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On the zeros of the Scorer functions

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Abstract

Asymptotic approximations are developed for zeros of the solutions Gi(z) and Hi(z) of the inhomogeneous Airy differential equation $w'' - zw = \pm\frac{1}{\pi}$. The solutions are also called Scorer functions. Tables are given with numerical values of the zeros.

1. Introduction

Scorer functions are particular solutions of the non-homogeneous Airy differential equation. Detailed information on these functions can be found in [1] and in references given in [2]. We summarize the properties that are needed in this paper.

We have for $z \in \mathbb{R}$

$$w'' - z\,w = -1/\pi \quad \text{with solution}$$

$$\mathrm{Gi}(z) = \frac{1}{\pi}\int_0^\infty \sin\left(zt + \frac{1}{3}t^3\right)dt, \tag{1}$$

and for $z \in \mathbb{C}$

$$w'' - z\,w = 1/\pi \quad \text{with solution} \quad \mathrm{Hi}(z) = \frac{1}{\pi}\int_0^\infty e^{zt - \frac{1}{3}t^3}\,dt. \tag{2}$$

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The solutions of the homogeneous Airy equation $w'' - zw = 0$ are denoted by $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$. They have the integral representations

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(zt + \frac{1}{3}t^3\right) dt,$$
$$\mathrm{Bi}(z) = \frac{1}{\pi}\int_0^\infty \sin\left(zt + \frac{1}{3}t^3\right) dt + \frac{1}{\pi}\int_0^\infty e^{zt-\frac{1}{3}t^3}\, dt, \qquad (3)$$

where we assume that z is real.

Initial values are

$$\mathrm{Gi}(0) = \frac{1}{2}\mathrm{Hi}(0) = \frac{1}{3}\mathrm{Bi}(0) = \frac{1}{\sqrt{3}}\mathrm{Ai}(0) = \frac{1}{3^{7/6}\Gamma(\frac{2}{3})},$$
$$\mathrm{Gi}'(0) = \frac{1}{2}\mathrm{Hi}'(0) = \frac{1}{3}\mathrm{Bi}'(0) = -\frac{1}{\sqrt{3}}\mathrm{Ai}'(0) = \frac{1}{3^{5/6}\Gamma(\frac{1}{3})}. \qquad (4)$$

From (1)–(3) it follows that

$$\mathrm{Gi}(z) + \mathrm{Hi}(z) = \mathrm{Bi}(z). \qquad (5)$$

Other relations that we need are (see [2,3])

$$\mathrm{Hi}(z) = e^{\pm 2\pi i/3}\mathrm{Hi}(ze^{\pm 2\pi i/3}) + 2e^{\mp \pi i/6}\mathrm{Ai}(ze^{\mp 2\pi i/3}), \qquad (6)$$

and

$$\mathrm{Gi}(z) = -e^{\pm 2\pi i/3}\mathrm{Hi}(ze^{\pm 2\pi i/3}) \pm i\mathrm{Ai}(z). \qquad (7)$$

Proofs follow easily by verifying that the right-hand sides satisfy the differential equations, and from the initial values.

We use the asymptotic expansions

$$\mathrm{Gi}(z) \sim \frac{1}{\pi z}\left[1 + \frac{1}{z^3}\sum_{s=0}^{\infty} \frac{(3s+2)!}{s!(3z^3)^s}\right], \quad z \to \infty, \quad |\mathrm{ph}\, z| \leqslant \frac{1}{3}\pi - \delta, \qquad (8)$$

$$\mathrm{Hi}(z) \sim -\frac{1}{\pi z}\left[1 + \frac{1}{z^3}\sum_{s=0}^{\infty} \frac{(3s+2)!}{s!(3z^3)^s}\right], \quad z \to \infty,$$
$$|\mathrm{ph}(-z)| \leqslant \frac{2}{3}\pi - \delta, \qquad (9)$$

δ being an arbitrary positive constant. These expansions follow from (1) and (2) and by using standard methods from asymptotics (Watson's lemma; see [4, page 112, 431]).

2. Qualitative properties of the real zeros of Gi(z) and Gi$'(z)$

From (2) we see that $\mathrm{Hi}(z) > 0$ and $\mathrm{Hi}'(z) > 0$ for real finite z. However, $\mathrm{Gi}(z)$ and $\mathrm{Gi}'(z)$ have real zeros. First we show that $\mathrm{Gi}(z)$ does not have positive zeros. Later, we study properties of the negative real zeros and we discuss the properties of the zeros of the derivative.

For studying qualitative properties of the zeros, relation 10.4.51 in [1].

$$W[\mathrm{Gi},\mathrm{Bi}](x) \equiv \mathrm{Gi}(z)\mathrm{Bi}'(z) - \mathrm{Gi}'(z)\mathrm{Bi}(z) = \frac{1}{\pi}\int_0^z \mathrm{Bi}(t)\,dt \tag{10}$$

will be useful, together with well-known properties on the interlacing of zeros of functions:

Lemma 1. *Let $f(x)$ and $g(x)$ be two continuously differentiable functions in an interval I. Let $W[f,g](x) = f(x)g'(x) - f'(x)g(x)$ be such that $W[f,g](x) \neq 0\ \forall x \in I$.*

Then, the zeros of $f(x)$ and $g(x)$ in I are simple. Furthermore, between two consecutive zeros of $f(x)$ there is exactly one zero of $g(x)$ (and vice versa).

As a consequence, if $g(x)$ has no zeros in I then $f(x)$ has at most one (simple) zero in I (and vice versa).

From Lemma 1 we can check that

Lemma 2. $\mathrm{Gi}(x)$ *is positive for* $x \geqslant 0$.

Proof. $\mathrm{Bi}(x) > 0$ for $x \geqslant 0$ (see [1], for example the series expansion in 10.4.3) and then (Eq. (10)) $W[\mathrm{Gi},\mathrm{Bi}](x) > 0$ for $x \geqslant 0$. From Lemma 1, $\mathrm{Gi}(x)$ can have at most one (simple) positive real zero, but $\mathrm{Gi}(0) > 0$ and $\mathrm{Gi}(x) > 0$ for large positive x (Eq. (9)); therefore $\mathrm{Gi}(x) > 0$ for $x \geqslant 0$. □

To consider negative values of z, we first remark that $\mathrm{Bi}(z)$ has an infinite number of negative zeros which we denote by $\{b_n\}$.

Lemma 3. $\mathrm{Gi}(b_n) < 0\ \forall n$.

Proof. From (5), and the fact that $\mathrm{Hi}(x) > 0$ we have $\mathrm{Bi}(x) > \mathrm{Gi}(x)$ for all real x. □

Lemma 4. $\mathrm{Gi}(x)$ *has exactly one simple zero in* $(b_1, 0)$.

Proof. $\mathrm{Gi}(b_1) < 0$ whereas $\mathrm{Gi}(0) > 0$ (see (4)); then $\mathrm{Gi}(x)$ has at least one zero in $(b_1, 0)$. Furthermore, $\mathrm{Bi}(x) > 0$ and then $W[\mathrm{Gi},\mathrm{Bi}](x) < 0$ in $(b_1, 0)$. Then, Lemma 1 implies that there is only one zero in this interval and that it is simple. □

Between $b_1 = -1.17371$ and $b_2 = -3.27109$, the function $\mathrm{Bi}(x)$ is negative, and so $\mathrm{Gi}(x)$ is negative in that interval ($\mathrm{Bi}(x) > \mathrm{Gi}(x)$). More generally, we have that:

Lemma 5. $\mathrm{Gi}(x)$ *has no zeros in the intervals* $[b_{2n+2}, b_{2n+1}]$, $n = 0, 1, \ldots$.

We are only left with the possibility of having zeros in intervals (b_{2n+1}, b_{2n}), $n = 1, 2, \ldots$, where $\mathrm{Bi}(x) > 0$. Numerical experiments show that these zeros are simple.

The proof that all real or complex zeros of $\mathrm{Gi}(z)$ and $\mathrm{Hi}(z)$ are simple does not follow from the inhomogeneous differential equations (1) and (2). Recall that for functions defined by homogeneous linear differential equations of second order, as the Airy functions, such a proof is trivial. The essential difference between these two cases is that the existence and uniqueness theorem for solutions of a linear second order homogeneous ODE guarantees that the only solution having a double zero at a point $x = x_0$ is the trivial solution; contrary, for (1) and (2), there is always one solution with a double zero at $x = x_0$ and it is not a trivial solution.

We can see this explicitly:

Lemma 6. *The function*

$$y(z) = \alpha(z_0)\mathrm{Ai}(z) + \beta(z_0)\mathrm{Bi}(z) + \mathrm{Gi}(z),$$

with $\alpha(z_0) = -\int_0^{z_0} \mathrm{Bi}(t)dt$, *and* $\beta(z_0) = \int_0^{z_0} \mathrm{Ai}(t)\,dt - \frac{1}{3}$ *is the solution of* $\omega'' - z\omega = -1/\pi$ *with a double zero at* z_0.

Proof. Solve the system $y(z_0) = 0$, $y'(z_0) = 0$ for α and β and use the Wronskian relations 10.4.10, 10.4.47 and 10.4.51 of [1]. □

In fact, $\int_0^x \mathrm{Ai}(t)\,dt$ is numerically seen to be negative for negative x, which indicates that the negative real zeros of $\mathrm{Gi}(x)$ are simple because $\beta(z_0) < 0\ \forall z_0 < 0$.

If there were any real double zero (which is not the case), it necessarily would be an extremum:

Lemma 7. *The double real zeros of a real solution of* $\omega'' - z\omega = \pm 1/\pi$ *are necessarily local extrema of the function.*

Proof. Let x_0 be a (double) zero of a solution $\omega(x)$. Then, using the differential equation, $\omega''(x_0) = \pm 1/\pi$. □

Lemma 8. *The number of simple zeros of* $\mathrm{Gi}(x)$ *in each interval* (b_{2n+1}, b_{2n}), $n = 1, 2, \ldots$, *is, at most, two.*

Proof. Given that $\mathrm{Gi}(x)$ is negative at the zeros of $\mathrm{Bi}(x)$ and that the double zeros, if any, are extrema of the function, we see that the number of simple zeros (if any) must be even. Let us show there can be no more than two simple zeros.

The fact that $\mathrm{Bi}(x) > 0$ in (b_{2n+2}, b_{2n+1}) implies that $\frac{d}{dx} W[\mathrm{Gi}, \mathrm{Bi}](x) > 0$, which means that $W[\mathrm{Gi}, \mathrm{Bi}](x)$ has at most one zero in (b_{2n+2}, b_{2n+1}). Then, if $\mathrm{Gi}(x)$ had $2n$ zeros, $n > 1$, at least two of these zeros would lie in an interval where $W[\mathrm{Gi}, \mathrm{Bi}](x)$ does not change sign; this would imply (Lemma 1) that there would be a zero of $\mathrm{Bi}(x)$ between these two zeros of $\mathrm{Gi}(x)$, but $\mathrm{Bi}(x) > 0$ in (b_{2n+2}, b_{2n+1}). □

In fact, numerical calculations show that in the intervals (b_{2n+1}, b_{2n}), $n = 1, 2, \ldots$, exactly two zeros of $\mathrm{Gi}(x)$ occur, which means that there are no double zeros of

$\mathrm{Gi}(x)$ and that $W[\mathrm{Gi},\mathrm{Bi}](x) = \int_0^x \mathrm{Bi}(t)\,dt$ has exactly one zero in the intervals (b_{2n+1}, b_{2n}). This, together with the monotony of $\int_0^x \mathrm{Bi}(t)\,dt$ in the intervals (b_{n+1}, b_n), indicates that:

Conjecture 1. *The real zeros of* $\mathrm{Bi}(x)$ *and* $\int_0^x \mathrm{Bi}(t)\,dt$ *are interlaced.*

We also propose that:

Conjecture 2. *There are exactly two zeros of* $\mathrm{Gi}(x)$ *in the intervals* (b_{2n+1}, b_{2n}).

We can prove that this holds for large negative zeros:

Lemma 9. *For large n each interval* (b_{2n+1}, b_{2n}) *has two zeros of* $\mathrm{Gi}(x)$.

Proof. This follows from known asymptotic estimates. $\mathrm{Bi}'(x)$ has negative zeros, denoted by b'_n. Then (see [1, p. 450]),

$$\mathrm{Bi}(b'_n) = (-1)^n \mathcal{O}(n^{-1/6}), \quad n \to \infty, \tag{11}$$

From (9) we see that $\mathrm{Hi}(b'_n) = \mathcal{O}(1/b'_n)$ as $n \to \infty$. Hence, $\mathrm{Gi}(b'_{2n}) = \mathrm{Bi}(b'_{2n}) - \mathrm{Hi}(b'_{2n})$ is positive for large values of n, and $\mathrm{Gi}(x)$ has at least two zeros in the interval (b_{2n+1}, b_{2n}). □

The fact that the real zeros of $\mathrm{Gi}(x)$ are simple is also supported by the fact that the zeros of $\mathrm{Gi}(x)$ and Gi' seem to be interlaced, which can be easily proved for large x using the forthcoming asymptotic expansions.

Conjecture 3. *The negative zeros of* $\mathrm{Gi}(x)$ *and* $\mathrm{Gi}'(x)$ *are interlaced.*

Assuming this conjecture to be true, together with the fact that, numerically, we observe that $g_1 > g'_1$ where g_1 and g'_1 are, respectively, the first negative zeros of $\mathrm{Gi}(x)$ and $\mathrm{Gi}'(x)$, we see that:

Lemma 10. *The negative zeros of* $\mathrm{Gi}'(x)$ *are simple.*

Proof. Differentiating the differential equation it is easy to see that the double zeros of $\mathrm{Gi}'(x)$ cannot be extrema of $\mathrm{Gi}(x)$: if x_0 is such that $\mathrm{Gi}'(x_0) = \mathrm{Gi}''(x_0) = 0$ then $\mathrm{Gi}'''(x_0) = x_0\mathrm{Gi}'(x_0) + \mathrm{Gi}(x_0) = \mathrm{Gi}(x_0) \neq 0$ (because we assume that the zeros of $\mathrm{Gi}(x)$ and $\mathrm{Gi}'(x)$ are interlaced). However, by this same assumption, between two zeros of $\mathrm{Gi}(x)$ there must be only one zero of $\mathrm{Gi}'(x)$ which, clearly, must be a local extrema and therefore cannot be a double zero of $\mathrm{Gi}'(x)$. □

We use an additional numerical fact to prove Lemma 12; also, the following result is used:

Lemma 11. *$W[\mathrm{Hi}',\mathrm{Gi}'](x)$ has, at most, one positive real zero.*

Proof. We use the differential equations and (5); with this:

$$W(x) \equiv W[\mathrm{Hi}',\mathrm{Gi}'](x) = \frac{1}{\pi}\mathrm{Bi}'(x) - \frac{x}{\pi}\int_0^x \mathrm{Bi}(t)\,dt.$$

Then, given that $W(0)>0$ and $W'(x) = -\int_0^x \mathrm{Bi}(t)\,dt<0$ for $x>0$, $W(x)$ has at most one positive zero. □

Indeed, we observe that such zero exists $x_0 \simeq 1.0653592469$.

Lemma 12. *$\mathrm{Gi}'(x)$ has exactly one positive real zero.*

Proof. Indeed, $\mathrm{Gi}'(0)>0$ (4) while (9) shows that $\mathrm{Gi}'(x)<0$ for large positive x, which implies that there must be at least one positive zero of $\mathrm{Gi}'(x)$. This, together with the fact that the double zeros of Gi' are not extrema, implies that there must be an odd number of positive zeros. Let us assume for the moment that all the positive zeros of $\mathrm{Gi}'(x)$ are simple; in this case, we show that there is only one positive zero.

$\mathrm{Gi}'(x)$ cannot have three or more simple positive zeros because, $W[\mathrm{Hi}',\mathrm{Gi}']$ has at most one positive zero (Lemma 11) and $\mathrm{Hi}'(x)>0$. The possible zero of $W[\mathrm{Hi}',\mathrm{Gi}'](x)$ cannot coincide with any zero of $\mathrm{Gi}'(x)$, because we are assuming by now that the zeros of $\mathrm{Gi}'(x)$ are simple; thus, if $\mathrm{Gi}'(x)$ had at least three zeros, at least two of them would lie in an interval where $W[\mathrm{Hi}',\mathrm{Gi}'](x)$ does not change sign. This is in contradiction with the fact that $\mathrm{Hi}'(x)>0$ (see Lemma 1).

On the other hand, the only possible double zero of $\mathrm{Gi}'(x)$ should coincide with the positive zero of $W[\mathrm{Hi}',\mathrm{Gi}'](x)$. However, it is numerically observed that

$\mathrm{Gi}'(x_0)<0$ being x_0 the positive zero of $W[\mathrm{Hi}',\mathrm{Gi}'](x)$. □

The numerical value of this isolated zero of $\mathrm{Gi}'(x)$, is $g' = 0.60907541707\ldots$.

3. Asymptotics of the negative zeros of $\mathrm{Gi}(z)$

We write $\mathrm{Gi}(-z) = \mathrm{Bi}(-z) - \mathrm{Hi}(-z)$, and use the asymptotic expansion of $\mathrm{Bi}(-z)$ as given in [1, p. 449] and of $\mathrm{Hi}(-z)$ that follows from (9). We write

$$\mathrm{Hi}(-z) = \frac{1}{\pi z} Ha(z), \quad Ha(z) \sim 1 - \sum_{s=0}^{\infty} \frac{h_s}{z^{3(s+1)}},$$
$$h_s = (-1)^s \frac{(3s+2)!}{s!3^s}, \tag{12}$$

$$\mathrm{Bi}(-z) = \frac{1}{\sqrt{\pi}\,z^{1/4}}\left[\cos\left(\zeta + \frac{1}{4}\pi\right)P(\zeta) + \frac{1}{\zeta}\sin\left(\zeta + \frac{1}{4}\pi\right)Q(\zeta)\right], \tag{13}$$

$$P(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s c_{2s}}{\zeta^{2s}}, \quad Q(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s c_{2s+1}}{\zeta^{2s}}, \tag{14}$$

$$\zeta = \frac{2}{3} z^{\frac{3}{2}}, \quad c_0 = 1, \quad c_s = \frac{\Gamma(3s+\frac{1}{2})}{54^s\, s!\, \Gamma(s+\frac{1}{2})}. \tag{15}$$

We explain the method by taking $Ha(\zeta) = P(\zeta) = 1$ and $Q(\zeta) = 0$. This gives for the equation $\mathrm{Gi}(-z) = \mathrm{Bi}(-z) - \mathrm{Hi}(-z) = 0$ a first equation

$$\cos\left(\zeta + \frac{1}{4}\pi\right) = \frac{1}{\sqrt{\pi}\, z^{3/4}}. \tag{16}$$

Using $z^{3/4} = \sqrt{3\zeta/2}$, we obtain

$$\cos\left(\zeta + \frac{1}{4}\pi\right) = \sqrt{\frac{2}{3\pi\zeta}}. \tag{17}$$

For large ζ solutions occur when the cosine function is small. We put

$$\zeta = \zeta_n + \varepsilon, \quad \zeta_n = \left(n - \frac{3}{4}\right)\pi, \quad n = 1, 2, 3, \ldots . \tag{18}$$

The equation for ε reads

$$\sin\varepsilon = \frac{c}{\sqrt{\zeta_n + \varepsilon}} = \frac{c\,t}{\sqrt{1+\varepsilon t^2}}, \quad t = 1/\sqrt{\zeta_n}, \quad c = (-1)^n\sqrt{2/(3\pi)}. \tag{19}$$

For small values of t this equation can be solved by substituting a power series $\varepsilon = \varepsilon_1 t + \varepsilon_2 t^2 + \cdots$, and the coefficients can be obtained by standard methods. For example, $\varepsilon_1 = c$. By using the asymptotic expansions for $Ha(z)$, $P(\zeta)$ and $Q(\zeta)$ a few extra technicalities are introduced. With the help of a computer algebra package the general coefficients ε_s are easy to calculate. Finally, we find for $z = (3\zeta/2)^{2/3}$, and for g_n, the zeros of $\mathrm{Gi}(z)$, the expansion

$$g_n \sim -\left(\frac{3}{2}\zeta_n\right)^{\frac{2}{3}}[1 + \varepsilon_1 t^3 + \varepsilon_2 t^4 + \cdots]^{2/3}, \quad n = 1, 2, 3, \ldots \tag{20}$$

or

$$g_n \sim -[(3\pi(4n-3)/8]^{\frac{2}{3}}[1 + \gamma_3 t^3 + \gamma_4 t^4 + \cdots],$$
$$t = \frac{1}{\sqrt{(n-3/4)\pi}}, \tag{21}$$

where

$$\gamma_3 = \frac{2c}{3}, \quad \gamma_4 = \frac{5}{108}, \quad \gamma_5 = \frac{c^3}{9}, \quad \gamma_6 = -\frac{4c^2}{9},$$
$$\gamma_7 = \frac{c(81c^4 - 1060)}{1620}, \quad \gamma_8 = -\frac{189c^4 + 20}{729}. \tag{22}$$

The expansion in (21) reduces to the expansion of the zeros b_n of $\mathrm{Bi}(z)$ if we take $c = 0$.

3.1. The real zeros of $\mathrm{Gi}'(z)$

For the real zeros of $\mathrm{Gi}'(z)$ we can use the same procedure. For this case we need

$$\mathrm{Hi}'(-z) = \frac{1}{\pi z^2}\widetilde{Ha}(z), \quad \widetilde{Ha}(z) \sim 1 - \sum_{s=0}^{\infty} \frac{\tilde{h}_s}{z^{3(s+1)}}, \quad \tilde{h}_s = (3s+4)h_s, \tag{23}$$

$$\mathrm{Bi}'(-z) = \frac{z^{1/4}}{\sqrt{\pi}}\left[\sin\left(\zeta + \frac{1}{4}\pi\right)R(\zeta) - \frac{1}{\zeta}\cos\left(\zeta + \frac{1}{4}\pi\right)S(\zeta)\right], \tag{24}$$

$$R(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s d_{2s}}{\zeta^{2s}}, \quad S(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s d_{2s+1}}{\zeta^{2s}}, \quad d_s = -\frac{6s+1}{6s-1}c_s, \tag{25}$$

where ζ and c_s are defined in (15). Using $\mathrm{Gi}'(-z) = \mathrm{Bi}'(-z) - \mathrm{Hi}'(-z)$ we obtain the equation for determining the zeros:

$$\sin\left(\zeta + \frac{1}{4}\pi\right) = \frac{1}{\zeta}\frac{S(\zeta)}{R(\zeta)}\cos\left(\zeta + \frac{1}{4}\pi\right) + \frac{1}{\sqrt{\pi}\,z^{9/4}}\frac{\widetilde{Ha}(z)}{R(\zeta)}. \tag{26}$$

Using $z^{9/4} = (3\zeta/2)^{3/2}$, we see that the main part of this equation is obtained by neglecting the term with the function $\widetilde{Ha}(z)$, but we can proceed in the same manner as before.

We put

$$\zeta = \zeta_n' + \varepsilon', \quad \zeta_n' = \left(n - \frac{1}{4}\right)\pi, \quad n = 1, 2, 3, \ldots, \tag{27}$$

and we can obtain for ε' an expansion. Finally, we obtain for g_n', the zeros of $\mathrm{Gi}'(z)$, the expansion

$$g_n' \sim -\left(\frac{3}{2}\zeta_n'\right)^{\frac{2}{3}}[1 + \varepsilon_3\tau^5 + \varepsilon_4\tau^6 + \cdots]^{2/3}, \quad n = 1, 2, 3, \ldots \tag{28}$$

or

$$g_n' \sim -[(3\pi(4n-1)/8]^{\frac{2}{3}}[1 + \gamma_4' t^4 + \gamma_5' t^5 + \cdots],$$
$$t = \frac{1}{\sqrt{(n-1/4)\pi}}, \tag{29}$$

where

$$\gamma_4' = -\frac{7}{108}, \quad \gamma_5' = \frac{2c}{3}, \quad \gamma_6' = \gamma_7' = 0,$$
$$\gamma_8' = \frac{35}{1458}, \quad \gamma_9' = -\frac{719c}{324}, \quad \gamma_{10}' = -\frac{10c^2}{9}. \tag{30}$$

This expansion reduces to that of b_n', the zeros of $\mathrm{Bi}'(z)$ if we take $c = 0$.

4. The complex zeros of Gi(z)

Gi(z) and Gi$'(z)$ have infinite many complex zeros $\{\chi_n\}$ just below the half-line $\mathrm{ph}\, z = \frac{1}{3}\pi$, and at the conjugate values. Asymptotic estimates can be obtained by using the connection formula (7) with z replaced with $ze^{\pi i/3}$. That is,

$$\mathrm{Gi}(ze^{\pi i/3}) = -e^{\pm 2\pi i/3}\mathrm{Hi}(-z) + i\mathrm{Ai}(ze^{\pi i/3}). \tag{31}$$

We write Hi$(-z)$ as in (12) and for Ai(z) we obtain from the standard asymptotic expansion of this Airy function

$$\mathrm{Ai}(ze^{\pi i/3}) = \frac{1}{2\sqrt{\pi}z^{1/4}} e^{-\pi i/12 - i\eta} A_a(\eta), \tag{32}$$

where (for c_s see (15))

$$\eta = \frac{2}{3} z^{\frac{3}{2}}, \quad A_a(\eta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s c_s}{(i\eta)^s}. \tag{33}$$

The equation for deriving the asymptotic expansion of χ_n for large n then reads

$$e^{i\eta} = \frac{1}{2}\sqrt{\pi} z^{3/4} e^{-\pi i/4} \frac{A_a(\eta)}{Ha(z)}. \tag{34}$$

We write

$$\eta = \eta_n - \frac{1}{2} i \ln(c\eta_n) + \varepsilon, \quad \eta_n = \left(2n - \frac{1}{4}\right)\pi, \quad c = \frac{3}{8}\pi, \tag{35}$$

and obtain for ε the equation

$$e^{i\varepsilon} = \sqrt{1 - i\delta t + \varepsilon t}\, \frac{A_a(\eta)}{Ha(z)}, \quad t = \frac{1}{\eta_n}, \quad \delta = \frac{1}{2}\ln(c\eta_n). \tag{36}$$

The next step is substituting a power series $\varepsilon = \varepsilon_1 t + \varepsilon_2 t^2 + \ldots$, considering δ as a fixed parameter. A few straightforward manipulations give the expansion

$$\chi_n \sim [3\pi(8n-1)/8]^{2/3} e^{\pi i/3}\left(1 + \frac{\gamma_1}{\eta_n} + \frac{\gamma_2}{\eta_n^2} + f\frac{\gamma_3}{\eta_n^3} + \cdots\right) \tag{37}$$

and the first few coefficients are

$$\gamma_1 = -\frac{2}{3} i\delta, \quad \gamma_2 = \frac{1}{108}(5 - 36\delta + 12\delta^2),$$

$$\gamma_3 = \frac{1}{162} i(-96 + 37\delta - 45\delta^2 + 8\delta^3),$$

$$\gamma_4 = \frac{1}{2916}(-944 + 43365\delta - 1182\delta^2 + 702\delta^3 - 84\delta^4). \tag{38}$$

4.1. The complex zeros of $\mathrm{Gi}'(z)$

For the complex zeros χ_n' of $\mathrm{Gi}'(z)$ we use (cf. (7))

$$\mathrm{Gi}'(ze^{\pi i/3}) = e^{\pi i/3}\mathrm{Hi}'(-z) + ie^{\pi i/3}\mathrm{Ai}'(ze^{\pi i/3}). \tag{39}$$

We need the expansion of $\mathrm{Hi}'(-z)$ given in (23) and

$$\mathrm{Ai}'(ze^{\pi i/3}) = -\frac{z^{1/4}}{2\sqrt{\pi}}e^{\pi i/12-i\eta}\widetilde{Aa}(\eta), \quad \widetilde{Aa}(\eta) \sim \sum_{s=0}^{\infty}\frac{(-1)^s d_s}{(i\eta)^s}, \tag{40}$$

where d_s is given in (25). We put

$$\eta = \eta_n' - i\delta' + \varepsilon', \quad \eta_n' = \left(2n + \frac{1}{4}\right)\pi, \quad c = \frac{3}{2}\left(\frac{\pi}{4}\right)^{\frac{1}{3}}, \tag{41}$$

and the equation for ε' reads

$$e^{i\varepsilon'} = (1 - i\delta' t + \varepsilon' t)^{3/2}\frac{\widetilde{Aa}(\eta)}{\widetilde{Ha}(\eta)}, \quad \delta' = \frac{3}{2}\ln(c\eta_n'). \tag{42}$$

The expansion for the zeros reads

$$\chi_n' \sim [3\pi(8n+1)/8]^{2/3}e^{\pi i/3}\left(1 + \frac{\gamma_1'}{\eta_n'} + \frac{\gamma_2'}{\eta_n'^2} + \frac{\gamma_3'}{\eta_n'^3} + \cdots\right),$$
$$n = 1, 2, 3, \ldots \tag{43}$$

and the first few coefficients are

$$\gamma_1' = -\frac{2}{3}i\delta', \quad \gamma_2' = \frac{1}{108}(-7 - 108\delta' + 12\delta'^2),$$
$$\gamma_3' = \frac{1}{324}i(-747 + 458\delta' - 270\delta'^2 + 16\delta'^3),$$
$$\gamma_4' = \frac{1}{5832}(-20029 + 43740\delta' - 16908\delta'^2 + 4212\delta'^3 - 168\delta'^4). \tag{44}$$

5. The complex zeros of Hi(z)

$\mathrm{Hi}(z)$ and $\mathrm{Hi}'(z)$ have infinite many complex zeros $\{\kappa_n\}$ just above the half-line $\mathrm{ph}\, z = \frac{1}{3}\pi$, and at the conjugate values. For $\mathrm{Hi}(z)$ we use (6) in the form

$$\mathrm{Hi}(ze^{\pi i/3}) = e^{2\pi i/3}\mathrm{Hi}(-z) + 2e^{-\pi i/6}\mathrm{Ai}(ze^{-\pi i/3}). \tag{45}$$

The analysis is analogous to the case for $\mathrm{Gi}(z)$ and gives (36) with i replaced by $-i$, also in $Aa(\eta)$, and c by $c = \frac{3}{2}\pi$. This gives for κ_n, the zeros of $\mathrm{Hi}(z)$,

$$\kappa_n \sim [3\pi(8n-1)/8]^{2/3}e^{\pi i/3}\left(1 + \frac{\overline{\gamma_1}}{\eta_n} + \frac{\overline{\gamma_2}}{\eta_n^2} + \frac{\overline{\gamma_3}}{\eta_n^3} + \cdots\right), \tag{46}$$

where η_n is given in (35), $\delta = \frac{1}{2}\ln(c\eta_n)$, with $c = \frac{3}{2}\pi$, and the first few γ_k are given in (38).

For $\mathrm{Hi}'(z)$ we find Eq. (42) with i replaced by $-i$, also in $\widetilde{Aa}(\eta)$, and $c = \frac{3}{2}\pi^{\frac{1}{3}}$. For κ'_n, the zeros of $\mathrm{Hi}'(z)$, we obtain

$$\kappa'_n \sim [3\pi(8n+1)/8]^{2/3} e^{\pi i/3}\left(1 + \frac{\overline{\gamma'_1}}{\eta'_n} + \frac{\overline{\gamma'_2}}{\eta'^2_n} + \frac{\overline{\gamma'_3}}{\eta'^3_n} + \cdots\right),$$
$$n = 1, 2, 3, \ldots \tag{47}$$

where η'_n is given in (41), $\delta = \frac{3}{2}\ln(c\eta'_n)$, with $c = \frac{3}{2}\pi^{\frac{1}{3}}$, and the first few γ'_k are given (44).

6. Numerical verifications and tables

Now we will illustrate the accuracy of the asymptotic approximations for the real and complex zeros of $\mathrm{Gi}(x)$, $\mathrm{Gi}'(x)$ (except the positive zero of $\mathrm{Gi}'(x)$) and the complex zeros of $\mathrm{Hi}(x)$ and $\mathrm{Hi}'(x)$. For the complex zeros, by complex conjugation, we only need to consider $\Im z > 0$.

We use the asymptotic approximations as starting values for a Newton method, obtaining convergence in all cases. The code [3] has been used for the calculations. The accuracy of the code is better than 10^{-12} and we expect that the zeros can be computed with at least 12 exact digits.

Table 1 shows the relative error of the asymptotic estimates.

Next we compare the approximate values of the first 10 zeros with the numerical values (see Tables 2–6).

Additionally, $G'(x)$ has a positive zero: $g' = 0.60907541707$.

In all cases, as could be expected, the asymptotic estimations are closer to the true value as larger zeros (in modulus) are considered. Furthermore, as commented, the

Table 1
Relative error of the modulus of the zeros from the asymptotic estimations, compared with numerical computations

n	Error$\|g_n\|$	Error$\|g'_n\|$	Error$\|\chi_n\|$	Error$\|\chi'_n\|$	Error$\|\kappa_n\|$	Error$\|\kappa'_n\|$
1	4×10^{-2}	5×10^{-3}	4×10^{-4}	2×10^{-3}	8×10^{-4}	3×10^{-3}
5	7×10^{-7}	1×10^{-4}	6×10^{-8}	2×10^{-6}	1×10^{-7}	3×10^{-6}
10	5×10^{-8}	2×10^{-5}	1×10^{-9}	6×10^{-8}	2×10^{-9}	9×10^{-8}
25	3×10^{-11}	2×10^{-6}	8×10^{-12}	6×10^{-10}	1×10^{-11}	9×10^{-10}
50	1×10^{-11}	3×10^{-7}	2×10^{-13}	2×10^{-11}	3×10^{-13}	2×10^{-11}
75	2×10^{-13}	1×10^{-7}	2×10^{-14}	2×10^{-12}	3×10^{-14}	3×10^{-12}
100	4×10^{-13}	6×10^{-8}	3×10^{-15}	4×10^{-13}	5×10^{-14}	6×10^{-13}
150	5×10^{-14}	2×10^{-8}	1×10^{-16}	5×10^{-14}	6×10^{-16}	7×10^{-14}
200	1×10^{-14}	1×10^{-8}	$<10^{-16}$	1×10^{-14}	2×10^{-16}	1×10^{-14}

The notation is as in the text. The number of non-zero coefficients of the asymptotic expansions considered is as follows: for $|g_n|$ we take 2 coefficients for $n = 1$ and 6 coefficients for the rest of values of n; for $|g'_n|$, $|\chi_n|$, $|\chi'_n|$, $|\kappa_n|$ and $|\kappa'_n|$ we take the first 4 non-zero coefficients.

Table 2
Asymptotic estimations of the first 10 negative real zeros of Gi(x) and Gi′(x) versus their numerical value (12 digits)

n	g_n (asymp.)	g_n (numer.)	g'_n (asymp.)	g'_n (numer.)
1	−0.70701728791 (2)	−0.73764033232	−2.26148803837 (4)	−2.24995421864
2	−3.40013324843 (2)	−3.39083150945	−4.08890415841 (4)	−4.08395408849
3	−4.75152465295 (3)	−4.75160079064	−5.50501788785 (4)	−5.50743021111
4	−6.22702978591 (5)	−6.22707083456	−6.78556344666 (4)	−6.78414405732
5	−7.33018484228 (5)	−7.33017070326	−7.93738558753 (4)	−7.93831371630
6	−8.53064462827 (5)	−8.53064781862	−9.02156063733 (4)	−9.02090166816
7	−9.50443871324 (5)	−9.50443547307	−10.0362185151 (4)	−10.0367106297
8	−10.5595675877 (5)	−10.5595678851	−11.0076119069 (4)	−11.0072288049
9	−11.4501841971 (5)	−11.4501830272	−11.9333405428 (4)	−11.9336474410
10	−12.4106527814 (5)	−12.4106527199	−12.8280143111 (4)	−12.8277622904

Between brackets, the number of the first non-zero coefficients taken in the calculation of the asymptotic expansion is given.

Table 3
Asymptotic estimations of the first 10 complex zeros of Gi(z) versus their numerical value (12 digits)

n	χ_n (asymptotic)	χ_n (numerical)
1	2.44433318205 + i 3.28043340740	2.44134455893 + i 3.28073610375
2	3.82724470205 + i 5.61364024656	3.82706907612 + i 5.61368067243
3	4.94973090968 + i 7.55292445144	4.94969805256 + i 7.55293472024
4	5.94054868777 + i 9.27655846564	5.94053866799 + i 9.27656211688
5	6.84659373818 + i 10.8567528445	6.84658973653 + i 10.8567544432
6	7.69146765566 + i 12.3317696540	7.69146576022 + i 12.3317704591
7	8.48916873952 + i 13.7249535559	8.48916772985 + i 13.7249540039
8	9.24886878556 + i 15.0518649809	9.24886819962 + i 15.0518652495
9	9.97699441862 + i 16.3235290835	9.97699405568 + i 16.3235292542
10	10.6782722198 + i 17.5481160856	10.6782719832 + i 17.5481161992

The expansion is calculated using the first 4 coefficients.

Table 4
Asymptotic estimations of the first 10 complex zeros of Gi′(z) versus their numerical value (12 digits)

n	χ'_n (asymptotic)	χ'_n (numerical)
1	3.73104015614 + i 3.20468169034	3.71910633591 + i 3.20254922301
2	5.05878908159 + i 5.49094064093	5.05721412684 + i 5.49107967331
3	6.14094636445 + i 7.40393823622	6.14051474537 + i 7.40403247457
4	7.09863883359 + i 9.11033272563	7.09847245342 + i 9.11038169520
5	7.97658092867 + i 10.6784124602	7.97650267337 + i 10.6784393595
6	8.79711673037 + i 12.1445154227	8.79707480551 + i 12.1445312817
7	9.57340329298 + i 13.5309191990	9.57337867420 + i 13.5309291376
8	10.3140187866 + i 14.8525432972	10.3140033118 + i 14.8525498458
9	11.0249563206 + i 16.1200042911	11.0249460691 + i 16.1200087871
10	11.7106171211 + i 17.3411995000	11.7106100405 + i 17.3412026935

The expansion is calculated using the first 4 coefficients.

Table 5
Asymptotic estimations of the first 10 complex zeros of Hi(z) versus their numerical value (12 digits)

n	κ_n (asymptotic)	κ_n (numerical)
1	1.31810666758 + i 3.93044374287	1.32022985770 + i 3.92618518472
2	2.71758115521 + i 6.25616826873	2.71776478546 + i 6.25594658531
3	3.86688975856 + i 8.18006876521	3.86692943374 + i 8.18002987309
4	4.88317281362 + i 9.88881442645	4.88318584624 + i 9.88880304438
5	5.81193606777 + i 11.4556905719	5.81194151066 + i 11.4556861557
6	6.67695905797 + i 12.9188986002	6.67696171311 + i 12.9188965531
7	7.49263240875 + i 14.3015594043	7.49263385258 + i 14.3015583321
8	8.26849226103 + i 15.6190187624	8.26849311165 + i 15.6190181486
9	9.01126446313 + i 16.8821242633	9.01126499607 + i 16.8821238875
10	9.7259145503 + i 18.0989041312	9.72591490090 + i 18.0989038885

The expansion is calculated using the first 4 coefficients.

Table 6
Asymptotic estimations of the first 10 complex zeros of Hi′(z) versus their numerical value (12 digits)

n	κ'_n (asymptotic)	κ'_n (numerical)
1	0.61539789841 + i 5.00682180461	0.62172976845 + i 4.99069463707
2	2.00101984737 + i 7.26100462042	2.00240099109 + i 7.25911069430
3	3.14666657916 + i 9.13725837677	3.14711339788 + i 9.13677671663
4	4.16377885499 + i 10.8089522433	4.16396557831 + i 10.8087759627
5	5.09551443130 + i 12.3455631789	5.09560639947 + i 12.3454833919
6	5.96454826183 + i 13.7832989849	5.96459897682 + i 13.7832574947
7	6.78472063073 + i 15.1440500844	6.78475098876 + i 15.1440262976
8	7.56527902355 + i 16.4423442321	7.56529836198 + i 16.4423295732
9	8.31279229101 + i 17.6884581451	8.31280522481 + i 17.6884485950
10	9.03213892761 + i 18.8900067299	9.03214792350 + i 18.8900002276

The expansion is calculated using the first 4 coefficients.

asymptotic estimations can be used as starting values to compute accurately the zeros of Scorer functions. The only exception is the positive real zero of Gi′(x), which cannot be estimated via the asymptotic expansions for the zeros.

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References

[1] M. Abramowitz, I. Stegun (Eds.), Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series, No. 55, US Government Printing Office, Washington, DC, 1964.

[2] A. Gil, J. Segura, N.M. Temme, On non-oscillating integrals for computing inhomogeneous Airy functions, Math. Comput 70 (2001) 1183–1194.

[3] A. Gil, J. Segura, N.M. Temme, GIZ, HIZ: two Fortran 77 routines for the computation of complex Scorer functions, ACM Trans. Math. Soft, in press.

[4] F.W.J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1997 (Reprinted by A.K. Peters).