frac{1}{2}} > H(n) - \log(n+\frac{1}{2}).$$

According to proposition 2.5 we have

$$H(n) - \log(n + \tfrac{1}{2}) > \gamma,$$

so that the only interesting thing in proposition 3.1 is the *monotonicity* of (101).

4. We conclude this note by proving the following remarkable identity

$$(118) \qquad 1 - \gamma = \sum_{n=2}^{\infty} (-1)^n \frac{[{}^2\log n]}{n+1},$$

where ${}^2\log n$ denotes the logarithm of n in the base 2 whereas [] denotes the greatest integer function.

Since the general term of the series in (118) tends to zero the convergence of this series follows from the convergence of

$$(119) \qquad S(N) \overset{\text{def}}{=} \sum_{n=2}^{2N+1} (-1)^n \frac{[{}^2\log n]}{n+1}, \qquad (N\to\infty).$$

It is easily seen that S(N) is increasing so that it suffices to prove the convergence of $S(2^N)$, $(N\to\infty)$.

Writing

$$(120) \qquad K(n) = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k},$$

we have

$$(121) \qquad K(2n) = H(2n) - H(n).$$

Now observe that

$$(122) \qquad S(2^N) = \sum_{n=2}^{2^{N+1}+1} (-1)^n \frac{[{}^2\log n]}{n+1} =$$

$$= \frac{1}{3} - \frac{1}{4} + 2(\frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}) + \ldots +$$

$$+ N\left(\frac{1}{2^N + 1} - \frac{1}{2^N + 2} + - \ldots + \frac{1}{2^N + 2^N - 1} - \frac{1}{2^{N+1}}\right) =$$

$$= \{K(4) - K(2)\} + 2\{K(8) - K(4)\} + \ldots + N\{K(2^{N+1}) - K(2^N)\} =$$

$$= -\{K(2) + (K(4) + \ldots + K(2^N)\} + N.K(2^{N+1}) =$$

$$= -\{(H(2) - H(1)) + (H(4) - H(2)) + \ldots + (H(2^N) - H(2^N))\} +$$

$$+ N . \{H(2^{N+1}) - H(2^N)\} =$$

$$= H(1) - H(2^N) + N . \{H(2^{N+1}) - H(2^N)\} =$$

$$= 1 + N . H(2^{N+1}) - (N+1) . H(2^N) =$$

$$= 1 + N . \{(N+1) \log 2 + \gamma + 0(\frac{1}{2^{N+1}})\} +$$

$$- (N+1) . \{N \log 2 + \gamma + 0(\frac{1}{2^N}) =$$

$$= 1 - \gamma + 0(\frac{N}{2^N}),$$

from which (118) is immediate. □

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ADDENDUM.

Just before the printing of this note we found that (118) is equivalent to a similar relation given by SANDHAM in the Amer. Math. Monthly, Vol. 56 (1949) p. 414.

A proof of SANDHAM's formula (by BARROW) may be found in the Amer. Math. Monthly, Vol. 58 (1951) p. 117.