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A NOTE ON TWO INTEGRALS RELATED WITH BESSEL FUNCTIONS

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A note on two integrals related with Bessel functions *)

by

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ABSTRACT

Two integrals are considered that can be expressed in terms of Bessel functions. Mellin transform techniques are used for proving the identities.

KEY WORDS & PHRASES: *Bessel function integrals, confluent hypergeometric function, Mellin transform*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this note we consider the integrals

$$G(r,\lambda) = \int_0^\infty \xi^5 \sin r\xi^3 \; e^{-\lambda^2\xi^2} d\xi$$

(1.1)

$$H(\rho,\lambda) = \int_0^\infty \xi^5 \; J_0(\rho\xi^3) e^{-\lambda^2\xi^2} d\xi.$$

We assume that r and ρ are real and $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda^2 > 0$; J_0 is the ordinary Bessel function. We consider first the integrals as given above, after that we treat more general forms of these integrals.

The functions $G(r,\lambda)$ and $H(\rho,\lambda)$ were presented to the author by dr. R.A. Pasmanter of Rijkswaterstaat (The Hague) who met them in his research on diffusion in highly turbulent fluids (see [3]). They occur as terms in expansions of the Green functions for the solutions of two and three dimensional turbulent diffusion problems. We write these functions first as confluent hypergeometric functions and we give expansions for the numerical evaluation. Then the relation with modified Bessel functions is described.

The functions G and H can be viewed as Laplace transforms or as sine and Hankel transforms. Standard tables do not give information about them and it was rather surprising to the author that the combination of ξ^5, ξ^3 and ξ^2 terms yielded a confluent hypergeometric function. Apparently other combinations of ξ-powers will do the same.

2. MELLIN TRANSFORM TECHNIQUES

The results of this note can be obtained in several ways. The Mellin-approach was found to be most convenient. The Mellin-transform of a function f is written as

$$\hat{f}(z) = \int_0^\infty t^{z-1} f(t)\,dt. \tag{2.1}$$

If f is continuous on $(0,\infty)$ and if we know that

$$f(t) = \begin{cases} O(t^{-\alpha}), & t \to 0^+ \\ O(t^{-\beta}), & t \to \infty \end{cases}$$

where α and β are real numbers ($\alpha < \beta$) then $\hat{f}$ exists for $\alpha < \operatorname{Re} z < \beta$, and perhaps for a wider domain of Re z.

The behaviour of $G(r,\lambda)$ and $H(\rho,\lambda)$ for $r,\rho \to 0^+$ is

$$G(r,\lambda) = O(r), \quad H(\rho,\lambda) = O(1),$$

as follows from the behaviour of $\sin(r\xi^3)$ and $J_0(\rho\xi^3)$ for $r,\rho \to 0$. For $r,\rho \to \infty$ we obtain by partial integration

$$G(r,\lambda) = \frac{-1}{3r}\int_0^\infty \xi^3 e^{-\lambda^2\xi^2}\, d\cos r\xi^3 =$$

$$= \frac{1}{r}\int_0^\infty \xi^2 \cos r\xi^3 e^{-\lambda^2\xi^2} d\xi - \frac{2\lambda^2}{3r}\int_0^\infty \xi^4 \cos r\xi^3 e^{-\lambda^2\xi^2} d\xi$$

$$= O(r^{-1}),$$

$$H(\rho,\lambda) = \frac{1}{3\rho}\int_0^\infty e^{-\lambda^2\xi^2} d[\xi^3 J_1(\rho\xi^3)] =$$

$$= \frac{2\lambda^2}{3\rho}\int_0^\infty \xi^4 J_1(\rho\xi^3) e^{-\lambda^2\xi^2} d\xi = O(\rho^{-1}),$$

where we used $\frac{d}{dz}\{z J_\nu(z)\} = z^\nu J_{\nu-1}(z)$. From this we conclude that the Mellin transforms of G and H with respect to r and ρ are

$$\text{(2.2)} \quad \begin{aligned} \hat{G}(z,\lambda) &= \int_0^\infty r^{z-1} G(r,\lambda)\,dr, \qquad -1 < \operatorname{Re} z < 1 \\ \hat{H}(z,\lambda) &= \int_0^\infty \rho^{z-1} H(\rho,\lambda)\,d\rho, \qquad 0 < \operatorname{Re} z < 1, \end{aligned}$$

where the ranges of Re z may be larger than indicated (for instance, by a refinement of the integration by parts process it follows that $H(\rho,\lambda) = \mathcal{O}(\rho^{-8/3})$, $\rho \to \infty$).

The next step consists in substituting (1.1) in (2.1). This leads to the question under which conditions is the equation

$$(2.3)\qquad \int_0^\infty \{\int_0^\infty f(x,y)\,dx\}dy = \int_0^\infty \{\int_0^\infty f(x,y)\,dy\}dx$$

correct. A well-known result of de la Valléé Poussin is (see for instance BROMWICH [2, §177]): (2.3) is correct, provided that both the integrals

$$\int_0^\infty |f(x,y)|\,dx, \qquad \int_0^\infty |f(x,y)|\,dy$$

are convergent and that either of the repeated integrals (2.3) converges absolutely. When using estimates for sin and J_0 in the form

$$(2.4)\qquad |\sin x| \le A\,\frac{x}{1+x}, \qquad |J_0(x)| \le \frac{B}{1+\sqrt{x}}, \qquad x \ge 0,$$

where A and B are assignable numbers, we can easily conclude that (1.1) can be substituted in (2.2) and that the order of integration may be interchanged. Using

$$\int_0^\infty r^{z-1}\sin r\xi^3\,dr = \xi^{-3z}\Gamma(z)\sin \tfrac{1}{2}\pi z, \qquad -1 < \mathrm{Re}\ z < 1,$$

$$\int_0^\infty \rho^{z-1}J_0(\rho\xi^3)\,d\rho = 2^{z-1}\xi^{-3z}\Gamma(\tfrac{1}{2}z)/\Gamma(1-\tfrac{1}{2}z), \qquad 0 < \mathrm{Re}\ z < 3/2,$$

we obtain

$$(2.5a)\qquad \begin{aligned} \hat{G}(z,\lambda) &= \Gamma(z)\sin\tfrac{1}{2}\pi z \int_0^\infty \xi^{5-3z}e^{-\lambda^2\xi^2}\,d\xi = \\ &= \tfrac{1}{2}\lambda^{-6+3z}\sin\tfrac{1}{2}\pi z\,\Gamma(z)\,\Gamma(3-\tfrac{3}{2}z), \end{aligned}$$

$$\hat{H}(z,\lambda) = 2^{z-1}\Gamma(\tfrac{1}{2}z)/\Gamma(1-\tfrac{1}{2}z)\int_0^\infty \xi^{5-3z}e^{-\lambda^2\xi^2}d\xi$$

(2.5b)

$$= \tfrac{1}{2}2^{z-1}\lambda^{-6+3z}\frac{\Gamma(\tfrac{1}{2}z)}{\Gamma(1-\tfrac{1}{2}z)}\Gamma(3-\frac{3}{2}z).$$

The inversion formula for (2.1) reads as follows

$$(2.6) \qquad f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} t^{-z}\hat{f}(z)dz, \qquad \alpha < \gamma < \beta.$$

For our Mellin transform pairs (2.2) and (2.5) we obtain

$$G(r,\lambda) = \frac{1}{4\pi i}\lambda^{-6}\int_{L_1} \lambda^{3z}r^{-z}\sin\tfrac{1}{2}\pi z\Gamma(z)\Gamma(3-\frac{3}{2}z)dz$$

(2.7)

$$H(\rho,\lambda) = \frac{1}{4\pi i}\lambda^{-6}\int_{L_2} \lambda^{3z}\rho^{-z}2^{z-1}\Gamma(\tfrac{1}{2}z)\Gamma(3-\frac{3}{2}z)/\Gamma(1-\tfrac{1}{2}z)dz$$

where the contours of integration L_1, L_2 are such that (compare (2.2))

on L_1: $-1 < \mathrm{Re}\ z < 1$,

on L_2: $0 < \mathrm{Re}\ z < 1$.

The integrals in (2.7) can be written in a standard form, "standard" for our purposes. We want to write them so that they are recognized as special cases of

$$(2.8) \qquad z^a\Gamma(a)\Gamma(1+a-b)U(a,b,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(-s)\Gamma(a+s)\Gamma(1+a-b+s)z^{-s}ds$$

$a \neq 0,-1,-2,\ldots$, $b-a \neq 1,2,\ldots$ and the contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$ and $\Gamma(1+a-b+s)$. The function $U(a,b,z)$ is a confluent hypergeometric function. To write (2.7) in the form (2.8) we need some properties of the gamma function, viz.

$$\Gamma(z+1) = z\Gamma(z)$$

$$\frac{\sin \pi z}{\pi z} = \frac{1}{\Gamma(1-z)\Gamma(1+z)}$$

(2.9)

$$\Gamma(2z) = \pi^{-\frac12}2^{2z-1}\Gamma(z)\Gamma(\tfrac12+z)$$

$$\Gamma(3z) = (2\pi)^{-1}3^{3z-\frac12}\Gamma(z)\Gamma(z+\tfrac13)\Gamma(z+\tfrac23)$$

When using this in the integrands of (2.7), we obtain

$$\sin \tfrac12\pi z\Gamma(z)\Gamma(3-\tfrac32 z) =$$

$$= \frac{\tfrac12\pi z 2^{z-1}\Gamma(\tfrac12 z)\Gamma(\tfrac12+\tfrac12 z)\,3^{\frac12(5-3z)}\,\Gamma(1-\tfrac12 z)\Gamma(\tfrac43-\tfrac12 z)\Gamma(\tfrac53-\tfrac12 z)}{\pi^{\frac12}2\pi\Gamma(1-\tfrac12 z)\Gamma(1+\tfrac12 z)}$$

$$= \pi^{-\frac12}2^{z-2}3^{\frac12(5-3z)}\,\Gamma(\tfrac12+\tfrac12 z)\Gamma(\tfrac43-\tfrac12 z)\Gamma(\tfrac53-\tfrac12 z),$$

$$\frac{2^{z-1}\Gamma(\tfrac12 z)\Gamma(3-\tfrac32 z)}{\Gamma(1-\tfrac12 z)}$$

$$= \frac{2^{z-1}\Gamma(\tfrac12 z)\,3^{\frac12(5-3z)}\,\Gamma(1-\tfrac12 z)\Gamma(\tfrac43-\tfrac12 z)\Gamma(\tfrac53-\tfrac12 z)}{2\pi\Gamma(1-\tfrac12 z)}$$

$$= \frac{1}{2\pi}\,2^{z-1}3^{\frac12(5-3z)}\,\Gamma(\tfrac12 z)\Gamma(\tfrac43-\tfrac12 z)\Gamma(\tfrac53-\tfrac12 z).$$

Furthermore we use in (2.7) the transformations

$$\tfrac12 + \tfrac12 z \to -s, \qquad \tfrac12 z \to -s,$$

respectively, which result in

$$(2.10)\quad \begin{aligned} G(r,\lambda) &= 2^{-3}3^{4}\pi^{-\frac{1}{2}}r\lambda^{-9}\,\frac{1}{2\pi i}\int_{L_1}\Gamma(-s)\Gamma(\tfrac{11}{6}+s)\Gamma(\tfrac{13}{6}+s)\zeta^{-s}ds \\ H(r,\lambda) &= 2^{-2}3^{5/2}\pi^{-1}\lambda^{-6}\,\frac{1}{2\pi i}\int_{L_2}\Gamma(-s)\Gamma(\tfrac{4}{3}+s)\Gamma(\tfrac{5}{3}+s)\eta^{-s}ds \end{aligned}$$

$$(2.11)\quad \zeta = \frac{4\lambda^6}{27r^2}, \qquad \eta = \frac{4\lambda^6}{27\rho^2},$$

and

$$\text{on } L_1:\ -1 < \operatorname{Re} s < 0,$$

$$\text{on } L_2:\ -\frac{1}{2} < \operatorname{Re} s < 0.$$

Finally, using (2.8) we arrive at

$$(2.12)\quad \begin{aligned} G(r,\lambda) &= \frac{105}{32}\sqrt{\pi}\, r\lambda^{-9}\zeta^{11/6}U(\tfrac{11}{6},\tfrac{2}{3},\zeta) \\ H(\rho,\lambda) &= \lambda^{-6}\eta^{4/3}U(\tfrac{4}{3},\tfrac{2}{3},\eta) \end{aligned}$$

with ζ and η given in (2.11).

From known results of confluent hypergeometric functions further properties of G and H can be derived, although some of them can be easily obtained from the definition (1.1). For instance, we mention the asymptotic expansion

$$z^a U(a,b,z) = \sum_{n=0}^{R-1} \frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n} + \mathcal{O}(|z|^{-R}),$$

as $z \to \infty$, $|\arg z| < 3\pi/2$. Expansions for small ζ and η follow from

$$U(a,b,z) = \frac{\pi}{\sin \pi b}\left\{\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right\}$$

with

$$M(a,b,z) = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(b)_n n!}.$$

This series converges for all finite $z \in \mathbb{C}$. In these formulas $(a)_n = \Gamma(a+n)/\Gamma(a)$.

Next we express G and H in terms of Bessel functions. We use the relation

$$(2.13) \qquad U(\nu+\tfrac{1}{2}, 2\nu+1, 2z) = \pi^{-\frac{1}{2}} e^{z} (2z)^{-\nu} K_\nu(z),$$

where K_ν is the modified Bessel function. In order to bring the functions in (2.12) into the form of the U-function in (2.13) we need some recurrence relations for the U-functions:

$$(2.14) \qquad \begin{aligned} a(1+a-b)U(a+1,b,z) &= aU(a,b,z) + zU'(a,b,z) \\ (1+a-b)U(a,b-1,z) &= (1-b)U(a,b,z) - zU'(a,b,z) \\ (b-a)U(a,b,z) &= zU(a,b+1,z) - U(a-1,b,z) \\ (b-a-1)U(a,b-1,z) &= (b+z-1)U(a,b,z) - zU(a,b+1,z) \end{aligned}$$

which can be found in the literature (see, f.i., ABRAMOWITZ & STEGUN [1, chapter 13]). They can also be derived from the integral

$$\Gamma(a)U(a,b,z) = \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt,$$

Re $z > 0$, Re $a > 0$.

By using (2.15) we obtain

$$U(\tfrac{11}{6}, \tfrac{2}{3}, \zeta) = -\tfrac{36}{35}[5(\zeta+\tfrac{2}{3})U(\tfrac{5}{6}, \tfrac{5}{3}, \zeta) + \zeta(6\zeta+5)U'(\tfrac{5}{6}, \tfrac{5}{3}, \zeta)]$$

$$U(\tfrac{4}{3}, \tfrac{2}{3}, \eta) = -[(6\eta+5)U(\tfrac{4}{3}, \tfrac{8}{3}, \eta) + \tfrac{3}{2}\eta(3\eta+2)U'(\tfrac{4}{3}, \tfrac{8}{3}, \eta)].$$

Hence G and H can be written in terms of

$$K_{1/3},\ K'_{1/3} \text{ and } K_{5/6},\ K'_{5/6}$$

respectively, and $K'_\nu(z)$ can be replaced by $\frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z)$. The result is

$$G(r,\lambda) = \frac{9}{16} r\lambda^{-9}\zeta^{3/2}e^{\frac{1}{2}\zeta}[3\zeta(6\zeta+5)K_{4/3}(\tfrac{1}{2}\zeta) - (18\zeta^2+45\zeta+20)K_{1/3}(\tfrac{1}{2}\zeta)]$$

$$H(\rho,\lambda) = \frac{1}{4}\pi^{-\frac{1}{2}}\lambda^{-6}\eta^{\frac{1}{2}}e^{\frac{1}{2}\eta}[3\eta(3\eta+2)K_{11/6}(\tfrac{1}{2}\eta) - (9\eta^2+30\eta+20)K_{5/6}(\tfrac{1}{2}\eta)].$$

Bessel functions of order $\frac{1}{3}$ are in fact Airy functions.

3. GENERALIZATIONS

Let us consider the integrals

$$G(r,\lambda;\alpha,\beta) = \int_0^\infty \xi^\alpha \sin r\xi^\beta e^{-\lambda^2\xi^2} d\xi$$

$$H_\nu(\rho,\lambda;\alpha,\beta) = \int_0^\infty \xi^\alpha J_\nu(\rho\xi^\beta)e^{-\lambda^2\xi^2} d\xi,$$

where the parameters are such that the integrals converge (the analysis will be rather formal here). The Mellin transforms are

$$\hat{G}(z,\lambda;\alpha,\beta) = \tfrac{1}{2}\lambda^{-\alpha-1+\beta z} \sin \tfrac{1}{2}\pi z\,\Gamma(z)\,\Gamma(\frac{\alpha+1-\beta z}{2})$$

$$\hat{H}_\nu(z,\lambda;\alpha,\beta) = \tfrac{1}{2}2^{z-1}\lambda^{-\alpha-1+\beta z} \frac{\Gamma(\frac{\alpha+1-\beta z}{2})\,\Gamma(\frac{\nu+z}{2})}{\Gamma(\frac{\nu+2-z}{2})}$$

and the inversion gives

$$G(r,\lambda;\alpha,\beta) = \frac{1}{2\pi i}\int_{L_1} r^{-z}\hat{G}(z,\lambda;\alpha,\beta)\,dz$$

$$H_\nu(\rho,\lambda;\alpha,\beta) = \frac{1}{2\pi i}\int_{L_2} \rho^{-z}\hat{H}_\nu(z,\lambda;\alpha,\beta)\,dz.$$

For specific combinations of α, β (and ν) the integrals may be reduced to combinations of (2.8). For $\beta = 2,3,4,\ldots$ it is possible to write them as a Meyer's G-function. For $\beta = 3$ we can use again

$$\Gamma\left(\frac{\alpha+1-\beta z}{2}\right) = \frac{1}{2\pi}\, 3^{(\alpha-3z)/2}\Gamma\left(\frac{\alpha+1}{6}-\tfrac{1}{2}z\right)\Gamma\left(\frac{\alpha+3}{6}-\tfrac{1}{2}z\right)\Gamma\left(\frac{\alpha+5}{6}-\tfrac{1}{2}z\right).$$

Hence if $\alpha = 3\nu + k$, then for $k = 1,3,5$, H_ν is a confluent hypergeometric function, for $k = 7,9,\ldots$ it is a combination of such functions. If $\alpha = 1,3,5$ then G is a U-function if $\alpha = 7,9,\ldots$ it is a linear combination of such functions.

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