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RECENT PROBLEMS FROM UNIFORM ASYMPTOTIC ANALYSIS OF INTEGRALS IN PARTICULAR IN CONNECTION WITH TRICOMI'S Ψ-FUNCTION

1. Tricomi's Ψ–function

Tricomi [44, p. 56] introduced the Ψ-function as the second solution of the confluent hypergeometric differential equation (also called Kummer's equation)

$$z\,\frac{d^2y}{dz^2} + (c-z)\,\frac{dy}{dz} - a\,y = 0\,. \tag{1.1}$$

Tricomi denoted the first solution by $\Phi(a,c;z)$, which in fact is a hypergeometric function, given by

$${}_1F_1(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\,\frac{z^n}{n!}\,, \tag{1.2}$$

with the usual condition $c \neq 0,-1,-2,\ldots$. ${}_1F_1(a,c;z)$ is an entire function of z. The symbol $(a)_n$ is the shifted factorial (Pochhammer's symbol)

$$(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1\,. \tag{1.3}$$

It is not difficult to verify that $z^{1-c}{}_1F_1(a-c+1,2-c;z)$ is also a solution of (1.1).

Tricomi denoted the second solution of the Kummer equation (1.1) by $\Psi(a,c;z)$(1). It is defined as a linear combination of the two ${}_1F_1$–solutions:

$$\begin{aligned}\Psi(a,c;z) = {} & \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a,c;z) + \\ & + \frac{\Gamma(c-1)}{\Gamma(a)}\,z^{1-c}\,{}_1F_1(a-c+1,2-c;z)\,.\end{aligned} \tag{1.4}$$

The Kummer equation (1.1) and the solutions ${}_1F_1(a,c;z)$ and $\Psi((a,c;z)$, which

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(1) Several notations for the Kummer functions are used in the literature; we prefer the notation ${}_1F_1(a,c;z)$ for the first solution; in honor of Tricomi, we use $\Psi(a,c;z)$ for the second solution.

are often called Kummer functions, arise in many problems of mathematical physics.

A different introduction of equation (1.1) is based on a limiting method applied to the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$, which is a solution of the differential equation

$$z(1-z)\,y'' + [c-(a+b+1)z]\,y' - ab\,y = 0\,, \tag{1.5}$$

and which has the series representation

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!}\,z^n, \quad |z|<1\,. \tag{1.6}$$

This equation has three regular singular points $z=0, z=1, z=\infty$. The Kummer functions arise when two of the regular singular points are allowed to merge into one singular point. Formally this process runs as follows. The function ${}_2F_1(a,b;c;z/b)$ has a regular singular point at $z=b$. Using the series in (1.6) it can be verified that the limit

$$\lim_{b\to\infty} {}_2F_1(a,b;c;z/b)$$

exists, and equals the series in (1.2). It can also verified that in the same limiting process the Gauss hypergeometric differential equation (1.5) transforms into (1.1). This explains the name *confluent hypergeometric functions* for the Kummer functions.

The basic integral representation reads

$${}_1F_1(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1}(1-t)^{c-a-1}\,dt\,, \tag{1.7}$$

where $\Re a > 0, \Re(c-a) > 0$ The second solution can also be defined by an integral

$$\Psi(a;c;z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt}\,t^{a-1}\,(1+t)^{c-a-1}\,dt\,, \tag{1.8}$$

which is valid if $\Re a > 0, \Re z > 0$. The Ψ–function is, in general, not analytic at the origin $z=0$. The integral can be used for analytic continuation with respect to z into the domain $\{|\mathrm{ph}\ z| < \pi, z \neq 0\}$, by turning the path of integration. If $a = 0,-1,-2,\ldots,$ $\Psi(a,c;z)$ is a polynomial in z, if $c-a-1=n$ (non-negative integer), $\Psi(a,c;z)$ can be expressed as a polynomial in z multiplied with z^{-a-n}.

There are remarkable functional relations:

$$\begin{aligned} {}_1F_1(a,c;z) &= e^z\,{}_1F_1(c-a,c;-z)\,, \\ \Psi(a,c;z) &= z^{1-c}\,\Psi(a-c+1,2-c;z)\,. \end{aligned} \tag{1.9}$$

Contour integrals are given by

$$(1.10)\qquad \begin{aligned} {}_1F_1(a,c;z) &= \frac{\Gamma(c)}{2\pi i}\int_{F} e^s\, s^{a-c}\,(s-z)^{-a}\,ds\,, \\ \Psi(a,c;z) &= \frac{\Gamma(c-a)}{2\pi i}\int_{\mathcal{L}_\Psi} e^s\, s^{a-c}\,(z-s)^{-a}\,ds\,, \end{aligned}$$

where, if $z>0$, $\mathcal{L}_F$ is a vertical line in the half plane $\Re s > z$, and $\mathcal{L}_\Psi$ is a vertical line that cuts the real axis between the origin and z. When z is complex, the contours need to be modified appropriately. In order to speed up convergence, the contours may be deformed into parabola shaped contours that terminate at $-\infty$. The contour integrals are more flexible in asymptotic analysis than the standard integrals given in (1.7) and (1.8).

1.1. *Special cases of the Kummer functions*

There are many special cases. We mention the most important ones.

- Error functions.
- Exponential integrals.
- Fresnel integrals.
- Incomplete gamma functions.
- Bessel functions.
- Orthogonal polynomials.
- Parabolic cylinder functions.
- Coulomb Wave Functions.
- Whittaker functions.

In fact, the Whittaker functions are a different notation of the confluent hypergeometric functions. The relations are

$$(1.11)\qquad \begin{aligned} M_{\kappa,\mu}(z) &= e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu}\,{}_1F_1\left(\frac{1}{2}+\mu-\kappa, 1+2\mu; z\right)\,, \\ W_{\kappa,\mu}(z) &= e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu}\,\Psi\left(\frac{1}{2}+\mu-\kappa, 1+2\mu; z\right)\,. \end{aligned}$$

$M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ satisfy the Whittaker equation

$$(1.12)\qquad w'' + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4}-\mu^2}{z^2}\right) w = 0\,.$$

There is a vast literature on Kummer functions. The books [13], [34] and [46] are exclusively devoted to the class of confluent hypergeometric functions or Whittaker functions. Especially in the first book many references are given to physical applications.

2. Asymptotic expansions of Laplace-type integrals

We mention a very useful result from the theory of asymptotics for Laplace integrals, known as Watson's Lemma.

Asuume that f is analytic inside a sector $\Omega\colon \alpha_1 < \mathrm{ph}\, t < \alpha_2$, where $\alpha_1 < 0$ and $\alpha_2 > 0$ and that, as $t \to 0+$ inside $\Omega_\delta : \alpha_1 + \delta < \mathrm{ph}\, t < \alpha_2 - \delta$,

$$f(t) \sim t^{\lambda-1} \sum_{n=0}^{\infty} a_n t^n, \quad \Re\lambda > 0. \tag{2.1}$$

and that the integral

$$F(z) = \int_0^\infty f(t) e^{-zt}\, dt \tag{2.2}$$

is convergent for sufficiently large values of $\Re z$. Then the integral (2.2), or its analytic continuation, has the asymptotic expansion

$$F(z) \sim \sum_{n=0}^{\infty} \Gamma(n+\lambda) \frac{a_n}{z^{n+\lambda}}, \quad z \to \infty, \tag{2.3}$$

(where $z^{n+\lambda}$ has its principal value) inside the sector

$$-\alpha_2 - \frac{1}{2}\pi + \delta \le \mathrm{ph}\, z \le -\alpha_1 + \frac{1}{2}\pi - \delta, \tag{2.4}$$

For a proof we refer to [27, p. 113], where more general conditions are assumed; see also [47].

When applying Watson's lemma in the theory of special functions, the condition in (2.1) often holds, because the function $f(t)$ is, up to the factor $t^{\lambda-1}$, usually an analytic function in a domain containing $[0, \infty)$.

Recall the definition of the Ψ–function in (1.8), where $f(t) = t^{a-1}(1+t)^{c-a-1}$. In this case $f(t)$ is analytic in the sector $|\mathrm{ph}\, t| < \pi$, and we obtain

$$\Psi(a,c;z) \sim z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (a-c+1)_n}{n!} (-z)^{-n}, \quad z \to \infty. \tag{2.5}$$

which holds for $|\mathrm{ph}\, z| < 3\pi/2$. By using the integral in (1.7) with a change of variable $t \to 1-t$, that is,

$${}_1F_1(a,c;z) = \frac{\Gamma(c)\, e^z}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{c-a-1} (1-t)^{a-1}\, dt, \tag{2.6}$$

we obtain for the ${}_1F_1$–function the result

$${}_1F_1(a,c;z) \sim \frac{\Gamma(c) e^z\, z^{a-c}}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!} z^{-n}, \quad z \to \infty, \tag{2.7}$$

which is valid in the sector $|\mathrm{ph}\, z| < \frac{1}{2}\pi$. The limited domain of validity is due to the singularity of the integrand in (2.6) at $t = 1$. To extend the domain we need a different integral. For example, we can replace the interval $(0,1)$ in (2.6) with two intervals $(0,\infty)$ and $(1,\infty)$, where the point at infinity can be chosen above the the branch line $(1,+\infty)$ or below, depending on the phase of z. The result is

$$\begin{aligned}\frac{1}{\Gamma(c)}\,{}_1F_1(a,c;z) \sim\ & \frac{e^z\,z^{a-c}}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(c-a)_n(1-a)_n}{n!}\,z^{-n}+\\ &+\frac{e^{\pm\pi i a}\,z^{-a}}{\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{(a)_n(1+a-c)_n}{n!}\,(-z)^{-n},\end{aligned} \tag{2.8}$$

where the upper sign is taken if $-\frac{1}{2}\pi < \operatorname{ph} z < \frac{3}{2}\pi$ and the lower sign if $-\frac{3}{2}\pi < \operatorname{ph} z < \frac{1}{2}\pi$. The first part is dominant when $\Re z > 0$ and corresponds with (2.7); the second part becomes dominant when z enters the half plane $\Re z < 0$.

2.1. *A class of polynomials introduced by Tricomi*

In [45] Tricomi introduced a class of polynomials. He used the polynomials in convergent and asymptotic expansions. The definition can be given by using Laguerre polynomials:

$$l_n(x) = (-1)^n\,L_n^{(x-n)}(x) = \sum_{k=0}^{n}(-1)^k\binom{x}{k}\frac{x^{n-k}}{(n-k)!}, \tag{2.9}$$

which, although closely related to the Laguerre polynomials, are essentially different from them. For instance, the degree of $l_n(x)$ is not n but the greatest integer in $\frac{1}{2}n$. The first few polynomials are

$$l_0(x)=1,\quad l_1(x)=0,\quad l_2(x)=-\frac{1}{2}x,\quad l_3(x)=-\frac{1}{3}x,\quad l_4(x)=\frac{1}{8}x^2-\frac{1}{4}x.$$

The polynomials show up in the generating function

$$e^{xz}\,(1-z)^x = \sum_{n=0}^{\infty} l_n(x)\,z^n,\quad |z|<1. \tag{2.10}$$

This relation is easily verified by expanding both the exponential and binomial function in the left-hand side, and by comparing the coefficients in the product with (2.9). There is a simple recursion relation:

$$(n+1)l_{n+1}(x) = nl_n(x) - xl_{n-1}(x),\quad n=1,2,\ldots, \tag{2.11}$$

which can be derived from the generating function.

Tricomi mentions two applications. First, for the ${}_1F_1$–function there is

$$\frac{1}{\Gamma(c)}\,{}_1F_1(a,c;x) = \sum_{n=0}^{\infty} l_n(-a)\,x^n\,J^*_{c+n-1}(-ax),$$

where

$$J_\nu^*(z) = z^{-\nu/2}\, J_\nu\left(2\sqrt{z}\right) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(n+\nu+1)\; n!},$$

which is an entire function of z. For the incomplete gamma function there is an asymptotic expansion:

$$\Gamma(\alpha+1, x) \sim -e^{-x}\, x^{\alpha+1} \sum_{n=0}^{\infty} n!\, l_n(\alpha)\, (\alpha - x)^{-n-1},$$

as $\zeta = \sqrt{x-\alpha}/\alpha \to \infty$, within the sector $-3\pi/4 < \mathrm{ph}(\zeta) < 3\pi/4$.

Also [2], [3], [4] and [33] used the polynomials in asymptotic problems. In [36] and [37] we used the polynomials for obtaining uniform asymptotic expansions of Laplace integrals. In Section 4 we consider a generalization of the Tricomi polynomials.

We explain how the polynomials defined in (2.9) can be used in uniform expansions of Laplace integrals and apply the method to the Tricomi Ψ−function and the ${}_1F_1$−function.

2.2. *Uniform expansions of Laplace-type integrals*

We consider the Laplace integral

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1}\, e^{-zt}\, f(t)\, dt, \tag{2.12}$$

where $\Re z > 0, \Re\lambda > 0$ and z is a large parameter. We are interested in the case that λ is large as well.

When λ is restricted to a bounded set in the complex plane, an expansion of $F_\lambda(z)$ can be obtained by using Watson's lemma. When we assume that f is analytic at $t = 0$ we obtain:

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad \Longrightarrow \quad F_\lambda(z) \sim \sum_{n=0}^{\infty} (\lambda)_n\, a_n z^{-n-\lambda}, \tag{2.13}$$

as $z \to \infty$ in the sector $|\mathrm{ph}\, z| \le \frac{1}{2}\pi - \delta < \frac{1}{2}\pi$.

The expansion (2.13) loses its asymptotic character when λ is large. For instance, if $\lambda = \mathcal{O}(z)$, then the ratio of consecutive terms in the asymptotic expansion satisfy

$$\frac{a_{n+1}}{a_n}\,\frac{n+\lambda}{z} = \mathcal{O}(1) \quad \text{if} \quad a_n \neq 0\,.$$

In [36] we have modified Watson's lemma to obtain an expansion in which large as well small values of λ are allowed. This expansion is obtained by expanding f at $t = \mu := \lambda/z$, at which point the dominant part of the integrand of (2.12), that is, $t^\lambda\, e^{-zt}$, attains its maximal value (considering real parameters at the moment). We write

$$f(t) = \sum_{n=0}^{\infty} a_n(\mu)(t-\mu)^n ,$$

and obtain by substituting this into (2.12) the formal result

$$F_\lambda(z) \sim \sum_{n=0}^{\infty} a_n(\mu) P_n(\lambda) z^{-n-\lambda}, \quad z \to \infty , \tag{2.14}$$

where

$$P_n(\lambda) = \frac{z^{n+\lambda}}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} (t-\mu)^n \, dt, \quad \mu = \lambda/z . \tag{2.15}$$

The functions $P_n(\lambda)$ are polynomials in λ. From (2.15) the recursion

$$P_{n+1}(\lambda) = n[P_n(\lambda) + \lambda P_{n-1}(\lambda)]$$

follows with initial values $P_0(\lambda) = 1, P_1(\lambda) = 0$. An explicit formula follows from expanding $(t-\mu)^n$ in (2.15), which gives

$$P_n(\lambda) = \sum_{k=0}^{n} \binom{n}{k} (\lambda)_k (-\lambda)^{n-k} .$$

Comparing these properties with those of the Tricomi polynomials $l_n(x)$, we find that

$$P_n(\lambda) = n! \, l_n(-\lambda), \quad n = 0, 1, 2, \ldots .$$

The nature of the expansion (2.14) is discussed in [36] and [37]. Under rather mild conditions on f it follows that the expansion (2.14) holds uniformly with respect to $\lambda \in [0, \infty)$, and in domains of the complex plane.

We can apply this method to Tricomi's Ψ–function for the case that $z \to \infty$, to obtain an alternative of (2.5). For the new expansion we write

$$f(t) = (1+t)^{c-a-1} = \sum_{n=0}^{\infty} a_n(\mu)(t-\mu)^n ,$$

where

$$a_n(\mu) = \binom{c-a-1}{n} (1+\mu)^{c-a-1-n}, \quad \text{and} \quad \mu = a/z .$$

This gives

$$\Psi(a, c; z) \sim \sum_{n=0}^{\infty} a_n(\mu) \, P_n(a) \, z^{-n-a}, \quad z \to \infty , \tag{2.16}$$

uniformly with respect to $a \in [0, \infty)$; c should be of comparable size of a. We need the condition $c - a = \mathcal{O}(1)$.

We see that for $\mu \to 0$ the expansion reduces to (2.5); if μ becomes large the asymptotic convergence improves. If $c = a$ the expansion becomes rather simple:

$$\Psi(a, a; z) \sim z^{1-a} \sum_{n=0}^{\infty} (-1)^n \frac{P_n(a)}{(z+a)^{n+1}}, \quad z \to \infty, \tag{2.17}$$

which is an expansion for the exponential integral. This example and (2.16) show quite well why large values of $\lambda = a$ are allowed: the degree of $P_n(a)$ equals $[n/2]$, and the effect of $P_n(a)$ is amply absorbed by the term $(z+a)^{-n-1}$. Another feature is that (2.14) holds for $\lambda \to \infty$, uniformly with respect to z, say $z \geq z_0 > 0$.

A similar method is available for ${}_1F_1(a, c; z)$ if we use the contour integral in (1.10). We have

$${}_1F_1(a+1, c; z) = \frac{z^{1-c} e^z \Gamma(c)}{2\pi i} \int_{\mathcal{L}} e^{zw} (1+w)^{a+1-c} w^{-a-1} \, dw .$$

Expanding

$$(1+w)^{a+1-c} = \sum_{n=0}^{\infty} b_n(\mu)(w-\mu)^n, \quad b_n(\mu) = \binom{a+1-c}{n} (1+\mu)^{a+1-c-n},$$

where $\mu = a/z$, we obtain

$${}_1F_1(a+1, c; z) \sim \frac{z^{a+1-c} e^z \Gamma(c)}{\Gamma(a+1)} \sum_{n=0}^{\infty} b_n(\mu) \, Q_n(a) \, z^{-n}, \tag{2.18}$$

where

$$Q_n(a) = \frac{z^{n-a}\Gamma(a+1)}{2\pi} \int_{\mathcal{L}} e^{zw} (w-\mu)^n w^{-a-1} \, dw .$$

By expanding $(w-\mu)^n$ it easily follows that $Q_n(a) = (-1)nP_n(-a)$. The expansion in (2.18) can be viewed as an alternative for (2.7), and holds for $z \to \infty$, uniformly with respect to $a \in [0, \infty)$, with $c - a = \mathcal{O}(1)$.

3. Uniform asymptotic expansions in terms of Bessel functions

Tricomi has derived several convergent expansions of the ${}_1F_1$–function in terms of Bessel functions that are useful for evaluating the function when the parameters are large. For example, we have

$${}_1F_1(a, c; z) = e^{\frac{1}{2}z}\Gamma(c)(\kappa z)^{(1-c)/2} \sum_{n=0}^{\infty} A_n(\kappa, c/2) \left(\frac{z}{4\kappa}\right)^{n/2} J_{c-1+n}\left(2\sqrt{\kappa z}\right), \tag{3.1}$$

where $\kappa = c/2 - a$ and the $A_n(\kappa, \lambda)$ are coefficients in the generating function

$$e^{2\kappa z} (1-z)^{\kappa-\lambda} (1+z)^{-\kappa-\lambda} = \sum_{n=0}^{\infty} A_n(\kappa, \lambda) \, z^n .$$

The series in (3.1) is convergent in the entire z–plane. Moreover, it can be used for the evaluation of ${}_1F_1(a, c; z)$ for large κ. For further details on these expansions we refer to [46].

The expansion in (3.1) may be compared with an asymptotic expansion of the Whittaker function $M_{\kappa,\mu}(x)$ (cf. (1.11)) as given in [27, p. 446]. Olver used the differential equation to derive an expansion in terms of J-Bessel functions, with the same argument $2\sqrt{\kappa z}$ as in (3.1), which is provided with error bounds for the remainder in the expansion. Several other expansions are given by Olver, also for the function $W_{\kappa,\mu}(x)$. In [26] an expansion for the Whittaker functions is given in terms of parabolic cylinder functions; Dunster has developed in [17] uniform expansions for the Whittaker functions in terms of Bessel functions and Airy functions. All these approaches are based on differential equations; they are valid for large domains of the complex parameters, and supplied with error bounds.

In [39] we have given an approach based on integral representations for obtaining a uniform asymptotic expansion in terms of the modified Bessel function $K_\nu(z)$, with an application to the Ψ-function. The standard form for deriving the expansion is the integral

$$F_\lambda(z, \alpha) = \int_0^\infty t^{\lambda-1} e^{-zt-\alpha/t} f(t) \, dt , \tag{3.2}$$

which reduces to a modified Bessel function in the case that f is a constant. We have

$$2(\alpha/z)^{\lambda/2} K_\lambda(2\sqrt{\alpha z}\,) = \int_0^\infty t^{\lambda-1} e^{-zt-\alpha/t} \, dt . \tag{3.3}$$

The integral in (3.2) is considered with $\alpha, \lambda \geq 0$ and large positive values of z. We have derived asymptotic expansions for $F_\lambda(z, \alpha)$ that hold uniformly with respect to both α and λ in the interval $[0, \infty)$. To handle the transition of the case $\alpha = 0$ to $\alpha > 0$, the modified Bessel function (3.3) is needed. Observe that when $\alpha = 0$ the essential singularity in the integrand of (3.2) disappears and that (3.2) becomes a more familiar Laplace integral, that can be expanded by using Watson's lemma.

For the Ψ-function we can derive the expansion

$$\begin{aligned} \Gamma(a) e^{-x/2} \Psi(a, c; x) =\sim {} & 2\beta^{1-c} K_{1-c}(2\beta a) \sum_{s=0}^{\infty} a_s \, a^{-s} + \\ & + 2\beta^{2-c} \left[K_{2-c}(2\beta a) - K_{1-c}(2\beta a)\right] \sum_{s=0}^{\infty} b_s \, a^{-s} , \end{aligned} \tag{3.4}$$

where β is given by

$$\beta = \frac{w_0 + \sinh w_0}{2}, \quad w_0 = \cosh^{-1}\left(1 + \frac{1}{2}x/a\right). \tag{3.5}$$

The coefficients a_s, b_s follow from an integration by parts procedure, and the expansion holds for $a \to \infty$, uniformly with respect to $x \in [0, \infty)$. The asymptotic nature of the expansion is discussed in [39], where also an expansion is considered in which c is no longer a fixed parameter.

4. The Tricomi-Carlitz polynomials

The Tricomi-Carlitz polynomials are defined by

$$t_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{x-\alpha}{k} \frac{x^{n-k}}{(n-k)!}. \tag{4.1}$$

The relation with the Laguerre polynomials reads:

$$t_n^{(\alpha)}(x) = (-1)^n L_n^{(x-\alpha-n)}(x), \tag{4.2}$$

and we observe that the class of polynomials $\{l_n(x)\}$ introduced in Section 2 follows from the present set by putting $\alpha = 0$. The new polynomials satisfy the recurrence

$$(n+1)t_{n+1}^{(\alpha)}(x) - (n+\alpha)\, t_n^{(\alpha)}(x) + x\, t_{n-1}^{(\alpha)}(x) = 0, \quad n \geq 1, \tag{4.3}$$

with initial values $t_0^{(\alpha)}(x) = 1,\ t_1^{(\alpha)}(x) = \alpha$. A few other values are

$$t_2^{(\alpha)}(x) = \frac{1}{2}\left(\alpha + \alpha^2 - x\right), \quad t_3^{(\alpha)}(x) = \frac{1}{6}\left(2\alpha + 3\alpha^2 + \alpha^3 - 2x - 3x\alpha\right). \tag{4.4}$$

[44] introduced the polynomials. Tricomi observed that $\{t_n^{(\alpha)}(x)\}$ is not a system of orthogonal polynomials, the recurrence relations failing to have the required form (cf. [35, p. 43]). However, Carlitz discovered ([14]) that if one sets

$$f_n^{(\alpha)}(x) = x^n\, t_n^{(\alpha)}(x)(x^{-2}), \tag{4.5}$$

then $\{f_n^{(\alpha)}(x)\}$ satisfies

$$(n+1) f_{n+1}^{(\alpha)}(x) - (n+\alpha)\, x\, f_n^{(\alpha)}(x) + f_{n-1}^{(\alpha)}(x) = 0, \quad n \geq 1, \tag{4.6}$$

with initial values $f_0^{(\alpha)}(x) = 1,\ f_1^{(\alpha)}(x) = \alpha x$. A few other values are

$$f_2^{(\alpha)}(x) = \frac{1}{2}\left[\alpha(1+\alpha)x^2 - 1\right],$$

$$f_3^{(\alpha)}(x) = \frac{1}{6}x\left(-2 + 2\alpha x^2 - 3\alpha + 3\alpha^2 x^2 + \alpha^3 x^2\right).$$

There is a generating function for $f_n^{(\alpha)}(x)$:

$$e^{w/x+(1-\alpha x^2)/x^2 \ln(1-xw)} = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) w^n. \quad |wx| < 1. \tag{4.7}$$

If $x = 0$ this reduces to

$$e^{-\frac{1}{2}w^2} = \sum_{n=0}^{\infty} f_{2n}^{(\alpha)}(0)\, w^{2n},$$

giving

$$f_{2n}^{(\alpha)}(0) = (-1)^n 2^{-n}/n!, \quad f_{2n+1}^{(\alpha)}(0) = 0, \quad n = 0, 1, 2, \ldots .$$

Carlitz proved that for $\alpha > 0$, $\{f_n^{(\alpha)}(x)\}$ satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} f_m^{(\alpha)}(x)\, f_n^{(\alpha)}(x)\, d\psi^{(\alpha)}(x) = \frac{2\, e^{\alpha}}{(n+\alpha)\, n!}\, \delta_{mn}, \tag{4.8}$$

where $\psi^{(\alpha)}(x)$ is the step function whose jumps are

$$d\,\psi^{(\alpha)}(x) = \frac{(k+\alpha)^{k-1}\, e^{-k}}{k!} \quad \text{at} \quad x = x_k = \pm\frac{1}{\sqrt{k+\alpha}}, \qquad k = 0, 1, 2, \ldots . \tag{4.9}$$

The values x_k play a special role in the generating function because for these $x-$values we have

$$e^{w/x_k}\,(1 - x_k w)^k = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x_k)\, w^n,$$

and now the series converges for all values of w.

For further generalizations of the Tricomi-Carlitz polynomials we refer to [14] and [16]; [15] gives a brief treatment of the polynomials $t_n^{(\alpha)}(x)$. Goh and Wimp establish in [19] and [20] the asymptotic behavior of the Tricomi-Carlitz polynomials and discuss their zero distribution. They observe that the polynomials $f_n(x/\sqrt{\alpha})$ have all zeros in the interval $[-1, 1]$. They use in their second paper a probabilistic approach for improving their earlier results concerning the asymptotic distribution of the zeros of the polynomials $f_n^{(\alpha)}(x)$. Saddle point methods are used to study the asymptotics for $f_n^{(\alpha)}(x)$ in the complex plane.

In this section we describe a method how to obtain an asymptotic representation of the Tricomi polynomials in terms of the Hermite polynomial. We concentrate on large values of the parameter α and $n = \mathcal{O}(\alpha)$; for x we assume $-1/\sqrt{\alpha} < x < 1/\sqrt{\alpha}$, the interval of the zeros. The distribution of the zeros of $f_n^{(\alpha)}(x)$ can be obtained by using the zeros of the Hermite polynomials. The role of the Hermite polynomials can be shown by observing that

$$\lim_{\alpha\to\infty} f_n^{(\alpha)}\left(\frac{x\sqrt{2}}{\alpha}\right) = \frac{2^{-n/2}}{n!}\, H_n(x). \tag{4.10}$$

This follows from the generating function given in (4.7). Replacing x in the left-hand side with $x\sqrt{2}/\alpha$ yields, if $\alpha \to \infty$, $\exp(x\sqrt{2}\,w + \frac{1}{2}w^2)$. This is, up to some scaling, the generating function of the Hermite polynomials, which reads

$$e^{2xz-z^2} = \sum_{n=0}^{\infty} H_n(x)\,\frac{z^n}{n!}\,. \tag{4.11}$$

Although the Tricomi-Carlitz polynomials can be expressed in terms of the Laguerre polynomials (see (4.2) and (4.5)) it is not possible to use existing results on Laguerre polynomials from the literature to describe the asymptotics of $f_n^{(\alpha)}(x)$; this is due to the peculiar role and position of the parameters n and x in (4.2). In particular, the Tricomi-Carlitz polynomials $f_n^{(\alpha)}(x)$ do not satisfy a differential equation. Hence, the powerful results obtained in [17], [18] and [27] for the Whittaker functions cannot be used in the present case.

5. Hermite-type expansions of the Tricomi-Carlitz polynomials

We take the generating function (4.7) as starting point, and use the Cauchy-type integral:

$$f_n^{(\alpha)}(x) = \frac{1}{2\pi i}\int_C e^{w/x+(1-\alpha x^2)/x^2\,\ln(1-xw)}\,\frac{dw}{w^{n+1}}\,. \tag{4.12}$$

The contour C is a circle around the origin with radius less than $1/|x|, x \neq 0$.

Our approach for the Tricomi-Carlitz polynomials is earlier discussed in [38]. We summarize the main steps of this publication. In [22] a similar approach is used for Meixner polynomials; also in this case a differential equation is not available. The same problem occurs for the Charlier and Pollaczek polynomials, which are considered in [9] and [10], respectively, and for which Airy functions and Bessel functions are used as main approximants. For more details on these publication we refer to Section 5.

Rescaling the parameters in (4.12) by writing

$$x = \xi/\sqrt{\alpha}\,, \quad n = \nu\alpha, \quad w = s\sqrt{\alpha}$$

we obtain

$$f_n^{(\alpha)}(x) = \frac{\alpha^{-n/2}}{2\pi i}\int_C e^{\alpha\,\phi(s)}\,\frac{ds}{s}\,, \tag{4.13}$$

where

$$\phi(s) = \frac{s}{\xi} + \frac{1-\xi^2}{\xi^2}\,\ln(1-\xi s) - \nu\ln s\,. \tag{4.14}$$

The saddle points are given by

$$s_{1,2} = \frac{\xi(\nu+1) \pm \sqrt{\xi^2(\nu+1)^2 - 4\nu}}{2}\,. \tag{4.15}$$

If

$$-\frac{2\nu}{\sqrt{\nu+1}} < \xi < \frac{2\nu}{\sqrt{\nu+1}}$$

the saddle points are complex, and for these values of ξ the zeros of $f_n^{(\alpha)}(x)$ occur. In that case the saddle point are located on the circle with radius $\sqrt{\nu}$.

Comparing the behavior of the saddle points of the integral in (4.13) we observe that the situation is quite analogous to the behavior of the saddle points for various values of x and n of the Cauchy-type integral that defines the Hermite polynomials, viz.

$$H_n(x) = \frac{n!}{2\pi i} \int_C e^{2xz-z^2} \frac{dw}{w^{n+1}}, \tag{4.16}$$

which follows from (4.11). Due to this analogy, the integral in (4.13) can be approximated in terms of Hermite polynomials.

Before giving a few details on the saddle point analysis we give a first result. If $n \ll \alpha$ the complex saddle points given in (4.15) are close to the origin. For small values of s the phase function $\phi(s)$ can be approximated by

$$\phi_0(s) = \xi s - \frac{1}{2}(1-\xi^2)\, s^2 - \nu \ln s.$$

Substituting this into (4.13) and using (4.16) we obtain for $|\xi/\sqrt{\alpha}| < 1$ the approximation

$$f_n^{(\alpha)}(x) = \left(\frac{1-\alpha x^2}{2}\right)^{n/2} \frac{1}{n!} \left[H_n\left(\frac{\alpha x}{\sqrt{2(1-\alpha x^2)}}\right) + \varepsilon_n^{(\alpha)}(x)\right], \tag{4.17}$$

where we expect that $|\varepsilon_n^{(\alpha)}(x)|$ is small if $\alpha \gg n$. Observe that the limit in (4.14) follows from (4.17) if indeed $\lim_{\alpha\to\infty} \varepsilon_n^{(\alpha)}(x) = 0$.

Computing the zeros of $f_n^{(\alpha)}(x)$ for $n = 10, \alpha = 50$ with the help of (4.17) and the zeros of $H_{10}(x)$ gives a maximal absolute error of 0.0054 for the zeros of $f_{10}^{(50)}(x)$ and a relative error of about 5%.

To obtain an optimal approximation we first use a different scaling of the parameters for (4.12). This time we introduce the parameters ξ, ν, s by writing

$$x = \xi/\sqrt{\alpha - \frac{1}{2}}, \quad n + \frac{1}{2} = \nu(\alpha - \frac{1}{2}), \quad w = s\sqrt{\alpha - \frac{1}{2}}, \tag{4.18}$$

which yields

$$f_n^{(\alpha)}(x) = \frac{(\alpha - \frac{1}{2})^{-n/2}}{2\pi i} \int_C e^{(\alpha-\frac{1}{2})\phi(s)} \frac{ds}{\sqrt{s(1-\xi s)}}, \tag{4.19}$$

where $\phi(s)$ is given in (4.14) and the saddle points in (4.15) (now with different

ξ and ν as given in (4.18))(2). Next we substitute

$$\phi(s) = \psi(t) + A\,, \tag{4.20}$$

which in fact is a conformal mapping of the s-plane to the t-plane, where

$$\psi(t) = 2\eta t - \nu \ln t - \frac{1}{2}t^2 .$$

The quantities A and η follow from the condition that the saddle points in the s–plane correspond to the saddle points

$$t_{1,2} = \eta \pm \sqrt{\eta^2 - \nu} \tag{4.21}$$

in the t-plane. Using the transformation (4.20), we obtain from (4.19) the representation

$$f_n^{(\alpha)}(x) = \frac{(\alpha - \frac{1}{2})^{-n/2}\, e^{(\alpha-\frac{1}{2})A}}{2\pi i} \int_C e^{(\alpha-\frac{1}{2})\psi(t)}\, f(t)\, \frac{dt}{\sqrt{t}}\,, \tag{4.22}$$

where

$$f(t) = \frac{\sqrt{t}}{\sqrt{s(1-\xi s)}}\, \frac{ds}{dt}\,. \tag{4.23}$$

Evaluating the equation $\psi(t_1) - \psi(t_2) = \phi(s_1) - \phi(s_2)$, which defines the quantity η, we obtain

$$\begin{aligned} 2\eta\sqrt{\nu-\eta^2} + 2\nu \arcsin\frac{\eta}{\sqrt{\nu}} &= \frac{\sqrt{W}}{\xi} + \\ &- 2\frac{1-\xi^2}{\xi^2} \arcsin\frac{\xi\sqrt{W}}{2\sqrt{1-\xi^2}} + 2\nu \arcsin\frac{\xi(\nu+1)}{2\sqrt{\nu}}\,. \end{aligned} \tag{4.24}$$

where $W = 4\nu - \xi^2(\nu+1)^2$. The relation in (4.24) holds for $-2\sqrt{\nu}/(\nu+1) \le \xi \le 2\sqrt{\nu}/(\nu+1)$; the corresponding η–interval is $[-\sqrt{\nu}\,, \sqrt{\nu}\,]$.

Replacing the function $f(t)$ in (4.22) with a constant c_0, we obtain,

$$f_n^{(\alpha)}(x) = c_0\, e^{(\alpha-\frac{1}{2})A}\, \frac{2^{-n/2}}{n!} \left[H_n\left(\eta\sqrt{2\alpha-1}\right) + \varepsilon_n^{(\alpha)}(x) \right]. \tag{4.25}$$

In Table 4.1 we give the zeros x_k of $f_n^{(\alpha)}(x)$ for $n = 10, \alpha = 50$ and compare the zeros with approximations x_k^a obtained from this asymptotic formula. That is, let (for $k = 1, 2, \ldots, 10$) h_k be the zeros of $H_{10}(x)$. Define $\eta_k = h_k/\sqrt{2\alpha-1}$, and invert the relation in (4.24) to obtain ξ_k. Then the approximations of the zeros are given by $x_k^a = \xi_k/\sqrt{\alpha - \frac{1}{2}}$. We observe that the approximations for

(2)This choice of the phase function, which leaves the term $1/\sqrt{s(1-\xi s)}$ as part of the integrand, is not very obvious; also, the role of the large parameter $\alpha - \frac{1}{2}$ instead of α is not obvious. We refer to [43] for more details on this point.

these values of n and α are quite satisfactory; at least 5 significant decimal digits can be obtained in this way.

TABLE 4.1. – *Comparing the zeros of $f_n^{(\alpha)}(x)$ for $n = 10, \alpha = 50$ with approximations based on the zeros of $H_n(x)$. We show $x_k, k = 1, 2, \ldots, 10$ (the zeros of $f_{10}^{(50)}(x)$) with their approximations x_k^a, and the absolute and relative errors.*

k	x_k	x_k^a	abs. error	rel. error
1	−0.0855233907	−0.0855230252	0.36×10^{-6}	0.42×10^{-5}
2	−0.0650754635	−0.0650753259	0.13×10^{-6}	0.21×10^{-5}
3	−0.0460298897	−0.0460298453	0.44×10^{-7}	0.96×10^{-6}
4	−0.0274857009	−0.0274856920	0.89×10^{-8}	0.32×10^{-6}
5	−0.0091433976	−0.0091433973	0.32×10^{-9}	0.35×10^{-7}
6	0.0091433976	0.0091433973	0.32×10^{-9}	0.35×10^{-7}
7	0.0274857009	0.0274856920	0.89×10^{-8}	0.32×10^{-6}
8	0.0460298897	0.0460298454	0.44×10^{-7}	0.96×10^{-6}
9	0.0650754635	0.0650753259	0.13×10^{-6}	0.21×10^{-5}
10	0.0855233907	0.0855230252	0.36×10^{-6}	0.42×10^{-5}

Furhter details on the Hermite-type approximation of the Tricomi-Carlitz polynomials will be given in [43].

6. OTHER RECENT RESULTS ON UNIFORM EXPANSIONS OF INTEGRALS

In this paper we have concentrated on results for functions related to the Tricomi Ψ-function. This function is also an important topic in the recent interest in the Stokes phenomenon. In this section we mention a few aspects of the Stokes phenomenon; in particular we discuss shortly Olver's work on the $\Psi-$function in connection with this topic. There are several other recent publications in which uniform asymptotic expansions are derived by using integrals; we mention a series of papers by Wong and co-workers on certain orthogonal polynomials.

5.1. *Expansions in connection with the Stokes phenomenon*

In [5] Berry gave the Stokes phenomenon a new interpretation. This phenomenon is related with the different asymptotic expansions a function may have in certain sectors in the complex plane, and with the changing of constants multiplying asymptotic series when the complex variable crosses certain lines (also called Stokes lines). Berry explained that the constants are in fact rapidly changing smooth functions, which can be approximated in terms of the error function. His approach was followed by a series of papers by himself and other writers. At the same time interest arose in earlier work by Stieltjes, Airey and Dingle to re-expand remainders in asymptotic expansions and to improve the accuracy obtainable from asymptotic expansions by considering exponentially small terms.

The Stokes phenomenon and the topic of exponentially asymptotics are connected with uniform expansions of integrals, in particular, with approximations which are uniformly valid with respect to variations in the phase of the large parameter. We mention the contributions on a better understanding of the asymptotics of the gamma function by [6], [12] and [31]. More general papers are [7], [8] and [21]. For applications to the Ψ-function we mention [24] and [25]. In [11] new results for the modified K-Bessel function have been given. In [23] a method has been devised for estimating the optimal remainder in an asymptotic approximation which is uniform with respect to variations in the phase of the large parameter. In [28] and [29] interesting results for Tricomi's Ψ-function have been derived; see also [30]. An introductory paper on the Stokes phenomenon and exponential asymptotics is [32].

5.2. *Orthogonal polynomials*

In a series of papers, Wong and his co-workers have derived uniform asymptotic approximations for orthogonal polynomials that do not satisfy a differential equation, and for which integral methods are used. In these papers conformal mappings have been used that are of the same kind as the one given in (4.20) and in [38].

[1] In [22] the Meixner polynomials have been considered, which can be defined by the generating function

$$\left(1-\frac{\omega}{c}\right)^{x}(1-\omega)^{-x-\beta}=\sum_{n=0}^{\infty} m_n(x;\beta,c)\frac{\omega^n}{n!}\,. \tag{5.1}$$

There is a relation with the Gauss hypergeometric function:

$$m_n(x;\beta,c)=(\beta)_n\,{}_2F_1(-n,-x;\beta;1-1/c)\,.$$

Two infinite asymptotic expansions are derived for $m_n(n\alpha;\beta,c)$. One holds uniformly for $0<\varepsilon\le\alpha\le 1+a$, and the other holds uniformly for $1-b\le\alpha\le M<\infty$, where a and b are two small positive quantities. The main approximants are parabolic cylinder functions, which are in fact Hermite polynomials.

[2] [9] gives expansions for Charlier polynomials, which follow from the generating function

$$e^{-aw}(1+w)^x=\sum_{n=0}^{\infty} C_n^{(a)}(x)\,\frac{w^n}{n!},\qquad |w|<1\,. \tag{5.2}$$

There is a relation with the Laguerre polynomials and the Tricomi-Carlitz polynomials, because the Charlier polynomials can be written as

$$C_n^{(a)}(x) = n!\, L_n^{(x-n)}(a)\,. \tag{5.3}$$

(cf. (4.2) and (4.5)). An infinite asymptotic expansion is derived for $C_n^{(a)}(n\beta)$, as $n \to \infty$, which holds uniformly for $0 < \varepsilon \le \beta \le M < \infty$. The results are in terms of the $J-$Bessel function. Considering a as the large parameter gives an asymptotic problem as treated in the previous section, with approximations in terms of Hermite polynomials.

[3] [10] treats the Pollaczek polynomials, which are defined by the generating function

$$\left(1 - we^{i\theta}\right)^{-1/2+ih(\theta)} \left(1 - we^{-i\theta}\right)^{-1/2-ih(\theta)} = \sum_{n=0}^{\infty} P_n(x;a,b)\, w^n\,,$$

where

$$h(\theta) = \frac{a\cos\theta + b}{2\sin\theta}, \quad a > \pm b\,.$$

An asymptotic expansion is derived for $P_n(\cos(t/\sqrt{n});a,b)$, as $n \to \infty$, which holds uniformly for $0 < \varepsilon \le t \le M < \infty$. The results are in terms of Airy functions. A discussion on approximations of the zeros of $P_n(\cos(t/\sqrt{n});a,b)$ is included.

[4] In [42] incomplete gamma functions are considered for negative values of the parameters; the results can be used for complex values also, and complement earlier results that concentrate on positive values of the parameters, again with extension to complex values.

[5] [40] gives a selection of recent problems in connection with uniform asymptotic methods for integrals.

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