Conference Applications of Mathematics 2013
in honor of the 70th birthday of Karel Segeth.
Jan Brandts, Sergey Korotov, Michal Křížek,
Jakub Šístek, and Tomáš Vejchodský (Eds.),
Institute of Mathematics AS CR, Prague 2013

ON THE COMPUTATION OF MOMENTS OF THE PARTIAL NON-CENTRAL CHI-SQUARE DISTRIBUTION FUNCTION

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Abstract

Properties satisfied by the moments of the partial non-central chi-square distribution function, also known as Nuttall Q-functions, and methods for computing these moments are discussed in this paper. The Nuttall Q-function is involved in the study of a variety of problems in different fields, as for example digital communications.

1. Introduction

The non-central chi-square distribution function of probability appears in many applications. For example, in radar communications it appears when computing the detection of signals in noise using a square-law detector. Its cumulative distribution function is also known as the generalized Marcum $Q-$ function, which is defined by using the integral representation

$$Q_\mu(x,y) = x^{\frac{1}{2}(1-\mu)} \int_y^{+\infty} t^{\frac{1}{2}(\mu-1)} \mathrm{e}^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right)\, dt, \tag{1}$$

where $\mu > 0$ and $I_\mu(z)$ is the modified Bessel function.

In radar problems, if the signal-to-noise power ratio is x for the sum of μ independent samples of the output of a square-law detector, this integral gives the probability of that the sum will be y or more.

The complementary function of the generalized Marcum $Q-$ function is given by

$$P_\mu(x,y) = x^{\frac{1}{2}(1-\mu)} \int_0^{y} t^{\frac{1}{2}(\mu-1)} \mathrm{e}^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right)\, dt, \tag{2}$$

*Former address: CWI, 1098 XG Amsterdam, The Netherlands and the following relation holds

$$P_\mu(x,y) + Q_\mu(x,y) = 1. \tag{3}$$

Methods and an algorithm for computing the functions $P_\mu(x,y)$ and $Q_\mu(x,y)$ are described in [2].

The ηth moment of the partial non-central chi-square distribution function is given by

$$Q_{\eta,\mu}(x,y) = x^{\frac{1}{2}(1-\mu)} \int_y^{+\infty} t^{\eta+\frac{1}{2}(\mu-1)} \mathrm{e}^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right)\, dt. \tag{4}$$

In this manuscript, we give properties satisfied by the moments of the partial non-central chi-square distribution functions and discuss methods for computing these moments, also known as Nuttall Q-functions [4]. There are several applications where these functions are involved as for example, the analysis of the outage probability of wireless communication systems with a minimum signal power constraint [5], to mention just one example within the telecommunications field.

2. Properties

The Maclaurin series for the modified Bessel function reads

$$I_\mu(z) = \left(\tfrac{1}{2}z\right)^\mu \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^n}{n!\,\Gamma(\mu+n+1)}. \tag{5}$$

By substituting this expression in the integral representation, we obtain the series expansion for the ηth moment of the non-central chi-square distribution function:

$$Q_{\eta,\,\mu}(x,y) = \mathrm{e}^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\Gamma(\eta+\mu+n,\, y)}{\Gamma(\mu+n)}. \tag{6}$$

This expansion is given in terms of one of the standard incomplete gamma functions defined by

$$\Gamma(\mu,x) = \int_x^{+\infty} t^{\mu-1}\mathrm{e}^{-t}\, dt. \tag{7}$$

Introducing the factor $\Gamma(\eta+\mu+n)$ in (6), the expansion can be also given in terms of the incomplete gamma function ratio $Q_\mu(y)$, defined by

$$Q_\mu(x) = \frac{\Gamma(\mu,x)}{\Gamma(\mu)}, \tag{8}$$

and for which algorithms are given in [3].

The expansion for the ηth moment of the non-central chi-square distribution function in terms of incomplete gamma function ratios is given by

$$Q_{\eta,\,\mu}(x,y) = \mathrm{e}^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\Gamma(\eta+\mu+n)}{\Gamma(\mu+n)} Q_{\eta+\mu+n}(y). \tag{9}$$

The series representation can be computed by using the algorithms for the incomplete gamma ratios described in [3]. The recurrence relation

$$Q_{\eta+\mu+1}(y) = Q_{\eta+\mu}(y) + \frac{y^{\eta+\mu}\mathrm{e}^{-y}}{\Gamma(\eta+\mu+1)}, \tag{10}$$

is stable for $Q_\mu(y)$ in the forward direction, so the evaluation of the terms in the series for this function in (9) is rather easy.

A recurrence relation for the moments of the non-central chi-squared distribution function can be obtained considering integration by parts in the integral in (4), together with the relation $z^\mu I_{\mu-1}(z) = \frac{d}{dz}\left(z^\mu I_\mu(z)\right)$. This gives

$$Q_{\eta,\mu}(x,y) = Q_{\eta,\mu+1}(x,y) - \eta Q_{\eta-1,\mu+1} - \left(\frac{y}{x}\right)^{\mu/2} y^\eta \mathrm{e}^{-x-y} I_\mu(2\sqrt{xy}). \tag{11}$$

When $\eta = 0$, this recurrence reduces to a first order difference equation for the Marcum-Q function (see, for instance, [6] [1]). The recurrence relation given in (11) can be used for testing, and it can be also used for computation, as we describe later.

3. Computing moments using the series expansion

The series expansion given in (6) has been tested by using the recurrence relation of (11) written in the form

$$\frac{Q_{\eta,\mu+1}(x,y)}{Q_{\eta,\mu}(x,y) + \eta Q_{\eta-1,\mu+1} + \left(\frac{y}{x}\right)^{\mu/2} y^\eta \mathrm{e}^{-x-y} I_\mu(2\sqrt{xy})} = 1. \tag{12}$$

The deviations from 1 of the left-hand side of (12) (in absolute value) will measure the accuracy of the tested methods. The series expansion has been implemented in the Fortran 90 module **NuttallF**. This module uses another module (**IncgamFI**) for the computation of the gamma function ratios. We have tested the parameter region $(\eta,\ \mu,\ x,\ y) \in (1,\ 50) \times (1,\ 50) \times (0,\ 20) \times (0,\ 20)$. The tests show that an accuracy better than 10^{-12} in this region can be obtained with the series expansion.

When μ or $\mu+n$ are large, it is convenient to use approximations for the ratio of gamma functions appearing in the expression, in order to avoid the appearance of overflow problems sooner than expected. In the case $\mu+n \to \infty$ we have:

$$\frac{\Gamma(\eta+\mu+n)}{\Gamma(\mu+n)} \sim (\mu+n)^\eta. \tag{13}$$

The following table shows some values of moments of the chi-square distribution function computed with the series expansion and the corresponding values obtained

[1]We note that a factor e^{-y} is missing in [6, Eq. (1.4)].

η	μ	x	y	$Q_{\eta,\mu}(x,y)$	$Q_{\eta,\mu}(x,y)$ with Maple
1	1	0.1	1.5	0.6644091427683566	0.66440914276835656
5	10	0.1	1.5	252472.22699183668	252472.226991836658
50	30	0.1	1.5	$1.1944632251434243\,10^{+86}$	$1.19446322514344860\,10^{+86}$
1	1	1.2	5	0.5457546041478581	0.54575460414785805
5	10	1.2	5	419098.1927146542	419098.192714654143
50	30	1.2	5	$6.809314196073125\,10^{+86}$	$6.80931419607285639\,10^{+86}$
1	1	5	10	1.4822515303982464	1.48225153039824667
5	10	5	10	1654969.264263704	1654969.26426370245
50	30	5	10	$1.1734657613338925\,10^{+89}$	$1.17346576133388184\,10^{+89}$

Table 1: Values of the moments of the chi-square distribution function for different choices of the parameters η, μ x and y. The values shown are obtained with the series expansion and with the direct computation of the integral representation using Maple with 50 digits.

with the direct computation of the integral representation using Maple with 50 digits (the results shown in the table correspond to the first 18 digits obtained with these computations). The computation of the series expansion has been implemented in the double precision Fortran 90 module **NuttallF**. As can be seen, an agreement of minimum 14-15 digits is obtained in all cases, which is consistent with the expected accuracy of the double precision Fortran 90 module.

In some cases, Maple fails to compute the integral and acceleration can be obtained by suitably truncating the improper integral and changing the variable of integration. We notice that, as before commented, the modified Bessel function is exponentially increasing for large arguments and then the integrand in (4) can be estimated by $t^{\gamma}\mathrm{e}^{-(\sqrt{t}-\sqrt{x})^2}$, $\gamma=\eta+(\mu-1)/2$ which is related to a Gaussian centered $t=x$. The maximum value of this function is attained at $t=(\sqrt{x}+\sqrt{x+4\gamma})^2/4$ and integrating around this value with a sufficiently wide interval is enough. This truncated integral over finite interval $[a,b]$ can be then transformed with a linear change to an integral in $[-1,1]$ and the convergence is further accelerated by considering the change of variable $t=\tanh(u)$, particularly if the trapezoidal rule is used for evaluating the integral (see [1, §5.4.2]). These modifications are observed to speed up the computation of the integrals using Maple, particularly for the last value in Table 1 for which Maple does not appear to be able to converge to an accurate value.

4. Computing moments by recursion

If we write the recurrence relation (11) as

$$Q_{\eta,\mu+1}(x,y)=Q_{\eta,\mu}(x,y)+\eta Q_{\eta-1,\mu+1}+\left(\frac{y}{x}\right)^{\mu/2}y^{\eta}\mathrm{e}^{-x-y}I_{\mu}(2\sqrt{xy}),\qquad(14)$$

then it is clear that we have a numerically stable relation because all the terms in the right hand side are positive.

Now, assume that the moments of order zero (Marcum functions) $Q_{0,\mu}$ are known for $\mu = 1, 2, \ldots, N$ (or for a sequence of real values μ_i, $i=1, \ldots N$, with $\mu_{i+1} - \mu_i = 1$). If $Q(1, \mu)$ is also known, the relation (15) can be used to compute $Q(1, \mu + 1)$; therefore, starting from the value $Q(1, 1)$ we can compute $Q(1, \mu)$, $\mu = 1, 2, \ldots, N$ in a stable way. In the same way, after determining $Q(1, \mu)$, $\mu = 1, 2, \ldots, N$ and if $Q(2, 1)$ is know, we can compute $Q(1, \mu)$, $\mu = 1, 2, \ldots N$ and so on.

It is worth mentioning that the inhomogeneous recurrence has to be applied with care, particularly the inhomogeneous term. As x and/or y becomes large the Bessel function increases exponentially; therefore we have the product of a small exponential times an exponentially large function and because of the bad conditioning of the exponentials, this translates into larger relative errors; additionally, the exponentials may overflow/underflow. Part of this error can be avoided by considering the scaled Bessel function $\tilde{I}_\nu(x) = e^{-x} I_\nu(x)$. In terms of this function

$$Q_{\eta,\mu+1}(x, y) = Q_{\eta,\mu}(x, y) + \eta Q_{\eta-1,\mu+1} + \left(\frac{y}{x}\right)^{\mu/2} y^\eta \mathrm{e}^{-(\sqrt{x}-\sqrt{y})^2} \tilde{I}_\mu(2\sqrt{xy}). \tag{15}$$

An alternative way of computing with recurrences is considering a homogeneous equation, which we can be constructed from the inhomogeneous equation writing

$$\begin{aligned} Q_{\eta,\mu+2} - Q_{\eta,\mu+1} - \eta Q_{\eta-1,\mu+2} = \; & c_{\mu+1}(Q_{\eta,\mu+1} - Q_{\eta,\mu} - \eta Q_{\eta-1,\mu+1}), \\ c_{\mu+1} = \; & \sqrt{\frac{y}{x}} \frac{I_{\mu+1}(2\sqrt{xy})}{I_\mu(2\sqrt{xy})}. \end{aligned} \tag{16}$$

Then, if $Q(\eta - 1, \mu)$ is know $\mu = 1, 2, \ldots N$, we can compute $Q(\eta, \mu)$ $\mu = 1, 2, \ldots N$, starting from $Q(\eta, 1)$ and $Q(\eta, 2)$ with the recurrence

$$Q_{\eta,\mu+2} = (1 + c_{\mu+1})Q_{\eta,\mu+1} - c_{\mu+1}Q_{\eta,\mu} + \eta Q_{\eta-1,\mu+2} - \eta c_{\mu+1}Q_{\eta-1,\mu+1}. \tag{17}$$

The advantage of this recurrence is that the overflow problems are reduced because ratios of Bessel functions appear instead of Bessel functions themselves. Also, for computing these ratios, continued fraction representations can be used. In Table 2 the use of the recurrence relation for computing $Q_{2,N}$ is tested for several values of N. The values of x and y are fixed to 2 and 3, respectively. The table shows the relative error obtained when comparing the value obtained with the recurrence relation and the direct computation using the series expansion of (9):

$$E_r = \left| 1 - \frac{Q^S_{2,N}(2, 3)}{Q^R_{2,N}(2, 3)} \right|. \tag{18}$$

The continued fraction for the ratio of Bessel functions is computed using the modified Lentz algorithm [7] and [1, §6.6.2].

N	Relative error (18)
10	$2.96\,10^{-16}$
20	$4.84\,10^{-16}$
30	$1.67\,10^{-15}$
40	$1.02\,10^{-15}$
50	$9.02\,10^{-15}$
60	$3.09\,10^{-15}$

Table 2: Test of the application of the recurrence relation given in (17). The relative errors are obtained when comparing the value obtained with the recurrence relation (17) and the direct computation using the series expansion of (9).

Acknowledgements

This work was supported by *Ministerio de Ciencia e Innovación*, project MTM2009-11686 and *Ministerio de Economía y Competitividad*, project MTM2012-34787.

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