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# On the Zeros of the Scorer Functions

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## ABSTRACT

Asymptotic approximations are developed for zeros of the solutions  $\text{Gi}(z)$  and  $\text{Hi}(z)$  of the inhomogeneous Airy differential equation  $w'' - zw = \pm \frac{1}{\pi}$ . The solutions are also called Scorer functions. Tables are given with numerical values of the zeros.

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## 1. INTRODUCTION

Scorer functions are particular solutions of the non-homogeneous Airy differential equation. Detailed information on these functions can be found in [1] and in references given in [2]. We summarize the properties that are needed in this paper.

We have for  $z \in \mathbb{R}$

$$w'' - zw = -1/\pi, \quad \text{with solution} \quad \text{Gi}(z) = \frac{1}{\pi} \int_0^\infty \sin\left(zt + \frac{1}{3}t^3\right) dt, \quad (1.1)$$

and for  $z \in \mathbb{C}$

$$w'' - zw = 1/\pi, \quad \text{with solution} \quad \text{Hi}(z) = \frac{1}{\pi} \int_0^\infty e^{zt - \frac{1}{3}t^3} dt. \quad (1.2)$$

The solutions of the homogeneous Airy equation  $w'' - zw = 0$  are denoted by  $\text{Ai}(z)$  and  $\text{Bi}(z)$ . They have the integral representations

$$\begin{aligned} \text{Ai}(z) &= \frac{1}{\pi} \int_0^\infty \cos\left(zt + \frac{1}{3}t^3\right) dt, \\ \text{Bi}(z) &= \frac{1}{\pi} \int_0^\infty \sin\left(zt + \frac{1}{3}t^3\right) dt + \frac{1}{\pi} \int_0^\infty e^{zt - \frac{1}{3}t^3} dt, \end{aligned} \quad (1.3)$$

where we assume that  $z$  is real.

Initial values are

$$\text{Gi}(0) = \frac{1}{2}\text{Hi}(0) = \frac{1}{3}\text{Bi}(0) = \frac{1}{\sqrt{3}}\text{Ai}(0) = \frac{1}{3^{7/6}\Gamma(\frac{2}{3})}, \quad (1.4)$$

$$\text{Gi}'(0) = \frac{1}{2}\text{Hi}'(0) = \frac{1}{3}\text{Bi}'(0) = -\frac{1}{\sqrt{3}}\text{Ai}'(0) = \frac{1}{3^{5/6}\Gamma(\frac{1}{3})}.$$

From (1.1), (1.2) and (1.3) it follows that

$$\text{Gi}(z) + \text{Hi}(z) = \text{Bi}(z). \quad (1.5)$$

Other relations that we need are (see [2] and [3])

$$\text{Hi}(z) = e^{\pm 2\pi i/3}\text{Hi}\left(ze^{\pm 2\pi i/3}\right) + 2e^{\mp \pi i/6}\text{Ai}\left(ze^{\mp 2\pi i/3}\right). \quad (1.6)$$

and

$$\text{Gi}(z) = -e^{\pm 2\pi i/3}\text{Hi}\left(ze^{\pm 2\pi i/3}\right) \pm i\text{Ai}(z). \quad (1.7)$$

Proofs follow easily by verifying that the right-hand sides satisfy the differential equations, and from the initial values.

We use the asymptotic expansions

$$\text{Gi}(z) \sim \frac{1}{\pi z} \left[ 1 + \frac{1}{z^3} \sum_{s=0}^{\infty} \frac{(3s+2)!}{s!(3z^3)^s} \right], \quad z \rightarrow \infty, \quad |\text{ph } z| \leq \frac{1}{3}\pi - \delta, \quad (1.8)$$

$$\text{Hi}(z) \sim -\frac{1}{\pi z} \left[ 1 + \frac{1}{z^3} \sum_{s=0}^{\infty} \frac{(3s+2)!}{s!(3z^3)^s} \right], \quad z \rightarrow \infty, \quad |\text{ph}(-z)| \leq \frac{2}{3}\pi - \delta, \quad (1.9)$$

$\delta$  being an arbitrary positive constant. These expansions follow from (1.1) and (1.2) and by using standard methods from asymptotics (Watson's lemma; see [4], page 112 and page 431).

## 2. QUALITATIVE PROPERTIES OF THE REAL ZEROS OF $\text{Gi}(z)$ AND $\text{Gi}'(z)$

From (1.2) we see that  $\text{Hi}(z) > 0$  and  $\text{Hi}'(z) > 0$  for real finite  $z$ . However,  $\text{Gi}(z)$  and  $\text{Gi}'(z)$  have real zeros. First we show that  $\text{Gi}(z)$  does not have positive zeros. Later, we study properties of the negative real zeros and we discuss the properties of the zeros of the derivative.

For studying qualitative properties of the zeros, the relation 10.4.51 in [1].

$$W[\text{Gi}, \text{Bi}](x) \equiv \text{Gi}(z)\text{Bi}'(z) - \text{Gi}'(z)\text{Bi}(z) = \frac{1}{\pi} \int_0^z \text{Bi}(t) dt \quad (2.1)$$

will be useful, together with well-known properties on the interlacing of zeros of functions:

**Lemma 1** *Let  $f(x)$  and  $g(x)$  be two continuously differentiable functions in an interval  $I$ . Let  $W[f, g](x) = f(x)g'(x) - f'(x)g(x)$  be such that  $W[f, g](x) \neq 0 \forall x \in I$ .*

*Then, the zeros of  $f(x)$  and  $g(x)$  in  $I$  are simple. Furthermore, between two consecutive zeros of  $f(x)$  there is exactly one zero of  $g(x)$  (and vice-versa).*

*As a consequence, if  $g(x)$  has no zeros in  $I$  then  $f(x)$  has at most one (simple) zero in  $I$  (and vice-versa).*

From Lemma 1 we can check that

**Lemma 2**  $\text{Gi}(x)$  is positive for  $x \geq 0$ .

**Proof.**  $\text{Bi}(x) > 0$  for  $x \geq 0$  (see [1], for example the series expansion in 10.4.3) and then (Eq. (2.1))  $W[\text{Gi}, \text{Bi}](x) > 0$  for  $x \geq 0$ . From Lemma 1,  $\text{Gi}(x)$  can have at most one (simple) positive real zero, but  $\text{Gi}(0) > 0$  and  $\text{Gi}(x) > 0$  for large positive  $x$  (Eq. (1.9)); therefore  $\text{Gi}(x) > 0$  for  $x \geq 0$ . ■

To consider negative values of  $z$ , we first remark that  $\text{Bi}(z)$  has an infinite number of negative zeros which we denote by  $\{b_n\}$ .

**Lemma 3**  $\text{Gi}(b_n) < 0 \forall n$ .

**Proof.** From (1.5), and the fact that  $\text{Hi}(x) > 0$  we have  $\text{Bi}(x) > \text{Gi}(x)$  for all real  $x$ . ■

**Lemma 4**  $\text{Gi}(x)$  has exactly one simple zero in  $(b_1, 0)$ .

**Proof.**

$\text{Gi}(b_1) < 0$  whereas  $\text{Gi}(0) > 0$  (see (1.4)); then  $\text{Gi}(x)$  has at least one zero in  $(b_1, 0)$ . Furthermore  $\text{Bi}(x) > 0$  and then  $W[\text{Gi}, \text{Bi}](x) < 0$  in  $(b_1, 0)$ . Then, Lemma 1 implies that there is only one zero in this interval and that it is simple. ■

Between  $b_1 = -1.17371$  and  $b_2 = -3.27109$ , the function  $\text{Bi}(x)$  is negative, and so  $\text{Gi}(x)$  is negative in that interval ( $\text{Bi}(x) > \text{Gi}(x)$ ). More generally, we have that:

**Lemma 5**  $\text{Gi}(x)$  has no zeros in the intervals  $[b_{2n+2}, b_{2n+1}]$ ,  $n = 0, 1, \dots$

We are only left with the possibility of having zeros in intervals  $(b_{2n+1}, b_{2n})$ ,  $n = 1, 2, \dots$ , where  $\text{Bi}(x) > 0$ . Numerical experiments show that these zeros are simple.

The proof that all real or complex zeros of  $\text{Gi}(z)$  and  $\text{Hi}(z)$  are simple does not follow from the inhomogeneous differential equations (1.1) and (1.2). Recall that for functions defined by homogeneous linear differential equations of second order, as the Airy functions, such a proof is trivial. The essential difference between these two cases is that the existence and uniqueness theorem for solutions of a linear second order homogeneous ODE guarantees that the only solution having a double zero at a point  $x = x_0$  is the trivial solution; contrary, for (1.1) and (1.2), there is always one solution with a double zero at  $x = x_0$  and it is not a trivial solution.

We can see this explicitly:

**Lemma 6** The function

$$y(z) = \alpha(z_0)\text{Ai}(z) + \beta(z_0)\text{Bi}(z) + \text{Gi}(z),$$

with  $\alpha(z_0) = -\int_0^{z_0} \text{Bi}(t)dt$ , and  $\beta(z_0) = \int_0^{z_0} \text{Ai}(t)dt - \frac{1}{3}$  is the solution of  $\omega'' - z\omega = -1/\pi$  with a double zero at  $z_0$

**Proof.** Solve the system  $y(z_0) = 0$ ,  $y'(z_0) = 0$  for  $\alpha$  and  $\beta$  and use the Wronskian relations 10.4.10, 10.4.47 and 10.4.51 of [1]. ■

In fact,  $\int_0^x \text{Ai}(t)dt$  is numerically seen to be negative for negative  $x$ , which indicates that the negative real zeros of  $\text{Gi}(x)$  are simple because  $\beta(z_0) < 0 \forall z_0 < 0$ .

If there were any real double zero (which is not the case), it necessarily would be an extremum:

**Lemma 7** The double real zeros of a real solution of  $\omega'' - z\omega = \pm 1/\pi$  are necessarily local extrema of the function.

**Proof.** Let  $x_0$  be a (double) zero of a solution  $\omega(x)$ . Then, using the differential equation,  $\omega''(x_0) = \pm 1/\pi$ . ■

**Lemma 8** *The number of simple zeros of  $\text{Gi}(x)$  in each interval  $(b_{2n+1}, b_{2n})$ ,  $n = 1, 2, \dots$ , is, at most, two.*

**Proof.** Given that  $\text{Gi}(x)$  is negative at the zeros of  $\text{Bi}(x)$  and that the double zeros, if any, are extrema of the function, we see that the number of simple zeros (if any) must be even. Let us show there can be no more than two simple zeros.

The fact that  $\text{Bi}(x) > 0$  in  $(b_{2n+2}, b_{2n+1})$  implies that  $\frac{d}{dx}W[\text{Gi}, \text{Bi}](x) > 0$ , which means that  $W[\text{Gi}, \text{Bi}](x)$  has at most one zero in  $(b_{2n+2}, b_{2n+1})$ . Then, if  $\text{Gi}(x)$  had  $2n$  zeros,  $n > 1$ , at least two of these zeros would lie in an interval where  $W[\text{Gi}, \text{Bi}](x)$  does not change sign; this would imply (Lemma 1) that there would be a zero of  $\text{Bi}(x)$  between these two zeros of  $\text{Gi}(x)$ , but  $\text{Bi}(x) > 0$  in  $(b_{2n+2}, b_{2n+1})$ . ■

In fact, numerical calculations show that in the intervals  $(b_{2n+1}, b_{2n})$ ,  $n = 1, 2, \dots$ , exactly two zeros of  $\text{Gi}(x)$  occur, which means that there are no double zeros of  $\text{Gi}(x)$  and that  $W[\text{Gi}, \text{Bi}](x) = \int_0^x \text{Bi}(t)dt$  has exactly one zero in the intervals  $(b_{2n+1}, b_{2n})$ . This, together with the monotony of  $\int_0^x \text{Bi}(t)dt$  in the intervals  $(b_{n+1}, b_n)$ , indicates that:

**Conjecture 1** *The real zeros of  $\text{Bi}(x)$  and  $\int_0^x \text{Bi}(t)dt$  are interlaced.*

and we also propose that

**Conjecture 2** *There are exactly two zeros of  $\text{Gi}(x)$  in the intervals  $(b_{2n+1}, b_{2n})$ .*

We can prove that this holds for large negative zeros:

**Lemma 9** *For large  $n$  each interval  $(b_{2n+1}, b_{2n})$  has two zeros of  $\text{Gi}(x)$ .*

**Proof.** This follows from known asymptotic estimates.  $\text{Bi}'(x)$  has negative zeros, denoted by  $b'_n$ . Then (see [1], page 450),

$$\text{Bi}(b'_n) = (-1)^n \mathcal{O}\left(n^{-1/6}\right), \quad n \rightarrow \infty, \quad (2.2)$$

From (1.9) we see that  $\text{Hi}(b'_n) = \mathcal{O}(1/b'_n)$  as  $n \rightarrow \infty$ . Hence,  $\text{Gi}(b'_n) = \text{Bi}(b'_n) - \text{Hi}(b'_n)$  is positive for large values of  $n$ , and  $\text{Gi}(x)$  has at least two zeros in the interval  $(b_{2n+1}, b_{2n})$ . ■

The fact that the real zeros of  $\text{Gi}(x)$  are simple is also supported by the fact that the zeros of  $\text{Gi}(x)$  and  $\text{Gi}'$  seem to be interlaced, which can be easily proved for large  $x$  using the forthcoming asymptotic expansions.

**Conjecture 3** *The negative zeros of  $\text{Gi}(x)$  and  $\text{Gi}'(x)$  are interlaced.*

Assuming this conjecture to be true, together with the fact that, numerically, we observe that  $g_1 > g'_1$  where  $g_1$  and  $g'_1$  are, respectively, the first negative zeros of  $\text{Gi}(x)$  and  $\text{Gi}'(x)$ , we see that:

**Lemma 10** *The negative zeros of  $\text{Gi}'(x)$  are simple*

**Proof.** Differentiating the differential equation it is easy to see that the double zeros of  $\text{Gi}'(x)$  can not be extrema of  $\text{Gi}(x)$ : if  $x_0$  is such that  $\text{Gi}'(x_0) = \text{Gi}''(x_0) = 0$  then  $\text{Gi}'''(x_0) = x_0\text{Gi}'(x_0) + \text{Gi}(x_0) = \text{Gi}(x_0) \neq 0$  (because we assume that the zeros of  $\text{Gi}(x)$  and  $\text{Gi}'(x)$  are interlaced). However, by this same assumption, between two zeros of  $\text{Gi}(x)$  there must be only one zero of  $\text{Gi}'(x)$  which, clearly, must be a local extrema and therefore can not be a double zero of  $\text{Gi}'(x)$ . ■

We use an additional numerical fact to prove Lemma 12; also, the following result is used:

**Lemma 11**  *$W[\text{Hi}', \text{Gi}'](x)$  has, at most, one positive real zero.*

**Proof.** We use the differential equations and (1.5); with this:

$$W(x) \equiv W[\text{Hi}', \text{Gi}'](x) = \frac{1}{\pi} \text{Bi}'(x) - \frac{x}{\pi} \int_0^x \text{Bi}(t) dt$$

Then, given that  $W(0) > 0$  and  $W'(x) = -\int_0^x \text{Bi}(t) dt < 0$  for  $x > 0$ ,  $W(x)$  has at most one positive zero. ■

Indeed, we observe that such zero exists  $x_0 \simeq 1.0653592469$ .

**Lemma 12**  $\text{Gi}'(x)$  has exactly one positive real zero.

**Proof.** Indeed,  $\text{Gi}'(0) > 0$  (1.4) while (1.9) shows that  $\text{Gi}'(x) < 0$  for large positive  $x$ , which implies that there must be at least one positive zero of  $\text{Gi}'(x)$ . This, together with the fact that the double zeros of  $\text{Gi}'$  are not extrema, implies that there must be an odd number of positive zeros. Let us assume for the moment that all the positive zeros of  $\text{Gi}'(x)$  are simple; in this case, we show that there is only one positive zero.

$\text{Gi}'(x)$  can not have three or more simple positive zeros because,  $W[\text{Hi}', \text{Gi}']$  has at most one positive zero (Lemma 11) and  $\text{Hi}'(x) > 0$ . The possible zero of  $W[\text{Hi}', \text{Gi}'](x)$  can not coincide with any zero of  $\text{Gi}'(x)$ , because we are assuming by now that the zeros of  $\text{Gi}'(x)$  are simple; thus, if  $\text{Gi}'(x)$  had at least three zeros, at least two of them would lie in an interval where  $W[\text{Hi}', \text{Gi}'](x)$  does not change sign. This is in contradiction with the fact that  $\text{Hi}'(x) > 0$  (see Lemma 1).

On the other hand, the only possible double zero of  $\text{Gi}'(x)$  should coincide with the positive zero of  $W[\text{Hi}', \text{Gi}'](x)$ . However, it is numerically observed that

$\text{Gi}'(x_0) < 0$  being  $x_0$  the positive zero of  $W[\text{Hi}', \text{Gi}'](x)$ . ■

The numerical value of this isolated zero of  $\text{Gi}'(x)$ , is  $g' = 0.60907541707\dots$

### 3. ASYMPTOTICS OF THE NEGATIVE ZEROS OF $\text{Gi}(z)$

We write  $\text{Gi}(-z) = \text{Bi}(-z) - \text{Hi}(-z)$ , and use the asymptotic expansion of  $\text{Bi}(-z)$  as given in [[1], page 449] and of  $\text{Hi}(-z)$  that follows from (1.9). We write

$$\text{Hi}(-z) = \frac{1}{\pi z} Ha(z), \quad Ha(z) \sim 1 - \sum_{s=0}^{\infty} \frac{h_s}{z^{3(s+1)}}, \quad h_s = (-1)^s \frac{(3s+2)!}{s! 3^s}, \quad (3.1)$$

$$\text{Bi}(-z) = \frac{1}{\sqrt{\pi} z^{1/4}} \left[ \cos(\zeta + \frac{1}{4}\pi) P(\zeta) + \frac{1}{\zeta} \sin(\zeta + \frac{1}{4}\pi) Q(\zeta) \right], \quad (3.2)$$

$$P(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s c_{2s}}{\zeta^{2s}}, \quad Q(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s c_{2s+1}}{\zeta^{2s}}, \quad (3.3)$$

$$\zeta = \frac{2}{3} z^{\frac{3}{2}}, \quad c_0 = 1, \quad c_s = \frac{\Gamma(3s + \frac{1}{2})}{54^s s! \Gamma(s + \frac{1}{2})}. \quad (3.4)$$

We explain the method by taking  $Ha(\zeta) = P(\zeta) = 1$  and  $Q(\zeta) = 0$ . This gives for the equation  $\text{Gi}(-z) = \text{Bi}(-z) - \text{Hi}(-z) = 0$  a first equation

$$\cos(\zeta + \frac{1}{4}\pi) = \frac{1}{\sqrt{\pi} z^{3/4}}. \quad (3.5)$$

Using  $z^{3/4} = \sqrt{3\zeta/2}$ , we obtain

$$\cos(\zeta + \frac{1}{4}\pi) = \sqrt{\frac{2}{3\pi\zeta}}. \quad (3.6)$$

For large  $\zeta$  solutions occur when the cosine function is small. We put

$$\zeta = \zeta_n + \varepsilon, \quad \zeta_n = (n - \frac{3}{4})\pi, \quad n = 1, 2, 3, \dots \quad (3.7)$$

The equation for  $\varepsilon$  reads

$$\sin \varepsilon = \frac{c}{\sqrt{\zeta_n + \varepsilon}} = \frac{ct}{\sqrt{1 + \varepsilon t^2}}, \quad t = 1/\sqrt{\zeta_n}, \quad c = (-1)^n \sqrt{2/(3\pi)}. \quad (3.8)$$

For small values of  $t$  this equation can be solved by substituting a power series  $\varepsilon = \varepsilon_1 t + \varepsilon_2 t^2 + \dots$ , and the coefficients can be obtained by standard methods. For example,  $\varepsilon_1 = c$ . By using the asymptotic expansions for  $Ha(z)$ ,  $P(\zeta)$  and  $Q(\zeta)$  a few extra technicalities are introduced. With the help of a computer algebra package the general coefficients  $\varepsilon_s$  are easy to calculate. Finally we find for  $z = (3\zeta/2)^{2/3}$ , and for  $g_n$ , the zeros of  $Gi(z)$ , the expansion

$$g_n \sim - \left(\frac{3}{2}\zeta_n\right)^{\frac{2}{3}} [1 + \varepsilon_1 t^3 + \varepsilon_2 t^4 + \dots]^{2/3}, \quad n = 1, 2, 3, \dots, \quad (3.9)$$

or

$$g_n \sim - [(3\pi(4n-3)/8)^{\frac{2}{3}} [1 + \gamma_3 t^3 + \gamma_4 t^4 + \dots]], \quad t = \frac{1}{\sqrt{(n-3/4)\pi}}, \quad (3.10)$$

where

$$\begin{aligned} \gamma_3 &= \frac{2c}{3}, \quad \gamma_4 = \frac{5}{108}, \quad \gamma_5 = \frac{c^3}{9}, \quad \gamma_6 = -\frac{4c^2}{9}, \\ \gamma_7 &= \frac{c(81c^4 - 1060)}{1620}, \quad \gamma_8 = -\frac{189c^4 + 20}{729}. \end{aligned} \quad (3.11)$$

The expansion in (3.10) reduces to the expansion of the zeros  $b_n$  of  $Bi(z)$  if we take  $c = 0$ .

### 3.1 The real zeros of $Gi'(z)$

For the real zeros of  $Gi'(z)$  we can use the same procedure. For this case we need

$$Hi'(-z) = \frac{1}{\pi z^2} \widetilde{Ha}(z), \quad \widetilde{Ha}(z) \sim 1 - \sum_{s=0}^{\infty} \frac{\widetilde{h}_s}{z^{3(s+1)}}, \quad \widetilde{h}_s = (3s+4)h_s \quad (3.12)$$

$$Bi'(-z) = \frac{z^{1/4}}{\sqrt{\pi}} \left[ \sin(\zeta + \frac{1}{4}\pi) R(\zeta) - \frac{1}{\zeta} \cos(\zeta + \frac{1}{4}\pi) S(\zeta) \right], \quad (3.13)$$

$$R(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s d_{2s}}{\zeta^{2s}}, \quad S(\zeta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s d_{2s+1}}{\zeta^{2s}}, \quad d_s = -\frac{6s+1}{6s-1} c_s \quad (3.14)$$

where  $\zeta$  and  $c_s$  are defined in (3.4). Using  $Gi'(-z) = Bi'(-z) - Hi'(-z)$  we obtain the equation for determining the zeros:

$$\sin(\zeta + \frac{1}{4}\pi) = \frac{1}{\zeta} \frac{S(\zeta)}{R(\zeta)} \cos(\zeta + \frac{1}{4}\pi) + \frac{1}{\sqrt{\pi} z^{9/4}} \frac{\widetilde{Ha}(z)}{R(\zeta)}. \quad (3.15)$$



Using  $z^{9/4} = (3\zeta/2)^{3/2}$ , we see that the main part of this equation is obtained by neglecting the term with the function  $\widetilde{Ha}(z)$ , but we can proceed in the same manner as before.

We put

$$\zeta = \zeta'_n + \varepsilon', \quad \zeta'_n = (n - \frac{1}{4})\pi, \quad n = 1, 2, 3, \dots, \quad (3.16)$$

and we can obtain for  $\varepsilon'$  an expansion. Finally we obtain for  $g'_n$ , the zeros of  $\text{Gi}'(z)$ , the expansion

$$g'_n \sim - \left( \frac{3}{2} \zeta'_n \right)^{\frac{2}{3}} [1 + \varepsilon_3 \tau^5 + \varepsilon_4 \tau^6 + \dots]^{2/3}, \quad n = 1, 2, 3, \dots, \quad (3.17)$$

or

$$g'_n \sim - [(3\pi(4n-1)/8)^{\frac{2}{3}} [1 + \gamma'_4 t^4 + \gamma'_5 t^5 + \dots]], \quad t = \frac{1}{\sqrt{(n-1/4)\pi}}, \quad (3.18)$$

where

$$\begin{aligned} \gamma'_4 &= -\frac{7}{108}, & \gamma'_5 &= \frac{2c}{3}, & \gamma'_6 &= \gamma'_7 = 0, \\ \gamma'_8 &= \frac{35}{1458}, & \gamma'_9 &= -\frac{719c}{324}, & \gamma'_{10} &= -\frac{10c^2}{9}. \end{aligned} \quad (3.19)$$

This expansion reduces to that of  $b'_n$ , the zeros of  $\text{Bi}'(z)$  if we take  $c = 0$ .

#### 4. THE COMPLEX ZEROS OF $\text{Gi}(z)$

$\text{Gi}(z)$  and  $\text{Gi}'(z)$  have infinite many complex zeros  $\{\chi_n\}$  just below the half line  $\text{ph } z = \frac{1}{3}\pi$ , and at the conjugate values. Asymptotic estimates can be obtained by using the connection formula (1.7) with  $z$  replaced with  $ze^{\pi i/3}$ . That is,

$$\text{Gi}\left(ze^{\pi i/3}\right) = -e^{\pm 2\pi i/3} \text{Hi}(-z) + i \text{Ai}\left(ze^{\pi i/3}\right). \quad (4.1)$$

We write  $\text{Hi}(-z)$  as in (3.1) and for  $\text{Ai}(z)$  we obtain from the standard asymptotic expansion of this Airy function

$$\text{Ai}\left(ze^{\pi i/3}\right) = \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\pi i/12 - i\eta} A_a(\eta), \quad (4.2)$$

where (for  $c_s$  see (3.4))

$$\eta = \frac{2}{3} z^{\frac{3}{2}}, \quad A_a(\eta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s c_s}{(i\eta)^s}. \quad (4.3)$$

The equation for deriving the asymptotic expansion of  $\chi_n$  for large  $n$  then reads

$$e^{i\eta} = \frac{1}{2} \sqrt{\pi} z^{3/4} e^{-\pi i/4} \frac{A_a(\eta)}{Ha(z)}. \quad (4.4)$$

We write

$$\eta = \eta_n - \frac{1}{2} i \ln(c\eta_n) + \varepsilon, \quad \eta_n = (2n - \frac{1}{4})\pi, \quad c = \frac{3}{8}\pi, \quad (4.5)$$

and obtain for  $\varepsilon$  the equation

$$e^{i\varepsilon} = \sqrt{1 - i\delta t + \varepsilon t} \frac{A_a(\eta)}{Ha(z)}, \quad t = \frac{1}{\eta_n}, \quad \delta = \frac{1}{2} \ln(c\eta_n). \quad (4.6)$$

The next step is substituting a power series  $\varepsilon = \varepsilon_1 t + \varepsilon_2 t^2 + \dots$ , considering  $\delta$  as a fixed parameter. A few straightforward manipulations give the expansion

$$\chi_n \sim [3\pi(8n-1)/8]^{2/3} e^{\pi i/3} \left( 1 + \frac{\gamma_1}{\eta_n} + \frac{\gamma_2}{\eta_n^2} + \frac{\gamma_3}{\eta_n^3} + \dots \right) \quad (4.7)$$

and the first few coefficients are

$$\begin{aligned} \gamma_1 &= -\frac{2}{3}i\delta, & \gamma_2 &= \frac{1}{108}(5 - 36\delta + 12\delta^2), \\ \gamma_3 &= \frac{1}{162}i(-96 + 37\delta - 45\delta^2 + 8\delta^3), \\ \gamma_4 &= \frac{1}{2916}(-944 + 4365\delta - 1182\delta^2 + 702\delta^3 - 84\delta^4). \end{aligned} \quad (4.8)$$

#### 4.1 The complex zeros of $\mathbf{Gi}'(z)$

For the complex zeros  $\chi'_n$  of  $\mathbf{Gi}'(z)$  we use (cf. (1.7))

$$\mathbf{Gi}'(ze^{\pi i/3}) = e^{\pi i/3} \mathbf{Hi}'(-z) + ie^{\pi i/3} \mathbf{Ai}'(ze^{\pi i/3}). \quad (4.9)$$

We need the expansion of  $\mathbf{Hi}'(-z)$  given in (3.12) and

$$\mathbf{Ai}'(ze^{\pi i/3}) = -\frac{z^{1/4}}{2\sqrt{\pi}} e^{\pi i/12 - i\eta} \widetilde{Aa}(\eta), \quad \widetilde{Aa}(\eta) \sim \sum_{s=0}^{\infty} \frac{(-1)^s d_s}{(i\eta)^s}. \quad (4.10)$$

where  $d_s$  is given in (3.14). We put

$$\eta = \eta'_n - i\delta' + \varepsilon', \quad \eta'_n = (2n + \frac{1}{4})\pi, \quad c = \frac{3}{2}(\frac{\pi}{4})^{\frac{1}{3}}, \quad (4.11)$$

and the equation for  $\varepsilon'$  reads

$$e^{i\varepsilon'} = (1 - i\delta't + \varepsilon't)^{3/2} \frac{\widetilde{Aa}(\eta)}{\widetilde{Ha}(\eta)}, \quad \delta' = \frac{3}{2} \ln(c\eta'_n). \quad (4.12)$$

The expansion for the zeros reads

$$\chi'_n \sim [3\pi(8n+1)/8]^{2/3} e^{\pi i/3} \left( 1 + \frac{\gamma'_1}{\eta'_n} + \frac{\gamma'_2}{\eta'^2_n} + \frac{\gamma'_3}{\eta'^3_n} + \dots \right), \quad n = 1, 2, 3, \dots \quad (4.13)$$

and the first few coefficients are

$$\begin{aligned} \gamma'_1 &= -\frac{2}{3}i\delta', & \gamma'_2 &= \frac{1}{108}(-7 - 108\delta' + 12\delta'^2), \\ \gamma'_3 &= \frac{1}{324}i(-747 + 458\delta' - 270\delta'^2 + 16\delta'^3), \\ \gamma'_4 &= \frac{1}{5832}(-20029 + 43740\delta' - 16908\delta'^2 + 4212\delta'^3 - 168\delta'^4). \end{aligned} \quad (4.14)$$

### 5. THE COMPLEX ZEROS OF $\text{Hi}(z)$

$\text{Hi}(z)$  and  $\text{Hi}'(z)$  have infinite many complex zeros  $\{\kappa_n\}$  just above the half line  $\text{ph } z = \frac{1}{3}\pi$ , and at the conjugate values. For  $\text{Hi}(z)$  we use (1.6) in the form

$$\text{Hi}\left(ze^{\pi i/3}\right) = e^{2\pi i/3}\text{Hi}(-z) + 2e^{-\pi i/6}\text{Ai}\left(ze^{-\pi i/3}\right). \quad (5.1)$$

The analysis is analogous to the case for  $\text{Gi}(z)$  and gives (4.6) with  $i$  replaced by  $-i$ , also in  $Aa(\eta)$ , and  $c$  by  $c = \frac{3}{2}\pi$ . This gives for  $\kappa_n$ , the zeros of  $\text{Hi}(z)$ ,

$$\kappa_n \sim [3\pi(8n-1)/8]^{2/3} e^{\pi i/3} \left(1 + \frac{\overline{\gamma}_1}{\eta_n} + \frac{\overline{\gamma}_2}{\eta_n^2} + \frac{\overline{\gamma}_3}{\eta_n^3} + \dots\right) \quad (5.2)$$

where  $\eta_n$  is given in (4.5),  $\delta = \frac{1}{2}\ln(c\eta_n)$ , with  $c = \frac{3}{2}\pi$ , and the first few  $\gamma_k$  are given in (4.8).

For  $\text{Hi}'(z)$  we find equation (4.12) with  $i$  replaced by  $-i$ , also in  $\widetilde{Aa}(\eta)$ , and  $c = \frac{3}{2}\pi^{\frac{1}{3}}$ . For  $\kappa'_n$ , the zeros of  $\text{Hi}'(z)$ , we obtain

$$\kappa'_n \sim [3\pi(8n+1)/8]^{2/3} e^{\pi i/3} \left(1 + \frac{\overline{\gamma}'_1}{\eta'_n} + \frac{\overline{\gamma}'_2}{\eta'^2_n} + \frac{\overline{\gamma}'_3}{\eta'^3_n} + \dots\right), \quad n = 1, 2, 3, \dots \quad (5.3)$$

where  $\eta'_n$  is given in (4.11),  $\delta = \frac{3}{2}\ln(c\eta'_n)$ , with  $c = \frac{3}{2}\pi^{\frac{1}{3}}$ , and the first few  $\gamma'_k$  are given (4.14).

### 6. NUMERICAL VERIFICATIONS AND TABLES

Now we will illustrate the accuracy of the asymptotic approximations for the real and complex zeros of  $\text{Gi}(x)$ ,  $\text{Gi}'(x)$  (except the positive zero of  $\text{Gi}'(x)$ ) and the complex zeros of  $\text{Hi}(x)$  and  $\text{Hi}'(x)$ . For the complex zeros, by complex conjugation, we only need to consider  $\Im z > 0$ .

We use the asymptotic approximations as starting values for a Newton method, obtaining convergence in all cases. The code [3] has been used for the calculations. The accuracy of the code is better than  $10^{-12}$  and we expect that the zeros can be computed with at least 12 exact digits.

Table 1 shows the relative error of the asymptotic estimates.

$n$	$\text{Error} g_n $	$\text{Error} g'_n $	$\text{Error} \chi_n $	$\text{Error} \chi'_n $	$\text{Error} \kappa_n $	$\text{Error} \kappa'_n $
1	$4 \cdot 10^{-2}$	$5 \cdot 10^{-3}$	$4 \cdot 10^{-4}$	$2 \cdot 10^{-3}$	$8 \cdot 10^{-4}$	$3 \cdot 10^{-3}$
5	$7 \cdot 10^{-7}$	$1 \cdot 10^{-4}$	$6 \cdot 10^{-8}$	$2 \cdot 10^{-6}$	$1 \cdot 10^{-7}$	$3 \cdot 10^{-6}$
10	$5 \cdot 10^{-8}$	$2 \cdot 10^{-5}$	$1 \cdot 10^{-9}$	$6 \cdot 10^{-8}$	$2 \cdot 10^{-9}$	$9 \cdot 10^{-8}$
25	$3 \cdot 10^{-11}$	$2 \cdot 10^{-6}$	$8 \cdot 10^{-12}$	$6 \cdot 10^{-10}$	$1 \cdot 10^{-11}$	$9 \cdot 10^{-10}$
50	$1 \cdot 10^{-11}$	$3 \cdot 10^{-7}$	$2 \cdot 10^{-13}$	$2 \cdot 10^{-11}$	$3 \cdot 10^{-13}$	$2 \cdot 10^{-11}$
75	$2 \cdot 10^{-13}$	$1 \cdot 10^{-7}$	$2 \cdot 10^{-14}$	$2 \cdot 10^{-12}$	$3 \cdot 10^{-14}$	$3 \cdot 10^{-12}$
100	$4 \cdot 10^{-13}$	$6 \cdot 10^{-8}$	$3 \cdot 10^{-15}$	$4 \cdot 10^{-13}$	$5 \cdot 10^{-14}$	$6 \cdot 10^{-13}$
150	$5 \cdot 10^{-14}$	$2 \cdot 10^{-8}$	$1 \cdot 10^{-16}$	$5 \cdot 10^{-14}$	$6 \cdot 10^{-16}$	$7 \cdot 10^{-14}$
200	$1 \cdot 10^{-14}$	$1 \cdot 10^{-8}$	$< 10^{-16}$	$1 \cdot 10^{-14}$	$2 \cdot 10^{-16}$	$1 \cdot 10^{-14}$

**Table 1.** Relative error of the modulus of the zeros from the asymptotic estimations, compared with numerical computations. The notation is as in the text. The number of non-zero coefficients of the asymptotic expansions considered is as follows: for  $|g_n|$  we take 2 coefficients for  $n = 1$  and 6 coefficients for the rest of values of  $n$ ; for  $|g'_n|$ ,  $|\chi_n|$ ,  $|\chi'_n|$ ,  $|\kappa_n|$  and  $|\kappa'_n|$  we take the first 4 non zero coefficients.

We now compare the approximate values of the first 10 zeros with the numerical values.

$n$	$g_n$ (asyp.)	$g_n$ (numer.)	$g'_n$ (asyp.)	$g'_n$ (numer.)
1	-0.70701728791 (2)	-0.73764033232	-2.26148803837 (4)	-2.24995421864
2	-3.40013324843 (2)	-3.39083150945	-4.08890415841 (4)	-4.08395408849
3	-4.75152465295 (3)	-4.75160079064	-5.50501788785 (4)	-5.50743021111
4	-6.22702978591 (5)	-6.22707083456	-6.78556344666 (4)	-6.78414405732
5	-7.33018484228 (5)	-7.33017070326	-7.93738558753 (4)	-7.93831371630
6	-8.53064462827 (5)	-8.53064781862	-9.02156063733 (4)	-9.02090166816
7	-9.50443871324 (5)	-9.50443547307	-10.0362185151 (4)	-10.0367106297
8	-10.5595675877 (5)	-10.5595678851	-11.0076119069 (4)	-11.0072288049
9	-11.4501841971 (5)	-11.4501830272	-11.9333405428 (4)	-11.9336474410
10	-12.4106527814 (5)	-12.4106527199	-12.8280143111 (4)	-12.8277622904

**Table 2.** Asymptotic estimations of the first 10 negative real zeros of  $\text{Gi}(x)$  and  $\text{Gi}'(x)$  versus their numerical value (12 digits). Between brackets, the number of the first non zero coefficients taken in the calculation of the asymptotic expansion is given.

Additionally,  $G'(x)$  has a positive zero:  $g' = 0.60907541707$ .

$n$	$\chi_n$ (asymptotic)	$\chi_n$ (numerical)
1	$2.44433318205 + i3.28043340740$	$2.44134455893 + i3.28073610375$
2	$3.82724470205 + i5.61364024656$	$3.82706907612 + i5.61368067243$
3	$4.94973090968 + i7.55292445144$	$4.94969805256 + i7.55293472024$
4	$5.94054868777 + i9.27655846564$	$5.94053866799 + i9.27656211688$
5	$6.84659373818 + i10.8567528445$	$6.84658973653 + i10.8567544432$
6	$7.69146765566 + i12.3317696540$	$7.69146576022 + i12.3317704591$
7	$8.48916873952 + i13.7249535559$	$8.48916772985 + i13.7249540039$
8	$9.24886878556 + i15.0518649809$	$9.24886819962 + i15.0518652495$
9	$9.97699441862 + i16.3235290835$	$9.97699405568 + i16.3235292542$
10	$10.6782722198 + i17.5481160856$	$10.6782719832 + i17.5481161992$

**Table 3.** Asymptotic estimations of the first 10 complex zeros of  $\text{Gi}(z)$  versus their numerical value (12 digits). The expansion is calculated using the first 4 coefficients.

$n$	$\chi'_n$ (asymptotic)	$\chi'_n$ (numerical)
1	$3.73104015614 + i3.20468169034$	$3.71910633591 + i3.20254922301$
2	$5.05878908159 + i5.49094064093$	$5.05721412684 + i5.49107967331$
3	$6.14094636445 + i7.40393823622$	$6.14051474537 + i7.40403247457$
4	$7.09863883359 + i9.11033272563$	$7.09847245342 + i9.11038169520$
5	$7.97658092867 + i10.6784124602$	$7.97650267337 + i10.6784393595$
6	$8.79711673037 + i12.1445154227$	$8.79707480551 + i12.1445312817$
7	$9.57340329298 + i13.5309191990$	$9.57337867420 + i13.5309291376$
8	$10.3140187866 + i14.8525432972$	$10.3140033118 + i14.8525498458$
9	$11.0249563206 + i16.1200042911$	$11.0249460691 + i16.1200087871$
10	$11.7106171211 + i17.3411995000$	$11.7106100405 + i17.3412026935$

**Table 4.** Asymptotic estimations of the first 10 complex zeros of  $\text{Gi}'(z)$  versus their numerical value (12 digits). The expansion is calculated using the first 4 coefficients.

$n$	$\kappa_n$ (asymptotic)	$\kappa_n$ (numerical)
1	$1.31810666758 + i3.93044374287$	$1.32022985770 + i3.92618518472$
2	$2.71758115521 + i6.25616826873$	$2.71776478546 + i6.25594658531$
3	$3.86688975856 + i8.18006876521$	$3.86692943374 + i8.18002987309$
4	$4.88317281362 + i9.88881442645$	$4.88318584624 + i9.88880304438$
5	$5.81193606777 + i11.4556905719$	$5.81194151066 + i11.4556861557$
6	$6.67695905797 + i12.9188986002$	$6.67696171311 + i12.9188965531$
7	$7.49263240875 + i14.3015594043$	$7.49263385258 + i14.3015583321$
8	$8.26849226103 + i15.6190187624$	$8.26849311165 + i15.6190181486$
9	$9.01126446313 + i16.8821242633$	$9.01126499607 + i16.8821238875$
10	$9.7259145503 + i18.0989041312$	$9.72591490090 + i18.0989038885$

**Table 5.** Asymptotic estimations of the first 10 complex zeros of  $\text{Hi}(z)$  versus their numerical value (12 digits). The expansion is calculated using the first 4 coefficients.

$n$	$\kappa'_n$ (asymptotic)	$\kappa'_n$ (numerical)
1	$0.61539789841 + i5.00682180461$	$0.62172976845 + i4.99069463707$
2	$2.00101984737 + i7.26100462042$	$2.00240099109 + i7.25911069430$
3	$3.14666657916 + i9.13725837677$	$3.14711339788 + i9.13677671663$
4	$4.16377885499 + i10.8089522433$	$4.16396557831 + i10.8087759627$
5	$5.09551443130 + i12.3455631789$	$5.09560639947 + i12.3454833919$
6	$5.96454826183 + i13.7832989849$	$5.96459897682 + i13.7832574947$
7	$6.78472063073 + i15.1440500844$	$6.78475098876 + i15.1440262976$
8	$7.56527902355 + i16.4423442321$	$7.56529836198 + i16.4423295732$
9	$8.31279229101 + i17.6884581451$	$8.31280522481 + i17.6884485950$
10	$9.03213892761 + i18.8900067299$	$9.03214792350 + i18.8900002276$

**Table 6.** Asymptotic estimations of the first 10 complex zeros of  $\text{Hi}'(z)$  versus their numerical value (12 digits). The expansion is calculated using the first 4 coefficients.

In all cases, as could be expected, the asymptotic estimations are closer to the true value as larger zeros (in modulus) are considered. Furthermore, as commented, the asymptotic estimations can be used as starting values to compute accurately the zeros of Scorer functions. The only exception is the positive real zero of  $\text{Gi}'(x)$ , which can not be estimated via the asymptotic expansions for the zeros.

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