STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1951 - 028

Report on a manuscript entitled
"CONFORMAL RELATIVITY"

by R. Ingraham

by
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1951

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Rapport
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1951-028

Report on a manuscript entitled "Conformal Relativity" by R.Ingraham.

1. The title of the MS is incorrect. This is no conformal relativity. An n-dimensional conformal geometry is the geometry in a space where a (symmetric) fundamental tensor is given to within an arbitrary factor or in other words a tensor density of weight $-\frac{2}{n}$. The definition given on page 24 is wrong. None of the spaces occuring in the manuscript have a conformal geometry. In order to get a conformal geometry it would have been necessary to introduce an $\Gamma_{\mu\lambda}$ with index $\{-\frac{1}{2}, \frac{2}{n+2}, 0, 0, 0\}$ but taking the index $\{-2, 0, 0, 0, 0\}$ conformal geometry is from the beginning excluded. The correct title is "Relativity with six coordinates". It will be necessary to inform the reader that though the notations are used of conformal geometry with 4+2 supernumerary coordinates, as developed in Sch.-H. 1936, this is only done as an artifice to get six coordinates and not to deal with conformal geometry. Hoffmann's paper of 1948 uses a (restricted) conformal geometry but the point of view of the MS is entirely Riemannian and has nothing to do with conformal geometry neither in the sense of Veblen nor in the sense of Sch.-H. The passage in the introduction concerning conformal geometry must be changed and the whole section VIII must be dropped because the first part is based on a misconception and of the second part it is not clear from which point of view this part could be interesting. In the case of conformal geometry we have a definite problem, if we start from a conformal four-dimensional space, a six-dimensional space with fields $g_{\mu\lambda}$ and $\Gamma^{K}_{\mu\lambda}$ can be constructed and this can be done in several ways. So the problem is to prove that these possible fields are "not essentially different". But in the case of the MS the six coordinates are quite artificial and there seems to be no problem connected with them. The "uniqueness" of the H_{n+1} with $g_{\mu\lambda}$ that plays such an important role in conformal geometry seems to be here quite unimportant. In fact, in section VIII no definite problem is formulated.

If possible it would be desirable to make clear why conformal geometry is not used, because every reader seeing six coordinates will expect a conformal geometry. The answer given on p. 24 is not sufficient. It would be very enlightening if there could be an answer to the question what happens if the geometry is taken just as in Sch.-H. 1936. Then it would be more clear why in the MS quite another point of view is taken.

p. 2 2. Section IV is unnecessarily complicated. The curvature affinor $N_{\nu\mu\lambda}{}^{\kappa}$ must be defined immediately by the formula

2.1) $$N_{\nu\mu\lambda}{}^{\kappa} = 2\partial_{[\nu}\Gamma^{\kappa}_{\mu]\lambda} + 2\Gamma^{\kappa}_{[\nu|\rho|}\Gamma^{\rho}_{\mu]\lambda}$$

for holonomic coordinates and

2.2) $$N_{ikl}{}^{a} = 2\partial_{[k}\Gamma^{a}_{l]i} + 2\Gamma^{a}_{[k|e|}\Gamma^{e}_{l]i} + 2\Gamma^{e}_{ik}\Gamma^{a}_{el}$$

for general coordinates. A proof is not necessary because these things have been published long ago (Einf. I 1935, p. 110) and the proof in the MS using the device of two different connections is certainly not preferable. It is not allowed to give $N_{ikl}{}^{a}$ a meaning other than just (2.2). The expressions $R_{ikl}{}^{a}$, R_{ik} and $\mathfrak{R}_{ik}$ are entirely superfluous.

3. There is a difference between the notations of Schrödinger and those in the MS and in both there is an inconsequence. Schrödinger (51 A 13) has in four dimensions a not symmetric affinor density g^{ik} of weight + 1 and its determinant $g = \det(g^{ik})$ of weight $-2+4=2$. From these quantities he gets the affinor $g^{ik} = g^{ik}$ and its inverse g_{ik} with $g_{ij}g^{jk} = A^{k}_{i}$ and $g = \det(g_{ik})$. But later (51 A 16) he introduces not the symmetric part of g_{ik} but another symmetric quantity defined by

3.1) $$S^{ik} = \frac{\sigma^{(ik)}}{\sqrt{-S}} \quad ; \quad S = \mathrm{Det}(\sigma^{(ik)}) \quad ; \quad \boxed{S^{ik} \neq g^{(ik)}}$$

and its inverse S_{ik}:

3.2) $$S_{ij}S^{jk} = A^{k}_{i} \quad ; \quad \boxed{S_{ik} \neq g_{(ik)}}$$

The raising and lowering by means of S_{ik} is in contradiction with the definition of g_{ik} from g^{ik}.

In the MS there is a general affinor $S_{\lambda\kappa}$ in six dimensions. From this is defined $S^{\kappa\lambda}$ and S

3.3) $$S^{\mu\kappa}S_{\mu\lambda} = A^{\kappa}_{\lambda}$$

3.4) $$S = \mathrm{Det}(S_{\mu\lambda}) \qquad \text{(weight +2)}$$

Then in the variation integral the affinor density $\sqrt{S}\,S^{\kappa\lambda}$ is used. But later the symmetric part of $S_{\mu\lambda}$ is introduced

3.5) $$h_{\mu\lambda} = S_{(\mu\lambda)}$$

and its inverse $h^{\kappa\lambda}$

3.6) $$h^{\kappa\mu} h_{\lambda\mu} = H^{\kappa}_{\lambda}$$

and raising and lowering is defined by means of $h_{\lambda\kappa}$. But this is in contradiction with the definition of $S^{\mu\lambda}$ from $S_{\lambda\kappa}$. Both authors agree that raising and lowering should be done by means of the symmetric part. So the mistake is that g^{ik} and g^{kl} in (51 A 13) and $S_{\mu\lambda}$ and $S^{\mu\lambda}$ in the MS got the same kernel.

We propose to start with a non-symmetrically affinor $S_{\lambda\kappa}$ in sux dimensions and to call its inverse $S^{\kappa\lambda}$:

3.7) $$S^{\kappa\mu} S_{\mu\lambda} = H^{\kappa}_{\lambda}$$

Note the place of the indices. This is a minor detail but it is a good principle to choose notations in accordance with matrix calculus. Then we have a density of weight + 2

3.8) $$\mathcal{S} = Det\,(S_{\lambda\kappa})$$

an affinor density of weight + 1

3.9) $$\mathcal{S}^{\kappa\lambda} = \sqrt{\mathcal{S}}\, S^{\kappa\lambda}$$

a symmetric part of $S_{\lambda\kappa}$ called $h_{\lambda\kappa}$

3.10) $$h_{\lambda\kappa} = S_{(\lambda\kappa)}$$

its inverse $h^{\kappa\lambda}$:

3.11) $$h^{\kappa\mu} h_{\mu\lambda} = H^{\kappa}_{\lambda}$$ $\boxed{h^{\kappa\lambda} \neq S^{(\kappa\lambda)}!}$

and its determinant

3.12) $$f = Det\,(h_{\lambda\kappa})$$ (weight + 2)

p.4 **Table of notations:**

Schrödinger (4 dim.)	MS (6 dim.)	Proposal[x]
$\mathfrak{g}^{hi}$	$S^{\kappa\lambda}\sqrt{S}$	$\mathfrak{g}^{\kappa\lambda} = \sqrt{\mathfrak{g}}\, S^{\kappa\lambda}$
$g = Det(\mathfrak{g}^{hi}) = Det(g_{ik})$	$S = Det(S_{\lambda\kappa})$	$\mathfrak{g} = Det(S_{\lambda\kappa})$
$g^{hi} \overset{def}{=} \frac{1}{\sqrt{-g}}\mathfrak{g}^{hi}$	$S_{\lambda\kappa} : S^{\mu\kappa}S_{\mu\lambda} = A^{\kappa}_{\lambda}$	$Det(\mathfrak{g}^{\kappa\lambda}) = \mathfrak{g}^{n/2}$
$g_{ik} : g^{kj}g_{ij} = A^{k}_{i}$	$h_{\lambda\kappa} = S_{(\lambda\kappa)}$	$S_{\lambda\kappa} : S^{\kappa\mu}S_{\mu\lambda} = A^{\kappa}_{\lambda}$
$S^{hi} = \frac{\mathfrak{g}^{(hi)}}{\sqrt{-S}} \neq g^{(hi)}$	$h^{\kappa\lambda}\ h^{\mu\kappa}h_{\mu\lambda} = A^{\kappa}_{\lambda}$	$\gamma_{\lambda\kappa} : \mathfrak{g}^{\mu\kappa\rho}\gamma_{\mu\lambda} = A^{\kappa}_{\lambda}$
$S = Det(\mathfrak{g}^{(hi)})$	$h^{\kappa\lambda} \neq S^{(\kappa\lambda)}$	$h_{\lambda\kappa} = S_{(\lambda\kappa)}$
$S_{ik}\ S^{jk}S_{ik} = A^{j}_{i}$	$Det(h_{\lambda\kappa})$ not used	$h^{\kappa\lambda}\ h^{\mu\kappa}h_{\mu\lambda} = A^{\kappa}_{\lambda}$
$S_{ik} \neq g_{ik}$		$h^{\kappa\lambda} \neq S^{(\kappa\lambda)}$
$g^{hi} \neq S^{hj}S^{ik}g_{jk}$	$S^{\kappa\lambda} \neq h^{\kappa\rho}h^{\lambda\tau}S_{\rho\tau}$	$\mathfrak{f} = Det(h_{\lambda\kappa})$ if needed
inconsistence	inconsistence	no inconsistence

4. The variation is partly effected for $S_{[\mu\lambda]} \neq 0$ and the point where the MS makes the simpler assumption $S_{[\mu\lambda]} = 0$ is not very clearly marked. If for instance at the end of a section it is said that no use is made of the assumed symmetry, the careful reader will have to re-read the whole section.

As to the general case $S_{[\mu\lambda]} \neq 0$ we remark that the MS does not contain anything new. It is stated that the procedure is a "straightforward generalization of Schrödinger" but by reading Schrödinger one sees that these is nothing original or generalized in the MS. The dimension is changed into 6 and homogeneous coordinates have been taken. However, this makes no difference at all for the variation. The fact that independent variation of $S_{\mu\lambda}$ and $\Gamma^{\kappa}_{\mu\lambda}$ leads to Riemannian geometry in the case $S_{[\mu\lambda]} = 0$ is very interesting but by no means new because it has been proved before by Schrödinger. Now the case $S_{[\mu\lambda]} \neq 0$ is used only in a very short passage on p.32 concerning possible generalizations. So it would be quite sufficient only to mention the results of the variation and to state that this can be proved and has been proved by Schrödinger. However, in Schrödinger not all aspects came to light as was found out by Mr Nijenhuis. Therefore it would

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x) The difference between notations in the MS and those of Schrödinger should be mentioned in a footnote.

p.5 be advisable to put in the MS his modified way of treatment that follows here. It is much simpler than the very complicated methods of the MS.

From the variation of $\Gamma^{\kappa}_{\mu\lambda}$ in

4.1) $$\int \mathfrak{H}^{\mu\lambda}\, \delta N_{\mu\lambda}\, (dx)$$

we get immediately

4.2) $$\nabla_\kappa \mathfrak{H}^{\mu\lambda} - A^{\mu}_{\kappa} \nabla_\rho \mathfrak{H}^{\rho\lambda} - 2\mathfrak{H}^{\mu\rho\lambda} S_{\rho\kappa}{}^{\sigma} + \mathfrak{H}^{\sigma\lambda} S_{\rho\sigma}{}^{\rho} A^{\mu}_{\kappa} - 2\mathfrak{H}^{\rho\sigma\lambda} S_{\kappa\rho}{}^{\mu} = 0$$

and this can be written in the form

4.3) $$\mathfrak{H}^{\mu\lambda}{}_{\kappa} - A^{\mu}_{\kappa} \mathfrak{H}^{\sigma\lambda}{}_{\sigma} = 0$$

where

4.4) $$\mathfrak{H}^{\mu\lambda}{}_{\kappa} = \bar{\nabla}_\kappa \bar{\mathfrak{H}}^{\mu\lambda}_{+} - \frac{2}{n-1} A^{\mu}_{\kappa} S_{\nu\rho}{}^{\rho} \mathfrak{H}^{\mu\lambda}$$

In this expression μ and λ have the usual meaning and the $\bar{}$ -sign above the $\mathfrak{H}$ denotes that in the term $-\Gamma_\kappa \mathfrak{H}^{\mu\lambda}$ the $\Gamma_\kappa = \Gamma_{\nu\rho}^{\rho}$ have to be substituted by $\Gamma^{\rho}_{\rho\kappa}$. From (4.3) we get

4,5) $$\mathfrak{H}^{\mu\rho\lambda}{}_{\rho} - n\, \mathfrak{H}^{\mu\sigma\lambda}{}_{\sigma} = 0$$

Hence $\mathfrak{H}^{\mu\rho\lambda}{}_{\rho} = 0$ and consequently

4.6) $$\nabla_\kappa \bar{\mathfrak{H}}^{\mu\lambda}_{+} - \frac{2}{n-1} A^{\mu}_{\kappa} S_{\nu\rho}{}^{\rho} \mathfrak{H}^{\nu\lambda} = 0$$

or

$$\partial_\kappa \mathfrak{H}^{\mu\lambda} + \Gamma^{\mu}_{\sigma\kappa} \mathfrak{H}^{\sigma\rho\lambda} + \Gamma^{\lambda}_{\kappa\rho} \mathfrak{H}^{\mu\nu\rho} - \Gamma^{\rho}_{\rho\kappa} \mathfrak{H}^{\mu\nu\lambda} -$$

4.7) $$- \frac{2}{n-1} A^{\mu}_{\kappa} S_{\nu\rho}{}^{\rho} \mathfrak{H}^{\nu\nu\lambda} = 0$$

or

4.8) $$\partial_\kappa \mathfrak{H}^{\mu\lambda} = P^{\mu\lambda\,\tau\sigma}_{\kappa\,\rho}\, \Gamma^{\rho}_{\tau\sigma}$$

where

$$P^{\mu\lambda\,\tau\sigma}_{\kappa\ \rho} \stackrel{\text{def}}{=} \mathfrak{g}^{\mu\omega}\mathfrak{g}^{\lambda(\tau}\left(-A^{\sigma)\nu}_{\kappa\rho\omega} + \frac{2}{n-1}\,A^{\nu[\sigma\tau]}_{\kappa\rho\omega}\right) +$$

4.9)

$$+\ \mathfrak{g}^{\mu[\lambda}\mathfrak{g}^{\omega]\nu}\left(-A^{\lambda\tau\sigma}_{\rho\kappa\omega} + A^{\tau\sigma\lambda}_{\rho\kappa\omega}\right)$$

If $\Gamma^{\ \rho\ \kappa}_{\mu\ \lambda}$ is a solution of (4.8) it follows from (4.9) that $\Gamma^{\ \rho\ \kappa}_{\mu\ \lambda} - \Gamma^{\ \rho\ \kappa}_{\lambda\ \mu} + \rho_\mu A^{\rho\ \kappa}_{\ \lambda}$ is also a solution if ρ_μ is an arbitrary vector. That means that the $\rho^{\tau\sigma}$ -rank of P is $\leq n^3 - n$. This is confirmed by the identity $P^{\mu\lambda\ \tau\sigma}_{\kappa\ \ \sigma} = 0$. Then, however, also the A^{λ} -rank is $\leq n^3 - n$, and that is confirmed by

4.10) $$P^{\mu\lambda]\ \tau\sigma}_{\kappa\ \ \rho} = 0$$

The equation (4.8) can therefore only have solutions if

4.11) $$\partial_\mu \mathfrak{g}^{[\mu\lambda]} = 0 .$$

Note that this is an <u>invariant</u> condition because $\mathfrak{g}^{[\mu\lambda]}$ is a bivector density of weight $+1$. The condition is also sufficient if the $\rho^{\tau\sigma}$ -rank of P equals $n^3 - n$.

It is easily proved that the two parts in which are split up in (4.9) each have exactly the $\rho^{\tau\sigma}$ -rank Moreover we know that the equations (4.8) have exactly one solution if $\mathfrak{g}^{\mu\lambda}$ is symmetric. Hence the $\rho^{\tau\sigma}$ -rank of $P^{\mu\lambda\ \tau\sigma}_{\kappa\ \ \rho}$ is exactly $n^3 - n$ for all values of $\mathfrak{g}^{\mu\lambda}$ lying in a sufficiently small neighbourhood of any set of symmetric values[1]. It is an algebraic problem to find the n.e.s. conditions to be imposed on the field $\mathfrak{g}^{\mu\lambda}$ in order that the foresaid $\rho^{\tau\sigma}$ -rank be exactly $n^3 - n$. Now we have for the solutions $\Gamma^{\ \rho\ \kappa}_{\mu\ \lambda}$ and $\Gamma^{\ \rho\ \kappa}_{\lambda\ \mu}$ above:

4.12) $$S_{\mu\lambda}{}^{\lambda} = S'_{\mu\lambda}{}^{\lambda} + \tfrac{1}{2}(n-1)\rho_\mu .$$

and that means that if $\Gamma^{\ \rho\ \kappa}_{\mu\ \lambda}$ is a solution, always a solution can be found for which $S_{\mu\lambda}{}^{\lambda} = 0$. <u>Hence it is convenient to introduce the condition</u> $S_{\mu\lambda}{}^{\lambda} = 0$. Then we have the

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1) This remark was also made by Mr Schrödinger in a letter to Mr Nijenhuis of 18.11.1951.

P.) equations

a) $\partial_\kappa \mathcal{G}^{\mu\lambda} = \mathcal{P}^{\mu\lambda}_{\kappa\rho}{}^{\tau\sigma} \Gamma^{\rho}_{\tau\sigma}$

4.13)

b) $\Gamma^{\rho}_{\rho\lambda} = \Gamma^{\rho}_{\lambda\rho}$

by which the $\Gamma^{\kappa}_{\mu\lambda}$ are completely determined.

Taking the solution satisfying $S_{\mu\lambda} = 0$ (the geometrical meaning of which is that the connection behaves like a symmetricone with respect to scalar densities!!) we get from (4.6)

4.14) $\nabla_\mu \mathcal{G}^{\underline{+}\,\kappa\lambda} = \nabla_\mu \mathcal{G}^{\underline{+}\,\kappa\lambda} = 0$

(where the $\underline{+}$ -sign above $\mathcal{G}$ can be omitted because of (4.13 b)) and

4.15) $\nabla_\mu \mathcal{G}_{\underline{+}\,\lambda\kappa} = 0$

Note that if (4.6) is satisfied the equation $\nabla_\kappa \mathcal{G}^{\mu\lambda} = 0$ has as a consequence that $S_{\mu\rho}{}^{\rho} = 0$ and that the same holds for the equation $\nabla_\mu \mathcal{G}^{\underline{+}\,\kappa\lambda} = 0$ if $\mathcal{G}^{(\kappa\lambda)}$ has the highest rank. This latter remark was made already by Einstein and Straus, cf. Schrödinger.

Now taking N for the dimension ($N = n+2 = 6$ in the MS) the weight of $\mathcal{G}$ is 2 and because of

4.16) $$\mathcal{G}^{\kappa\lambda} = \frac{\partial \log (\mathrm{Det}\, \mathcal{G}_{\lambda\kappa})}{\partial \mathcal{G}_{\lambda\kappa}} = \frac{\partial \log \mathcal{G}^{-\frac{1}{2}(N-2)}}{\partial \mathcal{G}_{\lambda\kappa}}$$

we have

$$\partial_\mu \log \mathcal{G}^{-\frac{1}{2}(N-2)} = \frac{\partial \log \mathcal{G}^{-\frac{1}{2}(N-2)}}{\partial \mathcal{G}^{2}_{\lambda\kappa}} \partial_\mu \mathcal{G}_{\lambda\kappa} = -\mathcal{G}_{\lambda\kappa} \partial_\mu \mathcal{G}^{\kappa\lambda} =$$

4.17)

$$= -\mathcal{G}_{\lambda\kappa} \left(\nabla_\mu \mathcal{G}^{\underline{+}\,\kappa\lambda} - \Gamma^{\kappa}_{\rho\mu} \mathcal{G}^{\rho\lambda} - \Gamma^{\lambda}_{\mu\rho} \mathcal{G}^{\kappa\rho} + \Gamma^{\rho}_{\mu\rho} \mathcal{G}^{\kappa\lambda} \right) =$$

$$= \Gamma^{\kappa}_{\kappa\mu} + \Gamma^{\kappa}_{\mu\kappa} - N\Gamma_\mu = (2-N)\,\Gamma_\mu$$

Hence

4.18) $$\nabla_\mu \mathcal{G} = \partial_\mu \mathcal{G} - (2-N)\,\Gamma_\mu \mathcal{G} = 0.$$

and consequently

4.19) $$\nabla_\mu S^{\underline{+}\,\kappa\lambda} = 0 \quad ; \quad \nabla_\mu S_{\underline{+}\,\lambda\kappa} = 0$$

p.8 Of course from the foregoing (from (4.1) on) only the most essential points and the results need to appear in the MS. $\bar{\Gamma}^{\kappa}_{\mu\lambda}$, $H^{\mu\lambda}_{\tau}$ H_{τ}, $\bar{H}_{\tau}$ and the whole complicated argumentation with two different connections can be avoided and therefore must be dropped.

If $S_{\lambda\kappa}$ is symmetric we have both $\nabla_\mu S_{\lambda\kappa} = 0$ and $\nabla_\mu S_{\lambda\kappa} = 0$ and the difference equals $-4 S_{\mu\lambda}{}^{\rho} S_{\kappa\rho} = 0$, hence $S_{\mu\lambda} = 0$. Or in another way:

$$\partial_\mu S_{\lambda\kappa} - \Gamma_{\mu\lambda}^{\rho} S_{\rho\kappa} - \Gamma_{\kappa\mu}^{\rho} S_{\lambda\rho} = 0$$

4.20)
$$\partial_\lambda S_{\kappa\mu} - \Gamma_{\lambda\kappa}^{\rho} S_{\rho\mu} - \Gamma_{\mu\lambda}^{\rho} S_{\kappa\rho} = 0$$

$$\partial_\kappa S_{\mu\lambda} - \Gamma_{\kappa\mu}^{\rho} S_{\rho\lambda} - \Gamma_{\lambda\kappa}^{\rho} S_{\mu\rho} = 0$$

from which we get immediately

4.21)
$$\Gamma_{\mu\lambda}^{\kappa} = \left\{ {\kappa \atop \mu\lambda} \right\} \quad , \quad S_{\mu\lambda}{}^{\kappa} = 0$$

and (Einf. I p. 83)

4.22)
$$\Gamma_{cb}^{a} = \left\{ {a \atop cb} \right\} - P_{cb}^{a} + S_{cd} S^{ae} P_{be}^{d} + S_{bd} S^{ae} P_{ce}^{d}$$

If it is agreed upon as in MS p. 8 to write $+P_{cb}^{a}$ instead of $-P_{cb}^{a}$ we may even write

4.23)
$$\Gamma_{cb}^{a} = \left\{ {a \atop cb} \right\} + P_{cb}{}^{a} + P_{b}{}^{a}{}_{c} + P_{c}{}^{a}{}_{b}$$

(form. MS (5.21), MS p. 13) but the well-known formula (4.22) need not be derived and certainly not in the very inpractical way followed in the MS. (The readers are expected to know more difficult facts.) Also the formula MS (5.18) though perhaps useful for other purposes and the quantity $J_{\mu\rho}{}^{\tau}$ are here quite superfluous and represent only very undesirable complications.

The variation of the $S^{\mu\lambda}$ can be dealt with for $\delta^{[\mu\lambda]} \neq 0$ without any difficulty:

$$\delta\left(\mathfrak{H}^{\mu\lambda} N_{\mu\lambda}\right) =$$

4.24)
$$= \mathfrak{H}^{1/2} N_{\mu\lambda} \delta S^{\mu\lambda} + N_{\mu\lambda} S^{\mu\lambda} \tfrac{1}{2} \mathfrak{H}^{-1/2} \frac{\partial \mathfrak{H}}{\partial S_{\rho\sigma}} \delta S_{\rho\sigma} =$$

$$= \mathfrak{H}^{1/2} N_{\mu\lambda} \delta S^{\mu\lambda} - \tfrac{1}{2} N_{\rho\sigma} S^{\rho\sigma} \mathfrak{H}^{1/2} S_{\lambda\mu} \delta S^{\mu\lambda} =$$

$$= \mathfrak{H}^{1/2} \left(N_{\mu\lambda} - \tfrac{1}{2} N S_{\lambda\mu} \right) \delta S^{\mu\lambda} \quad , \quad N = N_{\rho\sigma} S^{\rho\sigma}$$

p.9 and if the other variation is worked out for the more general case there is no reason not to do, is here.

From § 14 on the MS seems to take $\sum \lambda^{\alpha} = 0$. Here $G^{(\alpha\alpha)}$ is introduced for $G^{cd} X_c X_d$. This can only be done if it is said here explicitly that $G^{(\alpha\alpha)}$ will indeed be used later on as a component with respect to a very special anholonomic system. Otherwise it is not allowed to write a thing that is no component in the form of a component. It is strongly advis to write G^{00} with poetical indices and not to put the reader for unnecessary complications and the printer for a more difficult and more expensive notation.

The problem is now to write out the equation

$$G_{,c} - X_c X_b G^{00} = 0$$

in terms of Γ^a_{cl}, S_{al}, Ω^a_{cb} and their derivatives. This is a quite straightforward and well-known process and in order to perform it it is neither necessary nor desirable to introduce expressions as $\vec{R}_{cd}$, R_{cd}, $\vec{R}$ and J^{cb} (this latter quantity in quite another sense as on page 13!). Moreover the resulting equation MS (5.27) has no sense at all because it contains the covariant derivative of a geometric object Ω^a_{cb} that is no quantity. This is in no circumstances allowed! It is not to be seen why a process that contains no difficulty at all and only needs a certain amount of work is not performed in the ordinary way.

5. The most serious difficulties arise in section VI. By means of

5.1) $$x^m = f^m(X^k) \quad , \quad m = 1,2,3,4$$

four functions of the X^k are introduced and because these functions are homogeneous of degree zero they represent a system of ∞^4 X_2's in the X_6 such that each X_2 is built up by ∞^1 of the ∞^5 rays (preferred curves) of the X_6 that constitute the H_5. The x^m may be considered as coordinates of the X_4 arising from reduction of the X_6 with respect to these X_2's. Each point of this X_4 represents an X_5 in X_6. Next to these X_2's an X_5 in X_6 is introduced in two ways, first by its parametric equations

5.2) $$X^h = X^h(\xi^\alpha) \quad , \quad \alpha = 0,1,\ldots,4,$$

and then as it seems by its null form

p.10 5.3) $S_{\mu\lambda} X^{\mu} X^{\lambda} = -1$

but this is not quite sure, perhaps it could be meant that (5.3) holds throughout the x_6 ?

Anyhow, the x_5 is certainly built up by ∞^4 of the rays.

Now each x_2 of (5.1) intersects this x_5 in one ray. Hence from now on the x^m can be considered also as the coordinates of an x_4 each point of which represents only one ray lying on the x_- defined by (5.2).,

Now we have (cf. Sch.-H. I p. 600)

5.4)

a) $A^{\kappa}_{\lambda} \overset{def}{=} \partial_{\lambda} X^{\kappa}$

b) $B^{\mu}_{\lambda} \overset{def}{=} \partial_{\lambda} X^{\kappa}$

c) $E^{k}_{\beta} \overset{def}{=} \partial_{\beta} x^{k}$

These quantities do not occur in the MS and because the variables are de fac to not used it is not necessary to introduce them. We only use them here to make things perfectly clear. (definition of the manuscript).

d) $H^{k}_{\lambda} \overset{def}{=} \partial_{\lambda} x^{k}$

e) $A^{k}_{i} \overset{def}{=} \partial_{i} x^{k}$

and from this we see that it is highly objectionable to use for two quite different quantities as A and A the same kernel. It would be even better to give H^{k} and H^{k} different kernels but as the x^{κ} may be interpreted as supernumerary coordinates in x_4 the notation A^{k} A^{k} is not incorrect.

After introduction of the special anholonomic system (a) we have $A^{m} * A^{m}$ but is no reason to take from the beginning the kernel $A^{\lambda}_{(i)}$ where A should be used. Incorrect notations of this kind may seem innocent but in fact they use to be sources of the most disagreeable and unexpected mistakes.

Instead of using a lot of A's in several formulae it would be much better to introduce the bivector density

5.5) $\mathfrak{H}^{\kappa\lambda} = A^{\kappa}_{[0} A^{[1 \ldots 4]}_{1 \ldots 4} A^{\lambda}_{5]} \; ; \quad \{-1, +1, -1\}$

and the bivector

5.6) $W^{\kappa\lambda} = \varphi \frac{\sqrt{g}}{\sqrt{5}} \mathfrak{H}^{\kappa\lambda}$

5.6)

with a scalar factor φ such that

5.7) $$Y^\kappa \overset{def}{=} W^{\kappa\lambda} X_\lambda$$

satisfies

5.7) $$S_{\pm\lambda} Y^\lambda Y^\lambda = \varepsilon \quad (\varepsilon = \pm 1)$$

Then it is not necessary to introduce first φ^κ, and the equations MS (5.9) get the simple form

$$A^m_\kappa Y^\kappa = A^m_\kappa W^{\kappa\lambda} X_\lambda = 0$$

5.8)

$$X_\kappa Y^\kappa = X_\kappa X_\lambda W^{\kappa\lambda} = 0$$

$X_{\kappa\lambda}$ and $Y_{\kappa\lambda}$ can not be defined by (6.12). Because if they were defined in this way, $X_{\kappa\lambda}$ and $Y_{\kappa\lambda}$ would only be abbreviations of $\Omega^0_{\kappa\lambda}$ and $\Omega^5_{\kappa\lambda}$ and because $\Omega^{a}_{\kappa\lambda}$ is not defined and $\Omega^0_{\kappa\lambda}$, $\Omega^5_{\kappa\lambda}$ having no meaning/we could never speak of $X_{\kappa\lambda}$ and $Y_{\kappa\lambda}$. It should be done as follows.

(Ω is not a quantity!)

The expressions

$$A^{\kappa\lambda}_{\alpha\beta} \partial_{[\kappa} X_{\lambda]} \overset{*}{=} A^{\kappa\lambda}_{\alpha\beta} \Omega^0_{[\kappa\lambda]}$$

5.9)

$$A^{\kappa\lambda}_{\alpha\beta} \partial_{[\kappa} Y_{\lambda]} \overset{*}{=} A^{\kappa\lambda}_{\alpha\beta} \partial_{[\kappa} \Omega^5_{\lambda]}$$

are components with respect to (a) of the bivectors $\partial_{[\kappa} X_{\lambda]}$ and $\partial_{[\kappa} Y_{\lambda]}$ of the now fixed covariant vectorfields X_λ and Y_λ. For these bivectors we introduce the kernels X and Y

$$X_{\kappa\lambda} = -\partial_{[\kappa} X_{\lambda]}$$

5.10)

$$Y_{\kappa\lambda} = \varepsilon \partial_{[\kappa} Y_{\lambda]}$$

and then we have as an incidental result:

$$X_{\alpha\beta} \overset{*}{=} \Omega^0_{\alpha\beta}$$

5.11) $$Y_{\alpha\beta} \overset{*}{=} \Omega^5_{\alpha\beta}$$

The introduction of $\langle\ \rangle$ in (6.22), is highly objectionable. For three indices it could be introduced and used in cases where we really want an expression for $3[\kappa\lambda\mu] + 3(\kappa\lambda\mu)$

p.12 and also in those cases (certain algebraic problems connected with the splitting up of affinors) it would be better to define $\langle\ \rangle = [\ ] + (\ )$ instead of $\langle\ \rangle = \tfrac{1}{2}[\ ] + (\ )$. But in all cases where either $(\ )$ or $[\ ]$ over three indices lead to zero there is no excuse for the introduction of a new sign. The MS is for the average reader already very difficult and it can not be tolerated that the text is made still more difficult by introducing quite unnecessary complications. There can be no reason to write $-2\Sigma\varphi_m(\partial_{\langle\xi}H^m_{\mu)}\ldots$ where $-6\Sigma\varphi_m(\partial_{[\xi}H^m_{\mu]}\ldots$ is simpler and clearer. To this formula must also be remarked that according to a general convention the working of $\partial_\lambda, \nabla_\lambda$ etc. as differentiation symbols reaches till the first closing bracket whose corresponding opening bracket stands at the left hand side of the symbol. In the average this rule diminishes the number of necessary brackets but in MS (6.22) some brackets must be put in.

If φ is a function of the x^m, the operator ∂_m has a sense and because the H^m_λ and H^m_ξ are defined also, $H^m_\lambda\partial_m$ has a sense and also $H^m_\xi\partial_m$. The X_4 is a set of ∞^4 X_2's in X_6 and $\varphi = \varphi(x^m)$ means that to every X_2 corresponds one definite value of φ. Hence φ may be considered as a very special function of the x^κ that is constant on each of the ∞^4 X_2's. In an analytic way φ is a function of the x^m and the x^m are functions of the X^κ, hence

5.12) $$\partial_\lambda\varphi = \frac{\partial\varphi}{\partial X^\lambda} = \frac{\partial\varphi}{\partial x^m}\frac{\partial x^m}{\partial X^\lambda} = H^m_\lambda\partial_m\varphi$$

and

5.13) $$\partial_\xi\varphi = H^\lambda_\xi\partial_\lambda\varphi = H^m_\xi\partial_m\varphi, \quad H^m_\xi = H^\lambda_\xi A^m_\lambda.$$

X^κ and Y^κ being two vectors in the tangent plane of the local X_2, it is quite clear that $\partial_0\varphi = -X^\lambda\partial_\lambda\varphi = 0$ and $\partial_S\varphi = \Sigma Y^\lambda\partial_\lambda\varphi = 0$.

If φ is a function of the X^κ that is not constant on every preferred X_2, it is not a function of the x^m and it is no longer true that $\partial_\xi\varphi = H^m_\xi\partial_m\varphi$. If φ happens to be homogeneous of degree zero we have $X^\kappa\partial_\kappa\varphi = 0$ but also in this case in general $Y^\lambda\partial_\lambda\varphi \neq 0$. Hence

(5.14) $$\partial_\xi\varphi = H^m_\xi\partial_m\varphi + \Sigma Y^\lambda\partial_\lambda\varphi \neq H^m_\xi\partial_m\varphi$$

Now according to definition the H^m_λ are functions of the X^κ homogeneous of degree -1 and $g^{mn} = A^m_\mu A^n_\nu h^{\mu\nu}$ are functions of the X^κ, homogeneous of degree 0. But the g^{mn} are not constant on every preferred X_2 and accordingly they are no functions of the X^m. The same holds for $\chi_{mn}, \gamma_{mn}, \varphi_m, \xi_m$.

p.13 This is <u>very serious</u> because it means that the computations on pages 18-21 are incorrect and the results on p. 21 at least doubtful. It seems probable that for these results in some way a correct formulation can be found but this whole passage must be revised thoroughly.

It has to be remarked that the difficulty mentioned here cannot be removed by assuming that the χ_5 has already been chosen in such a way that $y^4 \partial_4 \varphi$ gives no contribution to $\partial_m \varphi$ for then the tangent hyperplane of χ_5 would have to be spanned by the vectors $q^\kappa_0 = x^\kappa_0$, q^κ_m which is only possible if in all points of χ_5

$$-P^5_{0m} = 0 \quad , \quad -P^5_{mn} \pm y_{mn} = 0$$

This seems to be disagreeable because of the "medic" interpretation of y_{mn}.

6. In section IV and V a six dimensional integral occurs, that is an integral in the Π_5 considered as a six-dimensional space. But in IV it is maintained that this is a <u>four-dimensional</u> integral over the χ_4 defined by MS (6.1') and page MS 16. This can not be accepted without it being proved that really the six-dimensional integral can be connected with a definite four-dimensional one. Perhaps this is possible for integrals over a region in 6-space consisting of ∞^4 preferred χ_2's but this must be proved and it must be shown how the integrals are connected. It is possible that here a problem arises:
Dealing with six-dimensional integrals in χ_6 we may consider only those results in χ_6 derived by them that are in some way invariant for reduction of the χ_6 with respect to the ∞^4 χ_2's. These results may be interpreted as results in the χ_4 arising from the reduction. Now the question arises if these results in the χ_4 are invariant if the system of χ_2's is changed, that is if instead of the 4 functions y_m we take 4 arbitrary other functions. This could lead to a definite problem and the answer to it if found could possibly replace the latter part of section VIII. It is very important that this connection between 4 and 6 dimensional integrals is dealt with thoroughly because nobody can accept a six-dimensional integral as a four-dimensional one without an exact formulation of what is really meant.

7. The notation with function symbols as for instance $N_{\lambda\mu}(\Gamma)$ and $\Gamma_{\mu\nu}(N_{\lambda\mu})$ is very dangerous. A notation of this kind was introduced by Schrödinger (51 A 16 p. 481) as

p.14 a kind of abbreviation ad hoc. He did it with a kind of excuse and used it just once in a very simple case. As a general principle it should be agreed upon that for a function symbol not the same kernel is used as for the component symbol, e.g. in a simple case $v^\kappa = f^\kappa(\xi^\lambda)$. In order to get not too many kernels we may introduce for functions of the ξ^κ the following notation

$$v^\kappa = {}_{(\kappa)}v^\kappa(\xi^\lambda) = {}_{(\kappa')}v^\kappa(\xi^{\lambda'}) = {}_{(\kappa'')}v^\kappa(\xi^{\lambda''})$$

7.1) $$T^{\kappa\lambda} = {}_{(\kappa)}T^{\kappa\lambda}(\xi^\lambda) = {}_{(\kappa')}T^{\kappa'\lambda'}(\xi^{\lambda'}) = \text{etc.}$$

Where of course the index to the left is considered to belong to the kernel, this notation is consistent and can never lead to difficulties. But it is rather complicated. It can be abbreviated a little by assuming that for geometric objects wit indices all belonging to the same coordinate system the index to the left can be suppressed if the components with respect to a coordinate system are expressed as functions of the coordinates of this same system, for instance

7.2) $\Gamma_{\mu\lambda}^{\kappa}(\xi^\rho)$ instead of ${}_{(\kappa)}\Gamma_{\mu\lambda}^{\kappa}(\xi^\rho)$

We used this notation here already for investigations on general macroscopic and microscopic objects where an exact notation was necessary.

Now for objects that are to be expressed as functions of other objects it must be possible to establish a correct notation. But here a difficulty arises. If for instance the $N_{\mu\lambda}$ are written as functions of the $\Gamma_{\mu\lambda}^{\kappa}$ and the $\partial_\nu\Gamma_{\mu\lambda}^{\kappa}$ the transformed components $N_{\mu'\lambda'}$ are the same functions of the $\Gamma_{\mu'\lambda'}^{\kappa'}$ and $\partial_{\nu'}\Gamma_{\mu'\lambda'}^{\kappa'}$. So if we write

7.3) $$N_{\mu\lambda} = \phi_{\mu\lambda}(\Gamma_{\sigma\tau}^{\rho})$$

and

7.4) $$N_{\mu'\lambda'} = \phi_{\mu'\lambda'}(\Gamma_{\sigma'\tau'}^{\rho'})$$ (κ') holonomic coordinate systems

it follows that $\phi_{\mu\lambda}$ and $\phi_{\mu'\lambda'}$ are expressions for the same functions:

7.5) $$\phi_{\mu\lambda}(\Gamma_{\sigma\tau}^{\rho}) = \mathfrak{A}_{\mu\lambda}^{\mu'\lambda'}\,\phi_{\mu'\lambda'}(\Gamma_{\sigma\tau}^{\rho})$$

p.15 but

7.6) $N_{\mu\lambda'} \neq \phi^{\kappa}_{\mu'\lambda'} N_{\mu\lambda}$!!

This difficulty does not occur if only functions of the ξ^{κ} are considered, infact $(\kappa)^{\mu\lambda} \neq \phi^{\mu\lambda}_{\kappa'(\kappa)} v^{\kappa}$. From (7.5) and (7.6) we see that it is certainly not allowed to write $N_{\mu\lambda}(\Gamma^{\rho}_{\sigma\tau})$ instead of $\phi_{\mu\lambda}(\Gamma^{\rho}_{\sigma\tau})$ and that it is necessary to give function symbols a very special form in order that they cannot be misinterpreted for component symbols. Now a consequent notation can be formed in three ways:

1. by using the symbol only for one definitely chosen coordinate system and never for another one. Then all difficulties vanish and even the kernel N can be used.

2. by using only $\phi_{\mu\lambda}$ and writing always $\phi^{\mu'\lambda'}\phi_{\mu\lambda}$ but never $\phi_{\mu'\lambda'}$.

3. by using $\phi_{\mu\lambda}$ and $\phi_{\mu'\lambda'}$ and remembering always that for instance ϕ_{12} and $\phi_{1'2'}$ are different symbols for the same function. In this case an abbreviation can be attained by dropping the indices inside the brackets

7.7) $N_{\mu\lambda} = \phi_{\mu\lambda}(\Gamma) \qquad N_{\mu'\lambda'} = \phi_{\mu'\lambda'}(\Gamma)$

Though not inconsequent, notations of this kind are so different from ordinary function symbolic that they should not be used if not absolutely necessary. As a matter of fact you find them nowhere except in Schrödingers paper. Now in the MS the connection Γ is not necessary and there remain only the connections $\Gamma^{\kappa}_{\mu\lambda}$, $\{^{\kappa}_{\mu\lambda}\}$, and in the asymmetric case perhaps $\Gamma^{\kappa}_{(\mu\lambda)}$. These can be written conveniently $\Gamma^{\kappa}_{\mu\lambda}$, $\overset{\circ}{\Gamma}{}^{\kappa}_{\mu\lambda}$ and if necessary $\overset{*}{\Gamma}{}^{\kappa}_{\mu\lambda}$ and the corresponding quantities $N_{\nu\mu\lambda}{}^{\kappa}$, $N_{\mu\lambda}$; $\overset{\circ}{N}{}_{\nu\mu\lambda}{}^{\kappa}$, $\overset{\circ}{N}{}_{\mu\lambda}$, $\overset{*}{N}{}_{\nu\mu\lambda}{}^{\kappa}$, $\overset{*}{N}{}_{\mu\lambda}$. In the same way $G_{\mu\lambda}$, $\overset{\circ}{G}{}_{\mu\lambda}$ and $\overset{*}{G}{}_{\mu\lambda}$ or better $M_{\mu\lambda}$, $\overset{\circ}{M}{}_{\mu\lambda}$ and $\overset{*}{M}{}_{\mu\lambda}$ in order to save the letter G for the Einstein tensor in the V_4 .

The abbreviations $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\beta}$, $\hat{R}_{\alpha\beta}$ represent no geometric objects. They have no geometric meaning and can better be dropped entirely, certainly on pages MS 14 and 15. They are used only for the special anholonomic coordinate system and only as a kind of abbreviations. If on some points such an abbreviation should be very useful, it could be tolerated as as temporary abbreviation only for this coordinate system but never in an end result (as MS 5.37). Of course other kernel letters, for instance $\phi_{\alpha\beta}$, $\hat{\phi}_{\alpha\beta}$, $\psi_{\alpha\beta}$ should be introduced because the kernel R , often connected with the curvature affinor, gives a false association. Some entirely-

P.16 ing letter, for instance a fat greek letter is the best notation for these queer and out of the way expressions. But the better way is not to use them at all. Tensor calculus was just made to avoid them, and gives always the means hereto.

8. We give here still a list of some changes not yet mentioned above that are not absolutely necessary but very desirable in order to make the MS as readable as possible:

1. For the alphabets take

$\kappa, \lambda, \mu, \nu, \rho, \sigma, \tau = 0, 1, \ldots 5$	for the X_6
$\alpha, \beta, \gamma, \delta, \varepsilon = 0, 1, \ldots 4$	for the X_5
$a, b, c, d, e, f, g = 0, 1, \ldots 5$	for anhol. coord. in X_6
$m, n, p, q, r = 1, 2, 3, 4$	for the holonomic coordinates in X_4 connected with the anhol. system (a).

The indices should be used mostly in a certain order: κ first contravariant, λ first covariant, μ first differentiating, ν second differentiating and so for the other alphabets. This saves a lot of time for the reader and for the printer and for the corrector of proofs. A function f of the coordinates that is invariant for coordinate transformations is known as a scalar and should not be called an invariant. There are lots of things invariant and this word should not be used where it is not necessary. The term "covariant" at MS p. 4 should be replaced by "invariant". The terms covariant differentiation and covariant tensor are in common use but equations are said to be invariant or in an invariant form.

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1st December 1951.

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