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Recent Problems from Uniform Asymptotic Analysis of Integrals

In Particular in Connection with Tricomi's $\Psi-$Function

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ABSTRACT

The paper discusses asymptotic methods for integrals, in particular uniform approximations. We discuss several examples, where we apply the results to Tricomi's $\Psi-$function, in particular we consider an expansion of Tricomi-Carlitz polynomials in terms of Hermite polynomials. We mention other recent expansions for orthogonal polynomials that do not satisfy a differential equation, and for which methods based on integral representations produce powerful uniform asymptotic expansions.

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1. Tricomi's $\Psi-$function

Tricomi (1954a, page 56) introduced the $\Psi-$function as the second solution of the confluent hypergeometric differential equation (also called Kummer's equation)

$$z\,\frac{d^2y}{dz^2}+(c-z)\,\frac{dy}{dz}-a\,y=0. \tag{1.1}$$

Tricomi denoted the first solution by $\Phi(a,c;z)$, which in fact is a hypergeometric function, given by

$${}_1F_1(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\,\frac{z^n}{n!}, \tag{1.2}$$

with the usual condition $c \neq 0, -1, -2, \ldots$. ${}_1F_1(a,c;z)$ is an entire function of z. The symbol $(a)_n$ is the shifted factorial (Pochhammer's symbol)

$$(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1. \tag{1.3}$$

It is not difficult to verify that $z^{1-c}{}_1F_1(a-c+1, 2-c; z)$ is also a solution of (1.1).

Tricomi denoted the second solution of the Kummer equation (1.1) by $\Psi(a,c;z)$ $^{*)}$, and it is defined as a linear combination of the two ${}_1F_1$–solutions:

$$\Psi(a,c;z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a,c;z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c}{}_1F_1(a-c+1, 2-c; z). \tag{1.4}$$

The Kummer equation (1.1) arises in many problems of mathematical physics. The confluent hypergeometric functions ${}_1F_1(a,c;z), \Psi(a,c;z)$ are also called Kummer functions.

A different introduction of equation (1.1) is based on a limiting method applied to the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$, which is a solution of the differential equation

$$z(1-z)\,y'' + [c-(a+b+1)z]\,y' - ab\,y = 0, \tag{1.5}$$

and which has the series representation

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!}\,z^n, \quad |z| < 1. \tag{1.6}$$

This equation has three regular singular points $z = 0, z = 1, z = \infty$. The Kummer functions arise when two of the regular singular points are allowed to merge into one singular point. Formally this process runs as follows. The function ${}_2F_1(a,b;c;z/b)$ has a regular singular point at $z = b$. Using the series in (1.6) it can be verified that the limit

$$\lim_{b\to\infty} {}_2F_1(a,b;c;z/b)$$

exists, and equals the series in (1.2). It can also be verified that in the same limiting process the Gauss hypergeometric differential equation (1.5) transforms into (1.1). This explains the name *confluent hypergeometric functions* for the Kummer functions.

The basic integral representation reads

$${}_1F_1(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt}t^{a-1}(1-t)^{c-a-1}\,dt, \quad \Re a > 0, \quad \Re(c-a) > 0. \tag{1.7}$$

The second solution can also be defined by an integral

$$\Psi(a;c;z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}\,t^{a-1}\,(1+t)^{c-a-1}\,dt, \quad \Re a > 0, \quad \Re z > 0. \tag{1.8}$$

$^{*)}$ Several notations for the Kummer functions are used in the literature; we prefer the notation ${}_1F_1(a,c;z)$ for the first solution; in honor of Tricomi, we use $\Psi(a,c;z)$ for the second solution.

This function is, in general, not analytic at the origin $z = 0$. The integral can be used for analytic continuation with respect to z into the domain $\{|\mathrm{ph}\ z| < \pi, z \neq 0\}$, by turning the path of integration. If $a = 0, -1, -2, \ldots$, $\Psi(a, c; z)$ is a polynomial in z, if $c - a - 1 = n$ (non-negative integer), $\Psi(a, c; z)$ can be expressed as a polynomial in z multiplied with z^{-a-n}.

There are a remarkable functional relations:

$$\begin{aligned} {}_1F_1(a, c; z) &= e^z\, {}_1F_1(c - a, c; -z), \\ \Psi(a, c; z) &= z^{1-c}\, \Psi(a - c + 1, 2 - c; z). \end{aligned} \tag{1.9}$$

Contour integrals are given by

$$\begin{aligned} {}_1F_1(a, c; z) &= \frac{\Gamma(c)}{2\pi i} \int_{F} e^s\, s^{a-c}\, (s - z)^{-a}\, ds, \\ \Psi(a, c; z) &= \frac{\Gamma(c - a)}{2\pi i} \int_{\mathcal{L}_\Psi} e^s\, s^{a-c}\, (z - s)^{-a}\, ds, \end{aligned} \tag{1.10}$$

where, if $z > 0$, $\mathcal{L}_F$ is a vertical line in the half plane $\Re s > z$, and $\mathcal{L}_\Psi$ is a vertical line that cuts the real axis between the origin and z. When z is complex, the contours need to be modified appropriately. In order to speed up convergence, the contours may be deformed into parabola shaped contours that terminate at $-\infty$. The contour integrals are more flexible in asymptotic analysis than the standard integrals given in (1.4) and (1.7).

1.1. Special cases of the Kummer functions

There are many special cases. We mention the most important ones, not only to demonstrate the importance of the Kummer function, but also for easy reference in later sections.

[1] *Error functions.* The definitions are

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \quad \operatorname{erfc} z = 1 - \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt. \tag{1.11}$$

The relations with the Kummer functions are

$$\operatorname{erf} z = z\, {}_1F_1\left(\tfrac{1}{2}, \tfrac{3}{2}; -z^2\right), \quad \operatorname{erfc} z = e^{-z^2}\, \Psi\left(\tfrac{1}{2}, \tfrac{1}{2}; z^2\right).$$

In physics the plasma dispersion function is used. The definition is

$$w(z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - z}\, dt, \quad \Im z > 0. \tag{1.12}$$

In asymptotics this function is important because of the pole at $t = z$ and the Gaussian function with the saddle point at the origin. It is not difficult to verify that

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz). \tag{1.13}$$

and that we have the symmetry relations

$$\operatorname{erf}(-z) = -\operatorname{erf} z, \quad \operatorname{erfc}(-z) = 2 - \operatorname{erfc} z, \quad w(-z) = 2e^{-z^2} - w(z). \tag{1.14}$$

[2] *Exponential integrals.* For $\nu \in \mathbb{C}$ we have

$$E_\nu(z) = \int_1^\infty \frac{e^{-zt}}{t^\nu}\, dt, \quad \Re z > 0. \tag{1.15}$$

The relation with the $\Psi-$function is

$$E_\nu(z) = e^{-z}\Psi(1, 2-\nu; z) = z^{\nu-1}e^{-z}\Psi(\nu, \nu; z). \tag{1.16}$$

The latter gives

$$E_\nu(z) = \frac{z^{\nu-1}e^{-z}}{\Gamma(\nu)} \int_0^\infty \frac{e^{-zt}t^{\nu-1}}{t+1}\, dt, \quad \Re z > 0.$$

[3] *Fresnel integrals.* These are

$$C(z) = \sqrt{\tfrac{2}{\pi}} \int_0^z \cos t^2\, dt, \quad S(z) = \sqrt{\tfrac{2}{\pi}} \int_0^z \sin t^2\, dt. \tag{1.17}$$

The t^2 in the circular functions suggests a relation with the error functions. Indeed we have:

$$C(z) + iS(z) = \frac{1+i}{2} \operatorname{erf} \frac{(1-i)z}{\sqrt{2}}.$$

[4] *Incomplete gamma functions.* The definitions are

$$\gamma(a, z) = \int_0^z t^{a-1}e^{-t}\, dt, \quad \Gamma(a, z) = \int_z^\infty t^{a-1}e^{-t}\, dt. \tag{1.18}$$

For $\gamma(a, z)$ we assume the condition $\Re a > 0$; with respect to z we assume $|\operatorname{ph} z| < \pi$. In probability theory these functions show up in connection with the gamma distribution. In this area of applications the normalizations

$$P(a, z) = \frac{\gamma(a, z)}{\Gamma(a)}, \quad Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)} \tag{1.19}$$

are frequently used, which satisfy $P(a, z) + Q(a, z) = 1$.
The relations with the Kummer functions are as follows:

$$\begin{aligned}
\gamma(a, z) &= a^{-1}z^a e^{-z}\,{}_1F_1(1, a+1; z) \\
&= a^{-1}z^a\,{}_1F_1(a, a+1; -z), \\
\Gamma(a, z) &= z^a e^{-z}\Psi(1, a+1; z) \\
&= e^{-z}\Psi(1-a, 1-a; z).
\end{aligned}$$

[5] *Bessel functions.* These functions arise when in ${}_1F_1(a,c;z)$ and $\Psi(a,c;z)$ the parameters satisfy $c=2a$. Two important relations are

$$\begin{aligned}
J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\, e^{-iz}\; {}_1F_1\left(\nu+\tfrac{1}{2}, 2\nu+1; 2iz\right), \\
K_\nu(z) &= \sqrt{\pi}\; e^{-z}\,(2z)^\nu\, \Psi\left(\nu+\tfrac{1}{2}, 2\nu+1; 2z\right).
\end{aligned} \tag{1.20}$$

The latter is a modified Bessel function.

[6] *Orthogonal polynomials.* The Hermite and Laguerre polynomials are special cases of the confluent hypergeometric functions. For these polynomials we have

$$\begin{aligned}
L_n^{(\alpha)}(x) &= \frac{(-1)^n}{n!}\Psi(-n,\alpha+1;x) = \binom{n+\alpha}{n}{}_1F_1(-n,\alpha+1;x) \\
&= \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \\
H_n(x) &= x\,2^n\,\Psi\left(\tfrac{1}{2}-\tfrac{1}{2}n, \tfrac{3}{2}; x^2\right) \\
&= n!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k}{k!\,(n-2k)!}(2x)^{n-2k}.
\end{aligned} \tag{1.21}$$

[7] *Parabolic cylinder functions.* The solutions of the differential equation

$$y'' + \left(z^2+pz+q\right)y = 0$$

are called parabolic cylinder functions or Weber parabolic cylinder functions. Another standard form is

$$y'' - \left(a+\tfrac{1}{4}z^2\right)y = 0. \tag{1.22}$$

All solutions are entire functions of z. The following even and odd solutions exist:

$$\begin{aligned}
y_1 &= \; e^{-z^2/4}{}_1F_1\left(\tfrac{1}{2}a+\tfrac{1}{4}, \tfrac{1}{2}; \tfrac{1}{2}z^2\right) = \; e^{z^2/4}{}_1F_1\left(-\tfrac{1}{2}a+\tfrac{1}{4}, \tfrac{1}{2}; -\tfrac{1}{2}z^2\right), \\
y_2 &= ze^{-z^2/4}{}_1F_1\left(\tfrac{1}{2}a+\tfrac{3}{4}, \tfrac{3}{2}; \tfrac{1}{2}z^2\right) = ze^{z^2/4}{}_1F_1\left(-\tfrac{1}{2}a+\tfrac{3}{4}, \tfrac{3}{2}; -\tfrac{1}{2}z^2\right).
\end{aligned}$$

The following pair is usually found in the literature:

$$\begin{aligned}
U(a,z) &= \sqrt{\pi}\;2^{-1/4-a/2}\left[\frac{y_1}{\Gamma\left(\frac{3}{4}+\frac{a}{2}\right)} - \frac{\sqrt{2}\,y_2}{\Gamma\left(\frac{1}{4}+\frac{a}{2}\right)}\right] \\
&= 2^{-3/4-a/2}e^{-\frac{1}{4}z^2}\,z\,U\left(\tfrac{3}{4}+\tfrac{a}{2}, \tfrac{3}{2}, \tfrac{1}{2}z^2\right), \\
V(a,z) &= \frac{1}{\pi}\Gamma\left(\tfrac{1}{2}+a\right)\left[\sin\pi a\,U(a,z)+U(a,-z)\right].
\end{aligned} \tag{1.23}$$

In the notation of Whittaker we have $D_\nu(z) = U\left(-\nu - \frac{1}{2}, z\right)$. When $a = -1/2, -3/2, -5/2, \ldots$ the Hermite polynomials arise:

$$H_n(z) = 2^{\frac{1}{2}n} e^{\frac{1}{2}z^2} U\left(-n - \tfrac{1}{2}, z\sqrt{2}\right) = 2^{\frac{1}{2}n} e^{\frac{1}{2}z^2} D_n(z\sqrt{2}). \tag{1.24}$$

[8] *Coulomb Wave Functions.* The differential equation

$$w'' + \left[1 - \frac{2\eta}{\rho} - \frac{\lambda(\lambda+1)}{\rho^2}\right] w = 0 \tag{1.25}$$

is a special form of Kummer's equation. It plays an important part in physics, in particular in quantum mechanics as a form of the Schrödinger equation in a central Coulomb field. The solutions of (1.25) are called Coulomb wave functions, and are usually denoted by $F_\lambda(\eta,\rho)$, $G_\lambda(\eta,\rho)$. We give the relations with the Kummer functions:

$$\begin{aligned}
F_\lambda(\eta,\rho) &= A\,{}_1F_1(\lambda+1-i\eta, 2\lambda+2, 2i\rho),\\
G_\lambda(\eta,\rho) &= iF_\lambda(\eta,\rho) + iB\,\Psi(\lambda+1-i\eta, 2\lambda+2, 2i\rho),\\
A &= \frac{|\Gamma(\lambda+1+i\eta)| e^{-\pi\eta/2 - i\rho}(2\rho)^{\lambda+1}}{2\Gamma(2\lambda+2)},\\
B &= e^{\pi\eta/2+\lambda\pi i - i\sigma_\lambda - i\rho}(2\rho)^{\lambda+1},\\
\sigma_\lambda &= \mathrm{ph}\{\Gamma(\lambda+1+i\eta)\} \quad \text{(the Coulomb phase shift)}.
\end{aligned}$$

The functions $F_\lambda(\eta,\rho)$ and $G_\lambda(\eta,\rho)$ are real for real values of η, $\rho > 0, \lambda \geq 0$.

[9] *Whittaker functions.* Finally we mention a different notation. In the literature an alternative pair for the confluent hypergeometric functions is given, called the Whittaker functions. The definitions are

$$\begin{aligned}
M_{\kappa,\mu}(z) &= e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu}\,{}_1F_1\left(\tfrac{1}{2}+\mu-\kappa, 1+2\mu; z\right),\\
W_{\kappa,\mu}(z) &= e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu}\,\Psi\left(\tfrac{1}{2}+\mu-\kappa, 1+2\mu; z\right).
\end{aligned} \tag{1.26}$$

$M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$ satisfy the Whittaker equation

$$w'' + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right) w = 0. \tag{1.27}$$

There is a vast literature on Kummer functions. The books of Buchholz (1969), Slater (1960) and Tricomi (1954a) are exclusively devoted to the class of confluent hypergeometric functions or Whittaker functions. Especially in the first book many references are given to physical applications.

2. Asymptotic expansions of Laplace-type integrals

We mention a very useful result from the theory of asymptotics for Laplace integrals, known as Watson's Lemma. First we give a definition of an asymptotic expansion.

2.1. Definition and example

Definition 1. Let F be function of a real or complex variable z; let $\sum_{n=0}^{\infty} a_n z^{-n}$ denote a (convergent or divergent) formal power series, of which the sum of the first n terms is denoted by $S_n(z)$; let

$$R_n(z) = F(z) - S_n(z).$$

That is,

$$F(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_{n-1}}{z^{n-1}} + R_n(z), \quad n = 0, 1, 2 \ldots, \tag{2.1}$$

where we assume that when $n = 0$ we have $F(z) = R_0(z)$. Next, assume that for each $n = 0, 1, 2, \ldots$ the following relation holds

$$R_n(z) = \mathcal{O}\left(z^{-n}\right), \quad \text{as} \quad z \to \infty \tag{2.2}$$

in some unbounded domain Δ. Then $\sum_{n=0}^{\infty} a_n z^{-n}$ is called an asymptotic expansion of the function $F(z)$ and we denote this by

$$F(z) \sim \sum_{n=0}^{\infty} a_n z^{-n}, \quad z \to \infty, \quad z \in \Delta. \tag{2.3}$$

This definition is due to Poincaré (1886). Analogous definitions can be given for $z \to 0$, and so on.

Observe that we do not assume that the infinite series $\sum_{n=0}^{\infty} a_n z^{-n}$ converges for certain $z-$values. This is not relevant in asymptotics; in the definition only a property of $R_n(z)$ is requested, with n fixed.

2.2. Watson's Lemma

Watson's lemma is usually the first step in asymptotics of integrals.

Theorem 2.1. *(Watson's lemma). Assume that:*

(i) *$f(t)$ is a real or complex function of the positive real variable t with a finite number of discontinuities and infinities.*

(ii) *As $t \to 0+$*

$$f(t) \sim t^{\lambda-1} \sum_{n=0}^{\infty} a_n t^n, \quad \Re\lambda > 0. \tag{2.4}$$

(iii) *The integral*

$$F(z) = \int_0^{\infty} f(t) e^{-zt}\, dt \tag{2.5}$$

is convergent for sufficiently large values of $\Re z$.

Then

$$F(z) \sim \sum_{n=0}^{\infty} \Gamma(n+\lambda) \frac{a_n}{z^{n+\lambda}}, \quad z \to \infty \tag{2.6}$$

in the sector

$$|\text{ph } z| \le \frac{1}{2}\pi - \delta(< \frac{1}{2}\pi),$$

where $z^{n+\lambda}$ *has its principal value.*

A larger $z-$sector can be obtained when we know that f is analytic in a certain domain of the complex plane. For example, when f is analytic in the sector $|\text{ph } t| < \pi/2$ and $f(t) = \mathcal{O}[\exp(\sigma|t|)]$ in that sector, for some number σ, then the asymptotic expansion in Watson's lemma holds in the sector

$$|\text{ph } z| \le \pi - \delta(< \pi).$$

For a proof we refer to OLVER (1974, page 113), where a more general condition (ii) is assumed.

When applying Watson's lemma in the theory of special functions, condition (i) often holds, since the function $f(t)$ is, up to the factor $t^{\lambda-1}$, usually an analytic function in a domain containing $[0,\infty)$. Compare the definition of the $\Psi-$function in (1.8), where $f(t) = t^{a-1}(1+t)^{c-a-1}$. In that case $f(t)$ is analytic in the sector $|\text{ph } t| < \pi$.

Next we formulate a second theorem in which a much larger domain than in the previous theorem for the phase of the large parameter z is possible. For a proof we refer to OLVER (1974, page 114).

Theorem 2.2. *Assume that:*

(i) *$f(t)$ is analytic inside a sector $\Omega\colon \alpha_1 < \text{ph } t < \alpha_2$, where $\alpha_1 < 0$ and $\alpha_2 > 0$.*

(ii) *For each $\delta \in (0, \frac{1}{2}\alpha_2 - \frac{1}{2}\alpha_1)$ (2.4) holds as $t \to 0$ in the sector*

$$\Omega_\delta\colon \alpha_1 + \delta < \text{ph } t < \alpha_2 - \delta;$$

for λ we again assume that $\Re\lambda > 0$.

(iii) *There is a real number σ such that $f(t) = \mathcal{O}(e^{\sigma|t|})$ as $t \to \infty$ in Ω_δ.*

Then the integral (2.5), or its analytic continuation, has the asymptotic expansion (2.6) in the sector

$$-\alpha_2 - \tfrac{1}{2}\pi + \delta \le \text{ph } z \le -\alpha_1 + \tfrac{1}{2}\pi - \delta. \tag{2.7}$$

Using Theorem 2.2 for the $\Psi-$function, we obtain

$$\Psi(a,c;z) \sim z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n(a-c+1)_n}{n!}(-z)^{-n}, \quad z \to \infty. \tag{2.8}$$

which holds for $|\text{ph } z| < 3\pi/2$. By using the integral in (1.7) with a change of variable $t \to 1-t$, that is,

$${}_1F_1(a,c;z) = \frac{\Gamma(c)\, e^z}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{c-a-1}(1-t)^{a-1}\, dt, \tag{2.9}$$

we obtain for the ${}_1F_1$−function the result

$$ {}_1F_1(a,c;z) \sim \frac{\Gamma(c) e^z\, z^{a-c}}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!}\, z^{-n}, \quad z\to\infty, \tag{2.10} $$

which is valid in the sector $|\mathrm{ph}\ z| < \frac12\pi$. The limited domain of validity is due to the singularity of the integrand in (2.9) at $t=1$. To extend the domain we need a different integral. For example, we can replace the interval $(0,1)$ in (2.9) with two intervals $(0,\infty)$ and $(1,\infty)$, where the point at infinity can be chosen above the the branch line $(1,+\infty)$ or below, depending on the phase of z. The result is

$$ \begin{aligned} \frac{1}{\Gamma(c)}\ {}_1F_1(a,c;z) \sim\ & \frac{e^z\, z^{a-c}}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!}\, z^{-n} \\ & + \frac{e^{\pm\pi i a}\, z^{-a}}{\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_n (1+a-c)_n}{n!}\, (-z)^{-n}, \end{aligned} \tag{2.11} $$

where the upper sign is taken if $-\frac12\pi < \mathrm{ph}\ z < \frac32\pi$ and the lower sign if $-\frac32\pi < \mathrm{ph}\ z < \frac12\pi$. The first part is dominant when $\Re z > 0$ and corresponds with (2.10); the second part becomes dominant when z enters the half plane $\Re z < 0$.

2.3. A class of polynomials introduced by Tricomi

In Tricomi (1951) a class of polynomials has been introduced. Tricomi used the polynomials in convergent and asymptotic expansions. The definition can be given by using Laguerre polynomials (see (1.21)):

$$ l_n(x) = (-1)^n\, L_n^{(x-n)}(x) = \sum_{k=0}^{n} (-1)^k \binom{x}{k} \frac{x^{n-k}}{(n-k)!}, \tag{2.12} $$

which, although closely related to the Laguerre polynomials, are essentially different from them. For instance, the degree of $l_n(x)$ is not n but the greatest integer in $\frac12 n$. The first few polynomials are

$$ l_0(x)=1,\quad l_1(x)=0,\quad l_2(x)=-\tfrac12 x,\quad l_3(x)=-\tfrac13 x,\quad l_4(x)=\tfrac18 x^2-\tfrac14 x. $$

The polynomials show up in the generating function

$$ e^{xz}\,(1-z)^x = \sum_{n=0}^{\infty} l_n(x)\, z^n, \quad |z|<1. \tag{2.13} $$

This relation is easily verified by expanding both the exponential and binomial function in the left-hand side, and by comparing the coefficients in the product with (2.12). There is a simple recursion relation:

$$ (n+1)l_{n+1}(x) = n l_n(x) - x l_{n-1}(x), \quad n=1,2,\ldots\,, \tag{2.14} $$

which can be derived from the generating function.

Tricomi mentions two applications. First, for the ${}_1F_1$−function there is

$$\frac{1}{\Gamma(c)}\,{}_1F_1(a,c;x) = \sum_{n=0}^{\infty} l_n(-a)\,x^n\,J^*_{c+n-1}(-ax),$$

where

$$J^*_\nu(z) = z^{-\nu/2}\,J_\nu\left(2\sqrt{z}\,\right) = \sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(n+\nu+1)\;n!},$$

which is an entire function of z. For the incomplete gamma function there is an asymptotic expansion:

$$\Gamma(\alpha+1,x) \sim -e^{-x}\,x^{\alpha+1}\sum_{n=0}^{\infty} n!\,l_n(\alpha)\,(\alpha-x)^{-n-1},$$

as $\zeta = \sqrt{x-\alpha}\,/(\alpha) \to \infty$, within the sector $-3\pi/4 < \mathrm{ph}(\zeta) < 3\pi/4$.

Also Berg (1959, 1962 and 1977) and Riekstins (1982) used the polynomials in asymptotic problems. In Temme (1983 and 1985) we used the polynomials for obtaining uniform asymptotic expansions of Laplace integrals. In Section 4 we consider a generalization of the Tricomi polynomials.

We explain how the polynomials defined in (2.12) can be used in uniform expansions of Laplace integrals and apply the method to the Tricomi Ψ−function and the ${}_1F_1$−function.

2.4. Uniform expansions of Laplace-type integrals

We consider the Laplace integral

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\,e^{-zt}\,f(t)\,dt, \tag{2.15}$$

where $\Re z > 0, \Re\lambda > 0$ and z is a large parameter. We are interested in the case that λ is large as well.

When λ is restricted to a bounded set in the complex plane, an expansion of $F_\lambda(z)$ can be obtained by using Watson's lemma. When we assume that f is analytic at $t=0$ we obtain by Theorem 1:

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad\Longrightarrow\quad F_\lambda(z) \sim \sum_{n=0}^{\infty}(\lambda)_n\,a_n z^{-n-\lambda}, \tag{2.16}$$

as $z\to\infty$ in the sector $|\mathrm{ph}\,z| \le \frac12\pi - \delta < \frac12\pi$.

The expansion (2.16) loses its asymptotic character when λ is large. For instance, if $\lambda = \mathcal{O}(z)$, then the ratio of consecutive terms in the asymptotic expansion satisfy

$$\frac{a_{n+1}}{a_n}\,\frac{n+\lambda}{z} = \mathcal{O}(1) \quad \text{if} \quad a_n \neq 0.$$

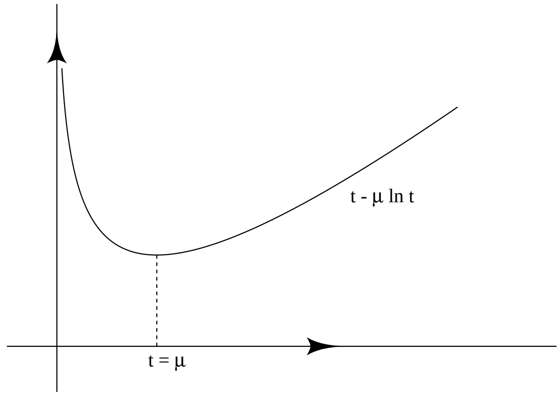

Figure 2.1. The function $\exp[-z(t-\mu\ln t)]$ has a saddle point at $t=\mu$.

In TEMME (1983) we have modified Watson's lemma to obtain an expansion in which large as well small values of λ are allowed. This expansion is obtained by expanding f at $t=\mu:=\lambda/z$, at which point the dominant part of the integrand of (2.15), that is, $t^\lambda\, e^{-zt}$, attains its maximal value (considering real parameters at the moment). We write

$$f(t)=\sum_{n=0}^\infty a_n(\mu)(t-\mu)^n,$$

and obtain by substituting this into (2.15) the formal result

$$F_\lambda(z)\sim\sum_{n=0}^\infty a_n(\mu)P_n(\lambda)z^{-n-\lambda},\quad z\to\infty, \tag{2.17}$$

where

$$P_n(\lambda)=\frac{z^{n+\lambda}}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\,e^{-zt}\,(t-\mu)^n\,dt,\quad \mu=\lambda/z. \tag{2.18}$$

The functions $P_n(\lambda)$ are polynomials in λ. From (2.18) the recursion $P_{n+1}(\lambda)=n[P_n(\lambda)+\lambda P_{n-1}(\lambda)]$ follows with initial values $P_0(\lambda)=1, P_1(\lambda)=0$. An explicit formula follows from expanding $(t-\mu)^n$ in (2.18), which gives

$$P_n(\lambda)=\sum_{k=0}^n\binom{n}{k}\,(\lambda)_k\,(-\lambda)^{n-k}.$$

Comparing these properties with those of the Tricomi polynomials $l_n(x)$, we find that

$$P_n(\lambda)=n!\,l_n(-\lambda),\quad n=0,1,2,\ldots\ .$$

The nature of the expansion (2.17) is discussed in TEMME (1983 and 1985). Under rather mild conditions on f it follows that the expansion (2.17) holds uniformly with respect to $\lambda\in[0,\infty)$, and in domains of the complex plane.

We can apply this method to Tricomi's $\Psi-$function for the case that $z \to \infty$, to obtain an alternative of (2.8). For the new expansion we write

$$f(t) = (1+t)^{c-a-1} = \sum_{n=0}^{\infty} a_n(\mu)(t-\mu)^n, \quad a_n(\mu) = \binom{c-a-1}{n} (1+\mu)^{c-a-1-n},$$

where $\mu = a/z$. This gives

$$\Psi(a,c;z) \sim \sum_{n=0}^{\infty} a_n(\mu)\, P_n(a)\, z^{-n-a}, \quad z \to \infty, \tag{2.19}$$

uniformly with respect to $a \in [0,\infty)$; c should be of comparable size of a. We need the condition $c - a = \mathcal{O}(1)$.

We see that for $\mu \to 0$ the expansion reduces to (2.8); if μ becomes large the asymptotic convergence improves. If $c = a$ the expansion becomes rather simple:

$$\Psi(a,a;z) \sim z^{1-a} \sum_{n=0}^{\infty} (-1)^n \frac{P_n(a)}{(z+a)^{n+1}}, \quad z \to \infty, \tag{2.20}$$

which is an expansion for the exponential integral (cf. (1.16)). This example and (2.19) show quite well why large values of $\lambda = a$ are allowed: the degree of $P_n(a)$ equals $[n/2]$, and the effect of $P_n(a)$ is amply absorbed by the term $(z+a)^{-n-1}$. Another feature is that (2.17) holds for $\lambda \to \infty$, uniformly with respect to z, say $z \geq z_0 > 0$.

A similar method is available for ${}_1F_1(a,c;z)$ if we use the contour integral in (1.10). We have

$${}_1F_1(a+1,c;z) = \frac{z^{1-c}\, e^z\, \Gamma(c)}{2\pi i} \int_{\mathcal{L}} e^{zw}\, (1+w)^{a+1-c}\, w^{-a-1}\, dw.$$

Expanding

$$(1+w)^{a+1-c} = \sum_{n=0}^{\infty} b_n(\mu)(w-\mu)^n, \quad b_n(\mu) = \binom{a+1-c}{n} (1+\mu)^{a+1-c-n}, \quad \mu = a/z,$$

we obtain

$${}_1F_1(a+1,c;z) \sim \frac{z^{a+1-c}\, e^z\, \Gamma(c)}{\Gamma(a+1)} \sum_{n=0}^{\infty} b_n(\mu)\, Q_n(a)\, z^{-n}, \tag{2.21}$$

where

$$Q_n(a) = \frac{z^{n-a}\Gamma(a+1)}{2\pi} \int_{\mathcal{L}} e^{zw}\, (w-\mu)^n\, w^{-a-1}\, dw.$$

By expanding $(w-\mu)^n$ it easily follows that $Q_n(a) = (-1)nP_n(-a)$. The expansion in (2.21) can be viewed as an alternative for (2.10), and holds for $z \to \infty$, uniformly with respect to $a \in [0,\infty)$, with $c - a = \mathcal{O}(1)$.

3. Uniform asymptotic expansions in terms of Bessel functions

Tricomi has derived several convergent expansions of the ${}_1F_1-$function in terms of Bessel functions that are useful for evaluating the function when the parameters are large. For example, we have

$$ {}_1F_1(a,c;z) = e^{\frac{1}{2}z}\Gamma(c)(\kappa z)^{(1-c)/2}\sum_{n=0}^{\infty} A_n(\kappa, c/2)\left(\frac{z}{4\kappa}\right)^{n/2} J_{c-1+n}\left(2\sqrt{\kappa z}\right), \tag{3.1} $$

where $\kappa = c/2 - a$ and the $A_n(\kappa,\lambda)$ are coefficients in the generating function

$$ e^{2\kappa z}(1-z)^{\kappa-\lambda}(1+z)^{-\kappa-\lambda} = \sum_{n=0}^{\infty} A_n(\kappa,\lambda)\, z^n. $$

The series in (3.1) is convergent in the entire $z-$plane. Moreover, it can be used for the evaluation of ${}_1F_1(a,c;z)$ for large κ. For further details on these expansions we refer to TRICOMI (1954a).

The expansion in (3.1) may be compared with an asymptotic expansion of the Whittaker function $M_{\kappa,\mu}(x)$ (cf. (1.26)) as given in OLVER (1974 & 1997, page 446). Olver used the differential equation to derive an expansion in terms of $J-$Bessel functions, with the same argument $2\sqrt{\kappa z}$ as in (3.1), which is provided with error bounds for the remainder in the expansion. Several other expansions are given by Olver, also for the function $W_{\kappa,\mu}(x)$. In OLVER (1980) an expansion for the Whittaker functions is given in terms of parabolic cylinder functions (cf. (1.23)). DUNSTER (1989) has developed uniform expansions for the Whittaker functions in terms of Bessel functions and Airy functions. All these approaches are based on differential equations; they are valid for large domains of the complex parameters, and supplied with error bounds.

In TEMME (1990a) we have given an approach based on integral representations for obtaining a uniform asymptotic expansion in terms of the modified Bessel function $K_\nu(z)$, with an application to the $\Psi-$function. The standard form for deriving the expansion is the integral

$$ F_\lambda(z,\alpha) = \int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t} f(t)\, dt, \tag{3.2} $$

which reduces to a modified Bessel function in the case that f is a constant. We have

$$ 2(\alpha/z)^{\lambda/2} K_\lambda(2\sqrt{\alpha z}\,) = \int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}\, dt. \tag{3.3} $$

The integral in (3.2) is considered with $\alpha, \lambda \ge 0$ and large positive values of z. We aim to derive asymptotic expansions for $F_\lambda(z,\alpha)$ that hold uniformly with respect to both α and λ in the interval $[0,\infty)$. To handle the transition of the case $\alpha = 0$ to $\alpha > 0$, the modified Bessel function (3.3) is needed. Observe that when $\alpha = 0$ the essential singularity in the integrand of (3.2) disappears and that (3.2) becomes a more familiar Laplace integral, that can be expanded by using Watson's lemma.

3.1. Construction of the formal series

The first step in constructing a uniform asymptotic expansion of (3.2) is the substitution

$$f(t) = a_0 + b_0(t-\beta) + (t - \beta^2/t)g(t), \tag{3.4}$$

where a_0, b_0 follow from substitution of $t = \pm\beta$. We have

$$a_0 = f(\beta), \quad b_0 = \frac{1}{2\beta}[f(\beta) - f(-\beta)].$$

Inserting (3.4) into (3.2) we obtain

$$F_\lambda(z,\alpha) = a_0 A_\lambda(z,\beta) + b_0 B_\lambda(z,\beta) + F_\lambda^{(1)}(z,\alpha),$$

where A_λ, B_λ are combinations of the modified Bessel functions introduced in (3.3). It is straightforward to verify that

$$A_\lambda(z,\beta) = 2\beta^\lambda K_\lambda(2\beta z), \quad B_\lambda(z,\beta) = 2\beta^{\lambda+1}[K_{\lambda+1}(2\beta z) - K_\lambda(2\beta z)]. \tag{3.5}$$

An integration by parts gives

$$\begin{aligned} F_\lambda^{(1)}(z,\alpha) &= -\frac{1}{z}\int_0^\infty t^\lambda g(t)\, d\, e^{-z(t+\beta^2/t)} \\ &= \frac{1}{z}\int_0^\infty t^{\lambda-1} e^{-z(t+\beta^2/t)} f_1(t)\, dt \end{aligned}$$

with

$$f_1(t) = t^{1-\lambda}\frac{d}{dt}[t^\lambda g(t)] = \lambda g(t) + t g'(t).$$

We see that $zF_\lambda^{(1)}(z,\alpha)$ is of the same form as $F_\lambda(z,\alpha)$. The above procedure can now be applied to $zF_\lambda^{(1)}(z,\alpha)$ and we obtain for (3.2) the formal expansion

$$F_\lambda(z,\alpha) \sim A_\lambda(z,\beta)\sum_{s=0}^\infty a_s z^{-s} + B_\lambda(z,\beta)\sum_{s=0}^\infty b_s z^{-s}, \quad \text{as} \quad z\to\infty, \tag{3.6}$$

where we define inductively $f_0 = f, g_0 = g$ and for $s = 1, 2, \ldots$

$$\begin{aligned} f_s(t) &= t^{1-\lambda}\frac{d}{dt}[t^\lambda g_{s-1}(t)] = a_s + b_s(t-\beta) + (t-\beta^2/t)g_s(t), \\ a_s &= f_s(\beta), \quad b_s = \frac{1}{2\beta}[f_s(\beta) - f_s(-\beta)]. \end{aligned} \tag{3.7}$$

For this procedure we need function values of f and derivatives at negative values, although the integral(3.2)is defined only for $t-$values in $[0,\infty)$. When we consider analytic functions f, as we do when dealing with special functions, we assume that f is analytic in a domain Ω in the complex plane that contains the real line.

3.2. Application to Tricomi's $\Psi-$function

We start the definition given in (3.7):

$$\Gamma(a)\Psi(a,c;x) = \int_0^\infty u^{a-1}(1+u)^{c-a-1}e^{-xu}\,du. \tag{3.8}$$

We consider a as the large parameter and x as a uniformity parameter in $[0,\infty)$; c is a fixed real parameter. We take $c \le 1$; the relation (3.8) can be used when $c > 1$.

First we give a preliminary transformation. The function $[u/(u+1)]^a$ assumes its maximal value (on $[0,\infty)$) at $u=\infty$. This function controls the asymptotic behavior of the integrand and, hence, we transform it to an exponential function by writing $u/(u+1) = \exp(-w)$. Then (3.8) becomes

$$\Gamma(a)\Psi(a,1-\lambda,x) = \int_0^\infty w^{\lambda-1}e^{-aw-x/(e^w-1)}\widetilde{f}(w)\,dw, \tag{3.9}$$

where

$$\widetilde{f}(w) = \left[\frac{1-e^{-w}}{w}\right]^{\lambda-1}. \tag{3.10}$$

We transform (3.9) into (3.2) with the help of the transformation

$$w + \frac{\nu}{e^w-1} = t + \frac{\beta^2}{t} + A, \tag{3.11}$$

where $\nu = x/a$ and β, A are to be determined. We compute them on the following condition on the mapping: the critical points of the $w-$function in (3.11) must correspond with the critical values of the $t-$function. Critical $w-$ and $t-$values are $\pm w_0$, $\pm t_0$, where

$$t_0 = \beta, \quad w_0 = \cosh^{-1}(1+\nu/2) = \ln\left(1+\frac{\nu+W_0}{2}\right), \quad W_0 = \sqrt{\nu^2+4\nu}. \tag{3.12}$$

It follows that

$$A = -\frac{\nu}{2}, \quad \beta = \frac{w_0+\sinh w_0}{2} = \frac{1}{2}\ln\left(1+\frac{\nu+W_0}{2}\right) + \frac{1}{4}W_0. \tag{3.13}$$

With these values of A,β the mapping $w \mapsto t$ is regular at $w = \pm w_0$ and at $w=0$. In fact it is regular in $(-\infty,\infty)$ and as a conformal mapping in a large domain Ω of the complex plane. We have the correspondences

$$t(\pm\infty) = \pm\infty, \quad t(\pm w_0) = \pm\beta, \quad t(0) = 0. \tag{3.14}$$

Using transformation (3.11) in (3.9), we obtain

$$\begin{aligned} F_\lambda(z,\alpha) = \Gamma(a)e^{-x/2}\Psi(a,c;x) &= \int_0^\infty t^{\lambda-1}e^{-at-\alpha/t}f(t)\,dt, \\ &\sim 2\beta^{1-c}K_{1-c}(2\beta a)\sum_{s=0}^\infty a_s\,a^{-s} + \\ &\quad 2\beta^{2-c}\left[K_{2-c}(2\beta a) - K_{1-c}(2\beta a)\right]\sum_{s=0}^\infty b_s\,a^{-s}, \end{aligned} \tag{3.15}$$

where $\alpha = z\beta^2$, $\lambda = 1 - c$ and β is defined in (3.13) with $\nu = x/a$. Furthermore, the coefficients a_s, b_s follow from the scheme given in (3.7) with

$$f(t) = \left(\frac{1 - e^{-w}}{t}\right)^{\lambda - 1} \frac{dw}{dt}, \quad \frac{dw}{dt} = \left(\frac{e^w - 1}{t}\right)^2 \frac{t^2 - \beta^2}{(e^w - 1)^2 - \nu e^w}. \tag{3.16}$$

The expansion in (3.15) holds for $a \to \infty$, uniformly with respect to $x \in [0, \infty)$. The asymptotic nature of the expansion is discussed in Temme (1990a), where also an expansion is considered in which c is no longer a fixed parameter.

We give the first coefficient $a_0(\beta)$ of the expansion in (3.15). A few calculations based on (3.16) and l'Hôpital's rule yield

$$\frac{dw}{dt}|_{t=\pm\beta} = \sqrt{2\tanh(\tfrac{1}{2}\mathrm{w}_0)/\beta}\ .$$

So we obtain

$$a_0(\beta) = \sqrt{2\tanh(\tfrac{1}{2}\mathrm{w}_0)/\beta}\ \left(\frac{\beta}{1 - e^{-w_0}}\right)^c, \quad b_0 = a_0\, \frac{1 - e^{-cw_0}}{2\beta}.$$

4. The Tricomi-Carlitz polynomials

The Tricomi-Carlitz polynomials are defined by

$$t_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{x - \alpha}{k} \frac{x^{n-k}}{(n-k)!}. \tag{4.1}$$

We obtain from (1.21) the relation with the Laguerre polynomials:

$$t_n^{(\alpha)}(x) = (-1)^n\, L_n^{(x-\alpha-n)}(x), \tag{4.2}$$

and we observe that the class of polynomials $\{l_n(x)\}$ introduced in Section 2 follows from the present set by putting $\alpha = 0$. The new polynomials satisfy the recurrence

$$(n+1)t_{n+1}^{(\alpha)}(x) - (n+\alpha)\, t_n^{(\alpha)}(x) + x\, t_{n-1}^{(\alpha)}(x) = 0, \quad n \ge 1, \tag{4.3}$$

with initial values $t_0^{(\alpha)}(x) = 1,\ t_1^{(\alpha)}(x) = \alpha$. A few other values are

$$t_2^{(\alpha)}(x) = \tfrac{1}{2}\left(\alpha + \alpha^2 - x\right), \quad t_3^{(\alpha)}(x) = \tfrac{1}{6}\left(2\alpha + 3\alpha^2 + \alpha^3 - 2x - 3x\alpha\right). \tag{4.4}$$

Tricomi (1948) introduced the polynomials. He observed that $\{t_n^{(\alpha)}(x)\}$ is not a system of orthogonal polynomials, the recurrence relations failing to have the required form (cf. Szegö (1975, page 43)). However, Carlitz (1958) discovered that if one sets

$$f_n^{(\alpha)}(x) = x^n\, t_n^{(\alpha)}(x)(x^{-2}), \tag{4.5}$$

then $\{f_n^{(\alpha)}(x)\}$ satisfies

$$(n+1)f_{n+1}^{(\alpha)}(x) - (n+\alpha)\,x\,f_n^{(\alpha)}(x) + f_{n-1}^{(\alpha)}(x) = 0, \quad n \geq 1, \tag{4.6}$$

with initial values $f_0^{(\alpha)}(x) = 1,\ f_1^{(\alpha)}(x) = \alpha\,x$. A few other values are

$$f_2^{(\alpha)}(x) = \tfrac{1}{2}\left[\alpha(1+\alpha)x^2 - 1\right] \quad f_3^{(\alpha)}(x) = \tfrac{1}{6}\,x\,\left(-2 + 2\alpha x^2 - 3\alpha + 3\alpha^2 x^2 + \alpha^3 x^2\right).$$

There is a generating function for $f_n^{(\alpha)}(x)$:

$$e^{w/x + (1-\alpha x^2)/x^2\,\ln(1-xw)} = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) w^n. \quad |wx| < 1. \tag{4.7}$$

If $x = 0$ this reduces to

$$e^{-\frac{1}{2}w^2} = \sum_{n=0}^{\infty} f_{2n}^{(\alpha)}(0)\,w^{2n},$$

giving

$$f_{2n}^{(\alpha)}(0) = (-1)^n\,2^{-n}/n!, \quad f_{2n+1}^{(\alpha)}(0) = 0, \quad n = 0,1,2,\ldots.$$

Carlitz proved that for $\alpha > 0$, $\{f_n^{(\alpha)}(x)\}$ satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} f_m^{(\alpha)}(x)\,f_n^{(\alpha)}(x)\,d\psi^{(\alpha)}(x) = \frac{2\,e^{\alpha}}{(n+\alpha)\,n!}\,\delta_{mn}, \tag{4.8}$$

where $\psi^{(\alpha)}(x)$ is the step function whose jumps are

$$d\,\psi^{(\alpha)}(x) = \frac{(k+\alpha)^{k-1}\,e^{-k}}{k!} \quad \text{at} \quad x = x_k = \pm\frac{1}{\sqrt{k+\alpha}}, \quad k = 0,1,2,\ldots. \tag{4.9}$$

The values x_k play a special role in the generating function because for these $x-$values we have

$$e^{w/x_k}\,(1 - x_k w)^k = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x_k)\,w^n,$$

and now the series converges for all values of w.

For further generalizations of the Tricomi-Carlitz polynomials the reader is referred to Askey & Ismail (1984) and Chihara & Ismail (1982); Chihara (1978) gives a brief treatment of the polynomials $t_n^{(\alpha)}(x)$. Goh & Wimp (1994 and 1997) establish the asymptotic behavior of the Tricomi-Carlitz polynomials and discuss their zero distribution. They observe that the polynomials $f_n(x/\sqrt{\alpha})$ have all zeros in the interval $[-1,1]$. They use in their second paper a probabilistic approach for improving their earlier results concerning the asymptotic distribution of the zeros of the polynomials $f_n^{(\alpha)}(x)$. Saddle point methods are used to study the asymptotics for $f_n^{(\alpha)}(x)$ in the complex plane.

In this section we describe a method how to obtain an asymptotic representation of the Tricomi polynomials in terms of the Hermite polynomial. We concentrate on large values of the parameter α and $n = \mathcal{O}(\alpha)$; for x we assume $-1/\sqrt{\alpha} < x < 1/\sqrt{\alpha}$, the interval of the zeros. The distribution of the zeros of $f_n^{(\alpha)}(x)$ can be obtained by using the zeros of the Hermite polynomials. The role of the Hermite polynomials can be shown by observing that

$$\lim_{\alpha\to\infty} f_n^{(\alpha)}\left(\frac{x\sqrt{2}}{\alpha}\right) = \frac{2^{-n/2}}{n!} H_n(x). \tag{4.10}$$

This follows from the generating function given in (4.7). Replacing x in the left-hand side with $x\sqrt{2}/\alpha$ yields, if $\alpha \to \infty$, $\exp(x\sqrt{2}\,w + \frac{1}{2}w^2)$. This is, up to some scaling, the generating function of the Hermite polynomials, which reads

$$e^{2xz-z^2} = \sum_{n=0}^{\infty} H_n(x)\,\frac{z^n}{n!}. \qquad (4.11).$$

Although the Tricomi-Carlitz polynomials can be expressed in terms of the Laguerre polynomials (see (4.2) and (4.5)) it is not possible to use existing results on Laguerre polynomials from the literature to describe the asymptotics of $f_n^{(\alpha)}(x)$; this is due to the peculiar role and position of the parameters n and x in (4.2).

Before treating the Tricomi-Carlitz polynomials we give a few details on the Laguerre polynomials. Tricomi has given several results; see TRICOMI (1949) and (1954b), with a summary of the results in TRICOMI (1954a) and in BUCHHOLZ (1969). Tricomi set $\nu = 4n + \alpha + 2$ and derived asymptotic formulas for x in each of the four regions:

(*i*) $x = \mathcal{O}(\nu^{1/3}$,

(*ii*) $a\nu \le x \le bx$,

(*iii*) $x - \nu = \mathcal{O}(\nu^{1/3}$,

(*iv*) $x \ge c\nu$,

where a, b, c are fixed and $0 < a < b < 1 < c$. Tricomi's results were later considerably improved by ERDÉLYI (1960). More precisely, Erdélyi gave two asymptotic formulas for $L_n^{(\alpha)}(\nu t)$, as $n \to \infty$, where t is real. One formula holds uniformly for $-\infty < t \le a$ and the other for $b \le t < \infty$, where a and b are two fixed numbers, $0 < b < a < 1$. These two intervals overlap and between them cover the entire $x-$axis. Erdélyi's method is based on the differential equation satisfied by the Laguerre polynomials. In FRENZEN & WONG (1988) it has been shown that the same results can be from their integral representations. Frenzen and Wong used the generating function to write the Laguerre polynomial as a Cauchy integral, and then applied the saddle point method to obtain expansions in terms of Airy and Bessel functions. In TEMME (1990b) we have mentioned several asymptotic forms of the Laguerre polynomials that are available in the literature for Whittaker functions, and which have been obtained by OLVER (1974) and (1980), and DUNSTER (1989) and (1990) by using differential equations. The Tricomi-Carlitz polynomials $f_n^{(\alpha)}(x)$ do not satisfy a differential equation. Hence, the powerful results obtained by Olver and Dunster for the Whittaker function cannot be used in the present case.

4.1. Hermite-type expansions of the Tricomi-Carlitz polynomials

We take the generating function (4.7) as starting point, and use the Cauchy-type integral:

$$f_n^{(\alpha)}(x) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{w/x+(1-\alpha x^2)/x^2\,\ln(1-xw)}\,\frac{dw}{w^{n+1}}. \tag{4.12}$$

The contour $\mathcal{C}$ is a circle around the origin with radius less than $1/|x|, x \neq 0$.

Our approach for the Tricomi-Carlitz polynomials is earlier discussed in TEMME (1986). We summarize the main steps of this publication. In JIN & WONG (1996) a similar approach is used for Meixner polynomials; also in this case a differential equation is not available. The same problem occurs for the Charlier and Pollaczek polynomials, which are considered in BO RUI & WONG (1994) and BO RUI & WONG (1996), respectively, and for which Airy functions and Bessel functions are used as main approximants. For more details on these publication we refer to Section 5.

Rescaling the parameters in (4.12) by writing

$$x = \xi/\sqrt{\alpha}\,, \quad n = \nu\alpha, \quad w = s\sqrt{\alpha}$$

we obtain

$$f_n^{(\alpha)}(x) = \frac{\alpha^{-n/2}}{2\pi i}\int_{\mathcal{C}} e^{\alpha\,\phi(s)}\,\frac{ds}{s}, \tag{4.13}$$

where

$$\phi(s) = \frac{s}{\xi} + \frac{1-\xi^2}{\xi^2}\,\ln(1-\xi s) - \nu\ln s. \tag{4.14}$$

The saddle points are given by

$$s_{1,2} = \frac{\xi(\nu+1) \pm \sqrt{\xi^2(\nu+1)^2 - 4\nu}}{2}. \tag{4.15}$$

If

$$-\frac{2\nu}{\sqrt{\nu+1}} < \xi < \frac{2\nu}{\sqrt{\nu+1}}$$

the saddle points are complex, and for these values of ξ the zeros of $f_n^{(\alpha)}(x)$ occur. In that case the saddle point are located on the circle with radius $\sqrt{\nu}$.

Comparing the behavior of the saddle points of the integral in (4.13) we observe that the situation is quite analogous to the behavior of the saddle points for various values of x and n of the Cauchy-type integral that defines the Hermite polynomials, viz.

$$H_n(x) = \frac{n!}{2\pi i}\int_{\mathcal{C}} e^{2xz-z^2}\,\frac{dw}{w^{n+1}}, \tag{4.16}$$

which follows from (4.11). Due to this analogy, the integral in (4.13) can be approximated in terms of Hermite polynomials.

Before giving a few details on the saddle point analysis we give a first result. If $n \ll \alpha$ the complex saddle points given in (4.15) are close to the origin. For small values of s the phase function $\phi(s)$ can be approximated by

$$\phi_0(s) = \xi s - \tfrac{1}{2}\,(1-\xi^2)\,s^2 - \nu\ln s.$$

Substituting this into (4.13) and using (4.16) we obtain for $|\xi/\sqrt{\alpha}\,| < 1$ the approximation

$$f_n^{(\alpha)}(x) = \left(\frac{1-\alpha x^2}{2}\right)^{n/2} \frac{1}{n!}\left[H_n\left(\frac{\alpha x}{\sqrt{2(1-\alpha x^2)}}\right) + \varepsilon_n^{(\alpha)}(x)\right], \tag{4.17}$$

where we expect that $|\varepsilon_n^{(\alpha)}(x)|$ is small if $\alpha \gg n$. Observe that the limit in (4.14) follows from (4.17) if indeed $\lim_{\alpha\to\infty} \varepsilon_n^{(\alpha)}(x) = 0$.

Computing the zeros of $f_n^{(\alpha)}(x)$ for $n = 10, \alpha = 50$ with the help of (4.17) and the zeros of $H_{10}(x)$ gives a maximal absolute error of 0.0054 for the zeros of $f_{10}^{(50)}(x)$ and a relative error of about 5%.

To obtain an optimal approximation we first use a different scaling of the parameters for (4.12). This time we introduce the parameters ξ, ν, s by writing

$$x = \xi/\sqrt{\alpha - \tfrac{1}{2}}, \quad n + \tfrac{1}{2} = \nu(\alpha - \tfrac{1}{2}), \quad w = s\sqrt{\alpha - \tfrac{1}{2}}, \tag{4.18}$$

which yields

$$f_n^{(\alpha)}(x) = \frac{(\alpha - \frac{1}{2})^{-n/2}}{2\pi i} \int_{\mathcal{C}} e^{(\alpha-\frac{1}{2})\phi(s)} \frac{ds}{\sqrt{s(1-\xi s)}}, \tag{4.19}$$

where $\phi(s)$ is given in (4.14) and the saddle points in (4.15) (now with different ξ and ν as given in (4.18)) $^{*)}$. Next we substitute

$$\phi(s) = \psi(t) + A, \tag{4.20}$$

which in fact is a conformal mapping of the $s-$plane to the $t-$plane, where

$$\psi(t) = 2\eta t - \nu \ln t - \tfrac{1}{2}t^2.$$

The quantities A and η follow from the condition that the saddle points in the $s-$plane correspond to the saddle points

$$t_{1,2} = \eta \pm \sqrt{\eta^2 - \nu} \tag{4.21}$$

in the $t-$plane. Using the transformation (4.20), we obtain from (4.19) the representation

$$f_n^{(\alpha)}(x) = \frac{(\alpha - \frac{1}{2})^{-n/2}\, e^{(\alpha-\frac{1}{2})A}}{2\pi i} \int_{\mathcal{C}} e^{(\alpha-\frac{1}{2})\,\psi(t)}\, f(t)\, \frac{dt}{\sqrt{t}}, \tag{4.22}$$

where

$$f(t) = \frac{\sqrt{t}}{\sqrt{s(1-\xi s)}}\, \frac{ds}{dt}. \tag{4.23}$$

$^{*)}$ This choice of the phase function, which leaves the term $1/\sqrt{s(1-\xi s)}$ as part of the integrand, is not very obvious; also, the role of the large parameter $\alpha - \frac{1}{2}$ instead of α is not obvious. We refer to TEMME (1997) for more details on this point.

Evaluating the equation $\psi(t_1) - \psi(t_2) = \phi(s_1) - \phi(s_2)$, which defines the quantity η, we obtain

$$\begin{aligned} 2\nu \operatorname{arccosh}\frac{\eta}{\sqrt{\nu}} - 2\eta\sqrt{\eta^2-\nu} = \\ -\frac{\sqrt{W}}{\xi} + 2\frac{1-\xi^2}{\xi^2}\operatorname{arcsinh}\frac{\xi\sqrt{W}}{2\sqrt{1-\xi^2}} + 2\nu \operatorname{arccosh}\frac{\xi(\nu+1)}{2\sqrt{\nu}}, \end{aligned} \tag{4.24}$$

where $W = \xi^2(\nu+1)^2 - 4\nu$. The relation in (4.24) holds for $2\sqrt{\nu}/(\nu+1) \le \xi < 1$, which correspond with $\eta-$values in $[\sqrt{\nu}, \eta_0)$, where η_0 is the value that corresponds with $\xi = 1$. For $-2\sqrt{\nu}/(\nu+1) \le \xi \le 2\sqrt{\nu}/(\nu+1)$, the $\eta-$interval becomes $[-\sqrt{\nu}, \sqrt{\nu}]$. In that case W is negative, and it is better to write

$$s_1 = \tfrac{1}{2}\left[\xi(\nu+1) - i\sqrt{-W}\right], \quad s_2 = \tfrac{1}{2}\left[\xi(\nu+1) + i\sqrt{-W}\right],$$

and

$$t_1 = \eta - i\sqrt{\nu - \eta^2}, \quad t_2 = \eta + i\sqrt{\nu-\eta^2}.$$

In this case the equation $\psi(t_1) - \psi(t_2) = \phi(s_1) - \phi(s_2)$ gives

$$\begin{aligned} 2\eta\sqrt{\nu-\eta^2} + 2\nu \arcsin\frac{\eta}{\sqrt{\nu}} = \\ \frac{\sqrt{-W}}{\xi} - 2\frac{1-\xi^2}{\xi^2}\arcsin\frac{\xi\sqrt{-W}}{2\sqrt{1-\xi^2}} + 2\nu\arcsin\frac{\xi(\nu+1)}{2\sqrt{\nu}}. \end{aligned} \tag{4.25}$$

The function η is an odd function of ξ. The first few coefficients in the Maclaurin expansion are given:

$$\eta = \frac{\nu+3}{6}\xi + \frac{8\nu^2+45\nu+135}{1620}\xi^3 + \frac{166\nu^3+1302\nu^2+4977\nu+14175}{408240}\xi^5 + \ldots.$$

When η is available, A follows by straightforward calculations from

$$A = \phi(s_1) - \psi(t_1) = \phi(s_2) - \psi(t_2). \tag{4.26}$$

Replacing this time the function $f(t)$ in (4.22) with a constant c_0, we obtain,

$$f_n^{(\alpha)}(x) = c_0\, e^{(\alpha-\frac{1}{2})A}\,\frac{2^{-n/2}}{n!}\left[H_n\left(\eta\sqrt{2\alpha-1}\right) + \varepsilon_n^{(\alpha)}(x)\right]. \tag{4.27}$$

In Table 4.1 we give the zeros x_k of $f_n^{(\alpha)}(x)$ for $n = 10, \alpha = 50$ and compare the zeros with approximations x_k^a obtained from this asymptotic formula. That is, let (for $k = 1, 2, \ldots, 10$) h_k be the zeros of $H_{10}(x)$. Define $\eta_k = h_k/\sqrt{2\alpha-1}$, and invert the relation in (4.25) to obtain ξ_k. Then the approximations of the zeros are given by

Table 4.1. Comparing the zeros of $f_n^{(\alpha)}(x)$ for $n = 10, \alpha = 50$ with approximations based on the zeros of $H_n(x)$. We show $x_k, k = 1, 2, \ldots, 10$ (the zeros of $f_{10}^{(50)}(x)$) with their approximations x_k^a, and the absolute and relative errors.

k	x_k	x_k^a	abs. error	rel. error
1	-0.0855233907	-0.0855230252	0.36×10^{-6}	0.42×10^{-5}
2	-0.0650754635	-0.0650753259	0.13×10^{-6}	0.21×10^{-5}
3	-0.0460298897	-0.0460298453	0.44×10^{-7}	0.96×10^{-6}
4	-0.0274857009	-0.0274856920	0.89×10^{-8}	0.32×10^{-6}
5	-0.0091433976	-0.0091433973	0.32×10^{-9}	0.35×10^{-7}
6	0.0091433976	0.0091433973	0.32×10^{-9}	0.35×10^{-7}
7	0.0274857009	0.0274856920	0.89×10^{-8}	0.32×10^{-6}
8	0.0460298897	0.0460298454	0.44×10^{-7}	0.96×10^{-6}
9	0.0650754635	0.0650753259	0.13×10^{-6}	0.21×10^{-5}
10	0.0855233907	0.0855230252	0.36×10^{-6}	0.42×10^{-5}

$x_k^a = \xi_k / \sqrt{\alpha - \frac{1}{2}}$. We observe that the approximations for these values of n and α are quite satisfactory; at least 5 significant decimal digits can be obtained in this way.

4.2. Hermite-type expansions

We give a few details on how to obtain a complete asymptotic expansion for Hermite-type expansions. More details on this method, and on the results of the previous subsection, can be found in a future publication TEMME (1997).

We consider integrals of the form

$$F_\kappa(\xi) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\kappa \Psi(t)} \, f(t) \, \frac{dt}{t}, \tag{4.28}$$

where

$$\Psi(t) = 2\xi t - \rho^2 \ln t - \tfrac{1}{2} t^2.$$

We assume that κ is a positive large parameter and that ρ is positive. The logarithmic function in Ψ assumes its principal value, which is real for positive values of t. The contour runs from $t = -\infty, \mathrm{ph}\, t = -\pi$, encircles the origin in positive direction, and terminates at $-\infty$, now with $\mathrm{ph}\, t = +\pi$.

The saddle points $t_{1,2}$ are given by

$$t_{1,2} = \xi \pm \sqrt{\xi^2 - \rho^2}\,.$$

For large values of κ the function $F_\kappa(\xi)$ defined in (4.28) can be expanded in terms of parabolic cylinder functions. This asymptotic expansion holds uniformly with respect to $\xi \in \mathbb{R}$ and $\rho \in [0, \infty)$. For certain values of κ and ρ the parabolic cylinder functions reduce to Hermite polynomials.

The procedure of obtaining the Hermite-type expansions runs as follows. We assume for constructing the expansion that f is an analytic function in a domain Ω of the complex plane that contains the saddle points and the contour $\mathcal{C}$.

We write

$$f(t) = \alpha_0 + \beta_0 t + (t - t_1)(t - t_2) g_0(t), \tag{4.29}$$

where α_0, β_0 follow from substituting $t = t_{1,2}$. That is,

$$\alpha_0 = \frac{t_2 f(t_1) - t_1 f(t_2)}{t_2 - t_1}, \quad \beta_0 = \frac{f(t_2) - f(t_1)}{t_2 - t_1}. \tag{4.30}$$

We obtain, on substituting (4.29) into (4.28) and integrating by parts the integral containing $g_0(t)$,

$$F_\kappa(\xi) = \alpha_0\, \mathcal{H}(\xi, \kappa, \rho) + \frac{\beta_0}{2\kappa}\, \mathcal{H}'(\xi, \kappa, \rho) + \frac{1}{\kappa} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\kappa \Psi(t)}\, f_1(t)\, \frac{dt}{t},$$

where

$$f_1(t) = t\, g_0'(t),$$

and $\mathcal{H}(\xi, \kappa, \rho)$ equals $F_\kappa(\xi)$ with $f(t)$ replaced by unity, that is,

$$\mathcal{H}(\xi, \kappa, \rho) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\kappa \Psi(t)}\, \frac{dt}{t},$$

and the prime in $\mathcal{H}'$ denotes the derivative with respect to ξ *). Repeating this procedure, we obtain for $N = 0, 1, 2, \ldots$

$$F_\kappa(\xi) = \mathcal{H}(\xi, \kappa, \rho) \sum_{s=0}^{N-1} \frac{\alpha_s}{\kappa^s} + \frac{\mathcal{H}'(\xi, \kappa, \rho)}{2\kappa} \sum_{s=0}^{N-1} \frac{\beta_s}{\kappa^s} + \frac{1}{\kappa^N} R_N(\xi, \kappa), \tag{4.31}$$

where

$$R_N(\xi, \kappa) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\kappa \Psi(t)}\, f_N(t)\, \frac{dt}{t},$$

and $\alpha_s, \beta_s, f_s(t)$ follow from the recursive scheme

$$\begin{aligned}
f_s(t) &= \alpha_s + \beta_s t + (t - t_1)(t - t_2) g_s(t), \\
f_{s+1}(t) &= t\, g_s'(t), \\
\alpha_s &= \frac{t_2 f_s(t_1) - t_1 f_s(t_2)}{t_2 - t_1}, \\
\beta_s &= \frac{f_s(t_2) - f_s(t_1)}{t_2 - t_1},
\end{aligned}$$

for $s = 0, 1, 2, \ldots$, with $f_0(t) = f(t)$.

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*) For convenience, we indicate three variables in $\mathcal{H}(\xi, \kappa, \rho)$, although it is a function of two variables.

The function $\mathcal{H}(\xi,\kappa,\rho)$ is a parabolic cylinder function. From ABRAMOWITZ & STEGUN (1964, page 687, formula 19.5.1) we obtain

$$\mathcal{H}(\xi,\kappa,\rho)=\frac{\kappa^{\frac12\kappa\rho^2}\,e^{\kappa\xi^2}}{\Gamma(\kappa\rho^2+1)}U(-\tfrac12-\kappa\rho^2,2\xi\sqrt{\kappa}\,).$$

If $\kappa\rho^2=n$ (non-negative integer) then $\mathcal{H}(\xi,\kappa,\rho)$ becomes a Hermite polynomial as in the previous subsection (cf. also formula 19.13.1 in ABRAMOWITZ & STEGUN):

$$\mathcal{H}(\xi,\kappa,\rho)=\frac{1}{n!}\left(\frac{n}{2\rho^2}\right)^{n/2}H_n\left(\xi\sqrt{2n}\,/\rho\right).$$

The above integration-by-parts technique is a variant of one given in BLEISTEIN (1966), and can be used in several other cases for obtaining uniform expansions of integrals (cf. WONG (1989)).

All coefficients α_s,β_s are well-defined and the functions f_s are analytic in the domain where $f_0=f$ is analytic. In order to show that (4.30) has a meaning as an asymptotic representation, an estimate of the remainder $R_N(\xi,\kappa)$ has to be given.

Remark 4.1. Under certain conditions on $f(t)$ in (4.28) the expansion in (4.31) becomes an expansion in negative powers of κ^2. That is, it may happen that $\beta_{2s}=\alpha_{2s+1}=0, s=0,1,2,\ldots$. The choice we made for the starting point (4.19), which produced a function f of the form as in (4.23), resulted in an expansion in which $\beta_0=0$, and in fact in highly accurate approximations of the zeros of $f_n^{(\alpha)}(x)$ as shown in Table 4.1.

5. Other recent results on uniform expansions of integrals

In this paper we have concentrated on results for functions related to the Tricomi $\Psi-$function. This function is also an important topic in the recent interest in the Stokes phenomenon. In this section we mention a few aspects of the Stokes phenomenon; in particular we discuss shortly Olver's work on the $\Psi-$function in connection with this topic. There are several other recent publications in which uniform asymptotic expansions are derived by using integrals; we mention a series of papers by Wong and co-workers on certain orthogonal polynomials.

In TEMME (1996) incomplete gamma functions are considered for negative values of the parameters; the results can be used for complex values also, and complement earlier results that concentrate on positive values of the parameters, again with extension to complex values.

TEMME (1995) gives a selection of recent problems in connection with uniform asymptotic methods for integrals.

5.1. Expansions in connection with the Stokes phenomenon

In BERRY (1989) the Stokes phenomenon has been given a new interpretation. This phenomenon is related with the different asymptotic expansions a function may have in certain sectors in the complex plane, and with the changing of constants multiplying asymptotic series when the complex variable crosses certain lines (also called Stokes lines). Berry explained that the constants are in fact rapidly changing smooth functions, which can be approximated in terms of the error function. His approach was followed by a series of papers by himself and other writers. At the same time interest arose in earlier work by Stieltjes, Airey and Dingle to re-expand remainders in asymptotic expansions and to improve the accuracy obtainable from asymptotic expansions by considering exponentially small terms.

The Stokes phenomenon and the topic of exponentially asymptotics are connected with uniform expansions of integrals, in particular, with approximations which are uniformly valid with respect to variations in the phase of the large parameter. We mention the contributions on a better understanding of the asymptotics of the gamma function by BERRY (1991), PARIS & WOOD (1992) and BOYD (1994). More general papers are HOWLS (1992), BERRY & HOWLS (1991) and (1994). For applications to the $\Psi-$function we mention OLDE DAALHUIS (1992) and (1993). In BOYD (1990) new results for the modified $K-$Bessel function have been given. In JONES (1990) a method has been devised for estimating the optimal remainder in an asymptotic approximation which is uniform with respect to variations in the phase of the large parameter. An introductory paper on the Stokes phenomenon and exponential asymptotics is PARIS & WOOD (1995).

In OLVER (1991a) and (1991b) Berry's approach is rigorously treated for integrals representing the $\Psi-$function (see also OLVER (1994)). Olver showed that the exponential integral

$$E_p(z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}\,dt = z^{p-1}\Gamma(1-p,z), \tag{5.1}$$

where $\Gamma(a,z)$ is the incomplete gamma function, plays an important role in Berry's smooth interpretation of the Stokes phenomena for certain integrals and special functions. OLVER (1991a) investigates $E_p(z)$ in particular at the Stokes lines $\mathrm{ph} z = \pm\pi$ and the results are used in OLVER (1991b) for the $\Psi-$function. We give a few details of Olver's results.

Let

$$F_p(z) = \frac{\Gamma(p)}{2\pi}\,\frac{E_p(z)}{z^{p-1}} \tag{5.2}$$

and $z = \rho e^{i\theta}, \alpha = n - \rho + p$ with ρ a large parameter, p fixed. Then

$$\begin{aligned} F_{n+p}(z) \sim (-1)^n i e^{-\rho\pi i}\Bigg[&\tfrac{1}{2}\mathrm{erfc}\left\{c(\theta)\sqrt{\tfrac{1}{2}\rho}\right\} \\ &- i\frac{e^{-\frac{1}{2}\rho\{c(\theta)\}^2}}{\sqrt{2\pi\rho}}\sum_{s=0}^{\infty}\left(\tfrac{1}{2}\right)_s g_{2s}(\theta,\alpha)\left(\tfrac{2}{\rho}\right)^s\Bigg], \end{aligned} \tag{5.3}$$

uniformly with respect to $\theta \in [-\pi+\delta, 3\pi-\delta]$ and bounded values of $|\alpha|$; δ denotes an arbitrarily small positive constant. Furthermore,

$$c(\theta) = \sqrt{2\left\{e^{i\theta} + i(\theta-\pi) + 1\right\}},$$

with the choice of branch of the square root that implies $c(\theta) \sim (\pi - \theta)$ as $\theta \to \pi$; the coefficients $g_{2s}(\theta,\alpha)$ are continuous functions of θ and α. A similar expansion for

$F_{n+p}(z)$ is given when $\theta \in [-3\pi+\delta, \pi-\delta]$. As Olver remarks, this expansion quantifies the Stokes phenomenon, that is, the rapid but smooth change in form of other expansions as θ passes through the common interval of validity of the other expansions.

By using these results OLVER (1991b) gives a detailed treatment of the $\Psi-$function. Define $R_n(a,b,z)$ by

$$\Psi(a, a-b+1; z) = z^{-a} \sum_{s=0}^{n-1} (-1)^s \frac{(a)_s (b))_s}{s!\, z^s} + R_n(a,b,z), \tag{5.4}$$

where

$$n = |z| - a - b + 1 + \alpha,$$

$|z|$ being large, a and b being fixed real or complex parameters, and $|\alpha|$ being bounded. Then

$$\begin{aligned} R_n(a,b,z) = (-1)^n 2\pi \frac{z^{b-1} e^z}{\Gamma(a)\Gamma(b)} \Bigg\{ & \sum_{s=0}^{m-1} (-1)^s \frac{(1-a)_s (1-b)_s}{s!} \frac{F_{n-s+a+b-1}(z)}{z^s} \\ & + (1-a)_m (1-b)_m R_{m,n}(a,b,z) \Bigg\}, \end{aligned}$$

where m is an arbitrary fixed integer, and

$$R_{m,n}(a,b,z) = \begin{cases} \mathcal{O}\left(e^{-z-|z|} z^{-m}\right), & \text{if } |\text{ph } z| \le \pi, \\ \mathcal{O}\left(z^{-m}\right), & \text{if } \pi \le |\text{ph } z| \le \frac{5}{2}\pi - \delta. \end{cases}$$

Furthermore, these sectors of validity are maximal. Observe that the expansion in (5.4) starts with the Poincaré-type expansion as given in (2.4). For other details on the expansion we refer to Olver's paper.

In later papers by Olver, Olde Daalhuis, etc., many results for the $\Psi-$function and other special functions are obtained by methods based on differential equations.

5.2. Orthogonal polynomials

In a series of papers, Wong and his co-workers have derived uniform asymptotic approximations for orthogonal polynomials that do not satisfy a differential equation, and for which integral methods are used. In these papers conformal mappings have been used that are of the same kind as the given in (4.20) and in TEMME (1986).

[1] In JIN & WONG (1996) the Meixner polynomials have been considered, which can be defined by the generating function

$$\left(1 - \frac{\omega}{c}\right)^x (1-\omega)^{-x-\beta} = \sum_{n=0}^{\infty} m_n(x;\beta,c) \frac{\omega^n}{n!}. \tag{5.5}$$

There is a relation with the Gauss hypergeometric function:

$$m_n(x;\beta,c) = (\beta)_n\, {}_2F_1(-n, -x; \beta; 1 - 1/c).$$

Two infinite asymptotic expansions are derived for $m_n(n\alpha;\beta,c)$. One holds uniformly for $0<\varepsilon\le\alpha\le 1+a$, and the other holds uniformly for $1-b\le\alpha\le M<\infty$, where a and b are two small positive quantities. The main approximants are parabolic cylinder functions, which are in fact Hermite polynomials.

[2] Bo Rui & Wong (1994) gives expansions for Charlier polynomials, which follow from the generating function

$$e^{-aw}\,(1+w)^x=\sum_{n=0}^{\infty}C_n^{(a)}(x)\,\frac{w^n}{n!},\quad |w|<1. \tag{5.6}$$

There is a relation with the Laguerre polynomials and the Tricomi-Carlitz polynomials, because the Charlier polynomials can be written as

$$C_n^{(a)}(x)=n!\,L_n^{(x-n)}(a). \tag{5.7}$$

(cf. (4.2) and (4.5)). An infinite asymptotic expansion is derived for $C_n^{(a)}(n\beta)$, as $n\to\infty$, which holds uniformly for $0<\varepsilon\le\beta\le M<\infty$. The results are in terms of the $J-$Bessel function. Considering a as the large parameter gives an asymptotic problem as treated in the previous section, with approximations in terms of Hermite polynomials.

[3] Bo Rui & Wong (1996) treats the Pollaczek polynomials, which are defined by the generating function

$$\left(1-we^{i\theta}\right)^{-1/2+ih(\theta)}\left(1-we^{-i\theta}\right)^{-1/2-ih(\theta)}=\sum_{n=0}^{\infty}P_n(x;a,b)\,w^n,$$

where

$$h(\theta)=\frac{a\cos\theta+b}{2\sin\theta},\quad a>\pm b.$$

An asymptotic expansion is derived for $P_n(\cos(t/\sqrt{n});a,b)$, as $n\to\infty$, which holds uniformly for $0<\varepsilon\le t\le M<\infty$. The results are in terms of Airy functions. A discussion on approximations of the zeros of $P_n(\cos(t/\sqrt{n});a,b)$ is included.

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