A three-point Taylor algorithm for three-point boundary value problems

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Abstract

We consider second-order linear differential equations $\varphi(x)y'' + f(x)y' + g(x)y = h(x)$ in the interval $(-1, 1)$ with Dirichlet, Neumann or mixed Dirichlet-Neumann boundary conditions given at three points of the interval: the two extreme points $x = \pm 1$ and an interior point $x = s \in (-1, 1)$. We consider $\varphi(x)$, $f(x)$, $g(x)$ and $h(x)$ analytic in a Cassini disk with foci at $x = \pm 1$ and $x = s$ containing the interval $[-1, 1]$. The three-point Taylor expansion of the solution $y(x)$ at the extreme points ± 1 and at $x = s$ is used to give a criterion for the existence and uniqueness of the solution of the boundary value problem. This method is constructive and provides the three-point Taylor approximation of the solution when it exists. We give several examples to illustrate the application of this technique.

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1 Introduction

Consider the second-order linear differential equation $\varphi(x)y'' + f(x)y' + g(x)y = h(x)$ in a real finite interval (a, b) with $\varphi(x) > 0$ and boundary data given at the extreme points $x = a$ and $x = b$ as well as at an interior point $x = c \in (a, b)$. By means of an affine change of the independent variable $x \to \frac{1}{2}[a + b + (b - a)x]$ or $x \to \frac{1}{2}[a + b - (b - a)x]$, the interval (a, b) is transformed into the interval $(-1, 1)$. After one of these changes of variables, the interior point $x = c$ is transformed into $x = s$ and we may consider, without loss of generality, $0 \leq s < 1$.

Then, without loss of generality, we consider the boundary value problem:

$$
\begin{cases}
\varphi(x)y'' + f(x)y' + g(x)y = h(x), & x \in (-1,1), \\
B \begin{pmatrix} y(-1) \\ y(s) \\ y(1) \\ y'(-1) \\ y'(s) \\ y'(1) \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, & 0 \le s < 1,
\end{cases}
\tag{1}
$$

with $\gamma_1, \gamma_2 \in \mathbb{R}$ and B a 2×6 rank-2 matrix which defines the (Dirichlet, Neumann or mixed) boundary conditions. A standard theorem for the existence and uniqueness of solution of (1) is based on the knowledge of the two-dimensional linear space of solutions of the equation $\varphi(x)y'' + f(x)y' + g(x)y = 0$ [5, Chap. 4, Sec. 1]. When φ, f, g and h are constants or in some other particular situations, it is possible to find a general solution of the equation (sometimes via the Green's function [5, Chap. 4], [13, Chaps. 1 and 3]). But, in general situations, this is not possible and that standard criterion for the existence and uniqueness of solution of (1) is not practical. Another well-known criterion for the existence and uniqueness of a solution of (1) is based on the Lax-Milgram theorem when (1) is an elliptic problem [6]. In any case, the determination of the existence and uniqueness of a solution of (1) requires a non-systematic detailed study of the problem, like for example the study of the eigenvalue problem associated to (1) [5, Chap. 4, Sec. 2], [13, Chap. 7].

In [11] we have considered the same problem, but with data given only at the two extreme points of the interval $x = \pm 1$. Using a two-point Taylor expansion [10] of the solution, we have given in [11] a simple algebraic criterion for the existence and uniqueness of the solution of the boundary value problem considered there. The purpose of this paper is to investigate the possible extension of the theory developed in [11] to the problem (1) defined by a boundary condition given at three points. A two-point Taylor expansion is not suitable for these kind of problems because the boundary data are given at three points.

Problem (1) or other more general (linear or non-linear) problems with boundary data given at three points may be used to design mathematical models for one-dimensional elasticity problems with constraints given at three points [15]. These kinds of problems have recently been analyzed by different authors from different points of view. For example, [7] considers a non-linear second order equation on $[0, \infty)$ with a particular Dirichlet datum at $x = 0$, $x = s > 0$ and a Neumann datum at $x = \infty$. Other authors [3], [16] also consider a non-linear second order differential equation on $[0, 1]$ with particular Dirichlet data given at $x = 0$, $x = s$ and $x = 1$, $0 < s < 1$. Existence and uniqueness aspects are analyzed in [1] and [2] for a similar problem on an interval $[a, b]$. In this paper we analyze a linear problem and consider more general boundary value data.

When φ, f, g and h are analytic in a disk with center at $x = 0$ and containing the interval $[-1, 1]$ with $\varphi(x) > 0$ in $[-1, 1]$, we may consider the initial value problem:

$$
\begin{cases}
\varphi(x)y'' + f(x)y' + g(x)y = h(x), & x \in (-1,1), \\
y(0) = y_0, y'(0) = y'_0,
\end{cases}
\tag{2}
$$

with $y_0, y'_0 \in \mathbb{R}$. Using the Frobenius method we can approximate the solution of this problem by its Taylor polynomial of degree N at $x = 0$, $y_N(x) = \sum_{n=0}^{N} c_k x^k$, where the coefficients c_k are affine functions of $c_0 = y_0$ and $c_1 = y'_0$. By imposing the boundary conditions given in (1) over $y_N(x)$, we obtain an algebraic linear system for y_0 and y'_0. The existence and uniqueness of the solution of this algebraic linear system gives information about the existence and uniqueness of the solution of (1). This procedure, although theoretically possible, has a difficult practical implementation since the data of the problem are given at $x = \pm 1$ and $x = s$, not at $x = 0$ [4], [14]. Moreover, when φ, f, g or h have a singularity close to the interval $[-1, 1]$ or φ vanishes at a point close to the interval $[-1, 1]$, the above mentioned disk does not contain the interval $[-1, 1]$ and the Taylor series of the solution $y(x)$ does not converge $\forall x \in [-1, 1]$.

In this case we can use a Taylor expansion of the solution at several points along the interval $[-1, 1]$ and match these expansions at intersecting disks [12, Sec. 7]. In this way, we obtain an approximation of the solution of (1) in the form of a piecewise polynomial in several subintervals of $[-1, 1]$. But this approximation is not uniform in the whole interval $[-1, 1]$ and the matching of the expansions translates into numerical errors.

The purpose of this paper is to improve these ideas using, not the standard Taylor expansion in the associated initial value problem (2), but a three-point Taylor expansion at the extreme points $x = \pm 1$ and at the interior point $x = s$ (see [9]) directly in the boundary value problem (1). In [10] we have shown that, when φ, f, g and h are analytic in a region containing the interval $[-1, 1]$, a two-point Taylor expansion of the solution $y(x)$ at the two extreme points of the interval ± 1 is useful to approximate the solution of a boundary value problem with Dirichlet data given at the extreme points of the interval. The convergence region for that two-point Taylor expansion is a Cassini disk [8] that avoids the possible singularities of the coefficient functions more efficiently than the standard Taylor disk [10]. The purpose of this paper is to investigate if a similar idea works for problem (1), that is, to essay a three-point Taylor expansion for the solution of (1) at the points $x = \pm 1$ and $x = s$. The generalization from two to three points is not trivial and requires further analysis.

In Section 2 we give an existence and uniqueness criterion of a solution of (1) based on the data of the problem (not on the knowledge of the general solution of the differential equation). Moreover, our method is constructive and provides a systematic algorithm to approximate the solution of (1) (when it exists). In Section 3 we consider the particular case of polynomial coefficients and we introduce some illustrative examples. Section 4 contains some final remarks.

2 Existence and uniqueness criterion

Assume that the coefficient functions φ, f, g, h in (1) are analytic and (with $\varphi(z) \neq 0$) in an open set $\Omega \subset \mathbb{C}$ containing a Cassini disk $\mathcal{D}_R = \{z \in \Omega \mid (z^2 - 1)(z - s)| < R\}$ with foci at $z = \pm 1$ and $z = s$ and Cassini's radius R, with $R <\mathrm{Inf}_{\mathbb{C}\setminus\Omega}\{|(z^2 - 1)(z - s)|\}$ [9]. To assure that the interval $[-1, 1]$ is contained inside the Cassini disk $\mathcal{D}_R$ (see Figure 1) we must also impose $R > R_0(s) := 2[3 - s^2 + s\sqrt{s^2 + 3}][\sqrt{s^2 + 3} + 2s]/27$. The positive number $R_0(s)$ is the maximum value of $|(x^2 - 1)(x - s)|$ for $x \in [-1, 1]$. It is attained at $x_0(s) := [s - \sqrt{s^2 + 3}]/3$. For $R > R_0(s)$ the Cassini disk is connected (as in Figure 1). For $R < R_0(s)$ it is disconnected and the interval $[-1, 1]$ is not contained in the disk (see [9] for further details).

Any solution $y(x)$ of the differential equation in (1) is analytic in the Cassini disk $\mathcal{D}_R$ where

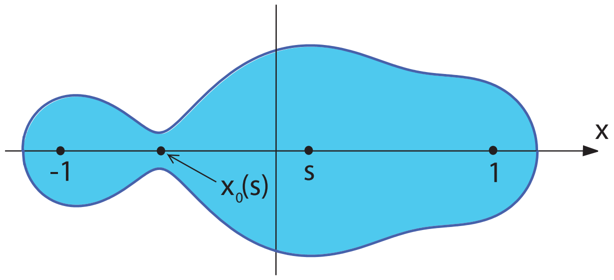

Figure 1: The Cassini disk $\mathcal{D}_R = \{z \in \mathbb{C} \mid |(z^2-1)(z-s)| < R\}$ with foci at $z = \pm 1$ and $z = s$ and radius $R > R_0(s)$ contains the real interval $[-1, 1]$.

the coefficient functions φ, f, g, h are analytic. This means that $y(x)$ can be represented in the form of a three-point Taylor expansion at the base points $x = \pm 1$ and $x = s$ [9]:

$$y(x) = \sum_{n=0}^{\infty} [\bar{a}_n + \bar{b}_n x + \bar{c}_n x^2][(x^2-1)(x-s)]^n, \quad x \in [-1, 1], \tag{3}$$

where $\bar{a}_n$, $\bar{b}_n$ and $\bar{c}_n$ are three sequences of complex numbers related to the derivatives of $y(x)$ at $x = \pm 1$ and $x = s$ [9]. This series is absolutely and uniformly convergent in the interval $[-1, 1]$.

For reasons that will be clear later, it may be convenient to re-scale the coefficients of the three-point Taylor expansion by a factor r^n, with $0 < r < R$ and define $a_n := \bar{a}_n r^n$, $b_n := \bar{b}_n r^n$ and $c_n := \bar{c}_n r^n$. Then we write $y(x)$ in the form:

$$y(x) = \sum_{n=0}^{\infty} [a_n + b_n x + c_n x^2] \left[\frac{(x^2-1)(x-s)}{r}\right]^n, \quad x \in [-1, 1]. \tag{4}$$

The derivatives of this series are also three-point Taylor series. For the first derivative $y'(x)$ we have:

$$y'(x) = \sum_{n=0}^{\infty} [a'_n + b'_n x + c'_n x^2] \left[\frac{(x^2-1)(x-s)}{r}\right]^n, \quad x \in [-1, 1], \tag{5}$$

where

$$\begin{aligned} a'_n &= (3n+1)b_n + snc_n - r^{-1}(n+1)(a_{n+1} + 3sb_{n+1} + s^2 c_{n+1}), \\ b'_n &= (3n+2)c_n - 2r^{-1}(n+1)(sa_{n+1} - b_{n+1} + sc_{n+1}), \\ c'_n &= r^{-1}(n+1)[3a_{n+1} + sb_{n+1} + (2+s^2)c_{n+1}]. \end{aligned} \tag{6}$$

For the second derivative $y''(x)$:

$$y''(x) = \sum_{n=0}^{\infty} [a''_n + b''_n x + c''_n x^2] \left[\frac{(x^2-1)(x-s)}{r}\right]^n, \quad x \in [-1, 1], \tag{7}$$

where

$$\begin{aligned} a''_n &= (3n+1)b'_n + snc'_n - r^{-1}(n+1)(a'_{n+1} + 3sb'_{n+1} + s^2 c'_{n+1}), \\ b''_n &= (3n+2)c'_n - 2r^{-1}(n+1)(sa'_{n+1} - b'_{n+1} + sc'_{n+1}), \\ c''_n &= r^{-1}(n+1)[3a'_{n+1} + sb'_{n+1} + (2+s^2)c'_{n+1}]. \end{aligned} \tag{8}$$

From (4) and (5) we have

$$\begin{pmatrix} y(-1) \\ y(s) \\ y(1) \\ y'(-1) \\ y'(s) \\ y'(1) \end{pmatrix} = T \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix}, \tag{9}$$

where T is the rank-6 matrix

$$T = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & s & s^2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2r^{-1}(1+s) & -2r^{-1}(1+s) & 2r^{-1}(1+s) \\ 0 & 1 & 2s & r^{-1}(s^2-1) & r^{-1}s(s^2-1) & r^{-1}s^2(s^2-1) \\ 0 & 1 & 2 & 2r^{-1}(1-s) & 2r^{-1}(1-s) & 2r^{-1}(1-s) \end{pmatrix}. \tag{10}$$

(The first six coefficients a_0, b_0, c_0, a_1, b_1, c_1 of the three-point Taylor expansion (4) are related to $y(-1)$, $y(s)$, $y(1)$, $y'(-1)$, $y'(s)$, $y'(1)$ by means of the matrix T^{-1}). Then, the boundary value problem (1) reads:

$$\begin{cases} \varphi(x)y'' + f(x)y' + g(x)y = h(x), \quad x \in (-1,1), \\ \tilde{R} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \end{cases} \tag{11}$$

with $\tilde{R} = BT$. Because we will use them later, denote by $R_{i,j}$, $i = 1, 2$, $j = 1, 2, 3, 4, 5, 6$, the entries of this matrix $\tilde{R}$. For simplicity in the exposition, we do not write them explicitly here, just observe that they are data of the problem because the boundary matrix B is a datum of the problem.

On the other hand (and as it happens in the standard Frobenius method for initial value problems), by introducing (4), (5) and (7) into the differential equation in (11), we find that the coefficients a_n, b_n and c_n of the three-point Taylor expansion (4) of the solution $y(x)$ of the differential equation in (11), satisfy a system of recursions of the form:

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} [A_{n,k}a_k + B_{n,k}b_k + C_{n,k}c_k] + J_n, \qquad n = 2, 3, 4, \dots, \\ b_n &= \sum_{k=0}^{n-1} [D_{n,k}a_k + E_{n,k}b_k + F_{n,k}c_k] + K_n, \qquad n = 2, 3, 4, \dots, \\ c_n &= \sum_{k=0}^{n-1} [G_{n,k}a_k + H_{n,k}b_k + I_{n,k}c_k] + L_n, \qquad n = 2, 3, 4, \dots, \end{aligned} \tag{12}$$

where the coefficients $A_{n,k}$, $B_{n,k}$, $\ldots$, L_n depend on the three-point Taylor coefficients of φ, f, g and h at $x = \pm 1$ and $x = s$. In general, as in the standard Frobenius method, the computation of the coefficients a_n, b_n and c_n involve the previous coefficients a_0, $b_0, c_0 \ldots, a_{n-1}, b_{n-1}$ and c_{n-1}. But when φ, f, g and h are polynomials, these recurrence relations are of finite order (say p) and the computation of the coefficients a_n, b_n and c_n only involve the previous $3p$ coefficients a_{n-p}, $b_{n-p}, c_{n-p}, \ldots, a_{n-1}$, b_{n-1} and c_{n-1}. We illustrate this situation with the following example.

Example 1. Consider the boundary value problem:

$$\begin{cases} (x^2+1)^2 y'' + 3x(x^2+1)y' + 2y = 0, & x \in (-1,1), \\ y(-1) + y(0) = y(0) + y(1) = 3/2. \end{cases} \tag{13}$$

We have $\varphi(x) = (x^2+1)^2$, $f(x) = 3x(x^2+1)$, $g(x) = 2$ and $h(x) = 0$. The function φ is nonvanishing in the Cassini disk $\mathcal{D}_R$ with foci at $x = \pm 1$ and $x = 0$ and $[-1,1] \subset \mathcal{D}_R$ for any R satisfying $R_0 = 2/(3\sqrt{3}) < R < 2$ (the function $\varphi(z)$ vanishes at $z = \pm i$ and, at these points, $|z(z^2-1)| = 2$). Then, in this example we may choose any $0 < r < 2$.

The three-point Taylor expansions of the coefficient functions are finite:

$$\varphi(x) = [1 + 0 \cdot x + 3 \cdot x^2] + [0 + r \cdot x + 0 \cdot x^2](x^2-1)x/r,$$

$$f(x) = [0 + 6 \cdot x + 0 \cdot x^2] + [3r + 0 \cdot x + 0 \cdot x^2](x^2-1)x/r, \qquad g(x) = [2 + 0 \cdot x + 0 \cdot x^2],$$

and then, the recursions in (12) are of order $p = 3$:

$$\begin{aligned} a_n'' + 3rb_{n-1}'' + r^2 c_{n-2}'' + 6rc_{n-1}' + 3ra_{n-1}' + 2a_n &= 0, \\ 4b_n'' + 3rc_{n-1}'' + ra_{n-1}'' + rc_{n-1}'' + 6a_n' + 6c_n' + 3rb_{n-1}' + 2b_n &= 0, \\ 3a_n'' + 4c_n'' + rb_{n-1}'' + 6b_n' + 3rc_{n-1}' + 2c_n &= 0, \end{aligned} \tag{14}$$

where a_n', b_n', c_n', a_n'', b_n'' and c_n'' are defined in (6) and (8). Choosing $r = 1$ and writing the recursions in terms of the original coefficients, we obtain for $n = 2, 3, 4, \ldots$ and $a_{-1} = b_{-1} = c_{-1} = 0$:

$$\begin{aligned} a_n = &- \frac{3(3n-8)(n-2)}{n(n-1)} b_{n-3} - \frac{30n^2 - 117n + 116}{n(n-1)} a_{n-2} - \frac{49n^2 - 173n + 152}{n(n-1)} c_{n-2} \\ &- \frac{15n-32}{n} b_{n-1}, \\ b_n = &- \frac{(3n-5)(3n-7)}{16n(n-1)} c_{n-3} - \frac{39n^2 - 125n + 102}{16n(n-1)} b_{n-2} - \frac{49n-62}{16n} a_{n-1} - \frac{16n-15}{4n} c_{n-1}, \\ c_n = &\frac{45(3n-8)(n-2)}{16n(n-1)} b_{n-3} + \frac{3(147n^2 - 575n + 572)}{16n(n-1)} a_{n-2} + \frac{348n^2 - 1246n + 1105}{8n(n-1)} c_{n-2} \\ &+ \frac{49n-109}{4n} b_{n-1}. \end{aligned} \tag{15}$$

□

As in the Frobenius method, the order of the recurrence relations is at least two, that is, $p \geq 2$. But, as a difference with the Frobenius method where we only have one recursion for the sequence of standard Taylor coefficients, here we have a system of three recurrence relations. In the standard Frobenius method designed for an initial value problem of the form:

$$\begin{cases} \varphi(x)y'' + f(x)y' + g(x)y = h(x), \\ y(0) = c_0, \quad y'(0) = c_1, \end{cases}$$

we seek for a solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then, the computation of the coefficients c_n for $n \geq 2$, only requires the initial seed c_0 and c_1, that are data of the problem.

The situation is different for the boundary value problem (11) when we look for a solution of the form (4). Since, in this case, we have a system of three recurrence relations instead of only one recursion, the computation of the coefficients a_n, b_n, c_n for $n \geq 2$ requires the initial seed a_0, a_1, b_0, b_1, c_0 and c_1. This does not mean that the linear space of solutions of the differential equation in (11) has dimension six, this space has of course dimension two. It is happening here that, apart from the two-dimensional linear space S of (true) solutions of the differential equation in (11), there is a bigger space of formal solutions W defined by:

$$W := \left\{ y(x) = \sum_{n=0}^{\infty} [a_n + b_n x + c_n x^2] \left[\frac{(x^2-1)(x-s)}{r} \right]^n, a_n, b_n, c_n \text{ given in (12)}, \right. \\ \left. a_0, b_0, c_0, a_1, b_1, c_1 \in \mathbb{R} \right\}. \tag{16}$$

Formally, all the three-point series in W are solutions of the differential equation in (11). But not all of them are convergent, only a subset: the two-dimensional linear space S of (true) solutions that may be identified as

$$S = \left\{ y \in W \;\middle|\; \sum_{n=0}^{\infty} [a_n + b_n x + c_n x^2] \left[\frac{(x^2-1)(x-s)}{r} \right]^n \text{ is uniformly convergent in } \mathcal{D}_R \right\}.$$

Any function y of S is a solution of the second order linear differential equation $\varphi(x)y'' + f(x)y' + g(x)y = h(x)$ with analytic coefficients in $\mathcal{D}_R$. This means that y is analytic in $\mathcal{D}_r$ and not only its three-point Taylor series highlighted in the definition of S, but also the derivatives of that three-point Taylor series converge in $\mathcal{D}_R$ [9]. Therefore, although not explicitly written above, this fact must be implicitly assumed in the definition of S. In order to give a more practical characterization of S, we must find a linear system of four independent equations for the parameters $a_0, b_0, c_0, a_1, b_1, c_1$. This is the purpose of the remaining of the section.

For a fixed $m \in \mathbb{N}$, $m \geq 2$, we define the vector:

$$v_n := (a_{n+2-m}, b_{n+2-m}, c_{n+2-m}, a_{n+3-m}, b_{n+3-m}, c_{n+3-m}, \ldots, a_n, b_n, c_n, a_{n+1}, b_{n+1}, c_{n+1}) \in \mathbb{R}^{3m},$$

with $a_{-k} = b_{-k} = c_{-k} = 0$ for $k \in \mathbb{N}$. In particular we have:

$$v_{m-2} = (a_0, b_0, c_0, a_1, b_1, c_1, \ldots, a_{m-1}, b_{m-1}, c_{m-1}) \quad \text{and} \quad v_0 = (0, 0, \ldots, 0, 0, a_0, b_0, c_0, a_1, b_1, c_1).$$

For $n = 0, 1, 2, \dots, m-2$, define the $(3m) \times (3m)$ matrix $M_n = (\omega_{i,j})$

$$M_n := \left(\begin{array}{ccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\
0 & \dots & 0 & A_{n+2,0} & B_{n+2,0} & C_{n+2,0} & \dots & \dots & A_{n+2,n+1} & B_{n+2,n+1} & C_{n+2,n+1} \\
0 & \dots & 0 & D_{n+2,0} & E_{n+2,0} & F_{n+2,0} & \dots & \dots & D_{n+2,n+1} & E_{n+2,n+1} & F_{n+2,n+1} \\
0 & \dots & 0 & G_{n+2,0} & H_{n+2,0} & I_{n+2,0} & \dots & \dots & G_{n+2,n+1} & H_{n+2,n+1} & I_{n+2,n+1}
\end{array} \right). \tag{17}$$

The only non-zero elements of this matrix are the corresponding to the entries $\omega_{i,i+3} = 1$, $i = 1, 2, 3, \dots, 3m-3$ and to the entries $\omega_{3m-2,k}$, $\omega_{3m-1,k}$, $\omega_{3m,k}$, $k = 3m-3n-5, \dots, 3m$. In particular we have:

$$M_0 = \left(\begin{array}{ccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \dots & 0 & 0 & A_{2,0} & B_{2,0} & C_{2,0} & A_{2,1} & B_{2,1} & C_{2,1} \\
0 & 0 & \dots & 0 & 0 & D_{2,0} & E_{2,0} & F_{2,0} & D_{2,1} & E_{2,1} & F_{2,1} \\
0 & 0 & \dots & 0 & 0 & G_{2,0} & H_{2,0} & I_{2,0} & G_{2,1} & H_{2,1} & I_{2,1}
\end{array} \right)$$

and

$$M_{m-2} = \left(\begin{array}{cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\
A_{m,0} & B_{m,0} & C_{m,0} & A_{m,1} & B_{m,1} & C_{m,1} & \dots & \dots & \dots & A_{m,m-1} & B_{m,m-1} & C_{m,m-1} \\
D_{m,0} & E_{m,0} & F_{m,0} & D_{m,1} & E_{m,1} & F_{m,1} & \dots & \dots & \dots & D_{m,m-1} & E_{m,m-1} & F_{m,m-1} \\
G_{m,0} & H_{m,0} & I_{m,0} & G_{m,1} & H_{m,1} & I_{m,1} & \dots & \dots & \dots & G_{m,m-1} & H_{m,m-1} & I_{m,m-1}
\end{array} \right).$$

We also need, for $n = 0, 1, 2, \dots, m-2$, to define the vector

$$c_n := (0, 0, \dots, 0, 0, J_{n+2}, K_{n+2}, L_{n+2}) \in \mathbb{R}^{3m}.$$

Then, the system of recurrence relations (12) can be written in a matrix form. For $n = 1, 2, 3, \dots, m-1$ we have:

$$v_n = M_{n-1} v_{n-1} + c_{n-1}.$$

To find the solution of this linear recurrence relation for the vector v_n, we define recurrently the following matrices:

$$\mathcal{M}_0 = M_0, \qquad \mathcal{M}_n = M_n \mathcal{M}_{n-1}, \qquad n = 1, 2, 3, \dots, m-2,$$

$$\mathcal{C}_0 = c_0, \qquad \mathcal{C}_n = M_n \mathcal{C}_{n-1} + c_n, \qquad n = 1, 2, 3, \ldots, m-2,$$

or

$$\mathcal{M}_n = \prod_{k=0}^{n} M_{n-k},$$

$$\mathcal{C}_n = c_n + \sum_{k=0}^{n-1} [M_n \cdot M_{n-1} \cdots M_{k+1}] c_k.$$

Then, we find

$$v_{m-1} = \mathcal{M}_{m-2} v_0 + \mathcal{C}_{m-2}$$

or, in an extended form:

$$\begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ a_m \\ b_m \\ c_m \end{pmatrix} = \begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ \mathcal{B}_{3m-2} \\ \mathcal{B}_{3m-1} \\ \mathcal{B}_{3m} \end{pmatrix} +$$

$$\begin{pmatrix} \star & \star & \star & \star & \star & \star & \star & \ldots & \star \\ \star & \star & \star & \star & \star & \star & \star & \ldots & \star \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \star & \star & \star & \star & \star & \star & \star & \ldots & \star \\ \star & \star & \star & \star & \star & \star & \star & \ldots & \star \\ \star & \ldots & \star & \mathcal{M}_{3m-2,3m-5} & \mathcal{M}_{3m-2,3m-4} & \mathcal{M}_{3m-2,3m-3} & \mathcal{M}_{3m-2,3m-2} & \mathcal{M}_{3m-2,3m-1} & \mathcal{M}_{3m-2,3m} \\ \star & \ldots & \star & \mathcal{M}_{3m-1,3m-5} & \mathcal{M}_{3m-1,3m-4} & \mathcal{M}_{3m-1,3m-3} & \mathcal{M}_{3m-1,3m-2} & \mathcal{M}_{3m-1,3m-1} & \mathcal{M}_{3m-1,3m} \\ \star & \ldots & \star & \mathcal{M}_{3m,3m-5} & \mathcal{M}_{3m,3m-4} & \mathcal{M}_{3m,3m-3} & \mathcal{M}_{3m,3m-2} & \mathcal{M}_{3m,3m-1} & \mathcal{M}_{3m,3m} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix}, \tag{18}$$

where the $\star$ denote real (unspecified) numbers.

At this point we meet the key point of the discussion. Take three different points x_1, x_2 and x_3 in $\mathcal{D}_R$ located at a common "Cassini's distance" r from the base points of the expansion: $|(x_k^2 - 1)(x_k - s)| = r < R$, $k = 1, 2, 3$. Define

$$u_n^{(k)} := (a_n + b_n x_k + c_n x_k^2) = (\bar{a}_n + \bar{b}_n x_k + \bar{c}_n x_k^2) r^n, \qquad k = 1, 2, 3.$$

The three numerical series $\sum_{n=0}^{\infty} u_n^{(k)}$, $k = 1, 2, 3$, are convergent, which means that $\lim_{n\to\infty} u_n^{(k)} = 0$ for $k = 1, 2, 3$. But the three sequences $\{u_n^{(k)}\}$ and the three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are related by an invertible matrix:

$$\begin{pmatrix} u_n^{(1)} \\ u_n^{(2)} \\ u_n^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}.$$

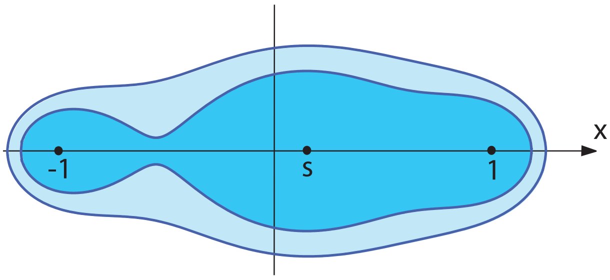

Figure 2: The Cassini disks $\mathcal{D}_R$ (blue) and $\mathcal{D}_{r_0}$ (dark blue) with foci at $z = \pm 1$ and $z = s$ and respective radius $R > r_0$.

This means that also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$ for any $0 < r < R$. (In particular, when $R > 1$, we can take $r = 1$ and then we have $\lim_{n\to\infty} \bar{a}_n = \lim_{n\to\infty} \bar{b}_n = \lim_{n\to\infty} \bar{c}_n = 0$.) These three limit conditions are necessary for the convergence of the three-point Taylor series in S, but they are also sufficient: consider a Cassini disk $\mathcal{D}_{r_0}$ of radius $r_0 < r < R$ inscribed in the Cassini disk $\mathcal{D}_R$ (see Figure 2). Any $x \in \mathcal{D}_{r_0}$ is located at a "Cassini's distance" $|(x^2-1)(x-s)| < r$ from the base points of the three-point Taylor expansion. Then, if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$, the three-point Taylor series

$$y(x) = \sum_{n=0}^{\infty} [a_n + b_n x + c_n x^2] \left[\frac{(x^2-1)(x-s)}{r}\right]^n$$

is convergent for any $x \in \mathcal{D}_{r_0}$ and therefore $y(x)$ is analytic in $\mathcal{D}_{r_0}$. But this function $y(x)$ is a solution of a differential equation with coefficients analytic in $\mathcal{D}_R$ and then $y(x)$ is analytic in $\mathcal{D}_R$ and the above series converges for any $x \in \mathcal{D}_R$.

Observe that when $R > 1$ we can take $r = 1$ and then the scaling r introduced in (4) is not necessary. But when $R \leq 1$, the scaling is necessary to argue that the rescaled coefficients tend to zero as $n \to \infty$.

On the other hand, observe that the coefficients of the (divergent) series contained in W and not in S must increase with n faster than any exponential α^n for any real α: consider again a Cassini disk $\mathcal{D}_\alpha$ of radius $\alpha > 0$ as small as one wishes. It must happen that the three-point Taylor series of that divergent series must diverge for any $x \in \mathcal{D}_\alpha$ or, otherwise, it would define an analytic function in $\mathcal{D}_\alpha$ and then in $\mathcal{D}_R$ (it would belong to S). This means that, when $n \to \infty$, $\bar{a}_n \alpha^n \to \infty$ for any $\alpha > 0$ (as small as one wishes). The same fact holds for the other two sequences of coefficients.

Therefore, a three-point Taylor series obtained from the above recurrence relations belongs to S if and only if $\lim_{n\to\infty}(a_n, b_n, c_n) = (0,0,0)$ for a certain $0 < r < R$.

For the forthcoming discussion it is more convenient to fix our attention in the last six columns of the matrix $\mathcal{M}$ displayed in (18), a sub-matrix of size $(3m) \times 6$, and see this sub-matrix as a matrix composed by m blocks of three rows (blocks of size 3×6, only the last block is detailed in formula (18)). The larger m is, the more blocks of three rows that sub-matrix contains. This vertical list of blocks of size 3×6 is, for finite m, a finite sequence of blocks, or also, three different finite sequences of rows. For reasons that we will show below, among these three sequences of rows, there must be four and only four subsequences that, in the limit

$m \to \infty$, become four independent rows. This means that at least one of the three sequences of rows of coefficients $\mathcal{M}_{3m-i,3m+j-6}$, $i = 0, 1, 2$, $j = 1, 2, 3, 4, 5, 6$ in the matrix $\mathcal{M}$ has not a limit when $m \to \infty$; there must be two subsequences of this row of coefficients having a limit. Then, taking the limit $m \to \infty$ into the equation (18) we must find

$$\begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \star \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \star & \star & \star & \star & \star & \star & \star & \dots & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \dots & \star & \star & \star \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \star & \star & \star & \star & \star & \star & \star & \dots & \star & \star & \star \\ \star & \star & \star & \star & \star & \star & \star & \dots & \star & \star & \star \\ \star & \star & \dots & \star & \star & R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} & R_{3,5} & R_{3,6} \\ \star & \star & \dots & \star & \star & R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & R_{4,5} & R_{4,6} \\ \star & \star & \dots & \star & \star & R_{5,1} & R_{5,2} & R_{5,3} & R_{5,4} & R_{5,5} & R_{5,6} \\ \star & \star & \dots & \star & \star & R_{6,1} & R_{6,2} & R_{6,3} & R_{6,4} & R_{6,5} & R_{6,6} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} \star \\ \star \\ \cdot \\ \cdot \\ \cdot \\ \star \\ \star \\ -\gamma_3 \\ -\gamma_4 \\ -\gamma_5 \\ -\gamma_6 \end{pmatrix},$$

where

$$\begin{aligned} R_{6-i,j} &:= \lim_{m\to\infty} \mathcal{M}_{3m-i,3m+j-6}, \quad i = 0, 1, 2, 3, \quad j = 1, 2, 3, 4, 5, 6; \\ -\gamma_{6-i} &:= \lim_{m\to\infty} \mathcal{B}_{3m-i}, \quad i = 0, 1, 2, 3. \end{aligned} \tag{19}$$

The above limits must be understood as limits of certain subsequences (four subsequences). Then, the four equations that we were looking for are given by

$$\begin{pmatrix} R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} & R_{3,5} & R_{3,6} \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & R_{4,5} & R_{4,6} \\ R_{5,1} & R_{5,2} & R_{5,3} & R_{5,4} & R_{5,5} & R_{5,6} \\ R_{6,1} & R_{6,2} & R_{6,3} & R_{6,4} & R_{6,5} & R_{6,6} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix} \tag{20}$$

We have stated a few lines above that these four equations must be linearly independent (and then they reduce the number of free parameters from six to two) and there are not more linearly independent rows in the limit $m \to \infty$ of the matrix $\mathcal{M}$ (that is, (20) has exactly four independent rows). The proof of this claim is as follows. Consider the initial value problem

$$\begin{cases} \varphi(x)y'' + f(x)y' + g(x)y = h(x), \\ y(-1) = y_{-1}, \quad y'(-1) = y'_{-1}, \end{cases} \tag{21}$$

with $y_{-1}, y'_{-1} \in \mathbb{R}$ and seek for a solution in the form (4). The coefficients $a_0, b_0, c_0, a_1, b_1, c_1$ of the three-point Taylor solution of this initial value problem are solutions of the linear system (20) and also of the two linear equations imposed by the initial conditions $y(-1) = y_{-1}$ and $y'(-1) = y'_{-1}$, that is,

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2r^{-1}(1+s) & -2r^{-1}(1+s) & 2r^{-1}(1+s) \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} & R_{3,5} & R_{3,6} \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & R_{4,5} & R_{4,6} \\ R_{5,1} & R_{5,2} & R_{5,3} & R_{5,4} & R_{5,5} & R_{5,6} \\ R_{6,1} & R_{6,2} & R_{6,3} & R_{6,4} & R_{6,5} & R_{6,6} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} y_{-1} \\ y'_{-1} \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix}.$$

If the rank of the coefficient matrix of this system was not six, then the initial value problem (21) would have more than one solution or no solution. This is impossible and then the four equations in (20) are linearly independent.

Joining the four equations in (20) with the two algebraic equations provided by the boundary conditions in (11), we find the following linear system of six equations and six unknowns:

$$\begin{pmatrix} R_{1,1} & R_{1,2} & R_{1,3} & R_{1,4} & R_{1,5} & R_{1,6} \\ R_{2,1} & R_{2,2} & R_{2,3} & R_{2,4} & R_{2,5} & R_{2,6} \\ R_{3,1} & R_{3,2} & R_{3,3} & R_{3,4} & R_{3,5} & R_{3,6} \\ R_{4,1} & R_{4,2} & R_{4,3} & R_{4,4} & R_{4,5} & R_{4,6} \\ R_{5,1} & R_{5,2} & R_{5,3} & R_{5,4} & R_{5,5} & R_{5,6} \\ R_{6,1} & R_{6,2} & R_{6,3} & R_{6,4} & R_{6,5} & R_{6,6} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix}. \tag{22}$$

At this point, we can formulate the following existence and uniqueness criterion for the boundary value problem (1).

The existence and uniqueness of solution of the boundary value problem (1) *is equivalent to the existence and uniqueness of solution of the linear system* (22)*. More precisely,*

- *When the linear system* (22) *has a unique solution* $(a_0, b_0, c_0, a_1, b_1, c_1)$*, the boundary value problem* (1) *has a unique solution given by* (4) *and* (12) *with* $(a_0, b_0, c_0, a_1, b_1, c_1)$ *the solution of* (22)*.*

- *When the linear system* (22) *has an infinite number of solutions (one or two of the parameters* $a_0, b_0, c_0,\ a_1, b_1, c_1$ *are free), the boundary value problem* (1) *has a one or a two-parametric family of solutions given by* (4) *and* (12) *with* $a_0, b_0, c_0, a_1, b_1, c_1$ *the solution of* (22)*.*

- *When the linear system* (22) *has no solution, the boundary value problem* (1) *has no solution.*

The two first equations of the system (22) are obtained from the boundary conditions (second line in (11)) and encode the boundary conditions. The other four equations are obtained from the differential equation in (11) and define the space of solutions of the differential equation: the subsystem obtained from the last four rows of the above system of equations determine the two-dimensional space of solutions of the differential equation. Then, the compatibility of the first two rows with that subsystem determine which (if any) of those solutions is also a solution of the boundary conditions. The two first rows of the system (22) are independent. The four last rows are also independent. This means that the dimension of the space of solutions of (1) is at most two.

If we denote by R the coefficient matrix of the system (22), $x := (a_0, b_0, c_0, a_1, b_1, c_1)$ the vector of unknowns and $\Lambda := (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6)$ the vector of independent terms, (22) can be written as the system $Rx = \Lambda$. In practice, the exact computation of the limits (19) is impossible and we must approximate them in the form:

$$\begin{aligned} R_{6-i,j} \simeq & \mathcal{M}_{3m-i,3m+j-6}, \quad i = 0,1,2,3, \quad j = 1,2,3,4,5,6; \\ -\gamma_i \simeq & \mathcal{B}_{3m-i}, \end{aligned} \tag{23}$$

for a large enough value of m. This means that, in practice, we work with an approximate system $R_m x_m = \Lambda_m$ instead of the system $Rx = \Lambda$. Then, the values of the coefficients x_m obtained from $R_m x_m = \Lambda_m$ are approximations of the exact coefficients a_0, b_0, c_0, a_1, b_1, c_1.

Also, in practice, we must apply the above existence and uniqueness criterion for the solution of (1) using the approximate linear system $R_m x_m = \Lambda_m$ instead of the exact system $Rx = \Lambda$. Nevertheless, the conclusions about existence and uniqueness are the same unless the ranks of the coefficient matrix R_m and/or of the augmented matrix $(R_m|\Lambda_m)$ sensibly depend on the precision in the computation of the approximate limits (23). In this case the above criterion is not conclusive from a practical point of view.

3 Polynomial coefficients

When the coefficient functions φ, f, g and h are polynomials, we can simplify the formulation of the above existence and uniqueness criterion. In general, as we have seen in the previous section, the computation of the coefficients (a_n, b_n, c_n) requires a matrix of size $(3m) \times (3m)$ with $m \geq n$. This means that we need matrices of increasing size to compute the coefficients (when n increases). In the case of polynomial coefficients, the situation is different. The recurrence relations (12) are of constant order p independent of n and the computation of the coefficients a_n, b_n and c_n involve only the previous $3p$ coefficients a_{n-p}, b_{n-p}, $c_{n-p}, \ldots, a_{n-1}$, b_{n-1}, and c_{n-1}. Thus, in this case, we do not need matrices of increasing size, but matrices of constant size $(3p) \times (3p)$.

The recurrence system (12) for polynomial coefficients is of the form

$$\begin{aligned} a_n &= \sum_{k=n-p}^{n-1} [A_{n,k} a_k + B_{n,k} b_k + C_{n,k} c_k] + J_n, \\ b_n &= \sum_{k=n-p}^{n-1} [D_{n,k} a_k + E_{n,k} b_k + F_{n,k} c_k] + K_n, \\ c_n &= \sum_{k=n-p}^{n-1} [G_{n,k} a_k + H_{n,k} b_k + I_{n,k} c_k] + L_n, \end{aligned} \tag{24}$$

for a certain $p \in \mathbb{N}$, $n = 0, 1, 2, \ldots$, with $a_{-k} = b_{-k} = c_{-k} = 0$, $n, k \in \mathbb{N}$. The discussion is identical to the one of the previous section, but we can eliminate the restriction $n \leq m$. Moreover, we can simplify the computations because now, the size of the matrices M_n does not depend on n. We can now define the matrices M_n of fixed size $(3p) \times (3p)$ in the form:

$$M_n := \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 1 \\ A_{n+2,n+2-p} & B_{n+2,n+2-p} & C_{n+2,n+2-p} & \ldots & \ldots & \ldots & A_{n+2,n+1} & B_{n+2,n+1} & C_{n+2,n+1} \\ D_{n+2,n+2-p} & E_{n+2,n+2-p} & F_{n+2,n+2-p} & \ldots & \ldots & \ldots & D_{n+2,n+1} & E_{n+2,n+1} & F_{n+2,n+1} \\ G_{n+2,n+2-p} & H_{n+2,n+2-p} & I_{n+2,n+2-p} & \ldots & \ldots & \ldots & G_{n+2,n+1} & H_{n+2,n+1} & I_{n+2,n+1} \end{pmatrix} \tag{25}$$

instead of the form (17), with $A_{n,-k} = B_{n,-k} = C_{n,-k} = D_{n,-k} = E_{n,-k} = F_{n,-k} = G_{n,-k} = H_{n,-k} = I_{n,-k} = 0$ for $k \in \mathbb{N}$. The computation of the system (22) is identical. The only difference is that now, the matrices $\mathcal{M}_m$ are of size $(3p) \times (3p)$ $\forall m \in \mathbb{N}$ and the vectors $\mathcal{C}_m \in \mathbb{R}^{3p}$ $\forall m \in \mathbb{N}$.

Example 2. As an example of boundary value problem with polynomial coefficients, we consider the problem defined in (13). The recurrence relations (14) are of order $p = 3$ and may be written in the form $v_{n+1} = M_n v_n$ with $v_n = (a_{n-1}, b_{n-1}, c_{n-1}, a_n, b_n, c_n, a_{n+1}, b_{n+1}, c_{n+1})$ and

$$M_n = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{-3n(3n-2)}{(n+1)(n+2)} & 0 & -\frac{30n^2+3n+2}{(n+1)(n+2)} & 0 & -\frac{49n^2+23n+2}{(n+1)(n+2)} & 0 & -\frac{15n-2}{n+2} & 0 \\ 0 & 0 & -\frac{(3n+1)(3n-1)}{16(n+1)(n+2)} & 0 & -\frac{39n^2+31n+8}{16(n+1)(n+2)} & 0 & -\frac{49n+36}{16(n+2)} & 0 & -\frac{16n+17}{4(n+2)} \\ 0 & \frac{45n(3n-2)}{16(n+1)(n+2)} & 0 & \frac{3(147n^2+13n+10)}{16(n+1)(n+2)} & 0 & \frac{348n^2+146n+5}{8(n+1)(n+2)} & 0 & \frac{44n-21}{4(n+2)} & 0 \end{pmatrix}.$$

For example, for $m = 10$, the linear system $R_m x_m = \Lambda_m$ reads

$$\begin{pmatrix} 2 & -1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 84.8507 & 0 & 752.266 & 0 & 1321.19 \\ -436.782 & 0 & 150.657 & 0 & 2048.44 & 0 \\ 0 & -183.796 & 0 & -1233.24 & 0 & -2041.26 \\ 591.279 & 0 & -458.166 & 0 & -3281.44 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system has a unique solution $(a_0, b_0, c_0, a_1, b_1, c_1) = (1, 0, -0.500007, 0, 0.250002, 0)$ and then, using the criterion obtained in Section 2, the boundary value problem (13) has a unique solution that can be approximated by the three-point Taylor polynomials:

$$y_n(x) = \sum_{k=0}^{n} [a_k + b_k x + c_k x^2] \left[(x^2 - 1)x\right]^k, \quad n = 0, 1, 2, \ldots. \tag{26}$$

For example, we obtain the following approximated polynomials for $n = 1, 3$ and 5:

$$\begin{aligned} \widetilde{y}_1(x) &= 1 - 0.500007x^2 + 0.250002x(x^2-1)x, \\ \widetilde{y}_3(x) &= \widetilde{y}_1(x) + (-0.249992 + 0.124994x^2)(x^2-1)^2 x^2 - 0.0624977x(x^2-1)^3 x^3, \\ \widetilde{y}_5(x) &= \widetilde{y}_3(x) + (0.0625236 - 0.0312701x^2)(x^2-1)^4 x^4 + 0.0156347x(x^2-1)^5 x^5. \end{aligned} \tag{27}$$

Table 1 and Figure 3 show a numerical experiment about the approximation supplied by (27) with $m = 10$. It can be observed the improvement in the approximation as n increases.

□

The following example shows the application of the above criterion to a boundary value problem containing parameters.

Example 3. Consider the boundary value problem:

$$\begin{cases} y'' - 2xy' - 2y = 2a, & x \in (-1, 1), \\ y(-1) + y(0) + y(1) = c, \\ y'(-1) + b\, y'(1) = 0, \end{cases} \tag{28}$$

n	$x = 0.1$	$x = 0.25$	$x = 0.5$	$x = 0.75$	$x = 0.9$
1	0.00245	0.013732	0.035153	0.026909	0.007300
3	6.0e−6	0.000188	0.001237	0.000730	0.000063
5	1.46e−8	2.5e−6	0.000042	0.000013	9.6e−6

Table 1: Numerical experiments about the relative errors in the approximation of the exact solution $y(x) = 1/(1+x^2)$ of (13) and the approximated three-point Taylor polynomials $\widetilde{y}_n(x)$, $n = 1, 3, 5$ given in (27) for different values of $x \in (-1, 1)$.

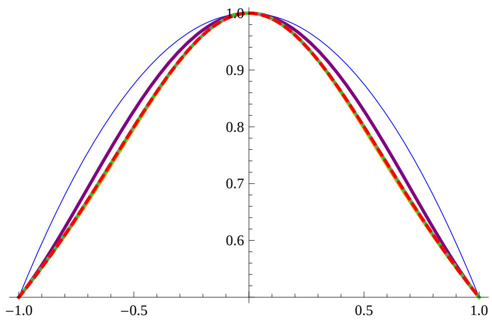

Figure 3: Plot of the exact solution $y(x) = 1/(1+x^2)$ (red and dashed) of (13) and several approximated three-point Taylor polynomials $\widetilde{y}_n(x)$ for $n = 0, 1, \ldots, 6$.

with a, b and c real parameters. We have $\varphi(x) = 1$, $f(x) = -2x$, $g(x) = -2$ and $h(x) = 2a$. These are entire functions and then we can take any $r > 0$; we take $r = 1$. The three-point Taylor expansions of these coefficient functions are finite:

$$\varphi(x) = [1+0\cdot x+0\cdot x^2], \quad f(x) = [0-2\cdot x+0\cdot x^2], \quad g(x) = [-2+0\cdot x+0\cdot x^2], \quad h(x) = [2a+0\cdot x+0\cdot x^2],$$

and then, the recursions (12) are of order $p = 2$. For $n = 2, 3, 4, \ldots$,

$$\begin{aligned} a_n =& \frac{2(3n-5)}{n(n-1)}\, a_{n-2} - \frac{9n^2-31n+28}{n(n-1)}\, c_{n-2} - \frac{3n-8}{n}\, b_{n-1} + a\, \delta_{n-2,0}, \\ b_n =& \frac{3n-4}{2n(n-1)}\, b_{n-2} - \frac{9n-16}{4n}\, a_{n-1} - \frac{6n-7}{2n}\, c_{n-1}, \\ c_n =& -\frac{3(3n-5)}{2n(n-1)}\, a_{n-2} + \frac{3(9n^2-29n+26)}{4n(n-1)}\, c_{n-2} - \frac{7}{2n}\, b_{n-1} - \frac{3a}{4}\, \delta_{n-2,0}, \end{aligned} \tag{29}$$

where $\delta_{k,j}$ is the Kronecker delta function. The recurrence relations (29) may be written in the form $v_{n+1} = M_n v_n + t\,\delta_{n,0}$ with $v_n = (a_n, b_n, c_n, a_{n+1},\, b_{n+1}, c_{n+1})$,

$$M_n = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{2(3n+1)}{(n+1)(n+2)} & 0 & -\frac{9n^2+5n+2}{(n+1)(n+2)} & 0 & -\frac{3n-2}{n+2} & 0 \\ 0 & \frac{3n+2}{2(n+1)(n+2)} & 0 & -\frac{9n+2}{4(n+2)} & 0 & -\frac{6n+5}{2(n+2)} \\ -\frac{3(3n+1)}{2(n+1)(n+2)} & 0 & \frac{3(9n^2+7n+4)}{4(n+1)(n+2)} & 0 & -\frac{7}{2(n+2)} & 0 \end{pmatrix},$$

and

$$t = \begin{pmatrix} 0 \\ 0 \\ 0 \\ a \\ 0 \\ -3a/4 \end{pmatrix}.$$

For $m = 10$, the linear system $R_m x_m = \Lambda_m$ is given by

$$\begin{pmatrix} 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & b+1 & 2(b-1) & 2(b+1) & 2(b-1) & 2(b+1) \\ 0 & -120.716 & 0 & -119.008 & 0 & 758.104 \\ 592.452 & 0 & -1043.52 & 0 & 1200.62 & 0 \\ 0 & 239.156 & 0 & 66.8993 & 0 & -1143.22 \\ -752.621 & 0 & 1529.81 & 0 & -1876.03 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \\ -592.452a \\ 0 \\ 752.621a \end{pmatrix}. \quad (30)$$

Applying now the criterion of Section 2, the existence and uniqueness of solution of (28) is equivalent to the existence and uniqueness of solution of (30), that, in this example, depends on the values of the parameters a, b and c in the following way.

- If $b \neq -1$, the system (30) has a unique solution and then (28) has a unique solution.
- If $b = -1$ and $3a + c = 0$, the system (30) has an infinite number of solutions and then (28) has an infinite number of solutions.
- If $b = -1$ and $3a + c \neq 0$, the system (30) has no solution and then (28) has no solution.

We next observe that this discussion (derived from our criterion) about existence and uniqueness of the solution of problem (28), is exactly the one provided by the standard criterion. (For this particular easy example the general solution is available and then it is possible to apply the standard criterion.) The general solution of the differential equation given in (28) is

$$y(x, c_1, c_2) = c_1 e^{x^2} \operatorname{erf}(x) + c_2 e^{x^2} - a.$$

The standard criterion of existence and uniqueness of solution depends on the existence of real numbers c_1 and c_2 that makes $y(x, c_1, c_2)$ compatible with the boundary conditions in (28). That is, it depends on the existence of a solution of the linear system

$$\begin{pmatrix} 0 & 2e+1 \\ \frac{[1 + e\sqrt{\pi}\operatorname{erf}(1)](b+1)}{\sqrt{\pi}} & (b-1)e \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3a+c \\ 0 \end{pmatrix}. \quad (31)$$

n	$x=-0.75$	$x=-0.5$	$x=-0.25$	$x=0.25$	$x=0.5$	$x=0.75$
1	0.071167	0.399822	0.036121	0.023652	0.072093	0.122509
4	5.5e−7	4.2e−6	8.2e−8	3.3e−8	2.7e−7	1.3e−7
7	1.1e−12	1.4e−11	1.1e−13	5.4e−14	2.7e−12	1.9e−12

n	$x=-0.75$	$x=-0.6$	$x=-0.25$	$x=0.25$	$x=0.5$	$x=0.75$
1	0.033427	0.120053	0.016097	0.0034221	0.003367	0.000205
4	7.7e−7	5.0e−6	2.0e−7	8.5e−8	6.9e−7	2.7e−7
7	4.9e−13	4.3e−12	9.7e−14	1.3e−14	2.0e−13	1.2e−14

Table 2: Numerical experiments about the relative errors in the approximation of the exact solution of (28) and the approximated three-point Taylor polynomials $\widetilde{y}_n(x)$, $n = 1, 4, 7$ for different values of $x \in (-1, 1)$. The first table corresponds to the values $a = c = 1, b = -2$ and the second one to the values $a = -c = -1, b = -2$.

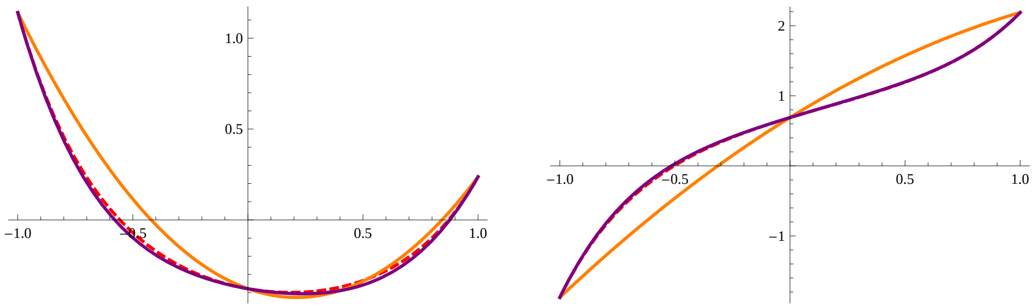

Figure 4: Plot of the exact solution $y(x) = \frac{\sqrt{\pi}(1-b)(3a+c)}{(2e+1)(1+e\sqrt{\pi}\mathrm{erf}(1))} e^{x^2+1}\mathrm{erf}(x) + \frac{(3a+c)}{2e+1} e^{x^2} - a$ (dashed red) of (28) and the two first approximated three-point Taylor polynomials $\widetilde{y}_n(x)$ for $n = 0, 1$ (orange and purple respectively) of (21) for $a = c = 1, b = 2$ (first graph), $a = -c = -1, b = -2$ (second graph).

The discussion about the existence and uniqueness of solution of (31) is just the discussion about the existence and uniqueness of solution of (30).

Table 2 and Figure 4 show a numerical experiment with $m = 10$ about the approximation supplied by the polynomials $\tilde{y}_n(x)$ to the solution of (28). It can be observed the improvement in the approximation as n increases.

$\square$

4 Final remarks

We have given at the end of Section 2 a straightforward and systematic criterion for the existence and uniqueness of solutions of three-point boundary value problems for second-order linear differential equations (1) when the coefficients of the differential equation are analytic functions inside a Cassini disk containing the domain of the differential equation. The criterion is very simple and establishes that the existence and uniqueness of the solution of the boundary value problem (1) is equivalent to the existence and uniqueness of the solution of the algebraic linear system $Rx = \Lambda$ given in (22). The last four entries of the system (22) are defined by the limits (19), whose exact computation is, in general, very difficult. In practice, these last four entries of the system (22) must be computed approximately in the form (23) and then, the solution of the system $Rx = \Lambda$, given by $x = (a_0, b_0, c_0, a_1, b_1, c_1)$, is approximated by the solution of the system $R_m x_m = \Lambda_m$. Also, in practice, we must apply the above existence and uniqueness criterion for the solution of (1) using the approximate linear system $R_m x_m = \Lambda_m$ instead of the exact linear system $Rx = \Lambda$. Nevertheless, the conclusions about the existence and uniqueness of solution are exact unless the ranks of the coefficient matrix R_m and/or of the augmented matrix $(R_m|\Lambda_m)$ sensibly depend on the precision in the computation of the approximate limits (23) (sensibly depend on m).

Formally, the criterion proposed in this paper is similar to the standard criterion based on the knowledge of the space of solutions: both criteria relate the existence and uniqueness of solution of the boundary value problem (1) to the existence and uniqueness of a solution of an algebraic linear system. As a difference with that standard criterion, our criterion does not require the knowledge of the general solution of the differential equation. This qualitative difference is very important when the general solution of the equation is not available. In this case, the standard criterion is not useful, whereas our criterion can be always applied (except in the case of sensible dependence of the ranks of the coefficient matrix R_m and/or of the augmented matrix $(R_m|\Lambda_m)$ with the precision of the computation discussed above). Moreover, the subsystem obtained from the last four rows of the full system $Rx = \Lambda$ determine the two-dimensional space of solutions of the differential equation. Then, the compatibility of the first two rows with that subsystem determine which (if any) of those solutions is also a solution of the boundary conditions.

We would like to remark here that the recursions (12) define a discrete dynamical system in $\mathbb{R}^6$ that determine the evolution, with respect to a discrete time variable n, of any vector in $\mathbb{R}^6$: from a starting vector $(a_0, b_0, c_0, a_1, b_1, c_1)$ at time $n = 0$, the recursions (12) give the vector $(a_n, b_n, c_n, a_{n+1}, b_{n+1}, c_{n+1})$ at any later time n. Then, we may identify the space W or formal solutions with the phase space $\mathbb{R}^6$ of that dynamical system. Moreover, the space S of true solutions may be identified with the (two-dimensional) stable variety at the origin of that discrete dynamical system: in general, for arbitrary starting point $(a_0, b_0, c_0, a_1, b_1, c_1)$, the asymptotic behavior of the solutions of (12) is divergent, that is, $\lim_{n\to\infty}(a_n, b_n, c_n, a_{n+1}, b_{n+1}, c_{n+1}) = \infty$ (most of the formal solutions of W are divergent). But, when the starting point $(a_0, b_0, c_0, a_1, b_1, c_1)$ is a solution of the last four rows of the system $Rx = \Lambda$ (it defines a true solution of S), the dynamics is convergent: $\lim_{n\to\infty}(a_n, b_n, c_n, a_{n+1}, b_{n+1}, c_{n+1}) = (0, 0, 0, 0, 0, 0, 0)$. That is, the origin of $\mathbb{R}^6$ behaves like a saddle point of the dynamical system having a two dimensional stable variety defined by the last four rows of the system $Rx = \Lambda$.

The analysis developed in [11] and the one introduced in this paper require the interval $[-1, 1]$ to be contained inside the Cassini disk of analyticity of the coefficient functions of the differential equation. This fact clearly depends on the proximity of the singularities of the coefficient functions to the interval $[-1, 1]$. But the shape of this Cassini disk depends on the number of base points that we choose for the multi-point Taylor expansion: the more base points we consider, the better we avoid those singularities (see [10] for a full explanation). This means that, in general, a three-point Taylor expansion is more convenient than a two-point Taylor expansion, a four-point Taylor expansion more convenient than a three-point expansion and so on. But the choice of the number of base points depends not only on the location of the singularities of the coefficient functions of the differential equation, but also on the number of points used for the definition of the boundary condition. We believe that the generalization from the three-point method presented in this paper to four or more base points is only a matter of computational complexity when the points used for the definition of the boundary condition are used as base points of the multi-point Taylor expansion. The discussion may be more complicated when the number of base points for the multi-point Taylor expansion is different from the number of points used to define the boundary conditions.

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