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On a Biorthogonal System associated with Uniform Asymptotic Expansions

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In a paper by Soni & Sleeman (1987), a family of polynomials is introduced. These polynomials are related to the coefficients in a uniform asymptotic expansion of a class of integrals. In this expansion parabolic cylinder functions (Weber functions) occur as basic approximants and the resulting series is of Bleistein type. In the present paper, a family of rational functions is introduced, and the two families form a biorthogonal system on a contour in the complex plane. The system can be viewed as a generalization of the families $\{z^n\}$ and $\{z^{-n-1}\}$, which occur in Taylor expansions and the Cauchy integrals of analytic functions. Explicit representations of the rational functions are given together with the rigorous estimates. These results are used to establish convergence of expansions of certain functions in terms of the polynomials and the rational functions. The main motivation to study this system stems from the above-mentioned problem on the asymptotic expansion of a class of integrals. It is shown how to use the system in order to construct bounds for the remainders in the asymptotic expansion. An instructive example is worked out in detail.

1. Introduction

Let $\{P_n(t)\}$ denote the system of polynomials defined as follows:

$$P_0(t) = 1, \tag{1.1}$$

$$P_1(t) = t/(\gamma + 1) \quad (\gamma > -1), \tag{1.2}$$

and, for $n \geqslant 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}[t^\gamma P_n(t)] = t^\gamma (t - b) P_{n-2}(t), \tag{1.3}$$

where b is an arbitrary complex number. In (1.3) we may assume that $-\pi < \arg t \leqslant \pi$. However, the polynomials are defined in the whole complex t-plane. For $n \geqslant 1$ and $-\pi < \arg t \leqslant \pi$,

$$P_{2n-1}(t) = \frac{t^{-\gamma}}{2^{n-1}\Gamma(n)} \int_0^t [t^2 - u^2 - 2b(t-u)]^{n-1} u^\gamma \, \mathrm{d}u, \tag{1.4}$$

$$P_{2n}(t) = \frac{t^{-\gamma}}{2^{n-1}\Gamma(n)} \int_0^t [t^2 - u^2 - 2b(t-u)]^{n-1} u^\gamma (u - b) \, \mathrm{d}u. \tag{1.5}$$

This system of polynomials for real t was developed by Soni & Sleeman [3] in connection with the uniform asymptotic expansion of integrals of the type

$$I(\eta)=\int_0^a t^\gamma(t-b)e^{-\eta(\frac{1}{2}t^2-bt)}g(t)\,dt, \tag{1.6}$$

when $0<a\leqslant\infty$, $|\eta|\to\infty$, $|\arg\eta|\leqslant\pi/2$ and the stationary point $t=b>0$ approaches the critical point $t=0$. In particular, if $g(t)$ is analytic in some neighbourhood of the origin and has the expansion

$$g(t)=\sum_{k=0}^{\infty} c_k P_k(t), \tag{1.7}$$

then the expansion coefficients $c_k=c_k(b,\gamma)$ are continuous functions of b near $b=0$ and, for $0\leqslant 2b<a$, $|\arg\eta|<\pi/2$, $|\eta|\to\infty$, $\gamma>0$,

$$I(\eta)\sim\Big(\sum_{k=0}^{\infty} c_{2k}\eta^{-k-1}\Big)\gamma W_{\gamma-1}(\eta)+\Big(\sum_{k=0}^{\infty} c_{2k+1}\eta^{-k-1}\Big)W_\gamma(\eta), \tag{1.8}$$

where

$$W_\gamma(\eta)=\int_0^\infty t^\gamma e^{-\eta(\frac{1}{2}t^2-bt)}\,dt. \tag{1.9}$$

The coefficients c_k are given in terms of the operator D defined by

$$(t-b)^{-1}\frac{d}{dt}=\mathrm{D},\qquad \mathrm{D}\mathrm{D}^{n-1}=\mathrm{D}^n, \tag{1.10}$$

and, for $n\geqslant 0$, $t>0$,

$$c_{2n}=t^{-\gamma}\mathrm{D}^n\Big\{t^\gamma\Big[g(t)-\sum_{k=0}^{2n-1} c_k P_k(t)\Big]\Big\}\Big|_{t=0}, \tag{1.11}$$

$$c_{2n+1}=t^{-\gamma}\frac{d}{dt}\mathrm{D}^n\Big\{t^\gamma\Big[g(t)-\sum_{k=0}^{2n} c_k P_k(t)\Big]\Big\}\Big|_{t=b} \tag{1.12}$$

(see [3]). Except perhaps for c_0 and c_1, the computation of the coefficients c_k by this process is, in general, quite complicated. To make matters worse, in many applications the integral (1.6) is arrived at by a change of variable which is defined only implicitly [1, 2: p. 347]. In this paper, we give an alternative approach based on the complex function theory. If we assume that $g(t)$ is analytic in some neighbourhood of the origin and if $\{\phi_n\}$ is defined by

$$\frac{1}{z-t}=\sum_{n=0}^{\infty}\phi_n(z)P_n(t), \tag{1.13}$$

then formally, for some appropriate contour C,

$$g(t)=\frac{1}{2\pi i}\int_C\frac{g(z)}{z-t}dz=\sum_{n=0}^{\infty}P_n(t)\Big(\frac{1}{2\pi i}\int_C\phi_n(z)g(z)\,dz\Big). \tag{1.14}$$

Now a comparison with (1.7) indicates that

$$\frac{1}{2\pi i}\int_C \phi_n(z)g(z)\,dz = c_n, \tag{1.15}$$

$$\frac{1}{2\pi i}\int_C \phi_n(z)P_m(z)\,dz = \delta_{n,m}. \tag{1.16}$$

We prove that the relations (1.13)–(1.16) do indeed hold and that the system $(\{P_n(z)\}, \{\phi_m(z)\})$ is a biorthogonal system which for $b=0$ reduces to the system $(\{z^n\}, \{z^{-m-1}\})$. In Section 2, we define the operators I_t and Θ_z and the functions $\phi_n(z)$ $(n=0, \ldots,)$. In Section 3, we state the main results, including the one mentioned above. These are proved in Sections 5 and 6. Some preliminary results are given in Section 4. An application of the expansion technique developed here to the uniform asymptotic expansions is discussed in the last section.

We conclude this section by describing Bleistein's method for (1.6) with $a=\infty$. Using integration by parts, we have

$$\eta I(\eta) = \int_0^\infty t^{\gamma-1}h(t)e^{-\eta(\frac{1}{2}t^2-bt)}\,dt, \qquad h(t) = t^{1-\gamma}\frac{d}{dt}[t^\gamma g(t)].$$

Next we write

$$h(t) = p_0 + p_1 t + t(t-b)f(t),$$

where $p_0 = h(0) = \gamma g(0)$ and $p_1 = [h(b)-h(0)]/b$. This gives, again after integrating by parts,

$$\eta I(\eta) = p_0 W_{\gamma-1}(\eta) + p_1 W_\gamma(\eta) + \frac{1}{\eta}\int_0^\infty t^{\gamma-1}h_1(t)e^{-\eta(\frac{1}{2}t^2-bt)}\,dt,$$

with

$$h_1(t) = t^{1-\gamma}\frac{d}{dt}[t^\gamma f(t)].$$

Repeating this procedure, we can generate more coefficients p_k, and hence more terms in Bleistein's expansion. Up to minor changes, p_k are the same as c_k in (1.7).

2. Notation and definitions

Polynomials $P_n(t)$

$P_n(t)$ denotes a polynomial of degree n and the sequence $\{P_n(t)\}$ satisfies (1.1)–(1.5). Furthermore,

$$P_n(0) = 0 \quad (n = 1, 2, \ldots). \tag{2.1}$$

For other properties of these polynomials, we will refer to [3].

The integral operator I

$$I_t f(t) = \int_0^t (u-b) f(u)\,\mathrm{d}u, \tag{2.2}$$

where t may be a complex number. Also,

$$(I_t)^1 = I_t, \tag{2.3}$$

$$(I_t)^n = I_t (I_t)^{n-1} \quad (n = 2, 3, \ldots). \tag{2.4}$$

By (1.3) and (2.1), for $-\pi < \arg t \leq \pi$, we can write

$$t^\gamma P_n(t) = I_t[t^\gamma P_{n-2}(t)] \qquad (n = 2, 3, \ldots), \tag{2.5}$$

$$t^\gamma P_{2n-1}(t) = (I_t)^{n-1}[t^\gamma P_1(t)] \quad (n = 2, 3, \ldots), \tag{2.6}$$

$$t^\gamma P_{2n}(t) = (I_t)^n [t^\gamma P_0(t)] \qquad (n = 1, 2, \ldots). \tag{2.7}$$

The differential operator Θ_z

$$\Theta_z f(z) = -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z-b} f(z)\right). \tag{2.8}$$

Also,

$$(\Theta_z)^1 = \Theta_z, \qquad (\Theta_z)^n = \Theta_z(\Theta_z)^{n-1} \quad (n = 2, 3, \ldots). \tag{2.9}$$

Definition of $\{\phi_n(z)\}$

The sequence $\{\phi_n(z)\}$ is defined as follows:

$$\phi_0(z) = 1/z, \tag{2.10}$$

$$\phi_1(z) = \frac{\gamma}{z(z-b)} + \frac{1}{(z-b)^2}. \tag{2.11}$$

$$\phi_{n+2}(z) = -z^\gamma \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\phi_n(z)}{z^\gamma (z-b)}\right) \quad (n = 0, 1, \ldots). \tag{2.12}$$

In (2.12), we may assume that $-\pi < \arg z \leq \pi$ $(z \neq 0, z \neq b)$, although we see by induction that $\phi_n(z)$ are meromorphic functions of z in the whole complex plane; their only singularities are poles at $z = 0$ and $z = b$. If we use the differential operator Θ_z, we can write

$$\phi_1(z) = z^\gamma \Theta_z(1/z^\gamma), \tag{2.13}$$

$$\phi_n(z) = z^\gamma \Theta_z[\phi_{n-2}(z)/z^\gamma] \quad (n = 2, 3, \ldots), \tag{2.14}$$

and, for $n = 1, \ldots$,

$$\phi_{2n-1}(z) = z^\gamma(\Theta_z)^n(1/z^\gamma), \qquad \phi_{2n}(z) = z^\gamma(\Theta_z)^n(1/z^{\gamma+1}). \tag{2.15}$$

The sequence $\{\phi_n(z)\}$ is defined uniquely by (2.10)–(2.12). The motivation to study these functions is provided by the following formal computations. Assume that $1/(z-t)$ can be expanded in terms of the polynomials $P_n(t)$,

$$\frac{1}{z-t} = \sum_{n=0}^{\infty} \phi_n(z) P_n(t), \tag{2.16}$$

and that the expansion is valid in particular at $t = 0$ and $t = b$. Substitute $t = 0$ and

$P_0(t)=1$ in (2.16). By (2.1), we obtain $\phi_0(z)=1/z$. Now multiply both sides by t^γ and differentiate term by term with respect to t. By (1.2) and (1.3), the resulting equation can be written as

$$(t-b)\sum_{n=2}^{\infty}\phi_n(z)P_{n-2}(t)=\frac{\gamma}{z(z-t)}+\frac{1}{(z-t)^2}-\phi_1(z). \qquad (2.17)$$

For $t=b$, we obtain $\phi_1(z)$ as defined in (2.11). When we substitute for $\phi_1(z)$ and simplify, the above equation reduces to

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_{n+2}(z)P_n(t)&=\left(\frac{\gamma}{z(z-b)}+\frac{1}{(z-b)^2}\right)\frac{1}{z-t}+\frac{1}{(z-b)(z-t)^2}\\&=\left(\frac{\gamma}{z(z-b)}+\frac{1}{(z-b)^2}\right)\sum_{n=0}^{\infty}\phi_n(z)P_n(t)-\frac{1}{z-b}\sum_{n=0}^{\infty}\phi_n'(z)P_n(t).\end{aligned} \qquad (2.18)$$

By comparing the coefficients of $P_n(t)$,

$$\phi_{n+2}(z)=\left(\frac{\gamma}{z(z-b)}+\frac{1}{(z-b)^2}\right)\phi_n(z)-\frac{1}{(z-b)}\phi_n'(z), \qquad (2.19)$$

which can be expressed as (2.12).

In Theorems 1–3 of the next section, we establish the validity of (2.16) when $|t|+2|b|<|z|$.

Definition of the operator D^n

$$\mathrm{D}^1=\mathrm{D}=(z-b)^{-1}\frac{\mathrm{d}}{\mathrm{d}z}, \qquad \mathrm{D}^n=\mathrm{D}\mathrm{D}^{n-1}. \qquad (2.20)$$

This operator is used by Soni & Sleeman [3] for computing the expansion coefficients in the expansions of type (1.7) as well as for the representation of the remainder in the asymptotic expansions of the Bleistein type [1]. Note that

$$\mathrm{D}^n f(z)=(z-b)^{-1}(-\Theta_z)^n[(z-b)f(z)]. \qquad (2.21)$$

Furthermore, if

$$f(t)=\sum_{k=0}^{\infty}c_kP_k(t),$$

then by (1.3) and (2.20) it follows that

$$t^{-\gamma}\mathrm{D}^n t^\gamma\left\{f(t)-\sum_{k=0}^{2n-1}c_kP_k(t)\right\}=\sum_{k=0}^{\infty}c_{k+2n}P_k(t). \qquad (2.22)$$

3. Statement of results

We will assume that $\gamma\geqslant 0$ unless specified otherwise.

THEOREM 1 *If $\phi_n(z)$ are defined by* (2.10)–(2.12), *then, for* $n=1,2,\ldots$,

$$\phi_{2n-1}(z)=\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{n}\frac{\Gamma(\gamma+n-k)\Gamma(n+k+1)}{k!\,(n-k)!\,2^kz^{n-k}(z-b)^{n+k}}, \qquad (3.1)$$

$$\phi_{2n}(z)=\frac{1}{\Gamma(\gamma+1)}\sum_{k=0}^{n}\frac{\Gamma(\gamma+n-k+1)\Gamma(n+k+1)}{k!\,(n-k)!\,2^kz^{n-k+1}(z-b)^{n+k}}. \qquad (3.2)$$

Furthermore, if $z \neq \xi b$ $(0 \leqslant \xi \leqslant 1)$, $-\pi < \arg z \leqslant \pi$, $\gamma > 0$, then

$$\phi_{2n-1}(z) = \frac{\Gamma(\gamma+2n+1)}{\Gamma(\gamma)2^n n!}(z-b)z^\gamma \int_0^1 \frac{u^{\gamma-1}(1-u^2)^n}{(z-b+bu)^{\gamma+2n+1}}\,du, \tag{3.3}$$

$$\phi_{2n}(z) = \frac{\Gamma(\gamma+2n+2)}{\Gamma(\gamma+1)2^n n!}(z-b)z^\gamma \int_0^1 \frac{u^{\gamma}(1-u^2)^n}{(z-b+bu)^{\gamma+2n+2}}\,du, \tag{3.4}$$

where the integration is along the straight line segment from 0 to 1. In (3.3) and (3.4), $\arg(z-b+bu)$ is chosen so that $\arg(z-b+bu) \to \arg z$ as $b \to 0$.

By (3.1) and (3.2), the only singularities of $\phi_n(z)$ are poles at $z=0$ and $z=b$. As $b \to 0$, these poles coalesce and then $\phi_n(z)$ has a pole of order $(n+1)$ at $z=0$. In fact, when $b=0$, $\phi_n(z) = k_n/z^{n+1}$, where k_n is a constant depending on γ and n.

THEOREM 2 *The series* $\sum_{n=0}^\infty P_n(t)\phi_n(z)$ *converges absolutely when* $0 < (|t| + |b|)/(|z| - |b|) < 1$ *and uniformly in the regions* Ω_t *and* Ω_z *which satisfy*

$$\sup_{t \in \Omega_t,\, z \in \Omega_z} \frac{|t|+|b|}{|z|-|b|} \leqslant 1-\delta < 1, \qquad |z| > |b|. \tag{3.5}$$

THEOREM 3 *Let* $|b| \leqslant \beta$. *Then, for all z and t such that* $|t| + 2\beta < |z|$,

$$(z-t)^{-1} = \sum_{n=0}^\infty P_n(t)\phi_n(z). \tag{3.6}$$

For $\beta = 0$, (3.6) reduces to the standard decomposition $(z-t)^{-1} = \sum_{n=0}^\infty t^n/z^{n+1}$ $(|t| < |z|)$. Theorem 3 can be proved by using a result of Soni & Sleeman [3] for real z and then extending it by analytic continuation. However, to make the paper self-contained, we give a proof which is of interest in itself.

THEOREM 4 *Let C be a simple closed contour, positively oriented, with* $z=0$ *and* $z=b$ *inside C. Then* $(\{P_m\}, \{\phi_n\})$ *is biorthogonal on C and*

$$\frac{1}{2\pi i}\int_C P_m(t)\phi_m(t)\,dt = 1. \tag{3.7}$$

The above theorem gives a straightforward extension of the biorthogonal system $(\{z^m\}, \{z^{-n-1}\})$.

The corresponding extensions of the Taylor and the Laurent series are given in the following theorems.

THEOREM 5 *Let* $f(t)$ *be regular in* $|t| \leqslant r$ *and let* $|b| \leqslant \beta < r$. *Then, for* $|t| + 2\beta < r$,

$$f(t) = \sum_{k=0}^\infty c_k P_k(t), \tag{3.8}$$

where

$$c_k = \frac{1}{2\pi i}\int_C f(w)\phi_k(w)\,dw \tag{3.9}$$

and C is the positively oriented contour $|w| = r$. Furthermore, if

$$R_n(t) = f(t) - \sum_{k=0}^{n-1} c_k P_k(t), \tag{3.10}$$

then

$$|R_n(t)| \leqslant \frac{r}{r - |t| - 2\beta} \left(\frac{|t| + \beta}{r - \beta} \right)^n \max_{|t|=r} |f(t)|. \tag{3.11}$$

Another bound for the remainder is given by

$$|R_{2m}(t)| \leqslant \frac{(|t| + \beta)^{2m}}{(2m)!} N \quad (m = 0, 1, \ldots), \tag{3.12}$$

where

$$N = \max_{w \in \triangle} |f^{(2m)}(w)|, \tag{3.13}$$

with $\triangle$ the triangle joining the points $w = 0$, b, t, respectively.

The next theorem is analogous to the Cauchy inequality.

THEOREM 6 *The expansion coefficients $c_k = c_k(b)$ in Theorem 5, are analytic functions of b in $|b| < r$ and for $|b| \leqslant \beta < r$ satisfy the following inequalities:*

$$|c_{2k-1}| \leqslant \frac{2^k \Gamma[\frac{1}{2}(\gamma + 1) + k] r}{\Gamma[\frac{1}{2}(\gamma + 1)](r - \beta)^{2k}} \max_{|z|=r} |f(z)| \tag{3.14}$$

and

$$|c_{2k}| \leqslant \frac{2^k \Gamma(\frac{1}{2}\gamma + k + 1) r}{\Gamma(\frac{1}{2}\gamma + 1)(r - \beta)^{2k+1}} \max_{|z|=r} |f(z)|. \tag{3.15}$$

Furthermore,

$$|c_{2k-1}| \leqslant \frac{2^k \Gamma[\frac{1}{2}(\gamma + 1) + k]}{\Gamma[\frac{1}{2}(\gamma + 1)]\Gamma(2k)} \max_{0 \leqslant \theta \leqslant 1} |f^{(2k-1)}(\theta b)|, \tag{3.16}$$

$$|c_{2k}| \leqslant \frac{2^k \Gamma(\frac{1}{2}\gamma + k + 1)}{\Gamma(\frac{1}{2}\gamma + 1)\Gamma(2k + 1)} \max_{0 \leqslant \theta \leqslant 1} |f^{(2k)}(\theta b)|. \tag{3.17}$$

THEOREM 7 *Let $f(t)$ be regular in the annulus $r \leqslant |t| \leqslant R$, $R - r > 4\beta$, $|b| \leqslant \beta$. Then for $r + 2\beta < |t| < R - 2\beta$*

$$f(t) = \sum_{k=0}^{\infty} c_k P_k(t) + \sum_{k=0}^{\infty} d_k \phi_k(t), \tag{3.18}$$

where

$$c_k = \frac{1}{2\pi \mathrm{i}} \int_C f(w) \phi_k(w)\, \mathrm{d}w,$$

$$d_k = \frac{1}{2\pi \mathrm{i}} \int_C f(w) P_k(w)\, \mathrm{d}w,$$

and C is the circle $|w| = r'$, oriented in the positive sense $r < r' < R$, $\beta < r'$.

In the last theorem, we provide two bounds for $\mathsf{D}^n t^\gamma R_{2n}(t)$, where R_n is defined in (3.10) and D in (2.20). For real t the second bound is given by Soni & Sleeman [3].

THEOREM 8 *If $f(t)$ is regular in $|t| \leqslant r$, then for $|t| < r$, $|b| \leqslant \beta < r$,*

$$|t^{-\gamma}\mathsf{D}^n t^\gamma R_{2n}(t)| \leqslant \frac{2^n\Gamma(\frac{1}{2}\gamma+n+1)r}{\Gamma(\frac{1}{2}\gamma+1)d^{2n+1}} |\max_{|t|=r} f(t)|, \tag{3.19}$$

where $R_{2n}(t)$ is the remainder after $2n$ terms, D is defined by (2.20), and $d = \min\{r-\beta, r-|t|\}$;

$$|t^{-\gamma}\mathsf{D}^n t^\gamma R_{2n}(t)| \leqslant \frac{2^n\Gamma(\frac{1}{2}\gamma+n+1)}{\Gamma(\frac{1}{2}\gamma+1)\Gamma(2n+1)} N, \tag{3.20}$$

where N is defined by (3.13).

4. Some preliminary results

LEMMA 1 *Let $f(u)$ be locally integrable in $[0,\infty)$ and of exponential order $\sigma \geqslant 0$ as $u \to \infty$. If*

$$F(z) = \int_0^\infty e^{-zu} f(u)\,du, \qquad \operatorname{Re} z > \sigma, \tag{4.1}$$

then

(*a*) *for* $\operatorname{Re} z > \max\{\operatorname{Re} b, \sigma\}$ $(n = 0, 1, \ldots)$,

$$(\Theta_z)^n(F(z)) = (z-b)\int_0^\infty e^{-(z-b)u} f_{2n}(u)\,du, \tag{4.2}$$

where

$$f_{2n}(u) = \frac{1}{n!\,2^n}\int_0^u e^{-bv}(u^2-v^2)^n f(v)\,dv; \tag{4.3}$$

(b) *for* $\operatorname{Re} z > \sigma$, $z \neq b$,

$$(\Theta_z)^n F(z) = \frac{1}{n!}\sum_{k=0}^n \binom{n}{k} \frac{2^{-k}(n+k)!}{(z-b)^{n+k}} (-1)^{n-k} F^{(n-k)}(z). \tag{4.4}$$

Proof

$$F(z) = \int_0^\infty e^{-(z-b)u} e^{-bu} f(u)\,du. \tag{4.5}$$

Using integration by parts,

$$F(z) = (z-b)\int_0^\infty e^{-(z-b)u} f_0(u)\,du, \tag{4.6}$$

where $f_0(u)$ is defined by (4.3). The integrated term vanishes at both ends. Then

$$-\frac{\mathrm{d}}{\mathrm{d}z}\frac{F(z)}{(z-b)}=\int_0^\infty \mathrm{e}^{-(z-b)u}uf_0(u)\,\mathrm{d}u. \tag{4.7}$$

Since

$$\int_0^u vf_0(v)\,\mathrm{d}v=\int_0^u v\int_0^v \mathrm{e}^{-bs}f(s)\,\mathrm{d}s\,\mathrm{d}v=\int_0^u \mathrm{e}^{-bs}f(s)\int_s^u v\,\mathrm{d}v\,\mathrm{d}s=\tfrac{1}{2}\int_0^u \mathrm{e}^{-bs}(u^2-s^2)f(s)\,\mathrm{d}s, \tag{4.8}$$

after an integration by parts (4.7) can be written as

$$\Theta_z F(z)=(z-b)\int_0^\infty \mathrm{e}^{-(z-b)u}f_2(u)\,\mathrm{d}u. \tag{4.9}$$

Now (4.2) and (4.3) follow by induction. To prove (4.4), substitute for $f_{2n}(u)$ in (4.2) and interchange the order of integration. Then

$$\begin{aligned}(\Theta_z)^n F(z)&=\frac{(z-b)}{n!\,2^n}\int_0^\infty \mathrm{e}^{-bv}f(v)\int_v^\infty \mathrm{e}^{-(z-b)u}(u^2-v^2)^n\,\mathrm{d}u\,\mathrm{d}v\\ &=\frac{(z-b)}{n!\,2^n}\int_0^\infty \mathrm{e}^{-2v}f(v)\int_0^\infty \mathrm{e}^{-(z-b)s}s^n(s+2v)^n\,\mathrm{d}s\,\mathrm{d}v.\end{aligned} \tag{4.10}$$

By using the binomial expansion for $(s+2v)^n$, we obtain

$$(\Theta_z)^n F(z)=\frac{(z-b)}{n!\,2^n}\sum_{k=0}^n\binom{n}{k}2^{n-k}\left(\int_0^\infty \mathrm{e}^{-zv}v^{n-k}f(v)\,\mathrm{d}v\right)\frac{\Gamma(n+k+1)}{(z-b)^{n+k+1}}. \tag{4.11}$$

The right-hand side can be written as in (4.4).

LEMMA 2 *Let $F(z)=z^{-\gamma}$ $(\gamma\geqslant 0)$. Then, for $z\notin\{0,b\}$, $-\pi<\arg z\leqslant\pi$,*

$$(\Theta_z)^n F(z)=\frac{1}{\Gamma(\gamma)}\sum_{k=0}^n\frac{(n+k)!\,\Gamma(\gamma+n-k)}{k!\,(n-k)!\,2^k z^{n-k+\gamma}(z-b)^{n+k}} \tag{4.12}$$

and

$$(\Theta_z)^n F(z)\,\big|_{z=1,b=0}=\frac{\Gamma(2n+1+\gamma)\Gamma(\frac{1}{2}\gamma)}{2^{n+1}\Gamma(n+1+\frac{1}{2}\gamma)\Gamma(\gamma)}. \tag{4.13}$$

Proof. In Lemma, 1, let $f(u)=u^{\gamma-1}/\Gamma(\gamma)$ $(\gamma>0)$, so that $F(z)=z^{-\gamma}$. Since

$$(-1)^{n-k}(z^{-\gamma})^{(n-k)}=\frac{\Gamma(\gamma+n-k)}{\Gamma(\gamma)z^{\gamma+n-k}}, \tag{4.14}$$

the relation (4.12) follows from (4.4). The restriction $\operatorname{Re} z>0$ can be removed by analytic continuation. Again, by (4.2) and (4.3),

$$(\Theta_z)^n F(z)\,|_{z=1,b=0} = \frac{1}{n!\,2^n\Gamma(\gamma)} \int_0^\infty \mathrm{e}^{-u} \int_0^u (u^2 - v^2)^n v^{\gamma-1}\,\mathrm{d}v\,\mathrm{d}u$$

$$= \frac{1}{n!\,2^n\Gamma(\gamma)} \int_0^\infty \mathrm{e}^{-u} u^{2n+\gamma}\,\mathrm{d}u \int_0^1 (1 - s^2)^n s^{\gamma-1}\,\mathrm{d}s$$

$$= \frac{\Gamma(2n+1+\gamma)\Gamma(\frac{1}{2}\gamma)}{2^{n+1}\Gamma(n+1+\frac{1}{2}\gamma)\Gamma(\gamma)}. \tag{4.15}$$

This proves (4.13). For $\gamma = 0$, take the limit in (4.12) and (4.13) as $\gamma \to 0^+$.

LEMMA 3 *Let* $F(z) = \{z^\gamma(z-t)\}^{-1}$ $(\gamma \geq 0)$. *For* $z \notin \{0, b, t\}$, $-\pi < \arg z \leq \pi$,

$$(\Theta_z)^n \left(\frac{1}{z^\gamma(z-t)}\right) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,2^k(z-b)^{n+k}} \sum_{l=0}^{n-k} \frac{\Gamma(l+\gamma)}{l!\,z^{l+\gamma}(z-t)^{n-k-l+1}}. \tag{4.16}$$

Furthermore,

$$(\Theta_z)^n \left(\frac{1}{z^\gamma(z-t)}\right)\Bigg|_{z=1,t=b=0} = \frac{\Gamma(2n+2+\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{2^{n+1}\Gamma(\gamma+1)\Gamma(n+\frac{3}{2}+\frac{1}{2}\gamma)}. \tag{4.17}$$

Proof. For $\gamma > 0$, apply Lemma 1. Let

$$f(u) = \frac{1}{\Gamma(\gamma)} \int_0^u v^{\gamma-1} \mathrm{e}^{(u-v)t}\,\mathrm{d}v. \tag{4.18}$$

For each t, $f(u)$ is of finite exponential order $\sigma = \sigma(t)$ and

$$F(z) = \frac{1}{z^\gamma(z-t)}, \qquad \operatorname{Re} z > \sigma. \tag{4.19}$$

Since

$$(-1)^{n-k}[F(z)]^{(n-k)} = \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{\Gamma(\gamma+l)(n-k-l)!}{\Gamma(\gamma) z^{\gamma+l}(z-t)^{n-k-l+1}}, \tag{4.20}$$

the relation (4.16) follows from (4.4). By analytic continuation, the equality holds for all z $(z \notin \{0, b, t\}$, $-\pi < \arg z \leq \pi)$. Again for $z = 1$, $t = b = 0$,

$$f(u) = u^\gamma/\Gamma(\gamma+1).$$

Hence, (4.17) follows from Lemma 2 by replacing γ by $(\gamma+1)$. As in Lemma 2, (4.16) and (4.17) hold for $\gamma = 0$ also.

LEMMA 4 *For* $z \notin \{0, b, t\}$, $-\pi < \arg z \leq \pi$, $-\pi < \arg t \leq \pi$, $\gamma \geq 0$,

$$I_t\Theta_z[(t/z)^\gamma(z-t)^{-1}] = (t/z)^\gamma[(z-t)^{-1} - P_0(t)\phi_0(z) - P_1(t)\phi_1(z)]. \tag{4.21}$$

Proof. Let

$$J(t) = \frac{\mathrm{d}}{\mathrm{d}t} I_t\Theta_z[(t/z)^\gamma(z-t)^{-1}] = t^\gamma(t-b)\Theta_z[z^{-\gamma}(z-t)^{-1}]. \tag{4.22}$$

By Lemma 3, for $n=1$,

$$J(t)=(t/z)^{\gamma}(t-b)\left(\frac{1}{(z-b)(z-t)^2}+\frac{1}{(z-b)^2(z-t)}+\frac{\gamma}{z(z-b)(z-t)}\right). \tag{4.23}$$

After some rearrangement of the terms, we can write

$$J(t)=(t/z)^{\gamma}\left(\frac{1}{(z-t)^2}-\frac{1}{(z-b)^2}+\frac{\gamma(t-b)}{z(z-b)(z-t)}\right). \tag{4.24}$$

By expressing the right-hand side of (4.21) as

$$K(t)=z^{-\gamma}\left[\frac{t^{\gamma+1}}{z(z-t)}-\frac{t^{\gamma+1}}{\gamma+1}\left(\frac{\gamma}{z(z-b)}+\frac{1}{(z-b)^2}\right)\right]$$

and differentiating with respect to t, we can easily verify that $J(t)=K'(t)$. The equality in (4.21) holds because the two sides are equal for $t=0$.

5. Proof of Theorems 1–4

Proof of Theorem 1. The representation (3.1) for $\phi_{2n-1}(z)$ follows from (2.15) and (4.12). For the representation for $\phi_{2n}(z)$, use (2.15) and apply (4.12) to $(\Theta_z)^n(z^{-\gamma-1})$. To obtain (3.3), proceed as follows. By (4.2) and (4.3), for $f(u)=u^{\gamma-1}/\Gamma(\gamma)$, $\operatorname{Re} z>\max\{\operatorname{Re} b, 0\}$, $\gamma>0$,

$$(\Theta_z)^n(z^{-\gamma})=\frac{(z-b)}{\Gamma(\gamma)2^n n!}\int_0^{\infty}e^{-(z-b)u}\int_0^u e^{-bv}(u^2-v^2)^n v^{\gamma-1}\,dv\,du. \tag{5.1}$$

By a change of variable,

$$\begin{aligned}(\Theta_z)^n(z^{-\gamma})&=\frac{z-b}{\Gamma(\gamma)2^n n!}\int_0^{\infty}e^{-(z-b)u}u^{2n+\gamma}\int_0^1 e^{-bus}s^{\gamma-1}(1-s^2)^n\,ds\,du,\\ &=\frac{z-b}{\Gamma(\gamma)2^n n!}\int_0^1(1-s^2)^n s^{\gamma-1}\int_0^{\infty}e^{-(z-b+bs)u}u^{2n+\gamma}\,du\,ds.\end{aligned} \tag{5.2}$$

The interchange of the order of integration is justified when $\operatorname{Re} z>\max\{\operatorname{Re} b, 0\}$ and the inner integral can be given explicitly. Now use (2.15) to obtain (3.3). The condition on z can be relaxed because the integral converges for all z except for $z=\xi b$ $(0\leq\xi\leq 1)$. The integral representation (3.4) can be obtained similarly.

Proof of Theorem 2. It is enough to prove that, for $|z|>|b|$,

$$\limsup_{n\to\infty}|P_n(t)\phi_n(z)|^{1/n}\leq\frac{|t|+|b|}{|z|-|b|}. \tag{5.3}$$

By (3.1),

$$\begin{aligned}|\phi_{2n-1}(z)|&\leq\frac{1}{(|z|-|b|)^{2n}}\frac{1}{\Gamma(\gamma)}\sum_{k=0}^{n}\frac{\Gamma(\gamma+n-k)(n+k)!}{k!\,(n-k)!\,2^k}\\ &=\frac{1}{(|z|-|b|)^{2n}}[\phi_{2n-1}(z)]_{z=1,b=0}.\end{aligned} \tag{5.4}$$

Again by (2.15) and (4.13),

$$|\phi_{2n-1}(z)| \leqslant \frac{\Gamma(\gamma+2n+1)\Gamma(\frac{1}{2}\gamma)}{2^{n+1}\Gamma(\frac{1}{2}\gamma+n+1)\Gamma(\gamma)(|z|-|b|)^{2n}}. \tag{5.5}$$

By the duplication formula for the Γ-function,

$$|\phi_{2n-1}(z)| \leqslant \frac{2^n\Gamma[\frac{1}{2}(\gamma+1)+n]}{\Gamma[\frac{1}{2}(\gamma+1)](|z|-|b|)^{2n}}. \tag{5.6}$$

Similarly,

$$|\phi_{2n}(z)| \leqslant \frac{2^n\Gamma(\frac{1}{2}\gamma+n+1)}{\Gamma(\frac{1}{2}\gamma+1)(|z|-|b|)^{2n+1}}. \tag{5.7}$$

Again, by (1.4) after a change of variable,

$$\begin{aligned} |P_{2n-1}(t)| &= \left|\frac{t^n}{2^{n-1}\Gamma(n)}\int_0^1 (1-v)^{n-1}[(1+v)t-2b]^{n-1}v^\gamma\,dv\right| \\ &\leqslant \frac{|t|^n}{2^{n-1}\Gamma(n)}\int_0^1 (1-v^2)^{n-1}(|t|+2|b|)^{n-1}v^\gamma\,dv \\ &\leqslant \frac{|t|(|t|+|b|)^{2n-2}}{2^{n-1}\Gamma(n)}\,\frac{1}{2}\,\frac{\Gamma[\frac{1}{2}(\gamma+1)]\Gamma(n)}{\Gamma[\frac{1}{2}(\gamma+1)+n]} \\ &\leqslant (|t|+|b|)^{2n-1}\frac{\Gamma[\frac{1}{2}(\gamma+1)]}{2^n\Gamma[\frac{1}{2}(\gamma+1)+n]}. \end{aligned} \tag{5.8}$$

To obtain a bound for $|P_{2n}(t)|$, use integration by parts in (1.5). For $\gamma>0$, the integrated term vanishes at both ends. Thus,

$$P_{2n}(t) = \frac{\gamma t^{-\gamma}}{2^n\Gamma(n+1)}\int_0^t [t^2-u^2-2b(t-u)]^n u^{\gamma-1}\,du. \tag{5.9}$$

By using the same technique as above in the computation of a bound for $|P_{2n-1}(t)|$,

$$|P_{2n}(t)| \leqslant \frac{(|t|+|b|)^{2n}\Gamma(\frac{1}{2}\gamma+1)}{2^n\Gamma(\frac{1}{2}\gamma+n+1)}. \tag{5.10}$$

For $\gamma=0$, this inequality can be obtained directly from (1.5). By (5.5) and (5.7), for $n=1, 2, \ldots$,

$$|P_{2n-1}(t)\phi_{2n-1}(z)| \leqslant \frac{(|t|+|b|)^{2n-1}}{(|z|-|b|)^{2n}}, \tag{5.11}$$

and by (5.6) and (5.9),

$$|P_{2n}(t)\phi_{2n}(z)| \leqslant \frac{(|t|+|b|)^{2n}}{(|z|-|b|)^{2n+1}}. \tag{5.12}$$

From (5.11) and (5.12) above, we obtain (5.3).

Proof of Theorem 3. We note that by (2.5) and (2.14), for $n=0, 1, \ldots$,

$$I_t\Theta_z[(t/z)^\gamma P_n(t)\phi_n(z)] = (t/z)^\gamma P_{n+2}(t)\phi_{n+2}(z). \tag{5.13}$$

By induction and Lemma 4, we obtain the equality

$$(I_t\Theta_z)^n[(t/z)^\gamma(z-t)^{-1}]=(t/z)^\gamma\left((z-t)^{-1}-\sum_{k=0}^{2n-1}P_k(t)\phi_k(z)\right). \tag{5.14}$$

Also,

$$(I_t\Theta_z)^n[(t/z)^\gamma(z-t)^{-1}]=(I_t)^n(\Theta_z)^n[(t/z)^\gamma(z-t)^{-1}]. \tag{5.15}$$

We will prove that, under certain restrictions on t and z, the right-hand side in (5.15) approaches zero as $n\to\infty$. Let $|z|>\max\{|t|,\beta\}$ and let

$$\rho=\min\{|z|-|t|,|z|-\beta\}. \tag{5.16}$$

By Lemma 3,

$$\begin{aligned}|(\Theta_z)^n\{z^{-\gamma}(z-t)^{-1}\}|&\leqslant\frac{|z|^{-\gamma}}{\Gamma(\gamma)\rho^{2n+1}}\sum_{k=0}^{n}\frac{(n+k)!}{k!\,2^k}\sum_{l=0}^{n-k}\frac{\Gamma(l+\gamma)}{l!}\\&=\frac{|z|^{-\gamma}}{\rho^{2n+1}}[(\Theta_z)^n[z^{-\gamma}(z-t)^{-1}]]_{z=1,t=b=0}\\&=\frac{|z|^{-\gamma}\Gamma(\gamma+2n+2)\Gamma[\frac{1}{2}(\gamma+1)]}{\rho^{2n+1}2^{n+1}\Gamma(\gamma+1)\Gamma(\frac{1}{2}\gamma+n+\frac{3}{2})}.\end{aligned} \tag{5.17}$$

Now use the duplication formula for the Γ-function so that

$$|(\Theta_z)^n[z^{-\gamma}(z-t)^{-1}]|\leqslant\frac{|z|^{-\gamma}2^n\Gamma(\frac{1}{2}\gamma+n+1)}{\rho^{2n+1}\Gamma(\frac{1}{2}\gamma+1)}. \tag{5.18}$$

An integral representation for $(I_t)^nf(t)$ is

$$(I_t)^nf(t)=\frac{1}{2^{n-1}\Gamma(n)}\int_0^t[t^2-u^2-2b(t-u)]^{n-1}(u-b)f(u)\,du. \tag{5.19}$$

Therefore, for $f(t)=(\Theta_z)^n[(t/z)^\gamma(z-t)^{-1}]$, after a change of variable,

$$\begin{aligned}&|(I_t)^n(\Theta_z)^n[(t/z)^\gamma(z-t)^{-1}]|\\&=\left|\frac{t^{n+\gamma}}{2^{n-1}\Gamma(n)}\int_0^1[t(1-v^2)-2b(1-v)]^{n-1}(\Theta_z)^n[z^{-\gamma}(z-tv)^{-1}]v^\gamma(tv-b)\,dv\right|\\&\leqslant\frac{|t|^{n+\gamma}|z|^{-\gamma}2^n\Gamma(\frac{1}{2}\gamma+n+1)}{2^{n-1}\Gamma(n)\rho^{2n+1}\Gamma(\frac{1}{2}\gamma+1)}\int_0^1[|t|(1-v^2)+2|b|(1-v)]^{n-1}(|t|v+|b|)v^\gamma\,dv.\end{aligned} \tag{5.20}$$

The last inequality has been obtained by using (5.18). Now use integration by parts. For $n\geqslant1$, $\gamma>0$, the integrated term vanishes at both ends. As in (5.8),

$$\begin{aligned}&|(I_t)^n(\Theta_z)^n[(t/z)^\gamma(z-t)^{-1}]|\\&\leqslant\frac{|t|^{n+\gamma}|z|^{-\gamma}\gamma\Gamma(\frac{1}{2}\gamma+n+1)}{\Gamma(n+1)\rho^{2n+1}\Gamma(\frac{1}{2}\gamma+1)}\int_0^1v^{\gamma-1}(1-v^2)^n\left(|t|+\frac{2|b|}{1+v}\right)^n dv\leqslant\frac{|t|^\gamma(|t|+|b|)^{2n}}{|z|^\gamma\rho^{2n+1}}.\end{aligned} \tag{5.21}$$

This inequality also holds for $\gamma=0$. It follows that the left-hand side in (5.14) approaches zero as $n\to\infty$ when $|t|+\beta<\min\{|z|-|t|,|z|-\beta\}$. Therefore,

$\sum_{n=0}^{\infty} P_n(t)\phi_n(z) = (z-t)^{-1}$ for all z and t which satisfy this condition and in particular, when t is bounded and z is large enough. Since $(z-t)$ is analytic in z and t and the series converges uniformly when $(|t|+\beta)/(|z|-\beta) \leqslant 1-\delta < 1$ for every $\delta > 0$, the equality holds when $|t|+\beta < |z|-\beta$ by analytic continuation.

Proof of Theorem 4. It can be shown in a straightforward manner that, on C,

(a) $P_m(t)$, $P_n(t)$ are orthogonal for $m, n = 0, 1, \ldots$;
(b) $\phi_m(t)$, $\phi_n(t)$ are orthogonal for $m, n = 0, 1, \ldots$;
(c) $P_m(t)$, $\phi_n(t)$ are orthogonal for $n \geqslant m+1$ $(m = 0, 1, \ldots)$.

Therefore, it is enough to show that $P_m(t)\phi_n(t)$ are orthogonal on C for $n < m$. Without loss of generality, we may assume that C is the positively oriented circle $|w| = r$ and that $|t| + 2|b| < r$. Then

$$P_m(t) = \frac{1}{2\pi i}\int_C \frac{P_m(w)}{w-t}\,dw = \sum_{n=0}^{\infty} P_n(t)\frac{1}{2\pi i}\int_C P_m(w)\phi_n(w)\,dw = \sum_{n=0}^{\infty} c_n P_n(t), \tag{5.22}$$

where

$$c_n = \frac{1}{2\pi i}\int_C P_m(w)\phi_n(w)\,dw. \tag{5.23}$$

The interchange of the order of summation and integration is justified by the absolute convergence of the series when w is on C. By (c) above,

$$P_m(t) = \sum_{n=0}^{m} c_n P_n(t). \tag{5.24}$$

Since $P_n(t)$ are linearly independent, it follows that $c_n = 0$ for $n = 0, 1, \ldots, m-1$ and $c_m = 1$. Hence $(\{P_m\}, \{\phi_n\})$ is biorthogonal on C and (3.7) holds.

6. Proof of Theorems 5–8

Proof of Theorem 5. As in the proof of Theorem 4,

$$f(t) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-t}\,dw = \sum_{k=0}^{\infty} P_k(t)\left(\frac{1}{2\pi i}\int_C f(w)\phi_k(w)\,dw\right). \tag{6.1}$$

The interchange of the order of summation and integration is justified by the absolute convergence of the series when $|t| + 2\beta < r$. This provides the expansion (3.8). Next,

$$\begin{aligned} R_n(t) &= f(t) - \sum_{k=0}^{n-1} c_k P_k(t) \\ &= \frac{1}{2\pi i}\int_C \left(\frac{1}{w-t} - \sum_{k=0}^{n-1} P_k(t)\phi_k(w)\right) f(w)\,dw = \frac{1}{2\pi i}\int_C \left(\sum_{k=n}^{\infty} P_k(t)\phi_k(w)\right) f(w)\,dw. \end{aligned} \tag{6.2}$$

Since by (5.11) and (5.12),

$$|P_k(t)\phi_k(w)| \leq \left(\frac{|t|+|b|}{|w|-|b|}\right)^k \left(\frac{1}{|w|-|b|}\right), \tag{6.3}$$

$$|R_n(t)| \leq \frac{1}{2\pi}\left(\frac{|t|+|b|}{r-|b|}\right)^n \frac{2\pi r}{r-|t|-2|b|} \max_{|w|=r} f(w). \tag{6.4}$$

The right-hand side is a maximum when $|b|=\beta$ and provides the bound (3.11). To obtain the bound (3.12), use the representation (5.14) in (6.2).

$$\begin{aligned} R_{2n}(t) &= \frac{1}{2\pi i}\int_C (z/t)^\gamma (I_t)^n (\Theta_z)^n [(t/z)^\gamma (z-t)^{-1}] f(z)\,dz \\ &= t^{-\gamma}(I_t)^n t^\gamma \frac{1}{2\pi i}\int_C z^\gamma (\Theta_z)^n [z^{-\gamma}(z-t)^{-1}] f(z)\,dz. \end{aligned} \tag{6.5}$$

By Lemma 3 and formulae (A.1) and (A.2) of the Appendix,

$$\begin{aligned} &\left|\frac{1}{2\pi i}\int_C z^\gamma (\Theta_z)^n [z^{-\gamma}(z-t)^{-1}] f(z)\,dz\right| \\ &= \left|\sum_{k=0}^{n} \frac{(n+k)!}{k!\,2^k\Gamma(\gamma)} \sum_{l=0}^{n-k} \frac{\Gamma(l+\gamma)}{l!} \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z^l (z-b)^{n+k}(z-t)^{n-l-k+1}}\right| \\ &\leq \frac{N}{(2n)!}(\Theta_z)^n [z^{-\gamma}(z-t)^{-1}]|_{z=l,t=b=0} = \frac{N\Gamma(2n+2+\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{(2n)!\,2^{n+1}\Gamma(\gamma+1)\Gamma(n+\frac{3}{2}+\frac{1}{2}\gamma)}. \end{aligned} \tag{6.6}$$

Now we use the same technique as in (5.20) and (5.21) to obtain an upper bound for $|R_{2n}|$ except that we use (6.6) for the inner integral in (6.5). This provides (3.12).

Proof of Theorem 6. By (3.1) and (3.9),

$$c_{2k-1} = \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{k} \frac{\Gamma(\gamma+k-l)\Gamma(k+l+1)}{l!\,(k-l)!\,2^l} \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k-l}(w-b)^{k+l}}\,dw. \tag{6.7}$$

Therefore $c_{2k-1}=c_{2k-1}(b)$ is an analytic function of b in $|b|<r$. By using the inequality (5.5) in (3.9),

$$|c_{2k-1}| \leq \frac{\Gamma(\gamma+2k+1)\Gamma(\frac{1}{2}\gamma)r}{2^{k+1}\Gamma(\frac{1}{2}\gamma+k+1)\Gamma(\gamma)(r-|b|)^{2k}} \max_{|w|=r}|f(w)|. \tag{6.8}$$

The right-hand side is a maximum when $|b|=\beta$ and we obtain (3.14). Furthermore, by (A.1),

$$\left|\frac{1}{2\pi i}\int_C \frac{f(w)}{w^{k-l}(w-b)^{k+l}}\,dw\right| \leq \frac{N}{(2k-1)!} \tag{6.9}$$

where $N=\max|f^{(2k-1)}(w)|$ when w is on the line segment joining $w=0$ and $w=b$. Therefore,

$$|c_{2k-1}| \leq \left(\frac{1}{\Gamma(\gamma)}\sum_{l=0}^{k} \frac{\Gamma(\gamma+k-l)\Gamma(k+l+1)}{l!\,(k-l)!\,2^l}\right)\frac{N}{(2k-1)!}. \tag{6.10}$$

The bound (3.16) follows by using (4.13).

The corresponding statements about c_{2k} are proved in a similar manner.

Proof of Theorem 7.

$$f(t)=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-t}\,dw-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-t}\,dw, \tag{6.11}$$

where C_1 and C_2 are the positively oriented contours $|w|=R$ and $|w|=r$, respectively. For $|t|+2\beta<R$,

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-t}\,dw=\sum_{k=0}^{\infty}P_k(t)\frac{1}{2\pi i}\int_{C_1}f(w)\phi_k(w)\,dw, \tag{6.12}$$

and, for $r+2\beta<|t|$,

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-t}\,dw=-\sum_{k=0}^{\infty}\phi_k(t)\int_{C_2}f(w)P_k(w)\,dw. \tag{6.13}$$

The series in (6.12) and (6.13) converge absolutely. Since $f(w)$ has no singularity in $r\leq|w|\leq R$, it follows that C_1 and C_2 can be replaced by C.

Proof of Theorem 8. By (2.22) and (3.9),

$$t^{-\gamma}D^n[t^\gamma R_{2n}(t)]=\sum_{k=0}^{\infty}P_k(t)\frac{1}{2\pi i}\int_C\phi_{k+2n}(w)f(w)\,dw. \tag{6.14}$$

Since, by (2.14),

$$\phi_{k+2n}(w)=w^\gamma(\Theta_w)^n[w^{-\gamma}\phi_k(w)], \tag{6.15}$$

$$\sum_{k=0}^{\infty}P_k(t)\phi_{k+2n}(w)=w^\gamma(\Theta_w)^n\left(w^{-\gamma}\sum_{k=0}^{\infty}P_k(t)\phi_k(w)\right)=w^\gamma(\Theta_w)^n[w^{-\gamma}(w-t)^{-1}]. \tag{6.16}$$

The interchange of the operator $(\Theta_w)^n$ and the summation is justified by the uniform convergence of the series when $|t|+2\beta<r-\delta\leq|w|$ when δ can be chosen arbitrarily small.

Furthermore, we can interchange the order of summation and integration in (6.14). Hence,

$$t^{-\gamma}D^n[t^\gamma R_{2n}(t)]=\frac{1}{2\pi i}\int_C w^\gamma(\Theta_w)^n[w^{-\gamma}(w-t)^{-1}]f(w)\,dw. \tag{6.17}$$

By (5.18),

$$|t^{-\gamma}D^n[t^\gamma R_{2n}(r)]|\leq\frac{2^n\Gamma(\frac{1}{2}\gamma+n+1)r}{\Gamma(\frac{1}{2}\gamma+1)d^{2n+1}}\max_{|w|=r}|f(w)|. \tag{6.18}$$

This proves (3.19). To prove (3.20), we use (4.16) for the expression $w^\gamma(\Theta_w)^n[w^{-\gamma}(w-t)^{-1}]$ and then (A.1). Since

$$\left|\frac{1}{2\pi i}\int_C\frac{dw}{(w-b)^{n+k}w^l(w-t)^{n-k-l+1}}\right|\leq\frac{N}{(2n)!}, \tag{6.19}$$

$$\begin{aligned}|t^{-\gamma}D^nt^\gamma R_{2n}(t)|&\leq\frac{1}{\Gamma(\gamma)}\left(\sum_{k=0}^{n}\frac{(n+k)!}{k!\,2^k}\sum_{l=0}^{n-k}\frac{\Gamma(l+\gamma)}{l!}\right)\frac{N}{(2n)!}\\&=\frac{\Gamma(2n+2+\gamma)\Gamma(\frac{1}{2}\gamma+\frac{1}{2})}{2^{n+1}\Gamma(\gamma+1)\Gamma(n+\frac{3}{2}+\frac{1}{2}\gamma)}\frac{N}{(2n)!}.\end{aligned} \tag{6.20}$$

The last expression above follows from (4.17). By using the duplication formula for the Γ-function, we obtain the inequality in (3.20).

7. An application to asymptotics

Let

$$I(\alpha, x) = \int_0^{\pi/2} e^{x(\cos\theta + \theta\sin\alpha)}\,d\theta, \tag{7.1}$$

where $x \to \infty$ and α is the uniformity parameter. There are saddle points at θ-values satisfying $\sin\theta = \sin\alpha$. The most important ones are $\theta_0 = \alpha$, $\theta_+ = \pi - \alpha$, $\theta_- = -\pi - \alpha$. We assume that $|\alpha| \leqslant \alpha_0 < \pi/2$ (α real and α_0 fixed). Then the saddle points $\theta_\pm$ are bounded away from the interval of integration. When α changes sign, the saddle point θ_0 leaves or enters the interval of integration and the asymptotic behaviour of the integral changes abruptly. Therefore, the point $\alpha = 0$ is of particular interest.

Olver [2: p. 346] has discussed the same integral. Here we use the theory developed in the previous chapters and in Soni & Sleeman [3] to give the first five terms of an asymptotic expansion of (7.1) which holds uniformly in $|\alpha| \leqslant \alpha_0$ (α real). We also provide some numerical error bounds. The expansion is expressed in terms of the integral

$$Q(x, \kappa) = \int_0^{\kappa} e^{-x(\frac{1}{2}t^2 - bt)}\,dt, \tag{7.2}$$

which is related to the error function erf(x). We have

$$Q(x, \kappa) = (\pi/2x)^{\frac{1}{2}} e^{\frac{1}{2}xb^2}\{\mathrm{erf}\,[b(\tfrac{1}{2}x)^{\frac{1}{2}}] + \mathrm{erf}\,[(\kappa - b)(\tfrac{1}{2}x)^{\frac{1}{2}}]\}.$$

7.1 *Transformation to Standard Form*

The integral (7.1) is reduced to the standard form (1.6) by means of the transformation

$$1 - \cos\theta - \theta\sin\alpha = \tfrac{1}{2}w^2 - bw, \tag{7.3}$$

where (we denote Olver's a by b, in accordance with the part that b plays in the previous sections)

$$b = \surd 2\alpha[(\alpha\sin\alpha + \cos\alpha - 1)/\alpha^2]^{\frac{1}{2}}. \tag{7.4}$$

In (7.3), θ and w are complex variables; $\theta = 0$ maps into $w = 0$ and $\theta = \alpha$ into $w = b$. By (7.4), b is an analytic function of α for $|\alpha| \leqslant \alpha_0$. When α is real, b is real and has the same sign as α. The transformation (7.3) has been discussed by Olver. By (7.4), we write

$$w - b = \surd 2(\theta - \alpha)\{[(\alpha - \theta)\sin\alpha + \cos\alpha - \cos\theta]/(\theta - \alpha)^2\}^{\frac{1}{2}}, \tag{7.5}$$

where the principal value of the square root is taken. The mapping (7.5) is analytic and one to one in a domain Ω_θ which includes the strip $|\mathrm{Re}\,\theta| \leqslant \pi/2$ in the θ-plane. The inverse map $\theta = \theta(w)$ is analytic on the corresponding domain Ω_w in the w-plane.

Olver considered the strip $|\mathrm{Re}\, \theta| \leqslant \pi/2$. However, to obtain realistic error bounds we use larger domains Ω_θ and Ω_w. Let $w_\pm$ be the images of $\theta_\pm = \pm\pi - \alpha$ (the 'other' saddle points). That is,

$$w_\pm = b \pm (4\cos\alpha + 4\alpha\sin\alpha \mp 2\pi\sin\alpha)^{\frac{1}{2}}, \tag{7.6}$$

where the square root is positive for the values of α considered here: $-\alpha_0 \leqslant \alpha \leqslant \alpha_0$. We take

$$\Omega_w = \mathbb{C} \backslash \{[w_+, +\infty) \cup (-\infty, w_-]\}.$$

Note that $\mathrm{d}\theta/\mathrm{d}w$, which can be obtained from (7.3), is singular at $w_\pm$. For that reason, we delete branch cuts on the real axis in the w-plane. The images of these branch cuts in the θ-plane are denoted by $S_\pm$ and defined by the equation

$$\tau \sin\alpha = \sin\sigma \sinh\tau, \qquad \theta = \sigma + \mathrm{i}\tau \qquad (\sigma, \tau \text{ real}). \tag{7.7}$$

It follows that $S_\pm$ are given by (see Fig. 1)

$$\sigma = \pm\pi - \arcsin\left(\frac{\tau\sin\alpha}{\sinh\tau}\right) \quad (\tau \in \mathbb{R}).$$

Using Olver's methods it follows that $w(\theta)$ and $\theta(w)$ are analytic in the domains Ω_θ and Ω_w, respectively. By the transformation (7.3),

$$\mathrm{e}^{-x} I(\alpha, x) = \int_0^K \mathrm{e}^{-x(\frac{1}{2}w^2 - bw)} g(w)\, \mathrm{d}w, \tag{7.8}$$

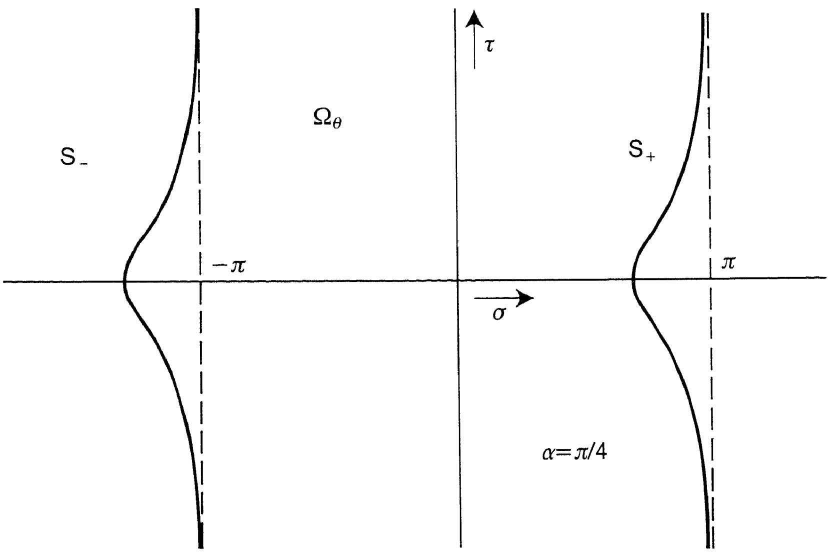

FIG. 1. Domain Ω_θ bounded by the curves $S_\pm$.

where

$$g(w) = \frac{d\theta}{dw} = \frac{w-b}{\sin\theta - \sin\alpha} \tag{7.9}$$

and, by (7.5),

$$\kappa = b + [(2\alpha - \pi)\sin\alpha + 2\cos\alpha]^{\frac{1}{2}}. \tag{7.10}$$

Note that $\theta(w)$, and therefore $g(w)$, has no singularities in Ω_w, which includes the real interval $[0, \kappa]$ and therefore the points $w = 0$, b in its interior. By (7.5), in the neighbourhood of $\theta = \alpha$,

$$w - b = (\theta - \alpha)(\cos\alpha)^{\frac{1}{2}}[1 - \tfrac{1}{3}\tan\alpha(\theta-\alpha) - \tfrac{1}{12}(\theta-\alpha)^2 + \tfrac{1}{60}\tan\alpha(\theta-\alpha)^3 + \tfrac{1}{360}(\theta-\alpha)^4 + O(\theta-\alpha)^5]^{\frac{1}{2}}, \tag{7.11}$$

and, in the neighbourhood of $\theta = 0$,

$$w = \theta\left[\frac{\sin\alpha}{b} + \left(\frac{\sin^2\alpha}{2b^3} - \frac{1}{2b}\right)\theta + O(\theta^2)\right]. \tag{7.12}$$

The expressions (7.11) and (7.12) will be used to compute expansion coefficients. In particular, by (7.9) and (7.11),

$$g(b) = 1/(\cos\alpha)^{\frac{1}{2}}. \tag{7.13}$$

Therefore,

$$e^{-x}I(\alpha, x) = [1/(\cos\alpha)^{\frac{1}{2}}]Q(x, \kappa) + J(\alpha, x), \tag{7.14}$$

where $Q(x, \kappa)$ is defined by (7.2) and

$$J(\alpha, x) = \int_0^\kappa e^{-x(\frac{1}{2}w^2 - bw)}(w-b)f(w)\,dw, \tag{7.15}$$

$$f(w) = [g(w) - g(b)]/(w-b) = (\sin\theta - \sin\alpha)^{-1} - (\cos\alpha)^{-\frac{1}{2}}/(w-b). \tag{7.16}$$

The integral in (7.15) has the standard form (1.6).

7.2 *Asymptotic Expansion of* $J(\alpha, x)$

We use the form of the asymptotic expansion given by Soni & Sleeman [3] when the upper limit of integration is bounded away from the saddle point. (This is different from the expansion given in [2: p. 348].) By (3.10), let

$$f(w) = \sum_{m=0}^{2n-1} c_m P_m(w) + R_{2n}(w). \tag{7.17}$$

Then, by [3: Theorem 6], for $\gamma = 0$,

$$J(x, \alpha) = W_0(x)\sum_{m=0}^{n-1} c_{2m}x^{-m} + W_1(x)\sum_{m=0}^{n-1} c_{2m-1}x^{-m} - A(x)\sum_{k=1}^{n} x^{-k}D^{k-1}[R_{2k}(\kappa)] + E^{(n)}, \tag{7.18}$$

where

$$E^{(n)}=x^{-n}\int_0^{\kappa} \mathrm{D}^n[R_{2n}(t)](t-b)\mathrm{e}^{-x(\frac{1}{2}t^2-bt)}\,\mathrm{d}t, \tag{7.19}$$

with D as defined in (2.20), and

$$W_i(x)=\int_0^{\kappa} t^i(t-b)\mathrm{e}^{-x(\frac{1}{2}t^2-bt)}\,\mathrm{d}t \quad (i=0,1), \tag{7.20}$$

$$A(x)=\mathrm{e}^{-x(\frac{1}{2}\kappa^2-b\kappa)}. \tag{7.21}$$

The functions $W_i(x)$ can be expressed in terms of A and Q of (7.2):

$$W_0(x)=\frac{1-A(x)}{x}, \qquad W_1(x)=\frac{Q(x,\kappa)-\kappa A(x)}{x}. \tag{7.22}$$

In (7.18), $E^{(n)}$ is the remainder and provides an error bound. The coefficients c_m are computed by using (3.9). Denote by C_w the contour in the w-plane which includes the interval $(0,\kappa)$ in its interior and which lies within Ω_w, the domain of analyticity of $\theta=\theta(w)$. Let C_θ be the image of C_w in the θ-plane. Then

$$\begin{aligned} c_m &= \frac{1}{2\pi \mathrm{i}}\int_{C_w} f(w)\phi_m(w)\,\mathrm{d}w \\ &= \frac{1}{2\pi \mathrm{i}}\int_{C_w}\left((\sin\theta-\sin\alpha)^{-1}-\frac{1}{(\cos\alpha)^{\frac{1}{2}}(w-b)}\right)\phi_m(w)\,\mathrm{d}w. \end{aligned}$$

By (3.1),

$$\int_{C_w}(w-b)^{-1}\phi_m(w)\,\mathrm{d}w=0 \quad (m=0,1,\ldots).$$

Therefore, by (7.9),

$$c_m=\frac{1}{2\pi \mathrm{i}}\int_{C_\theta}\frac{\phi_m[w(\theta)]}{w(\theta)-b}\,\mathrm{d}\theta.$$

We recall that in Soni & Sleeman [3] the coefficients c_m are defined as in (1.11) and (1.12). The above integral can be evaluated by the residue theorem. In particular, by (3.1) for $\gamma=0$,

$$\begin{aligned} \phi_0(w)&=\frac{1}{w}, & \phi_2(w)&=\frac{1}{w^2(w-b)}+\frac{1}{w(w-b)^2}, \\ \phi_1(w)&=\frac{1}{(w-b)^2}, & \phi_3(w)&=\frac{3}{(w-b)^4}. \end{aligned}$$

We use (7.11) and (7.12) to compute the residues in the above integral and obtain

$$\begin{aligned} c_0&=\frac{1}{b}\left(\frac{1}{(\cos\alpha)^{\frac{1}{2}}}-\frac{b}{\sin\alpha}\right), \\ c_1&=\frac{1}{(\cos\alpha)^{\frac{3}{2}}}\left(\tfrac{1}{8}+\tfrac{5}{24}\tan^2\alpha\right), \\ c_2&=\frac{1}{b(\cos\alpha)^{\frac{3}{2}}}\left(\tfrac{1}{8}+\tfrac{5}{24}\tan^2\alpha\right)-\frac{1}{b^3(\cos\alpha)^{\frac{1}{2}}}+\frac{1}{\sin^3\alpha}, \\ c_3&=\frac{1}{(\cos\alpha)^{\frac{5}{2}}}\left(\tfrac{9}{128}+\tfrac{77}{192}\tan^2\alpha+\tfrac{385}{1152}\tan^4\alpha\right). \end{aligned}$$

Thus, by (7.14) and (7.18),

$$e^{-x}I(\alpha, x) = \frac{1}{(\cos\alpha)^{\frac{1}{2}}} Q(x, \kappa) + W_0(x)(c_0 + c_2/x)$$
$$+ W_1(x)(c_1 + c_3/x) - A(x) \sum_{k=1}^{2} x^{-k} \mathrm{D}^{k-1}[R_{2k}(\kappa)] + E^{(2)}. \quad (7.23)$$

When κ is large with respect to b, the A-term is very small compared with the other terms. In fact, then it can be incorporated in the remainder $E^{(2)}$. In the present discussion, we consider the A-term as a significant contribution to the expansion, and we compute its two terms.

For $k = 1$, we have, using (2.20) and (7.17),

$$\mathrm{D}^0[R_2(\kappa)] = R_2(\kappa) = f(\kappa) - c_0 P_0(\kappa) - c_1 P_1(\kappa)$$
$$= 1/(1 - \sin\alpha) - \cos^{-\frac{1}{2}}(\alpha)(\kappa - b)^{-1} - c_0 - c_1\kappa,$$

and, for $k = 2$,

$$\mathrm{D}[R_4(\kappa)] = \frac{1}{\kappa - b}[f'(\kappa) - c_1 - c_2(\kappa - b) - c_3(\kappa^2 - b\kappa)],$$

where we used $P_2(w) = \frac{1}{2}w^2 - bw$ and $P_3(w) = \frac{1}{3}w^3 - \frac{1}{2}bw^2$. The value of $f'(\kappa)$ follows from (7.16) and is given by (observe that $w = \kappa \leftrightarrow \theta = \pi/2$)

$$f'(\kappa) = \frac{1}{(\cos\alpha)^{\frac{1}{2}}} \frac{1}{(\kappa - b)^2}.$$

All coefficients c_k are analytic functions in the domain of interest. As $\alpha \to 0$, by the relation (7.4), $c_0 \to 0$, $c_1 \to \frac{1}{8}$, $c_2 \to 0$, $c_3 \to \frac{9}{128}$, $\kappa \to \kappa_0 = \sqrt{2}$. Therefore,

$$\lim_{\alpha \to 0} e^{-x}I(\alpha, x) \approx \int_0^{\kappa_0} e^{-\frac{1}{2}xt^2}\, dt + \left(\frac{1}{8x} + \frac{9}{128x^2}\right)(\pi/2x)^{\frac{1}{2}}$$
$$\approx (\pi/2x)^{\frac{1}{2}}\left(1 + \frac{1}{8x} + \frac{9}{128x^2}\right) \quad \text{as } x \to \infty.$$

The expression on the right is the same as the first three terms in the asymptotic expansion of $e^{-x}I(0, x)$.

7.3 *Error Bounds*

To complete the discussion of the uniform asymptotic expansion of $e^{-x}I(x, \alpha)$, we compute bounds for the remainder $E^{(2)}$ in (7.23). We can use Theorem 8 (Section 3). However, this general result gives a rather rough estimate. In order to obtain sharper bounds and to show further techniques, we proceed as follows.

The expression $\mathrm{D}^2[R_4(t)]$ in (7.19) can be written as (see (6.17))

$$\mathrm{D}^2[R_4(t)] = \frac{1}{2\pi i} \int_{C_w} \Theta_w^2[(w - t)^{-1}] f(w)\, dw.$$

Since, by (2.8) and (2.9),

$$\Theta_w^2 \frac{1}{w-t} = \frac{2}{(w-b)^2(w-t)^3} + \frac{3}{(w-b)^3(w-t)^2} + \frac{3}{(w-b)^4(w-t)},$$

we obtain

$$\mathrm{D}^2[R_4(t)] = 3G_{5,1}(t) + 3G_{4,2}(t) + 2G_{3,3}(t), \tag{7.24}$$

where

$$G_{k,m}(t) = \frac{1}{2\pi i} \int_{C_\theta} \frac{d\theta}{(w-b)^k(w-t)^m}. \tag{7.25}$$

Here, C_θ is a contour in the θ-plane, for instance the image of C_w used in earlier expressions, and it encircles the points that correspond with $w=t$ and $w=b$ ($\theta=\alpha$).

To obtain error bounds for $E^{(2)}$ defined in (7.19), we need a bound for $G_{k,m}(t)$. In the Appendix, we derive the result

$$|G_{k,m}(t)| \leq \frac{B(\frac{1}{2}, \frac{1}{2}(k-1))}{\pi\sqrt{2}\,(2\cos\alpha)^{k/2}} [(t-w_-)^{-m}\beta(-\alpha)^{-(k-1)/2} + (w_+-t)^{-m}\beta(\alpha)^{-(k-1)/2}], \tag{7.26}$$

where $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ is the beta function, $w_\pm$ are defined in (7.6), and $\beta(\alpha)$ is given by

$$\beta(\alpha) = (2\alpha - \pi)\tan\alpha + 2. \tag{7.27}$$

Before constructing bounds for $E^{(2)}$, we remark that the coefficients c_1 and c_3 can be written in terms of the G-functions. We have

$$c_1 = G_{3,0}(t), \qquad c_3 = 3G_{5,0}(t), \tag{7.28}$$

where t does not play a part, since $m=0$. In Table 1, we give the ratios of c_1 and the right-hand side of (7.26) (with $k=3$, $m=0$), and the same for c_3. For α, we take the values

$$\alpha_j = (j/10)\pi/2 \quad (j=0,\dots,9). \tag{7.29}$$

It follows that the bounds are rather sharp for these cases.

Next, we construct two bounds for $E^{(2)}$. The first one follows from (7.19) from writing

$$|E^{(2)}| \leq x^{-2} \int_0^K |\mathrm{D}^2[R_4(t)](t-b)|\, e^{-x(\frac{1}{2}t^2 - bt)}\, dt \tag{7.30}$$

TABLE 1
$R_1 = \bar{G}_{3,0}/c_1$ and $R_2 = 3\bar{G}_{5,0}/c_3$, where $\bar{G}_{k,m}$ is the right-hand side of (7.26); j indicates the α_j-values given in (7.29)

j:	0	1	2	3	4	5	6	7	8	9
R_1:	1.28	1.27	1.26	1.23	1.21	1.19	1.17	1.16	1.15	1.15
R_3:	1.14	1.13	1.11	1.10	1.09	1.08	1.08	1.08	1.08	1.08

TABLE 2
Ratios of B(7.30) and (7.33) (bounds in (7.30) and (7.33), respectively) to 'exact' values of $E^{(2)}$; α_j are as in (7.29)

	x = 10		x = 25	
α_j	$B(7.30)/E^{(2)}$	$B(7.33)/E^{(2)}$	$B(7.30)/E^{(2)}$	$B(7.33)/E^{(2)}$
α_0	4.41	5.44	6.91	8.19
α_1	2.33	2.30	3.06	2.36
α_2	2.39	1.97	3.82	1.75
α_3	2.58	2.42	4.05	1.75
α_4	2.64	3.87	3.49	1.88
α_5	2.68	7.34	2.95	2.59
α_6	2.83	16.13	2.72	5.32
α_7	3.37	45.89	2.83	14.91
α_8	4.38	265.20	3.72	66.79
α_9	4.76	1038.00	27.11	2949.00

and by using the bounds in (7.26) for the G-terms (7.24). The second bound follows by using integration by parts in (7.19):

$$x^3E^{(2)} = \int_0^\kappa \frac{d}{dt}\,\mathrm{D}^2[R_4(t)]e^{-x(\frac{1}{2}t^2-bt)}\,dt + \mathrm{D}^2[R_4(0)] - \mathrm{D}^2[R_4(\kappa)]A(x), \qquad (7.31)$$

where $A(x)$ is given in (7.21). From (7.24) and (7.25), we derive

$$\frac{d}{dt}\,\mathrm{D}^2[R^4(t)] = 6G_{3,4}(t) + 6G_{4,3}(t) + 3G_{5,2}(t). \qquad (7.32)$$

So we obtain

$$x^3\,|E^{(2)}| \leq \int_0^\kappa \left|\frac{d}{dt}\,\mathrm{D}^2[R_4(t)]\right| e^{-x(\frac{1}{2}t^2-bt)}\,dt + |\mathrm{D}^2[R_4(0)]| + |\mathrm{D}^2[R_4(\kappa)]|\,A(x). \qquad (7.33)$$

By using the estimate (7.26) in the terms of (7.24) and (7.31), we have established bounds for $E^{(2)}$. In Table 2, we compare numerical values of the 'exact' value of $E^{(2)}$, defined in (7.19), (7.23), and (7.31), with the estimates that can be obtained from (7.30) and (7.33) (via (7.24) and (7.32)). We computed $E^{(2)}$ by using (7.23) and numerical quadrature for $I(\alpha, x)$.

Table 2 shows that in most cases the bound based on (7.30) gives a better estimate, especially for small values of α and α near $\pi/2$. The reason is that the integrated terms in (7.33) are taken with absolute values. This prevents any cancellation of the contributions. The situation is worse for larger values of α because, for $\sin\alpha > 2/\pi$ $(0<\alpha<\pi/2)$, $A(x)$ is exponentially increasing rather than exponentially decreasing. Furthermore, the upper bound for $|\mathrm{D}^2[R_4(\kappa)]|$ approaches infinity as $\alpha \to \pi/2$. All this is reflected, in particular, when $\alpha = \alpha_9$ and the bound (7.33) is used.

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Appendix

We will prove that, if $f(z)$ is an analytic function, regular within and on a simple closed contour C which has the points $z = a,\ b,\ c$ in its interior, then

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{l+1}(z-b)^{m+1}(z-c)^{n+1}}\,dz\right| \leq \frac{N}{(l+m+n+2)!}, \tag{A.1}$$

where

$$N = \max_{z\in\triangle} |f^{(l+m+n+2)}(z)|, \tag{A.2}$$

with $\triangle$ the triangle with vertices at $a,\ b,\ c$, respectively.

We will prove (A.1) when $a,\ b,\ c$ are all distinct. The inequality (A.1) holds even when $b = c$ or $a = b = c$. These cases can be considered directly or by taking the limit in (A.4) below.

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)(z-b)(z-c)}\,dz &= \frac{f(a)-f(c)}{(a-b)(a-c)} - \frac{f(b)-f(c)}{(a-b)(b-c)} \\
&= \frac{1}{(a-b)(a-c)}\int_c^a f'(u)\,du - \frac{1}{(a-b)(b-c)}\int_c^b f'(u)\,du \\
&= \frac{1}{(a-b)}\int_0^1 \{f'[c+(a-c)u] - f'[c+(b-c)u]\}\,du \\
&= \int_0^1\int_0^1 uf''[c+\{(b-c)+(a-b)v\}u]\,dv\,du.
\end{aligned} \tag{A.3}$$

Differentiating with respect to $a,\ b,\ c$ respectively $l,\ m,\ n$ times,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{f(z)l!\,m!\,n!}{(z-a)^{l+1}(z-b)^{m+1}(z-c)^{n+1}}\,dz \\
&= \int_0^1\int_0^1 f^{(l+m+n+2)}\{c+[(b-c)+(a-b)v]u\}u^{l+m+1}(1-u)^n v^l(1-v)^m\,dv\,du.
\end{aligned} \tag{A.4}$$

Therefore,

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_C \frac{f(z)}{(z-a)^{l+1}(z-b)^{m+1}(z-c)^{n+1}}\,dz\right| \\
&\leq N(l!\,m!\,n!)^{-1}\int_0^1 u^{l+m+1}(1-u)^n\,du\int_0^1 v^l(1-v)^m\,dv \leq \frac{N}{(l+m+n+2)!}.
\end{aligned} \tag{A.5}$$

Next we construct the bound given in (7.26). A suitable contour C_θ to obtain this result is given by $S_+ \cup S_-$, where $S_\pm$ are defined in (7.7) (see Fig. 1). Since w

is an analytic function of θ in Ω_θ and $|w| \to \infty$ as $|\mathrm{Im}\, \theta| \to \infty$, it follows that we can integrate along the boundary $S_+ \cup S_-$ of Ω_θ. On these curves, w is real, and if $\theta \in S_-$ then $w \in (-\infty, w_-]$ and if $\theta \in S_+$ then $w \in [w_+, +\infty)$. Note that, when θ is on the upper (lower) part of S_+, w is on the upper (lower) side of the cut $[w_+, +\infty)$, and similarly for S_-. From (7.25), we obtain

$$G_{k,m}(t) = \frac{1}{2\pi i}\int_{S_-} + \frac{1}{2\pi i}\int_{S_+} \frac{d\theta}{(w-b)^k (w-t)^m} = G^-_{k,m}(t) + G^+_{k,m}(t).$$

Recall that $0 \leqslant t \leqslant \kappa$. Hence, assuming for the moment that $0 \leqslant \alpha < \alpha_0$, we have

$$|(w-t)^{-m}| \leqslant (w_+ - t)^{-m} \quad \text{if } w \geqslant w_+,$$
$$|(w-t)^{-m}| \leqslant (t - w_-)^{-m} \quad \text{if } w \leqslant w_-.$$

Furthermore, if $\theta \in S_\pm$, it follows from (7.7) that

$$(w-b)^2 = 2[(\alpha - \sigma)\sin\alpha + \cos\alpha - \cos\sigma \cosh\tau].$$

We want to show that

$$(w-b)^2 \geqslant 2[(2\alpha - \pi\sin\alpha + \cos\alpha + \cos\alpha\cosh\tau] \quad \text{if } \theta \in S_+, \tag{A.6}$$
$$(w-b)^2 \geqslant 2[(2\alpha + \pi\sin\alpha + \cos\alpha + \cos\alpha\cosh\tau] \quad \text{if } \theta \in S_-. \tag{A.7}$$

To prove the first inequality, we have to show that the function

$$F(\tau) = \sin\alpha(\pi - \alpha - \sigma) - (\cos\sigma + \cos\alpha)\coth\tau$$

is non-negative for $\tau \in \mathbb{R}$. Using (7.7), we write

$$F(\tau) = \frac{\sinh\tau}{\tau}[(\phi - \alpha)\sin\phi + (\cos\phi - \cos\alpha)\tau\coth\tau],$$

where $\phi = \pi - \sigma$. Note that $0 \leqslant \phi \leqslant \alpha$. Since $\tau\coth\tau \geqslant 1$ if $\tau \in \mathbb{R}$, and $(\phi - \alpha)\sin\phi + \cos\phi - \cos\alpha$ is non-negative if $0 \leqslant \phi \leqslant \alpha \leqslant \alpha_0 < \pi/2$, it follows that $F(\tau) \geqslant 0$. This proves (A.6). The proof of (A.7) is similar. In that case, we take $\phi = -\sigma - \pi$; again, $0 \leqslant \phi \leqslant \alpha$.

With the above estimates, it is straightforward to construct bounds for $G_{k,m}(t)$. We integrate with respect to τ, using $d\theta/d\tau = d\sigma/d\tau + i$, where σ is an even function of τ. Hence, the contributions from $d\sigma/d\tau$ vanish. (Observe also that $G_{k,m}(t)$ is real when t and b are real.) Using (A.6) and (A.7), we obtain the estimates

$$|G^\pm_{k,m}(t)| \leqslant \pi^{-1}(2\cos\alpha)^{-k/2}|w_\pm - t|^{-m}\int_0^\infty \frac{d\tau}{[\beta(\pm\alpha) + \cosh\tau - 1]^{k/2}}.$$

The integral can be estimated by substituting $\sinh\tau/2 = u\,(\beta/2)^{\frac{1}{2}}$ (observe that β of (7.27) satisfies $0 \leqslant \beta(\alpha) \leqslant 2$ if $0 \leqslant \alpha \leqslant \pi/2$) and by replacing the quantity $(1 + \frac{1}{2}\beta u^2)$ in the resulting integral with the smaller quantity 1. This gives a beta integral and we obtain

$$|G^\pm_{k,m}(t)| \leqslant \frac{[\beta(\pm\alpha)]^{\frac{1}{2}}}{\pi\sqrt{2}}[2\beta(\pm\alpha)\cos\alpha]^{-k/2}|w_\pm - t|^{-m}B(\tfrac{1}{2}, \tfrac{1}{2}(k-1)),$$

which is the result in (7.26).