Asymptotic expansions for Riesz potentials of Airy functions and their products

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Abstract. Riesz potentials of a function are defined as fractional powers of the Laplacian. Asymptotic expansions for $x \to \pm\infty$ are derived for the Riesz potentials of the Airy function $Ai(x)$ and the Scorer function $Gi(x)$. Reduction formulas are provided that allow to compute Riesz potentials of the products of Airy functions $Ai^2(x)$ and $Ai(x)Bi(x)$, where $Bi(x)$ is the Airy function of the second type, via the Riesz potentials of $Ai(x)$ and $Gi(x)$. Integral representations are given for the function $A_2(a,b;x) = Ai(x-a)\,Ai(x-b)$ with $a, b \in \mathbf{R}$, and its Hilbert transform. Combined with the above asymptotic expansions they can be used for obtaining asymptotics of the Hankel transform of Riesz potentials of $A_2(a,b;x)$. The study of the above Riesz fractional derivatives can be used for establishing new properties of Korteweg-de Vries-type equations.

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1. Introduction

It is well known that fundamental solutions of equations of the Korteweg-de Vries (KdV henceforth) type are expressed in terms of the Airy function of the first type $Ai(x)$. Indeed, the fundamental solution of the linearized Cauchy problem for the classical Korteweg-de Vries equation,

$$u_t + u_{xxx} = -\left(u^2\right)_x,$$

can be written in the form

$$\mathcal{E}_0(x,t) = \frac{1}{\sqrt[3]{3t}} Ai\left(\frac{x}{\sqrt[3]{3t}}\right).$$

It was shown in [1] that for the close relative of KdV, the Ostrovsky equation,

$$u_t + u_{xxx} = \gamma \int_{-\infty}^{x} u\,\mathrm{d}y - \left(u^2\right)_x,$$

where $\gamma = const > 0$ is the rotation parameter, the corresponding fundamental solution can be represented in the form

$$\begin{aligned}\mathcal{E}(x,t) &= -\frac{1}{\sqrt[3]{3t}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty Ai\left(\frac{x+y}{\sqrt[3]{3t}}\right) J_0\left(2\sqrt{\gamma t y}\right)\,\mathrm{d}y \\ &= \frac{1}{\sqrt[3]{3t}}Ai\left(\frac{x}{\sqrt[3]{3t}}\right) - \frac{\sqrt{\gamma t}}{\sqrt[3]{3t}}\int_0^\infty Ai\left(\frac{x+y}{\sqrt[3]{3t}}\right)\frac{J_1\left(2\sqrt{\gamma t y}\right)}{\sqrt{y}}\,\mathrm{d}y,\end{aligned} \tag{1}$$

where $J_\nu(x)$ is the Bessel function of order ν.

Riesz potentials (sometimes also called Riesz fractional derivatives) of fundamental solutions are of great importance in studying global solvability, properties and the long-time behavior of the corresponding Cauchy problems (see [2, 3, 4, 5] and the references therein). In the current paper we are concerned with obtaining asymptotic expansions as $x \to \pm\infty$ of the Riesz potentials of the Airy function $Ai(x)$ and the Scorer function $Gi(x) = -HAi(x)$, where H is the Hilbert transform (see (5) below). Riesz fractional derivatives of these functions of order $\alpha = 1/2$ stand out as the highest Riesz potentials that are still uniformly bounded on the whole real axis (see [2, 3]). Moreover, all semi-integer derivatives of $Ai(x)$ and $Gi(x)$ can be expressed in terms of the products of Airy functions (see [5]). We also provide formulas that allow one to obtain asymptotic expansions of the products of Airy functions $Ai(x)Bi(x)$, $Ai^2(x)$ and $Ai\,(x-a)\,Ai\,(x-b)$ with $a,\ b \in R$. Here $Bi(x)$ is the Airy function of the second type.

The next statement was proved in [6]. It provides reduction formulas that allow to compute Riesz potentials of the products of Airy functions once $D_x^\alpha Ai(x)$ and $D_x^\alpha Gi(x)$ are known.

Theorem 1 *Riesz fractional derivatives of the products of Airy functions have the following representations for $\alpha > -1/2$ and $x \in \mathbf{R}$:*

$$\begin{aligned}D_x^\alpha\left\{Ai^2(x)\right\} = k_\alpha \Big[&\left(D^{\alpha-1/2}Ai\right)\left(2^{2/3}x\right) \\ &-\left(D^{\alpha-1/2}Gi\right)\left(2^{2/3}x\right)\Big]\end{aligned} \tag{2}$$

and

$$\begin{aligned}D_x^\alpha\left\{Ai(x)Bi(x)\right\} = k_\alpha \Big[&\left(D^{\alpha-1/2}Ai\right)\left(2^{2/3}x\right) \\ &+\left(D^{\alpha-1/2}Gi\right)\left(2^{2/3}x\right)\Big],\end{aligned} \tag{3}$$

where

$$k_\alpha = \frac{2^{2(\alpha-1)/3}}{\sqrt{2\pi}}. \tag{4}$$

2. Definitions

The Fourier transform of the function $f:\ \mathbf{R} \to \mathbf{R}$ is defined by the formula

$$\hat{f}(\xi) = \mathcal{F}\left\{f\right\}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x} f(x)\,\mathrm{d}x$$

and the inverse Fourier transform by

$$f(x)=\mathcal{F}^{-1}\left\{\hat{f}\right\}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\xi x}\hat{f}(\xi)\,\mathrm{d}\xi.$$

Introduce the Hankel transform of the function f by the formula (see [7, p. 316])

$$\tilde{f}(k)=\mathcal{H}_{x\to k}\left\{f\right\}(k)=\int_{0}^{\infty}f(x)J_m(kx)x\,\mathrm{d}x$$

and the corresponding inverse transform by

$$\mathcal{H}^{-1}_{k\to x}\left\{\tilde{f}\right\}(x)=\int_{0}^{\infty}\tilde{f}(k)J_m(kx)k\,\mathrm{d}k.$$

Introduce the Hilbert transform of the function f by the formula (see [8, p. 120])

$$H\left\{f\right\}(x)=\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f(y)}{y-x}\,\mathrm{d}y, \tag{5}$$

where $x\in\mathbf{R}$ and $P.V.$ denotes the Cauchy principal value of an integral. Notice that this definition differs by the opposite sign from the convolution-type definition of [9, p. 26]. According to our choice of the Fourier transform, $(\widehat{Hf})(\xi)=\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{f}(\xi)$. One can see that $H^2=-I$ on $L_p(\mathbf{R})$, $p\geq 1$, where I is the identity operator.

For $x\in\mathbf{R}^n$ Riesz potentials are defined via the Fourier transform (see [9, p. 117] and [10, p. 88])

$$\left((-\Delta)^{\alpha/2}f\right)^{\wedge}(\xi)=|\xi|^{\alpha}\hat{f}(\xi). \tag{6}$$

For α, $x\in\mathbf{R}$ define the Riesz potentials by

$$D^{\alpha}_{x}\left\{f(x)\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\xi|^{\alpha}\hat{f}(\xi)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi, \tag{7}$$

provided that the integral in the right-hand side exists. Notice that for any $a>0$

$$D^{\alpha}_{x}\left\{f(ax)\right\}=a^{\alpha}D^{\alpha}_{y}\left\{f(y)\right\}|_{y=ax}. \tag{8}$$

Introduce the function

$$A_2(a,b;x)=Ai\left(x-a\right)Ai\left(x-b\right). \tag{9}$$

This function appears in the studies of the Gelfand-Levitan-Marchenko equation (see [11, p. 408]), the second Painlevé equation (see [12, p. 134]) and the limit at the "edge of the spectrum" of the level spacing distribution functions obtained from scaling random models of Hermitian matrices in the Gaussian Unitary Ensemble ([13] and [14]).

3. Asymptotic expansions of Riesz potentials of the Airy and Scorer functions for $x\to+\infty$

The Riesz potentials of $Ai(x)$ and $Gi(x)$ can be written as

$$D^{\alpha}_{x}Ai(x)=\Re F(x),\quad D^{\alpha}_{x}Gi(x)=\Im F(x), \tag{10}$$

where $\Re f$ and $\Im f$ denote the real and imaginary parts of f, respectively, and

$$F(x)=\frac{1}{\pi}\int_{0}^{\infty}\xi^{\alpha}\mathrm{e}^{\mathrm{i}(x\xi+\frac{1}{3}\xi^3)}\,\mathrm{d}\xi. \tag{11}$$

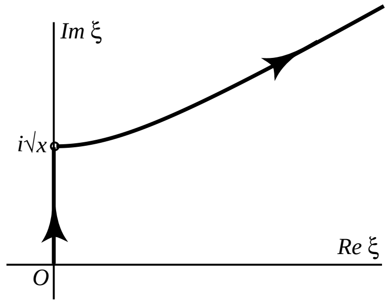

Figure 1. Modification of the path of integration giving the integral in (14) and an integral that is exponentially small.

Theorem 2 *The following asymptotic expansions hold for $\alpha > -1$ and $x \to +\infty$:*

$$D_x^\alpha Ai(x) \sim \frac{\cos(\pi(\alpha+1)/2)}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!} \frac{1}{x^{3k}} \quad , \tag{12}$$

where $\alpha \neq 0, 2, 4 \ldots$, and

$$D_x^\alpha Gi(x) \sim \frac{\sin(\pi(\alpha+1)/2)}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!} \frac{1}{x^{3k}}, \tag{13}$$

where $\alpha \neq 1, 3, 5 \ldots$.

Proof We use a representation of the integral in (11) similar to the one for $Gi(x)$ in (3.18) of [15]. To do so, notice that the exponential function in the integrand in (11) has a saddle point at $\xi = \mathrm{i}\sqrt{x}$. We integrate from the origin to this saddle point, and from there to ∞, inside the valley at $\infty \exp(\pi\mathrm{i}/6)$. The latter part can be neglected, because it is exponentially small compared with the first part. Therefore we have for large positive x

$$F(x) = \frac{\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}(\alpha+1)}}{\pi} \int_0^{\sqrt{x}} v^\alpha \mathrm{e}^{-xv+\frac{1}{3}v^3}\, \mathrm{d}v + \mathcal{O}\left(x^\alpha \mathrm{e}^{-\frac{2}{3}x^{3/2}}\right). \tag{14}$$

The asymptotic expansion follows from applying Watson's lemma (see [16, pp. 112–116]). We expand $\exp(\frac{1}{3}v^3) = \sum v^{3k}/(3^k\, k!)$, and integrate termwise (replacing the upper limit of the interval by ∞). As a result we obtain

$$F(x) \sim \frac{\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}(\alpha+1)}}{\pi} \sum_{k=0}^{\infty} \frac{1}{3^k\, k!} \int_0^\infty v^{\alpha+3k} \mathrm{e}^{-xv}\, \mathrm{d}v. \tag{15}$$

Evaluating these integrals we get

$$F(x) \sim \frac{\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}(\alpha+1)}}{\pi\, x^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+3k+1)}{3^k\, k!} \frac{1}{x^{3k}}, \qquad x \to +\infty. \tag{16}$$

Taking the real and imaginary parts of the last expression we deduce (12) and (13).

Remark. In order to recover the known asymptotic expansions for $\alpha = 0, 1, 2, \ldots$ we need to complement (12) and (13) with the corresponding exponentially decaying terms from (14), that is the real and imaginary parts of the integral from $\mathrm{i}\sqrt{x}$ to $\infty \exp(\pi\mathrm{i}/6)$.

4. Asymptotic expansions of Riesz potentials of the Airy and Scorer functions f for $x \to -\infty$

Theorem 3 *The following asymptotic expansions hold for $x \to -\infty$:*

$$\begin{aligned} D_x^\alpha Ai(x) \sim & \frac{|x|^{\frac{1}{2}\alpha-\frac{1}{4}} \cos\left(\frac{1}{4}\pi - \frac{2}{3}|x|^{3/2}\right)}{\sqrt{\pi}} \\ & - \frac{|x|^{\frac{1}{2}\alpha-\frac{1}{4}} \sin\left(\frac{1}{4}\pi - \frac{2}{3}|x|^{3/2}\right)(12\alpha^2 - 24\alpha + 5)}{\sqrt{\pi}48|x|^{3/2}} \\ & + \frac{\cos\left(\frac{1}{2}\pi(\alpha+1)\right)}{\pi\,|x|^{\alpha+1}} \left[\Gamma(\alpha+1) - \frac{\Gamma(\alpha+4)}{3|x|^3} + \mathcal{O}\left(\frac{1}{|x|^6}\right)\right] \end{aligned} \tag{17}$$

and

$$\begin{aligned} D_x^\alpha Gi(x) \sim & \frac{|x|^{\frac{1}{2}\alpha-\frac{1}{4}} \sin\left(\frac{1}{4}\pi - \frac{2}{3}|x|^{3/2}\right)}{\sqrt{\pi}} \\ & - \frac{|x|^{\frac{1}{2}\alpha-\frac{1}{4}} \cos\left(\frac{1}{4}\pi - \frac{2}{3}|x|^{3/2}\right)(12\alpha^2 - 24\alpha + 5)}{\sqrt{\pi}48|x|^{3/2}} \\ & - \frac{\sin\left(\frac{1}{2}\pi(\alpha+1)\right)}{\pi\,|x|^{\alpha+1}} \left[\Gamma(\alpha+1) - \frac{\Gamma(\alpha+4)}{3|x|^3} + \mathcal{O}\left(\frac{1}{|x|^6}\right)\right]. \end{aligned} \tag{18}$$

Proof We write

$$F(-x) = \frac{1}{\pi}\int_0^\infty \xi^\alpha e^{i(-x\xi+\frac{1}{3}\xi^3)}\, d\xi, \tag{19}$$

and assume that in the proof $x \to +\infty$. For the integral (19) there is a positive stationary point at $\xi = \sqrt{x}$, which gives a contribution to the asymptotic expansion, but there is also a contribution from the origin. To handle both contributions, we replace the original path of integration by two new contours, giving two integrals $F(-x) = F_1(-x)+F_2(-x)$, where F_j are defined by

$$\begin{aligned} F_1(-x) &= \frac{1}{\pi}\int_0^{-i\infty} \xi^\alpha e^{i(-x\xi+\frac{1}{3}\xi^3)}\, d\xi, \\ F_2(-x) &= \frac{1}{\pi}\int_{-i\infty}^{\infty e^{\pi i/6}} \xi^\alpha e^{i(-x\xi+\frac{1}{3}\xi^3)}\, d\xi. \end{aligned} \tag{20}$$

So, the contour for F_2 runs from the valley at $-i\infty$ to the valley at $\infty\exp(\pi i/6)$, and we can take the contour through the saddle point at $\xi = \sqrt{x}$. See Figure 2.

For F_1 we integrate by setting $\xi = -iv$, $v > 0$ and obtain

$$F_1(-x) = \frac{e^{-\frac{1}{2}i(\alpha+1)}}{\pi}\int_0^\infty v^\alpha e^{-(xv+\frac{1}{3}v^3)}\, dv. \tag{21}$$

Proceeding as for the integral in (14) we deduce that

$$F_1(-x) \sim \frac{e^{-\frac{1}{2}\pi i(\alpha+1)}}{\pi\, x^{\alpha+1}} \sum_{k=0}^\infty \frac{\Gamma(\alpha+3k+1)}{3^k\, k!}\,\frac{(-1)^k}{x^{3k}}, \tag{22}$$

as $x \to +\infty$.

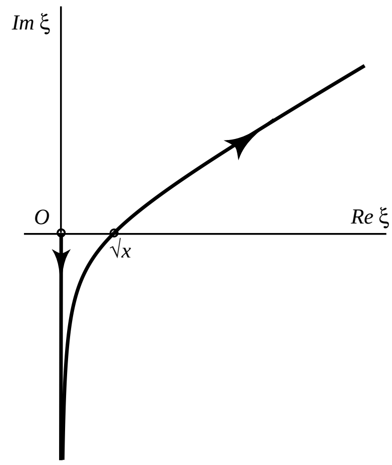

Figure 2. Modification of the path of integration giving the integrals in (20).

For F_2 we first write $\xi = \sqrt{x}\eta$, which gives

$$\begin{aligned} F_2(-x) &= \frac{x^{\frac{1}{2}(\alpha+1)}}{\pi} \int_{-\mathrm{i}\infty}^{\infty \mathrm{e}^{\pi \mathrm{i}/6}} \xi^\alpha \mathrm{e}^{-x\sqrt{x}\,\phi(\eta)}\, \mathrm{d}\eta, \\ \phi(\eta) &= \mathrm{i}\left(\eta - \frac{1}{3}\eta^3\right). \end{aligned} \tag{23}$$

We have $\phi(1) = \frac{2}{3}\mathrm{i}$ and $\phi''(1) = -2\mathrm{i}$. Performing the transformation

$$\phi(\eta) = \phi(1) + \frac{1}{2}\phi''(1)w^2,$$

that is

$$\begin{aligned} w^2 &= \frac{2}{3} - \eta + \frac{1}{3}\eta^3 = \frac{1}{3}(\eta+2)(\eta-1)^2, \\ w &= \sqrt{(\eta+2)/3}(\eta-1), \end{aligned} \tag{24}$$

We integrate in the neighborhood of the saddle point at $w = 0$ along the straight line through the origin which has an angle of $\frac{1}{4}\pi$ with the positive $w-$axis. This yields

$$F_2(-x) = \frac{x^{\frac{1}{2}(\alpha+1)}\, \mathrm{e}^{-\frac{2}{3}x\sqrt{x}\mathrm{i}}}{\pi} \int_{\infty \mathrm{e}^{-3\pi \mathrm{i}/4}}^{\infty \mathrm{e}^{\pi \mathrm{i}/4}} f(w) \mathrm{e}^{\mathrm{i}x\sqrt{x}w^2}\, \mathrm{d}w, \tag{25}$$

where

$$f(w) = \eta^\alpha \frac{\mathrm{d}\eta}{\mathrm{d}w}.$$

We expand $f(w) = \sum_{k=0}^\infty c_k w^k$ and deduce that

$$\begin{aligned} F_2(-x) \sim{}& \frac{x^{\frac{1}{2}(\alpha+1)}\, \mathrm{e}^{-\frac{2}{3}\mathrm{i}x\sqrt{x}}}{\pi} \\ &\times \sum_{k=0}^\infty c_{2k} \int_{\infty \mathrm{e}^{-3\pi \mathrm{i}/4}}^{\infty \mathrm{e}^{\pi \mathrm{i}/4}} w^{2k} \mathrm{e}^{\mathrm{i}x\sqrt{x}w^2}\, \mathrm{d}w. \end{aligned} \tag{26}$$

To evaluate the integrals we set $w = t\mathrm{e}^{\mathrm{i}\pi/4}$. This yields

$$\begin{aligned} \mathrm{e}^{\mathrm{i}(\frac{1}{4}+\frac{1}{2}\pi k)} &\int_{-\infty}^{\infty} t^{2k}\mathrm{e}^{-x\sqrt{x}t^2}\,\mathrm{d}t \\ &= \mathrm{e}^{\mathrm{i}(\frac{1}{4}\pi+\frac{1}{2}\pi k)}\Gamma\left(k+\frac{1}{2}\right)x^{-\frac{3}{2}(k+\frac{1}{2})}. \end{aligned} \tag{27}$$

So, we finally obtain

$$F_2(-x) \sim \frac{x^{\frac{1}{2}\alpha-\frac{1}{4}}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}-\frac{2}{3}\mathrm{i}x\sqrt{x}}}{\pi}\sum_{k=0}^{\infty} c_{2k}\frac{\mathrm{i}^k\,\Gamma(k+\frac{1}{2})}{x^{\frac{3}{2}k}}, \tag{28}$$

as $x \to +\infty$. A few first coefficients are

$$c_0 = 1, \quad c_2 = \frac{1}{24}\left(12\alpha^2 - 24\alpha + 5\right). \tag{29}$$

Taking the real and imaginary parts of (21) and (28) we obtain (17) and (18).

5. Applying the asymptotic results

The next statement was proved in [17].

Theorem 4 *The following representation holds for $x \in \mathbf{R}$, $a, b, \omega_1, \omega_2 \in \mathbf{R}$ and $\omega_1, \omega_2 \neq 0$:*

$$\begin{aligned} Ai\left(\frac{x-a}{\omega_1}\right)Ai\left(\frac{x-b}{\omega_2}\right) = &-\frac{2}{\Omega_1}\int_0^{\infty} J_0\left(2\left(\Omega_2 x + B\right)\eta\right) \\ &\times\frac{\mathrm{d}}{\mathrm{d}x}\left[Ai^2\left(\Omega_1 x - A + \eta^2\right)\right]\eta\,\mathrm{d}\eta, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \Omega_1 &= \frac{\omega_1+\omega_2}{2\omega_1\omega_2}, \qquad \Omega_2 = \frac{\omega_2-\omega_1}{2\omega_1\omega_2}, \\ A &= \frac{a\omega_1+b\omega_2}{2\omega_1\omega_2}, \qquad B = \frac{b\omega_1-a\omega_2}{2\omega_1\omega_2}. \end{aligned} \tag{31}$$

We list here several important corollaries that allow us to get the Hankel transforms of the function $A_2(a,b;x)$ and its Riesz fractional derivatives. Notice that

$$Ai\left(x-a\right)Ai\left(x-b\right) = Ai\left(x-Y-Z\right)Ai\left(x-Y+Z\right),$$

where

$$Y = \frac{a+b}{2} \qquad \text{and} \qquad Z = \frac{b-a}{2}. \tag{32}$$

Corollary 1 *The following formulas hold for $x \in \mathbf{R}$ and $a, b \in \mathbf{R}$:*

$$A_2(a,b;x) = -2\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\infty} Ai^2\left(x-Y+\eta^2\right)J_0\left(2Z\eta\right)\eta\,\mathrm{d}\eta \tag{33}$$

and

$$\begin{aligned} -H_x\left\{A_2(a,b;x)\right\} = -2&\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\infty} Ai\left(x-Y+\eta^2\right) \\ &\times Bi\left(x-Y+\eta^2\right)J_0\left(2Z\eta\right)\eta\,\mathrm{d}\eta. \end{aligned} \tag{34}$$

Proof Evidently, (33) is a particular case of (30) when $\omega_1 = \omega_2 = 1$. Taking the Hilbert transform of (33) with respect to x yields (34).

Corollary 2 *For* $\alpha,\ a,\ b \in \mathbf{R}$ *Riesz fractional derivatives of the function* $A_2(a,b;x)$ *are given by the formula*

$$D_x^\alpha \{A_2(a,b;x)\} = -\frac{2^{2(\alpha-1)/3}}{\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \Big[\Big(D_x^{\alpha-1/2}Ai\Big)\Big(2^{2/3}\big(x-Y+\eta^2\big)\Big) - \Big(D_x^{\alpha-1/2}Gi\Big)\Big(2^{2/3}\big(x-Y+\eta^2\big)\Big)\Big] J_0(2Z\eta)\,\eta\,\mathrm{d}\eta \tag{35}$$

and

$$H\{D_x^\alpha \{A_2(a,b;x)\}\} = \frac{2^{2(\alpha-1)/3}}{\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \Big[\Big(D_x^{\alpha-1/2}Ai\Big)\Big(2^{2/3}\big(x-Y+\eta^2\big)\Big) + \Big(D_x^{\alpha-1/2}Gi\Big)\Big(2^{2/3}\big(x-Y+\eta^2\big)\Big)\Big] J_0(2Z\eta)\,\eta\,\mathrm{d}\eta, \tag{36}$$

where the integrals in the right-hand sides exist at least in the sense of distributions.

Proof Follows from (33) and (34).

Corollary 3 *The following relations hold for* $\alpha > -\frac{1}{2}$*:*

$$2\mathcal{H}_{Z\to\zeta}\Big\{D_x^{\alpha-1}\left(Ai(x-Z)Ai(x+Z)\right)\Big\} = k_\alpha\Big[D^{\alpha-1/2}Ai\,(X) + D^{\alpha-1/2}Gi\,(X)\Big] \tag{37}$$

and

$$2\mathcal{H}_{Z\to\zeta}\Big\{D_x^{\alpha-1}H_x\left(Ai(x-Z)Ai(x+Z)\right)\Big\} = k_\alpha\Big[D^{\alpha-1/2}Ai\,(X) - D^{\alpha-1/2}Gi\,(X)\Big], \tag{38}$$

where k_α *is defined by (4) and* $X = 2^{2/3}\left(x + \frac{1}{4}\zeta^2\right)$.

Combining the asymptotic expansions (12), (13), (17) and (18) and Corollary 3 we can obtain asymptotic expansions of the Hankel transforms (37) and (38) for $x \to \pm\infty$ or $\zeta \to \infty$.

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