Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

MAS

Modelling, Analysis and Simulation

Modelling, Analysis and Simulation

Multi-point Taylor expansions of analytic functions

José L. López, Nico M. Temme

REPORT MAS-E0402 JANUARY 8, 2004

CWI is the National Research Institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organization for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3703

Multi-point Taylor expansions of analytic functions

ABSTRACT
Taylor expansions of analytic functions are considered with respect to several points, allowing confluence of any of them. Cauchy-type formulas are given for coefficients and remainders in the expansions, and the regions of convergence are indicated. It is explained how these expansions can be used in deriving uniform asymptotic expansions of integrals. The method is also used for obtaining Laurent expansions in several points as well as Taylor-Laurent expansions.

2000 Mathematics Subject Classification: 30B10, 30E20, 40A30.
Keywords and Phrases: multi-point Taylor expansions, Cauchy's theorem, analytic functions, multi-point Laurent expansions, uniform asymptotic expansions of integralrals
Note: Work carried out under project MAS1.2 Analysis, Asymptotics and Computing. This paper has been accepted for publication in *Transactions of the American Mathematical Society*.

Multi-point Taylor Expansions of Analytic Functions

José L. López[1] and Nico M. Temme[2]

[1] *Departamento de Matématica e Informática,*
Universidad Pública de Navarra, 31006-Pamplona, Spain
[2] *CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands*
e-mail: jl.lopez@unavarra.es, nicot@cwi.nl

ABSTRACT

Taylor expansions of analytic functions are considered with respect to several points, allowing confluence of any of them. Cauchy-type formulas are given for coefficients and remainders in the expansions, and the regions of convergence are indicated. It is explained how these expansions can be used in deriving uniform asymptotic expansions of integrals. The method is also used for obtaining Laurent expansions in several points as well as Taylor-Laurent expansions.

1991 Mathematics Subject Classification: 30B10, 30E20, 40A30.

Keywords & Phrases: multi-point Taylor expansions, Cauchy's theorem, analytic functions, multi-point Laurent expansions, uniform asymptotic expansions of integrals.

1. Introduction

In deriving uniform asymptotic expansions of a certain class of integrals one encounters the problem of expanding a function, that is analytic in some domain Ω of the complex plane, in several points. The first mention of the use of such expansions in asymptotics is given in [1], where Airy-type expansions are derived for integrals having two nearby (or coalescing) saddle points. This reference does not give further details about two-point Taylor expansions, because the coefficients in the Airy-type asymptotic expansion are derived in a different way. Other mentions of the use of such expansions in asymptotics is given in [7] and [5]. In [7], two-point Taylor expansions are used with applications to Airy-type expansions of parabolic cylinder functions. In [5] we used two-point Taylor expansions to derive convergent expansions of Charlier, Laguerre and Jacobi polynomials in terms of Gamma, Hermite and Chebyshev polynomials respectively.

To demonstrate an application in asymptotics of multi-Taylor expansions we may consider contour integrals of the form

$$I(\lambda;\alpha)=\int_{\mathcal{C}} g(z)e^{-\lambda f(z,\alpha)}\,dz, \tag{1}$$

where α is a vector of parameters, $\alpha=(\alpha_1,\ldots,\alpha_2)$ and the phase function $f(z,\alpha)$ has m saddle points z_1, z_2,...,z_m. The asymptotic behaviour of these integrals for large values of λ is determined by the saddle-point structure of the phase function [[9], chap. 7, sec. 6]. One method for obtaining an asymptotic expansion of this integral for large values of λ is based on expanding $g(z)$ at the saddle points of the phase function,

$$g(z)=\sum_{n=0}^{\infty}[a_0+a_1z+\ldots+a_{m-1}z^{m-1}](z-z_1)^n(z-z_2)^n\cdots(z-z_m)^n$$

and substitute this expansion into (1). When interchanging summation and integration, the result is a formal expansion in m series in terms of functions related with the functions

$$F_{n,k}(\lambda;\alpha)\equiv\int_{\mathcal{C}} z^k(z-z_1)^n(z-z_2)^n\cdots(z-z_m)^n e^{-\lambda f(z,\alpha)}\,dz, \qquad k=1,2,\ldots,m-1. \tag{2}$$

In [7], these functions $F_{n,k}(\lambda;\alpha)$ are the Airy functions, whereas in [5] these functions are the Gamma function, or the Hermite or Chebyshev polynomials.

In a future paper we will use multi-point Taylor expansions in the asymptotic analysis of integrals arising in diffraction theory, such as the Bessel function integral (see [3] and [4])

$$J(x,y)=\int_0^{\infty} tJ_0(yt)e^{i(t^4+xt^2)}\,dt, \tag{3}$$

which is related to the Pearcey function

$$\int_{-\infty}^{\infty} e^{i(\frac{1}{4}t^4+\frac{1}{2}xt^2+yt)}dt, \tag{4}$$

The Taylor-Laurent expansions will be used to study integrals with two saddle points and a pole of the integrand. Other applications in asymptotics include the study of Hermite-Padé approximations to the exponential function; in [2] integrals are considered with three saddle points.

In a recent paper [6] we have introduced the theory of two-point Taylor expansions, two-point Laurent expansions and two-point Taylor-Laurent expansions. The purpose of the present paper is to generalize that theory from 2 to m points, $m\geq 2$. We give details on the region of convergence and on representations in terms of Cauchy-type integrals of the coefficients and the remainders of the expansions. Earlier information

on this type of expansions is given in [8], Chapters 3 and 8. The theory of several-point Taylor expansions was already formulated in Chapter 3 of Walsh's book, although in a different setting. Chapter 8 of [8] presents also a theory of rational approximation of analytic functions, but is different from the theory of multi-point Laurent and Taylor-Laurent expansions presented here. Whereas the multi-point polynomial approximation of Chapter 3 may be reformulated as a multi-point Taylor approximation, the rational approximation of Chapter 8 can not be written as a multi-point Laurent or Taylor-Laurent approximation. For more details, see Section 5.

2. Multi-point Taylor expansions

We consider the Taylor expansion of an analytic function $f(z)$ in several points and give information on the coefficients and the remainder in the expansion. In what follows empty sums and derivatives of negative order must be understood as zero and empty products as one. We will deal with the following set of points:

Definition 1. *We define the set* $\mathcal{S} \equiv \{z_1, z_1, \ldots, z_1; z_2, z_2, \ldots, z_2, \ldots; z_p, z_p, \ldots, z_p\}$ *of* m *points consisting on* p *different points* z_1, z_2,..., z_p ($z_i \neq z_j$ *if* $i \neq j$), *each* z_j *repeated* m_j *times:* $m_1 + m_2 + \ldots + m_p = m$.

For clearity in the exposition, we first introduce the multi-point Taylor expansion for m different points z_1, z_2, $\ldots z_m$ ($m = p$, $m_j = 1$) in Theorem 1. In Theorem 2 we assume that the points z_1, z_2, $\ldots z_m$ may coalesce. We will need the following elementary lemma.

Lemma 1. *Given* z, w, $\in \mathbb{C}$, *take* m *different points* z_1, z_2,..., z_m *in* $\mathbb{C}$ *and define*

$$H_m(w, z; z_1, \ldots, z_m) \equiv \frac{\prod_{k=1}^{m}(w - z_k) - \prod_{k=1}^{m}(z - z_k)}{w - z}. \tag{5}$$

Then

$$H_m(w, z; z_1, \ldots, z_m) = \sum_{j=1}^{m} \frac{\prod_{k=1,k\neq j}^{m}(w - z_k) \prod_{k=1,k\neq j}^{m}(z - z_k)}{\prod_{k=1,k\neq j}^{m}(z_j - z_k)}. \tag{6}$$

Proof. The numerator of $H_m(w, z; z_1, \ldots, z_m)$ is a polynomial of degree m in the variable w that vanishes at $w = z$. Therefore, $H_m(w, z; z_1, \ldots, z_m)$ is a polynomial of degree $m - 1$ in the variable w. Let $P_m(w, z; z_1, \ldots, z_m)$ denote the function at the right-hand side of (6), which is also a polynomial of degree $m - 1$ in the variable w. Moreover, $H_m(z_s, z; z_1, \ldots, z_m) = \prod_{k=1,k\neq s}^{m}(z - z_k) = P_m(z_s, z; z_1, \ldots, z_m)$ for $s = 1$, $2,\ldots m$. Hence, $H_m(w, z; z_1, \ldots, z_m) = P_m(w, z; z_1, \ldots, z_m)$. □

Theorem 1. *Let* $f(z)$ *be an analytic function on an open set* $\Omega \subset \mathbb{C}$ *and* $\mathcal{S} \subset \Omega$ *with* $\mathcal{S}$ *consisting on* m *different points* ($m = p$). *Then,* $f(z)$ *has the multi-point Taylor expansion*

$$f(z) = \sum_{n=0}^{N-1} q_{n,m}(z) \prod_{k=1}^{m}(z - z_k)^n + r_N(z), \tag{7}$$

where $q_{n,m}(z)$ is the polynomial of degree $m-1$

$$q_{n,m}(z) \equiv \sum_{j=1}^{m} a_{n,j} \frac{\prod_{k=1,k\neq j}^{m}(z-z_k)}{\prod_{k=1,k\neq j}^{m}(z_j-z_k)} \tag{8}$$

and the coefficients $a_{n,j}$ of this polynomial are given by the Cauchy integral

$$a_{n,j} \equiv \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(w)\,dw}{(w-z_j)\prod_{k=1}^{m}(w-z_k)^n}. \tag{9}$$

The remainder term $r_N(z)$ is given by the Cauchy integral

$$r_N(z) \equiv \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(w)\,dw}{(w-z)\left[\prod_{k=1}^{m}(w-z_k)\right]^N}\left[\prod_{k=1}^{m}(z-z_k)\right]^N. \tag{10}$$

The contour of integration $\mathcal{C}$ is a simple closed loop which encircles the points z_1, z_2,...,z_m (for $a_{n,j}$) and z, z_1, z_2,...,z_m (for $r_N(z)$) in the counterclockwise direction and is contained in Ω (see Figure 1 (a)).

The expansion (7) is convergent for $z \in O_m$, where:

$$O_m \equiv \{z\in\Omega,\ \prod_{k=1}^{m}|z-z_k| < r\}, \qquad r \equiv \mathrm{Inf}_{w\in\mathcal{C}\setminus\Omega}\left\{\prod_{k=1}^{m}|w-z_k|\right\}. \tag{11}$$

That is, (7) is convergent for z inside the lemniscate $\prod_{k=1}^{m}|z-z_k| = r$ (see Figure 2; if $m=1$ this domain is a disk; if $m=2$ this domain is bounded by a Cassini oval). In particular, if $f(z)$ is an entire function ($\Omega = \mathcal{C}$), then the expansion (7) converges $\forall$ $z \in \mathcal{C}$.

Proof. By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(w)\,dw}{w-z}, \tag{12}$$

where $\mathcal{C}$ is the contour defined above (Figure 1 (a)). We write

$$\frac{1}{w-z} = \frac{H_m(w,z;z_1,\ldots,z_m)}{\prod_{k=1}^{m}(w-z_k)}\frac{1}{1-u}, \tag{13}$$

where $H_m(w,z;z_1,\ldots,z_m)$ is given in (5) and

$$u \equiv \frac{\prod_{k=1}^{m}(z-z_k)}{\prod_{k=1}^{m}(w-z_k)}. \tag{14}$$

Now we use Lemma 1 and introduce the right hand side of (6) and the expansion

$$\frac{1}{1-u} = \sum_{n=0}^{N-1} u^n + \frac{u^N}{1-u} \tag{15}$$

in (13) and this in (12). After straightforward calculations we obtain formulas (7)-(10).

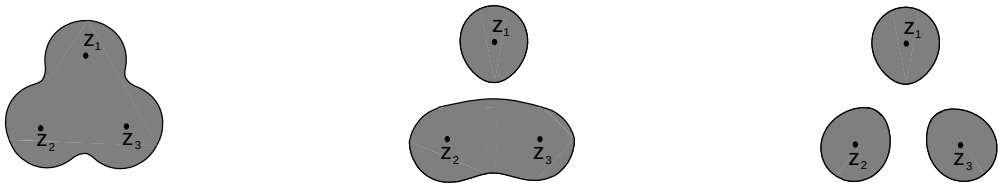
For any $z \in O_m$, we can take a contour $\mathcal{C}$ in Ω such that

$$\prod_{k=1}^{m} |z - z_k| < \prod_{k=1}^{m} |w - z_k|, \quad \forall w \in \mathcal{C}$$

(see Figure 1 (b)). On this contour $|f(w)|$ is bounded by some constant C: $|f(w)| \leq C$. Introducing these two bounds in (10) we see that $\lim_{N\to\infty} r_N(z) = 0$ and the proof follows. □

(a) (b)

Figure 1. *The case $m = 3$. (a) Contour $\mathcal{C}$ in the integrals (9) and (10). (b) For $z \in O_m$, we can take a contour $\mathcal{C}$ in Ω which contains O_m inside and therefore, $\prod_{k=1}^{m} |z - z_k| < \prod_{k=1}^{m} |w - z_k|$ $\forall\ w \in \mathcal{C}$.*



$r = r_a$ $r = r_b$ $r = r_c$

Figure 2. *Shape of the "lemniscate domain" O_m for $m = 3$. It depends on the size of the parameter r defined in (11). In these pictures $|z_2 - z_3| < |z_1 - z_3|$, $|z_1 - z_2|$ and $r_a > r_b > r_c$.*

We need the following lemma to consider the case of coalescing points in the set $\mathcal{S}$.

Lemma 2: *Given $z, w \in \mathbb{C}$, take m different points z_1, z_2, ..., z_m in $\mathbb{C}$, all different from w too. Let those m points to coalesce at z_m, say. Then*

$$\lim_{z_1,z_2,..,z_{m-1}\to z_m} \sum_{j=1}^{m} \frac{\prod_{k=1,k\neq j}^{m}(z - z_k)}{(w - z_j)\prod_{k=1,k\neq j}^{m}(z_j - z_k)} = \sum_{j=0}^{m-1} \frac{(z - z_m)^j}{(w - z_m)^{j+1}}.$$

Proof. We first note that the identity

$$\sum_{j=1}^{n} \prod_{l=1}^{j-1} (z_n - z_l) \prod_{l=j+1}^{n} (z_1 - z_l) = 0$$

holds for any set of points z_1, z_2,..., z_n, $n > 1$. It may be cheked in the following way: we take the two first terms of the sum, which gives $(z_n - z_2)(z_1 - z_3)(z_1 - z_4) \cdot$

$\cdot\cdot(z_1-z_{n-1})(z_1-z_n)$. Next we add to this the third term of the sum, which gives $(z_n-z_2)(z_n-z_3)(z_1-z_4)\cdots(z_1-z_{n-1})(z_1-z_n)$. We continue this process untill we add the $n-1$-th term of the sum, obtaining $(z_n-z_2)(z_n-z_3)(z_n-z_4)\cdots(z_n-z_{n-1})(z_1-z_n)$. But this is just the last term of the sum with opposite sign.

Using the above identity we have

$$\prod_{l=1}^{k-1}(z_s-z_l)\left[\sum_{j=k}^{s}\prod_{l=k}^{j-1}(z_s-z_l)\prod_{l=j+1}^{s}(z_k-z_l)\right]\prod_{l=s+1}^{m}(z_k-z_l)=0$$

for any $s=1,2,3,...,m$ and any $k=1,2,3,...,m$ with $k\neq s$. Then

$$\sum_{j=k}^{s}\prod_{l=1}^{j-1}(z_s-z_l)\prod_{l=j+1}^{m}(z_k-z_l)=\sum_{j=1}^{m}\prod_{l=1}^{j-1}(z_s-z_l)\prod_{l=j+1}^{m}(z_k-z_l)=0 \qquad (16)$$

for any $s,k=1,2,3,...,m$ with $k\neq s$.

Now, for every $s=1,2,3,...,m$, we define the following polynomials of degree $m-1$ in the variable z:

$$R_s(z)\equiv\prod_{l=1,l\neq s}^{m}(z-z_l), \qquad S_s(z)\equiv\sum_{j=1}^{m}\prod_{l=1}^{j-1}(z-z_l)\prod_{l=j+1}^{m}(z_s-z_l).$$

The zeros of $R_s(z)$ are z_k for $k=1,2,3,...,m$, $k\neq s$ and from (16), $S_s(z_k)=0$ for $k=1,2,3,...,m$, $k\neq s$. Moreover, the leading coeficient of $R_s(z)$ and of $S_s(z)$ coincide. Therefore, $R_s(z)=S_s(z)$ for $s=1,2,3,...,m$.

Finally, define the following polynomials of degree $m-1$ in the variable w:

$$P(w,z)\equiv\sum_{j=1}^{m}\frac{\prod_{k=1,k\neq j}^{m}(z-z_k)\prod_{k=1,k\neq j}^{m}(w-z_k)}{\prod_{k=1,k\neq j}^{m}(z_j-z_k)},$$

$$Q(w,z)\equiv\sum_{j=1}^{m}\prod_{k=1}^{j-1}(z-z_k)\prod_{k=j+1}^{m}(w-z_k).$$

For every $s=1,2,3,...,m$ we have $P(z_s,z)=R_s(z)$ and $Q(z_s,z)=S_s(z)$. But $R_s(z)=S_s(z)$ and therefore $P(w,z)=Q(w,z)$. Then,

$$\sum_{j=1}^{m}\frac{\prod_{k=1,k\neq j}^{m}(z-z_k)}{(w-z_j)\prod_{k=1,k\neq j}^{m}(z_j-z_k)}=\frac{P(w,z)}{\prod_{k=1}^{m}(w-z_k)}=\frac{Q(w,z)}{\prod_{k=1}^{m}(w-z_k)}=\sum_{j=1}^{m}\frac{\prod_{k=1}^{j-1}(z-z_k)}{\prod_{k=1}^{j}(w-z_k)}.$$

Taking the limits z_1, z_2,.., $z_{m-1}\to z_m$ in the left- and right-hand sides of these equalities, we obtain the desired result. □

Theorem 2. *Let $f(z)$ be an analytic function on an open set $\Omega \subset \mathcal{C}$ and $\mathcal{S} \subset \Omega$. Then, $f(z)$ has the multi-point Taylor expansion*

$$f(z) = \sum_{n=0}^{N-1} q_{n,m}(z) \prod_{k=1}^{p} (z - z_k)^{n m_k} + r_N(z), \tag{17}$$

where $q_{n,m}(z)$ is the polynomial of degree $m-1$

$$q_{n,m}(z) \equiv \sum_{j=1}^{p} \frac{\prod_{k=1,k\neq j}^{p} (z - z_k)^{m_k}}{\prod_{k=1,k\neq j}^{p} (z_j - z_k)^{m_k}} \sum_{l=0}^{m_j - 1} a_{n,j,l} (z - z_j)^l \tag{18}$$

and the coefficients $a_{n,j,l}$ of this polynomial are given by the Cauchy integral

$$a_{n,j,l} \equiv \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)\, dw}{(w - z_j)^{l+1} \prod_{k=1}^{p} (w - z_k)^{n m_k}}. \tag{19}$$

The remainder term $r_N(z)$ is given by the Cauchy integral

$$r_N(z) \equiv \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)\, dw}{(w - z) \left[\prod_{k=1}^{p} (w - z_k)^{m_k}\right]^N} \left[\prod_{k=1}^{p} (z - z_k)^{m_k}\right]^N. \tag{20}$$

The contour of integration $\mathcal{C}$ is a simple closed loop which encircles the points z_1, z_2,...,z_p (for $a_{n,j,l}$) and z, z_1, z_2,...,z_p (for $r_N(z)$) in the counterclockwise direction and is contained in Ω (see Figure 1 (a)).

The expansion (17) is convergent for $z \in O_p$:

$$O_p \equiv \{z \in \Omega,\ \prod_{k=1}^{p} |z - z_k|^{m_k} < r\}, \qquad r \equiv \mathrm{Inf}_{w \in \mathcal{C} \setminus \Omega} \left\{ \prod_{k=1}^{p} |w - z_k|^{m_k} \right\},$$

that is, inside the lemniscate $\prod_{k=1}^{p} |z - z_k|^{m_k} = r$. In particular, if $f(z)$ is an entire function ($\Omega = \mathcal{C}$), then the expansion (17) converges $\forall\ z \in \mathcal{C}$.

Proof. If all the points in $\mathcal{S}$ are different, we have from (8) and (9)

$$q_{n,m}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)\, dw}{\prod_{k=1}^{m} (w - z_k)^n} \sum_{j=1}^{m} \frac{\prod_{k=1,k\neq j}^{m} (z - z_k)}{(w - z_j) \prod_{k=1,k\neq j}^{m} (z_j - z_k)}. \tag{21}$$

This last sum may be also decomposed in the form

$$\sum_{j=1}^{m_1} \frac{\prod_{k=1,k\neq j}^{m} (z - z_k)}{(w - z_j) \prod_{k=1,k\neq j}^{m} (z_j - z_k)} + \sum_{j=m_1+1}^{m_2} \frac{\prod_{k=1,k\neq j}^{m} (z - z_k)}{(w - z_j) \prod_{k=1,k\neq j}^{m} (z_j - z_k)} + \ldots$$
$$\sum_{j=m_{p-1}+1}^{m} \frac{\prod_{k=1,k\neq j}^{m} (z - z_k)}{(w - z_j) \prod_{k=1,k\neq j}^{m} (z_j - z_k)}.$$

Now let the first m_1 points to coalesce to z_1, the second m_2 points to coalesce to z_2, and so on, and apply Lemma 2 to every one of the p sums above to obtain (17), (18) and (19). Equation (20) follows from (10). The proof of the convergence of (17) in the region O_p is a straightforward generalization of the correponding proof in Theorem 1. □

2.1. Explicit forms of the coefficients

Formula (19) is not appropriate for numerical computations. A more practical formula to compute the coefficients of the above multi-point Taylor expansion is given in the following proposition. First we have a definition:

Definition 2. *Let $f(w)$ be analytic at w; then for $n = 0, 1, 2, \ldots$ the differentential operator $D_w^n f(w)$ is defined by*

$$D_w^n f(w) = \frac{1}{n!}\frac{d^n}{dw^n} f(w).$$

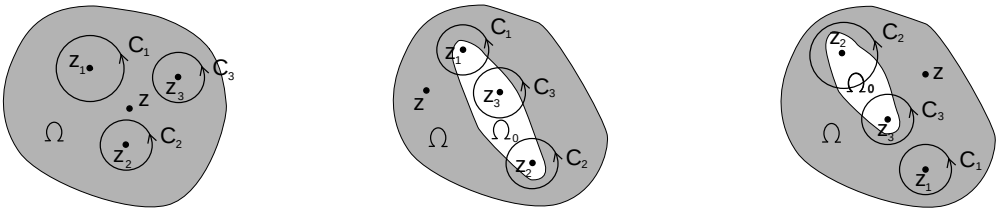
Proposition 1. *The coefficients $a_{n,j,l}$, for $n = 1, 2, 3, \ldots$, $j = 1, 2, \ldots, p$, $l = 0, 1, \ldots, m_j - 1$ in the expansion (17) are also given by the formula:*

$$\begin{aligned} a_{n,j,l} =& D_w^{nm_j+l}\left[\frac{f(w)}{\prod_{s=1,s\neq j}^{p}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_j} + \\ & \sum_{k=1,k\neq j}^{p} D_w^{nm_k-1}\left[\frac{f(w)}{(w-z_j)^{l+1}\prod_{s=1,s\neq k}^{p}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_k}. \end{aligned} \tag{22}$$

Proof. We deform the contour of integration $\mathcal{C}$ in equation (19) to any contour of the form $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_p$, also contained in Ω, where $\mathcal{C}_k$, $k = 1, 2, \ldots, p$, is a simple closed loop which encircles the point z_k in the counterclockwise direction and does not contain any other point z_j, $j = 1, 2, \ldots, p$, $j \neq k$ inside (see Figure 3 (a)). Then,

$$\begin{aligned} a_{n,j,l} =& \frac{1}{2\pi i}\sum_{k=1,k\neq j}^{p}\int_{\mathcal{C}_k}\frac{f(w)}{(w-z_j)^{l+1}\prod_{s=1,s\neq k}^{p}(w-z_s)^{nm_s}}\frac{dw}{(w-z_k)^{nm_k}} + \\ & \frac{1}{2\pi i}\int_{\mathcal{C}_j}\frac{f(w)}{\prod_{s=1,s\neq j}^{p}(w-z_s)^{nm_s}}\frac{dw}{(w-z_j)^{nm_j+l+1}}, \end{aligned}$$

from which equation (22) follows. □



(a) (b) (c)

Figure 3. *Integration contours $\mathcal{C}_k$ for $p = 3$ and $q = 1$. (a) The function $\prod_{s=1,s\neq k}^{p}(w - z_s)^{-nm_s} f(w)$ is analytic inside $\mathcal{C}_k$ for $k = 1, 2, \ldots, p$. (b) The functions $\prod_{s=1,s\neq k}^{p}(w - z_s)^{-nm_s} g_k(w)$ and $\prod_{s=1,s\neq k}^{p}(w - z_s)^{(n+1)m_s} g_k(w)$ are analytic inside $\mathcal{C}_k$ for $k = 1, 2, \ldots, p$. (c) The functions $\prod_{s=1,s\neq k}^{p}(w - z_s)^{-nm_s} g_k(w)$, $\prod_{s=1,s\neq k}^{q}(w - z_s)^{-nm_s} \prod_{s=q+1,s\neq k}^{p}(w - z_s)^{nm_s} g_k(w)$ and $\prod_{s=1,s\neq k}^{q}(w - z_s)^{-(n+1)m_s} \times \prod_{s=q+1,s\neq k}^{p}(w - z_s)^{(n+1)m_s} g_k(w)$ are analytic inside $\mathcal{C}_k$ for $k = 1, 2, \ldots, p$.*

2.2. Multi-point Taylor polynomials

In Theorem 1 we have assumed that the function $f(z)$ is analytic in Ω. If $f(z)$ is not analytic in Ω but has a finite number of derivatives at z_1, z_2,...,z_p, we can still define the multi-point Taylor polynomial of the function $f(z)$ at z_1, z_2,...,z_p in the following way:

Definition 3. *Let z be a real or complex variable. If $f(z)$ is $Nm_k - 1-$times differentiable at z_1, z_2,...,z_p, we define the multi-point Taylor polynomial of $f(z)$ at the points of $\mathcal{S}$ and degree $mN - 1$ as*

$$P_N(z) \equiv \sum_{n=0}^{N-1} q_{n,m}(z) \prod_{k=1}^{p}(z - z_k)^{nm_k},$$

where $q_{n,m}(z)$ is the polynomial of degree $m - 1$

$$q_{n,m}(z) \equiv \sum_{j=1}^{p} \frac{\prod_{k=1,k\neq j}^{p}(z - z_k)^{m_k}}{\prod_{k=1,k\neq j}^{p}(z_j - z_k)^{m_k}} \sum_{l=0}^{m_j-1} a_{n,j,l}(z - z_j)^l$$

and the coefficients $a_{n,j,l}$ are given in (22).

Proposition 2. *In the conditions of the above definition, define the remainder of the approximation of $f(z)$ by $P_N(z)$ at the points of $\mathcal{S}$ as*

$$r_N(z) \equiv f(z) - P_N(z).$$

Then, (i) $r_N(z) = o(z-z_k)^{Nm_k-1}$ as $z \to z_k$, $k = 1, 2, \ldots, p$. (ii) If $f(z)$ is Nm_k-times differentiable at z_k for some k, then $r_N(z) = \mathcal{O}(z - z_k)^{Nm_k}$ as $z \to z_k$.

Proof. The proof is trivial if $f(z)$ is analytic at every z_1, z_2,...,z_p by using (20). In any case, for real or complex variable, the proof follows by using l'Hôpital's rule and (22). □

Remark 1. Observe that the Taylor polynomial of $f(z)$ at the points of $\mathcal{S}$ and degree $mN-1$ is the Hermite's interpolation polynomial of $f(z)$ at z_1, z_2,...,z_p with data $f(z_k)$, $f'(z_k)$,...,$f^{(Nm_k-1)}(z_k)$, $k = 1, 2, \ldots, p$.

3. Multi-point Laurent expansions

In the standard theory for Taylor and Laurent expansions much analogy exists between the two types of expansions. For multi-point expansions, we have a similar agreement in the representations of coefficients and remainders.

Theorem 3. *Let Ω_0 and Ω be closed and open sets, respectively, of the complex plane, and $\Omega_0 \subset \Omega \subset \mathbb{C}$. Let $f(z)$ be an analytic function on $\Omega \setminus \Omega_0$ and z_1, z_2,...,$z_p \in \Omega_0$ (That is, $\mathcal{S} \in \Omega_0$). Then, for any $z \in \Omega \setminus \Omega_0$, $f(z)$ has the multi-point Laurent expansion*

$$f(z) = \sum_{n=0}^{N-1} q_{n,m}(z) \prod_{k=1}^{p} (z - z_k)^{nm_k} + \sum_{n=0}^{N-1} t_{n,m}(z) \prod_{k=1}^{p} (z - z_k)^{-(n+1)m_k} + r_N(z), \quad (23)$$

where $q_{n,m}(z)$ is the polynomial of degree $m-1$

$$q_{n,m}(z) \equiv \sum_{j=1}^{p} \frac{\prod_{k=1,k\neq j}^{p} (z - z_k)^{m_k}}{\prod_{k=1,k\neq j}^{p} (z_j - z_k)^{m_k}} \sum_{l=0}^{m_j-1} a_{n,j,l} (z - z_j)^l \quad (24)$$

and the coefficients $a_{n,j,l}$ of this polynomial are given by the Cauchy integral

$$a_{n,j,l} \equiv \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)\, dw}{(w - z_j)^{l+1} \prod_{k=1}^{p} (w - z_k)^{nm_k}}. \quad (25)$$

Also, $t_{n,m}(z)$ is the polynomial of degree $m-1$

$$t_{n,m}(z) \equiv \sum_{j=1}^{p} \frac{\prod_{k=1,k\neq j}^{p} (z - z_k)^{m_k}}{\prod_{k=1,k\neq j}^{p} (z_j - z_k)^{m_k}} \sum_{l=0}^{m_j-1} b_{n,j,l} (z - z_j)^l, \quad (26)$$

where the coefficients $b_{n,j,l}$ of this polynomial are given by the Cauchy integral

$$b_{n,j,l} \equiv \frac{1}{2\pi i} \int_{\Gamma_2} \prod_{k=1}^{p} (w - z_k)^{m_k(n+1)} \frac{f(w)\, dw}{(w - z_j)^{l+1}}. \quad (27)$$

The remainder term $r_N(z)$ is given by the Cauchy integrals

$$\begin{aligned} r_N(z) \equiv & \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)\, dw}{(w - z) \prod_{k=1}^{p} (w - z_k)^{Nm_k}} \prod_{k=1}^{p} (z - z_k)^{Nm_k} - \\ & \frac{1}{2\pi i} \int_{\Gamma_2} \prod_{k=1}^{p} (w - z_k)^{Nm_k} \frac{f(w)\, dw}{w - z} \frac{1}{\prod_{k=1}^{p} (z - z_k)^{Nm_k}}. \end{aligned} \quad (28)$$

In these integrals, the contours of integration Γ_1 and Γ_2 are simple closed loops contained in $\Omega \setminus \Omega_0$ which encircle the points z_1, z_2,...,z_p in the counterclockwise direction. Moreover, z is not inside Γ_2, whereas Γ_1 encircles Γ_2 and the point z (see Figure 4 (a)).

The expansion (23) is convergent for z inside the "lemniscate annulus" (see Figure 5)

$$A_p \equiv \{z \in \Omega \setminus \Omega_0, \;\; r_2 < \prod_{k=1}^{p} |z - z_k|^{m_k} < r_1\} \tag{29}$$

, where

$$r_1 \equiv \mathrm{Inf}_{w \in \mathcal{C} \setminus \Omega} \left\{ \prod_{k=1}^{p} |w - z_k|^{m_k} \right\}, \quad r_2 \equiv \mathrm{Sup}_{w \in \Omega_0} \left\{ \prod_{k=1}^{p} |w - z_k|^{m_k} \right\}. \tag{30}$$

Proof. By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w) dw}{w - z} - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)\, dw}{w - z}, \tag{31}$$

where Γ_1 and Γ_2 are the contours defined above. First we assume that the m points of the set $\mathcal{S}$ are all distinct and later we will let the first m_1 points to coalesce to z_1, the second m_2 points to coalesce to z_2 and so on. We substitute (13)-(14) into the first integral above and

$$\frac{1}{w - z} = -\frac{H_m(w, z; z_1, \ldots, z_m)}{\prod_{k=1}^{m}(z - z_k)} \frac{1}{1 - u}, \qquad u \equiv \frac{\prod_{k=1}^{m}(w - z_k)}{\prod_{k=1}^{m}(z - z_k)},$$

where $H_m(w, z; z_1, \ldots, z_m)$ is defined in (5), into the second one. Now we introduce the expansion (15) of the factor $(1 - u)^{-1}$ in both integrals in (31). Using (6) and after straightforward calculations we obtain

$$f(z) = \sum_{n=0}^{N-1} q_{n,m}(z) \prod_{k=1}^{m}(z - z_k)^n + \sum_{n=0}^{N-1} t_{n,m}(z) \prod_{k=1}^{m}(z - z_k)^{-n-1} + r_N(z),$$

where $q_{n,m}(z)$ is given by formulas (8) and (9) replacing the contour $\mathcal{C}$ by Γ_1. Also,

$$t_{n,m}(z) = \sum_{j=1}^{m} b_{n,j} \frac{\prod_{k=1, k \neq j}^{m}(z - z_k)}{\prod_{k=1, k \neq j}^{m}(z_j - z_k)}, \qquad b_{n,j} \equiv \frac{1}{2\pi i} \int_{\Gamma_2} \prod_{k=1}^{m}(w - z_k)^{n+1} \frac{f(w)\, dw}{w - z_j},$$

and

$$\begin{aligned} r_N(z) = & \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)\, dw}{(w - z) \prod_{k=1}^{m}(w - z_k)^N} \prod_{k=1}^{m}(z - z_k)^N - \\ & \frac{1}{2\pi i} \int_{\Gamma_2} \prod_{k=1}^{m}(w - z_k)^N \frac{f(w)\, dw}{w - z} \frac{1}{\prod_{k=1}^{m}(z - z_k)^N}. \end{aligned}$$

Now we write

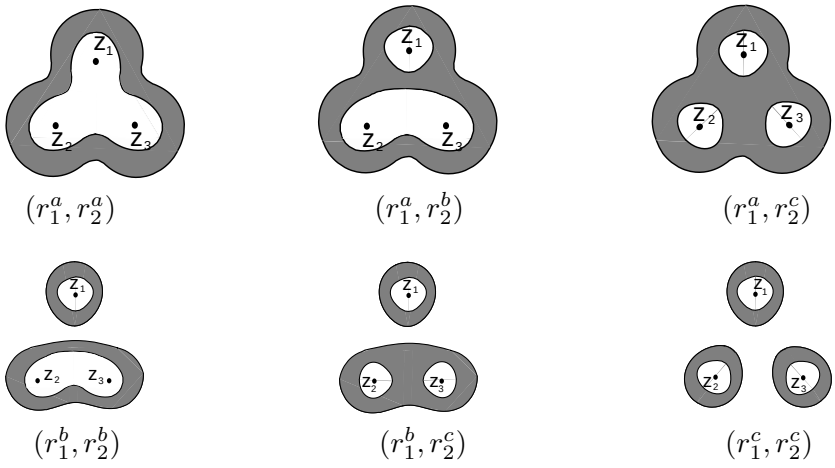
$$t_{n,m}(z) = \frac{1}{2\pi i} \int_{\Gamma_2} \prod_{k=1}^{m}(w - z_k)^{n+1} f(w)\, dw \sum_{j=1}^{m} \frac{\prod_{k=1, k \neq j}^{m}(z - z_k)}{(w - z_j) \prod_{k=1, k \neq j}^{m}(z_j - z_k)}$$

and repeat the steps following (21) in the proof of Theorem 2 for $q_{n,m}(z)$ and $t_{n,m}(z)$.

For any $z \in A_p$ we can take simple closed loops Γ_1 and Γ_2 in $\Omega \setminus \Omega_0$ such that $\prod_{k=1}^{p} |z - z_k|^{m_k} < \prod_{k=1}^{p} |w - z_k|^{m_k}$ $\forall\ w \in \Gamma_1$ and $\prod_{k=1}^{p} |z - z_k|^{m_k} > \prod_{k=1}^{p} |w - z_k|^{m_k}$ $\forall\ w \in \Gamma_2$ (see Figure 4 (b)). On these contours $|f(w)|$ is bounded by some constant C: $|f(w)| \leq C$. Introducing these bounds in (28) we see that $\lim_{N\to\infty} r_N(z) = 0$ and the proof follows. □

(a) (b)

Figure 4. *The case $p = 3$. (a) Contours Γ_1 and Γ_2 in the integrals (25), (27) and (28). (b) For $z \in A_p$, we can take a contour Γ_2 in Ω located between Ω_0 and A_p and a contour Γ_1 in Ω such that A_p is inside this contour. Therefore, $\prod_{k=1}^{p} |z - z_k|^{m_k} < \prod_{k=1}^{p} |w - z_k|^{m_k}$ $\forall\ w \in \Gamma_1$ and $\prod_{k=1}^{p} |w - z_k|^{m_k} < \prod_{k=1}^{p} |z - z_k|^{m_k}$ $\forall\ w \in \Gamma_2$.*



(r_1^a, r_2^a) (r_1^a, r_2^b) (r_1^a, r_2^c)

(r_1^b, r_2^b) (r_1^b, r_2^c) (r_1^c, r_2^c)

Figure 5. *Shape of the "lemniscate annulus" A_p for $p = 3$. It depends on the relative size of the parameters r_1 and r_2 defined in (30). The different forms are labeled by (r_1, r_2) with $r_1 > r_2$. In these pictures $|z_2 - z_3| < |z_1 - z_3|$, $|z_1 - z_2|$ and $r_2^a > r_2^b > r_2^c$.*

If the only singularities of $f(z)$ inside Ω_0 are just poles at z_1, z_2,...,z_p, then alternative formulas of (25) and (27) for computing the coefficients of the above multi-point Laurent expansion is given in the following proposition.

Proposition 3. *Suppose that $g_k(z) \equiv (z - z_k)^{\rho_k} f(z)$, $k = 1, 2, \ldots, p$ are analytic functions in Ω for certain ρ_1, ρ_2,...,$\rho_k \in \mathbb{N}$. Then, for $n = 0, 1, 2, \ldots$, coefficients $a_{n,j,l}$*

and $b_{n,j,l}$ in expansion (23) are also given by the formulas:

$$a_{n,j,l} = \sum_{k=1,k\neq j}^{p} D_w^{nm_k+\rho_k-1}\left[\frac{g_k(w)}{(w-z_j)^{l+1}\prod_{s=1,s\neq k}^{p}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_k} + D_w^{nm_j+\rho_j+l}\left[\frac{g_j(w)}{\prod_{s=1,s\neq j}^{p}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_j} \tag{32}$$

and

$$b_{n,j,l} = \sum_{k=1,k\neq j}^{p} D_w^{\rho_k-(n+1)m_k-1}\left[\frac{g_k(w)}{(w-z_j)^{l+1}}\prod_{s=1,s\neq k}^{p}(w-z_s)^{(n+1)m_s}\right]\Bigg|_{w=z_k} + D_w^{\rho_j-(n+1)m_j+l}\left[g_j(w)\prod_{s=1,s\neq j}^{p}(w-z_s)^{(n+1)m_s}\right]\Bigg|_{w=z_j}. \tag{33}$$

Proof. We deform both contours Γ_1 and Γ_2 of equations (25) and (27), respectively, to any contour of the form $\mathcal{C}_1\cup\mathcal{C}_2\cup\cdots\cup\mathcal{C}_p$ contained in Ω, where $\mathcal{C}_k$, $k=1,2,\ldots,p$ is a simple closed loop which encircles the point z_k in the counterclockwise direction and does not contain the point z_j $j=1,2,\ldots,p$, $j\neq k$ inside (see Figure 3 (b)). Then,

$$a_{n,j,l} = \frac{1}{2\pi i}\sum_{k=1,k\neq j}^{p}\int_{\mathcal{C}_k}\frac{g_k(w)}{(w-z_j)^{l+1}\prod_{s=1,s\neq k}^{p}(w-z_s)^{nm_s}}\frac{dw}{(w-z_k)^{nm_k+\rho_k}} + \frac{1}{2\pi i}\int_{\mathcal{C}_j}\frac{g_j(w)}{\prod_{s=1,s\neq j}^{p}(w-z_s)^{nm_s}}\frac{dw}{(w-z_j)^{nm_j+\rho_j+l+1}}$$

and

$$b_{n,j,l} = \frac{1}{2\pi i}\sum_{k=1,k\neq j}^{p}\int_{\mathcal{C}_k}\frac{\prod_{s=1,s\neq k}^{p}(w-z_s)^{(n+1)m_s}}{(w-z_j)^{l+1}}\frac{g_k(w)dw}{(w-z_k)^{\rho_k-(n+1)m_k}} + \frac{1}{2\pi i}\int_{\mathcal{C}_j}\prod_{s=1,s\neq j}^{p}(w-z_s)^{(n+1)m_s}\frac{g_j(w)dw}{(w-z_j)^{\rho_j-(n+1)m_j+l+1}}.$$

From here, equations (32) and (33) follow. □

Remark 2. Let z be a real or complex variable. Suppose that $g_k(z)\equiv(z-z_k)^{\rho_k}f(z)$ is ρ_k-1-times differentiable at every z_k in $\mathcal{S}$ for some $\rho_k\in\mathbb{N}$, $k=1,2,\ldots,p$. Define

$$g(z)\equiv f(z)-\sum_{n=0}^{M}t_{n,m}(z)\prod_{k=1}^{p}(z-z_k)^{-(n+1)m_k},$$

where $M \equiv \lfloor \mathrm{Max}\{(\rho_1-1)/m_1, (\rho_2-1)/m_2, \ldots, (\rho_p-1)/m_p\} \rfloor$ and $t_{n,m}(z)$ is the polynomial defined in (26) and (33). Then, the thesis of Proposition 2 holds for $f(z)$ replaced by $g(z)$. Moreover, if $\prod_{k=1}^{p}(z-z_k)^{\rho_k} f(z)$ is an analytic function in Ω, then the thesis of Theorem 2 applies to $g(z)$.

4. Multi-point Taylor-Laurent expansions

For multi-point expansions we have the possibility (that we do not have in the standard theory) of expanding in Taylor series in some points and in Laurent series in other points.

Theorem 4. *Let Ω_0 and Ω be closed and open sets, respectively, of the complex plane, and $\Omega_0 \subset \Omega \subset \mathcal{C}$. Let $f(z)$ be an analytic function on $\Omega \setminus \Omega_0$, $z_1, z_2, \ldots, z_q \in \Omega \setminus \Omega_0$ and $z_{q+1}, z_{q+2}, \ldots, z_p \in \Omega_0$ (q points are in $\Omega \setminus \Omega_0$ and $p-q$ points are in Ω_0). Write $s \equiv m_1 + m_2 + \cdots + m_q$. Then, for $z \in \Omega \setminus \Omega_0$, $f(z)$ has the Taylor-Laurent expansion*

$$\begin{aligned} f(z) = &\sum_{n=0}^{N-1} q_{n,m}(z) \prod_{k=1}^{p}(z-z_k)^{nm_k} + \sum_{n=0}^{N-1} t^{(1)}_{n,m}(z) \frac{\prod_{k=1}^{q}(z-z_k)^{nm_k}}{\prod_{k=q+1}^{p}(z-z_k)^{nm_k}} + \\ &\sum_{n=0}^{N-1} t^{(2)}_{n,m}(z) \frac{\prod_{k=1}^{q}(z-z_k)^{(n+1)m_k}}{\prod_{k=q+1}^{p}(z-z_k)^{(n+1)m_k}} + r_N(z), \end{aligned} \tag{34}$$

where $q_{n,m}(z)$ is the polynomial of degree $m-1$

$$q_{n,m}(z) \equiv \sum_{j=1}^{p} \frac{\prod_{k=1,k\neq j}^{p}(z-z_k)^{m_k}}{\prod_{k=1,k\neq j}^{p}(z_j-z_k)^{m_k}} \sum_{l=0}^{m_j-1} a_{n,j,l}(z-z_j)^l \tag{35}$$

and the coefficients $a_{n,j,l}$ of this polynomial are given by the Cauchy integral

$$a_{n,j,l} \equiv \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)\,dw}{(w-z_j)^{l+1} \prod_{k=1}^{p}(w-z_k)^{nm_k}}. \tag{36}$$

Also, $t^{(1)}_{n,m}(z)$ and $t^{(2)}_{n,m}(z)$ are the following polynomials of degrees $s-1$ and $m-s-1$ respectivelly,

$$t^{(1)}_{n,m}(z) \equiv -\sum_{j=1}^{q} \frac{\prod_{k=1,k\neq j}^{q}(z-z_k)^{m_k}}{\prod_{k=1,k\neq j}^{q}(z_j-z_k)^{m_k}} \sum_{l=0}^{m_j-1} b_{n,j,l}(z-z_j)^l, \tag{37}$$

where the coefficients $b_{n,j,l}$ of this polynomial are given by the Cauchy integral

$$b_{n,j,l} \equiv \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\prod_{k=q+1}^{p}(w-z_k)^{nm_k}}{\prod_{k=1}^{q}(w-z_k)^{nm_k}} \frac{f(w)\,dw}{(w-z_j)^{l+1}}. \tag{38}$$

$$t^{(2)}_{n,m}(z) \equiv \sum_{j=q+1}^{p} \frac{\prod_{k=q+1,k\neq j}^{p}(z-z_k)^{m_k}}{\prod_{k=q+1,k\neq j}^{p}(z_j-z_k)^{m_k}} \sum_{l=0}^{m_j-1} c_{n,j,l}(z-z_j)^l, \tag{39}$$

where the coefficients $c_{n,j,l}$ of this polynomial are given by the Cauchy integral

$$c_{n,j,l} \equiv \frac{1}{2\pi i}\int_{\Gamma_2} \frac{\prod_{k=q+1}^{p}(w-z_k)^{(n+1)m_k}}{\prod_{k=1}^{q}(w-z_k)^{(n+1)m_k}} \frac{f(w)\,dw}{(w-z_j)^{l+1}}. \tag{40}$$

The remainder term $r_N(z)$ is given by the Cauchy integrals

$$\begin{aligned} r_N(z) \equiv & \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(w)\,dw}{(w-z)\prod_{k=1}^{p}(w-z_k)^{Nm_k}} \prod_{k=1}^{p}(z-z_k)^{Nm_k} - \\ & \frac{1}{2\pi i}\int_{\Gamma_2} \frac{\prod_{k=q+1}^{p}(w-z_k)^{Nm_k}}{\prod_{k=1}^{q}(w-z_k)^{Nm_k}} \frac{f(w)dw}{w-z} \frac{\prod_{k=1}^{q}(z-z_k)^{Nm_k}}{\prod_{k=q+1}^{p}(z-z_k)^{Nm_k}}. \end{aligned} \tag{41}$$

In these integrals, the contours of integration Γ_1 and Γ_2 are simple closed loops contained in $\Omega \setminus \Omega_0$ which encircle Ω_0 in the counterclockwise direction. Moreover, the points z and z_1, z_2,...,z_q are not inside Γ_2, whereas Γ_1 encircles Γ_2 and the points z and z_1, z_2,...,z_q (see Figure 6 (a)).

The expansion (34) is convergent in the region (Figure 7)

$$D_{q,p} \equiv \{z \in \Omega \setminus \Omega_0,\ \prod_{k=1}^{p}|(z-z_k)|^{m_k} < r_1 \ \text{ and } \ \prod_{k=1}^{q}|(z-z_k)|^{m_k} < r_2 \prod_{k=q+1}^{p}|(z-z_k)|^{m_k}\} \tag{42}$$

where $r_1 \equiv Inf_{w\in\mathcal{C}\setminus\Omega}\{\prod_{k=1}^{p}|(w-z_k)|^{m_k}\}$ and
$r_2 \equiv Inf_{w\in\Omega_0}\left\{\prod_{k=1}^{q}|(w-z_k)|^{m_k}\prod_{k=q+1}^{p}|(w-z_k)^{-1}|^{m_k}\right\}$.

Proof. By Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(w)\,dw}{w-z}, \tag{43}$$

where Γ_1 and Γ_2 are the contours defined above.

First we assume that the m points of the set $\mathcal{S}$ are all distinct. Later we will let the first m_1 points coalesce to z_1, the second m_2 points to z_2, and so on. We substitute (13)-(14) into the first integral above and

$$\frac{1}{w-z} = \frac{F_m(w,z;z_1,\ldots,z_m)}{\prod_{k=1}^{s}(w-z_k)\prod_{k=s+1}^{m}(z-z_k)}\frac{1}{1-u}, \qquad u \equiv \frac{\prod_{k=1}^{s}(z-z_k)\prod_{k=s+1}^{m}(w-z_k)}{\prod_{k=1}^{s}(w-z_k)\prod_{k=s+1}^{m}(z-z_k)}$$

where

$$F_m(w,z;z_1,\ldots,z_m) \equiv \frac{\prod_{k=1}^{s}(w-z_k)\prod_{k=s+1}^{m}(z-z_k) - \prod_{k=1}^{s}(z-z_k)\prod_{k=s+1}^{m}(w-z_k)}{w-z}$$

into the second one. Next we introduce the expansion (15) of the factor $(1-u)^{-1}$ in both integrals in (43). We observe that $F_m(w,z;z_1,\ldots,z_m)$ may be written as

$$F_m(w,z;z_1,\ldots,z_m) = H_s(w,z;z_1,\ldots,z_p)\prod_{k=s+1}^{m}(z-z_k) - H_{m-s}(w,z;z_{p+1},\ldots,z_m)\prod_{k=1}^{s}(z-z_k),$$

where $H_m(w,z;z_1,\ldots,z_m)$ is defined in (5). Using this decomposition, equation (6), and after straightforward calculations we obtain

$$f(z) = \sum_{n=0}^{N-1} q_{n,m}(z)\prod_{k=1}^{m}(z-z_k)^n + \sum_{n=0}^{N-1} t_{n,m}^{(1)}(z)\frac{\prod_{k=1}^{s}(z-z_k)^n}{\prod_{k=s+1}^{m}(z-z_k)^n} + \sum_{n=0}^{N-1} t_{n,m}^{(2)}(z)\frac{\prod_{k=1}^{s}(z-z_k)^{n+1}}{\prod_{k=s+1}^{m}(z-z_k)^{n+1}} + r_N(z),$$

where $q_{n,m}(z)$ is given by formulas (8) and (9) replacing the contour $\mathcal{C}$ by Γ_1. Also,

$$t_{n,m}^{(1)}(z) = -\sum_{j=1}^{s} b_{n,j}\frac{\prod_{k=1,k\neq j}^{s}(z-z_k)}{\prod_{k=1,k\neq j}^{s}(z_j-z_k)}, \qquad b_{n,j} \equiv \frac{1}{2\pi i}\int_{\Gamma_2}\frac{\prod_{k=s+1}^{m}(w-z_k)^n}{\prod_{k=1}^{s}(w-z_k)^n}\frac{f(w)\,dw}{w-z_j},$$

$$t_{n,m}^{(2)}(z) = \sum_{j=s+1}^{m} c_{n,j}\frac{\prod_{k=s+1,k\neq j}^{m}(z-z_k)}{\prod_{k=s+1,k\neq j}^{m}(z_j-z_k)}, \quad c_{n,j} \equiv \frac{1}{2\pi i}\int_{\Gamma_2}\frac{\prod_{k=s+1}^{m}(w-z_k)^{n+1}}{\prod_{k=1}^{s}(w-z_k)^{n+1}}\frac{f(w)\,dw}{w-z_j}$$

and

$$r_N(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)\,dw}{(w-z)\prod_{k=1}^{m}(w-z_k)^N}\prod_{k=1}^{m}(z-z_k)^N - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{\prod_{k=s+1}^{m}(w-z_k)^N}{\prod_{k=1}^{s}(w-z_k)^N}\frac{f(w)\,dw}{w-z}\frac{\prod_{k=1}^{s}(z-z_k)^N}{\prod_{k=s+1}^{m}(z-z_k)^N}.$$

Now we write

$$t_{n,m}^{(1)}(z) = -\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\prod_{k=s+1}^{m}(w-z_k)^n}{\prod_{k=1}^{s}(w-z_k)^n}f(w)\,dw\sum_{j=1}^{s}\frac{\prod_{k=1,k\neq j}^{s}(z-z_k)}{(w-z_j)\prod_{k=1,k\neq j}^{s}(z_j-z_k)},$$

$$t_{n,m}^{(2)}(z) = \frac{1}{2\pi i}\int_{\Gamma_2}\frac{\prod_{k=s+1}^{m}(w-z_k)^{n+1}}{\prod_{k=1}^{s}(w-z_k)^{n+1}}f(w)\,dw\sum_{j=s+1}^{m}\frac{\prod_{k=s+1,k\neq j}^{m}(z-z_k)}{(w-z_j)\prod_{k=s+1,k\neq j}^{m}(z_j-z_k)},$$

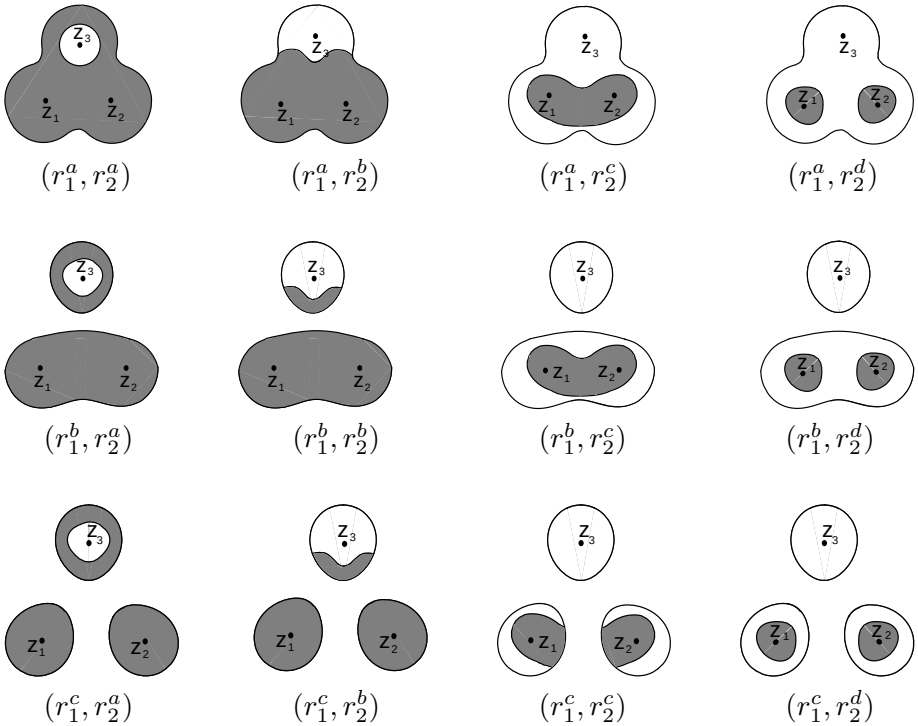
and repeat the steps following (21) in Theorem 2 for $q_{n,m}(z)$, $t_{n,m}^{(1)}(z)$ and $t_{n,m}^{(2)}(z)$.

For any z verifying (42), we can take simple closed loops Γ_1 and Γ_2 in $\Omega\backslash\Omega_0$ such that $\prod_{k=1}^{p}|z-z_k|^{m_k} < \prod_{k=1}^{p}|w-z_k|^{m_k}\ \forall\ w\in\Gamma_1$ and $\prod_{k=1}^{q}|w-z_k|^{m_k}\prod_{k=q+1}^{p}|z-z_k|^{m_k} >$

$\prod_{k=1}^{q}|z-z_k|^{m_k}\prod_{k=q+1}^{p}|w-z_k|^{m_k}\ \forall\ w\in\Gamma_2$ (see Figure 6 (b)). On these contours $|f(w)|$ is bounded by some constant C: $|f(w)|\leq C$. Introducing these bounds in (41) we see that $\lim_{N\to\infty} r_N(z)=0$ and the proof follows. □

(a) (b)

Figure 6. *The case $q=2$, $p=3$. (a) Contours Γ_1 and Γ_2 in the integrals (36), (38), (40) and (41). (b) For $z\in D_{q,p}$, we can take a contour Γ_2 located between Ω_0 and $D_{q,p}$ and a contour Γ_1 in Ω with $D_{q,p}$ inside this contour. Therefore, $\prod_{k=1}^{p}|z-z_k|^{m_k}<\prod_{k=1}^{p}|w-z_k|^{m_k}\ \forall\ w\in\Gamma_1$ and $\prod_{k=1}^{q}|z-z_k|^{m_k}\prod_{k=q+1}^{p}|w-z_k|^{m_k}<\prod_{k=1}^{q}|w-z_k|^{m_k}\prod_{k=q+1}^{p}|z-z_k|^{m_k}\ \forall\ w\in\Gamma_2$.*



(r_1^a, r_2^a) (r_1^a, r_2^b) (r_1^a, r_2^c) (r_1^a, r_2^d)

(r_1^b, r_2^a) (r_1^b, r_2^b) (r_1^b, r_2^c) (r_1^b, r_2^d)

(r_1^c, r_2^a) (r_1^c, r_2^b) (r_1^c, r_2^c) (r_1^c, r_2^d)

Figure 7. *The region $D_{q,p}$ defined in Theorem 4 is given by $D_{q,p}=O_p\bigcap B_{q,p}$, where O_p is the "lemniscate domain" of foci $z_1,\ldots,z_p$ and parameter r_1. Also, $B_{q,p}\equiv\{z\in\mathbb{C},\prod_{k=1}^{q}|(z-z_k)|^{m_k}<r_2\prod_{k=q+1}^{p}|(z-z_k)|^{m_k}\}$. This pictures show the topologically different forms of $D_{q,p}$ depending on the relative value of r_1 and r_2 when $q=2$ and $p=3$. The pictures are labeled with (r_1,r_2). In these pictures z_1 $|z_1-z_2|<|z_1-z_3|$, $|z_2-z_3|$ and $r_2^a>r_2^b>r_2^c>r_2^d$.*

If the only singularities of $f(z)$ inside Ω_0 are just poles at z_{q+1}, z_{q+2},...,z_p, then alternative formulas of (36), (38) and (40) for computing the coefficients of the above two-point Taylor-Laurent expansion is given in the following proposition.

Proposition 4. *Suppose that $g_k(z) \equiv (z-z_k)^{\rho_k} f(z)$ is an analytic function in Ω for certain $\rho_k \in \mathbb{N}$ and $k = q+1, q+2, \ldots, p$. Define $g_k(w) = f(w)$ for $k = 1, 2, 3, \ldots q$. Then the coefficients $a_{n,j,l}$, $b_{n,j,l}$ and $c_{n,j,l}$ in the expansion (34) are also given by the formulas:*

$$\begin{aligned} a_{n,j,l} = & \sum_{k=1,k\neq j}^{q} D_w^{nm_k-1}\left[\frac{f(w)}{(w-z_j)^{l+1}\prod_{s=1,s\neq k}^{p}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_k} + \\ & \sum_{k=q+1,k\neq j}^{p} D_w^{nm_k+\rho_k-1}\left[\frac{g_k(w)}{(w-z_j)^{l+1}\prod_{s=1,s\neq k}^{p}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_k} + \\ & D_w^{nm_j+\rho_j+l}\left[\frac{g_j(w)}{\prod_{s=1,s\neq j}^{p}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_j}, \end{aligned} \tag{44}$$

$$b_{n,j,l} = \sum_{k=q+1}^{p} D_w^{\rho_k-nm_k-1}\left[\frac{g_k(w)\prod_{s=q+1,s\neq k}^{p}(w-z_s)^{nm_s}}{(w-z_j)^{l+1}\prod_{s=1}^{q}(w-z_s)^{nm_s}}\right]\Bigg|_{w=z_k}. \tag{45}$$

$$\begin{aligned} c_{n,j,l} = & \sum_{k=q+1,k\neq j}^{p} D_w^{\rho_k-(n+1)m_k-1}\left[\frac{g_k(w)\prod_{s=q+1,s\neq k}^{p}(w-z_s)^{(n+1)m_s}}{(w-z_j)^{l+1}\prod_{s=1}^{q}(w-z_s)^{(n+1)m_s}}\right]\Bigg|_{w=z_k} + \\ & D_w^{\rho_j-(n+1)m_j+l}\left[\frac{g_j(w)\prod_{s=q+1,s\neq j}^{p}(w-z_s)^{(n+1)m_s}}{\prod_{s=1}^{q}(w-z_s)^{(n+1)m_s}}\right]\Bigg|_{w=z_j}. \end{aligned} \tag{46}$$

Proof. We deform both, the contour Γ_1 in equation (36) and the contour Γ_2 in equations (38) and (40) to any contour of the form $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_p$ contained in Ω, where $\mathcal{C}_k$, $k = 1, 2, ..., p$ is a simple closed loop which encircles the point z_k in the counterclockwise direction with z_j not inside $\mathcal{C}_k$, $j = 1, 2, \ldots, p$, $j \neq k$ (see Figure 3 (c)). Then,

$$\begin{aligned} a_{n,j,l} = & \frac{1}{2\pi i}\sum_{k=1,k\neq j}^{p}\int_{\mathcal{C}_k}\frac{g_k(w)}{(w-z_j)^{l+1}\prod_{s=1,s\neq k}^{p}(w-z_s)^{nm_s}}\frac{dw}{(w-z_k)^{nm_k+\rho_k}} + \\ & \int_{\mathcal{C}_j}\frac{g_j(w)}{\prod_{s=1,s\neq j}^{p}(w-z_s)^{nm_s}}\frac{dw}{(w-z_j)^{nm_j+\rho_j+l+1}}, \end{aligned}$$

$$b_{n,j,l} = \frac{1}{2\pi i}\sum_{k=q+1}^{p}\int_{\mathcal{C}_k}\frac{\prod_{s=q+1,s\neq k}^{p}(w-z_s)^{nm_s}}{(w-z_j)^{l+1}\prod_{s=1}^{q}(w-z_s)^{nm_s}}\frac{g_k(w)dw}{(w-z_k)^{\rho_k-nm_k}},$$

$$c_{n,j,l}=\sum_{k=q+1,k\neq j}^{p}\frac{1}{2\pi i}\int_{\mathcal{C}_k}\frac{\prod_{s=q+1,s\neq k}^{p}(w-z_s)^{(n+1)m_s}}{(w-z_j)^{l+1}\prod_{s=1}^{q}(w-z_s)^{(n+1)m_s}}\frac{g_k(w)dw}{(w-z_k)^{\rho_k-(n+1)m_k}}+$$
$$\frac{1}{2\pi i}\int_{\mathcal{C}_j}\frac{\prod_{s=q+1,s\neq j}^{p}(w-z_s)^{(n+1)m_s}}{\prod_{s=1}^{q}(w-z_s)^{(n+1)m_s}}\frac{g_j(w)dw}{(w-z_j)^{\rho_j-(n+1)m_j+l+1}}.$$

From here, equations (44), (45) and (46) follow. □

Remark 3. Let z be a real or complex variable and suppose that $(z-z_k)^{\rho_k}f(z)$ is ρ_k-1-times differentiable at z_k for certain $\rho_k\in\mathbb{N}$. Define

$$g(z)\equiv f(z)-\sum_{n=0}^{M}t_{n,m}^{(1)}(z)\frac{\prod_{k=1}^{q}(z-z_k)^{nm_k}}{\prod_{k=q+1}^{p}(z-z_k)^{nm_k}}-\sum_{n=0}^{M}t_{n,m}^{(2)}(z)\frac{\prod_{k=1}^{q}(z-z_k)^{(n+1)m_k}}{\prod_{k=q+1}^{p}(z-z_k)^{(n+1)m_k}},$$

where $M\equiv\lfloor\mathrm{Max}\{(\rho_{q+1}-1)/m_{q+1},(\rho_{q+2}-1)/m_{q+2},\ldots,(\rho_p-1)/m_p\}\rfloor$ and $t_{n,m}^{(1)}(z)$ and $t_{n,m}^{(2)}(z)$ are the polynomials defined in (37), (39), (45) and (46). Then, the thesis of Proposition 2 holds for $f(z)$ replaced by $g(z)$. Moreover, if $\prod_{k=q+1}^{p}(z-z_k)^{\rho_k}f(z)$ is an analytic function in Ω, then the thesis of Theorem 2 applies to $g(z)$.

5. Discussion and concluding remarks

In an earlier paper [6] we have discussed the theory of two-point Taylor expansions, two-point Laurent expansions and two-point Taylor-Laurent expansions. In the present paper we have generalized these two-point cases to multi-point cases. We have given details on the regions of convergence and on representations of the coefficients and the remainders of the expansions in terms of Cauchy-type integrals.

Multi-point Taylor expansions are related with topics from interpolation theory, in particular with the Newton interpolation theory with applications in numerical analysis. For example, applications can be found in initial and boundary value problems in connection with ordinary differential equations and in numerical quadrature of integrals.

From the point of view of interpolation theory detailed information on multi-point expansions can be found in [8], Chapters 3 and 8. The theory of several-point Taylor expansions is discussed in Chapter 3 of [8], although in a setting that is different from our approach. Our approach gives explicit Cauchy-type integrals of coefficients and remainders which cannot be found in Walsh's approach. In particular, we cannot find explicit formulas for the polynomials $q_{n,m}(z)$ of formula (15) as we have in (16)-(17). Knowledge of these explicit formulas is necessary to construct asymptotic expansions of integrals with several saddle points.

In addition to this, our Laurent and Taylor-Laurent expansions are new. They have a formal similarity with the rational approximations of Chapter 8 of [8]: they involve negative powers of z. But they are completely different. The rational approximations, in particular the Padé-type approximations $P_n(z)/Q_m(z)$ are of interpolatory type. These are generalizations of the Taylor polynomial at several points: a quotient of polynomials

instead of a polynomial. However, our expansions (21) or (32) have a different form and a different approximation property: they approach not only at regular points like Padé-type approximations but also at singular points of $f(z)$. And of course, the regions and convergence properties in [8] are different from ours.

Apart from applying the present results in problems from interpolation theory, in particular in problems from numerical analysis, we expect to find applications in asymptotic analysis of integrals, which application area is our main motivation; see [5]. In that paper certain orthogonal polynomials have been considered and we have given new convergent expansions that also have an asymptotic property for large values of a parameter (the degree n of the polynomials). Orthogonal polynomials and special functions can be studied when the variable and several parameters are large. In that case more than one or two so-called critical points occur that may give the main contributions to the integral, and expansions of analytic functions at these points gives again the possibility of constructing new convergent expansions with an asymptotic property. This method avoids the complicated conformal mapping of the phase function of the integral into a standard form (say a cubic or higher polynomial). In addition, when the critical points are multiple poles, Laurent-type expansions may be considered. A few application areas are mentioned in the Introduction, see the integral in (3), which we expect to approximate in terms of Airy functions and the Pearcey integral (4) and its derivative with respect to x and y.

6. Acknowledgements

The authors thank the referee for the comments on the first version of the paper.

J. L. López wants to thanks the saving bank *Caja Rural de Navarra* for its financial support. He also acknowledges the scientific and financial support of CWI in Amsterdam.

References

[1] C. Chester, B. Friedman, and F. Ursell, An extension of the method of steepest descent, *Proc. Cambridge Philos. Soc.*, **53** (1957) 599–611.

[2] K.A. Driver and N.M. Temme, On polynomials related with Hermite-Padé approximations to the exponential function, *J. Approx. Theory*, **95** (1998) 101–122.

[3] A.J.E.M. Janssen, On the asymptotics of some Pearcey-type integrals, *J. Phys. A: Math. Gen.*, **25** (1992) L823–L831.

[4] N.P. Kirk, J.N.L. Connor, P.R. Curtis and C.A. Hobbs, Theory of axially symmetric cusped focusing: numerical evaluation of a Bessoid integral by an adaptive contour algorithm, *J. Phys. A: Math. Gen.*, **33** (2000) 4797–4808.

[5] José L. Lopez and Nico M. Temme, Asymptotic expansions of Charlier, Laguerre and Jacobi polynomials. Accepted for publication in *The Proceedings of the Royal Society of Edinburgh A (Mathematics)*.

[6] José L. Lopez and Nico M. Temme, Two-point Taylor expansions of analytic functions. *Stud. Appl. Math*, **109** (2002) 297-311.

[7] Raimundas Vidunas and Nico M. Temme, Symbolic evaluation of coefficients in Airy-type asymptotic expansions. *J. Math. Anal. Appl.* 269 (2002) 317-331.

[8] J. L. Walsh, Interpolation and Approximation by rational functions in the complex domain, *American Mathematical Society, Providence*, 1969.

[9] R. Wong, Asymptotic Approximations of Integrals, *Academic Press, New York*, 1989.