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SUPERCOMPACTNESS AND WALLMAN SPACES

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PREFACE

In this treatise we mainly discuss supercompact spaces and supercompact superspaces of arbitrary topological spaces. The class of supercompact spaces was defined by DE GROOT [54]. This class naturally arose from investigations of DE GROOT & AARTS [57] on complete regularity and compactification theory.

The last years many people became interested in this part of the mathematical inheritance of DE GROOT (for a beautiful exposition of DE GROOT's topological works see BAAYEN & MAURICE [10] or BAAYEN [8]). Many conjectures of DE GROOT are proved now, new techniques have been developed and it is the author's expectation that this is still the beginning. Some of the best results of the last years are that

a) every compact metric space is supercompact (cf. STROK & SZYMAŃSKI [116]);
b) $\beta\mathbb{N}$ is not supercompact (cf. BELL [14]);
c) every compact metric space is regular supercompact (cf. VAN DOUWEN [42]);
d) supercompact spaces can be characterized by means of interval structures (cf. SCHRIJVER [24],[81]);
e) every connected space with a binary normal subbase has the fixed point property for continuous functions (cf. VAN DE VEL [118]).

This treatise consists of five chapters. In chapter 0 we present some notational conventions, some definitions and some simple results which are collected for easy reference throughout the remaining part of this monograph. Chapter I is captioned "supercompact spaces"; here we discuss supercompact spaces in general. The next chapter deals with superextensions, which are natural supercompact superspaces of topological spaces. Superextensions are constructed in about the same way as Wallman compactifications; we regard superextensions as (generalized) *Wallman spaces*. Chapter III contains the main results; among others, we show that the superextension of the closed unit interval is homeomorphic to the Hilbert cube, which proves a conjecture of DE GROOT [59]. The results of chapter IV deal with compactification theory. A final chapter is added to give a survey of some recent results.

Throughout this treatise, SCHRIJVER's interval structures are used extensively. Many good ideas are also taken from VERBEEK [119].

CONTENTS

CHAPTER 0

BASIC CONCEPTS

This short chapter contains some notational conventions and some simple facts for easy reference. In [A] some general remarks about subbases are made. Then, in [B], [C] and [D] we collect some notions from topology; our notation is standard, cf. DUGUNDJI [44], ENGELKING [48].

[A] General remarks about subbases

In this treatise all topological spaces under discussion are assumed to be T_1. If in a statement we write *Hausdorff* then this is to indicate that it is used essentially in the proof of the statement.

A *compactification* of a topological space X is a compact *Hausdorff* space αX in which X can be densely embedded. At two places we deviate from this convention, namely in the notes following theorem 2.2.4 and in corollary 2.2.6.

We often deal with subbases. A collection of closed subsets $\mathcal{S}$ of a topological space X is called a *closed subbase* provided that for each closed set $A \subset X$ and for each point $x \notin A$ there is a finite $\mathcal{F} \subset \mathcal{S}$ such that $x \notin \bigcup\mathcal{F} \supset A$. If $\mathcal{S}$ is a closed subbase for X then $\mathcal{U} = \{X\backslash S \mid S \in \mathcal{S}\}$ is called an *open subbase*. In this treatise "*subbase*" will always mean "closed subbase".

0.1. LEMMA. *Let* X *be a compact topological space and let* $\mathcal{S}$ *be a collection of closed subsets of* X *such that for all distinct* $x,y \in X$ *there is an* $S \in \mathcal{S}$ *such that* $x \notin S$ *and* $y \in \mathrm{int}_X(S)$. *Then* $\mathcal{S}$ *is a subbase for* X.

PROOF. Let A be a closed subset of X and let $x \in X\backslash A$. For each $a \in A$ let $S_a \in \mathcal{S}$ such that $x \notin S_a$ and $a \in \mathrm{int}_X(S_a)$. By the compactness of X there is a finite $F \subset A$ such that $A \subset \bigcup_{a\in F} S_a$. Clearly $x \notin \bigcup_{a\in F} S_a$. □

Let $\mathcal{S}$ be a collection of subsets of a set X. We will write $\vee.\mathcal{S}$ for the family of finite unions of elements of S and $\wedge.S$ for the family of finite intersections of elements of S. The family $\wedge.\vee.S = \vee.\wedge.S$ is closed both under finite intersections and finite unions; it is called the *ring* generated by S. If X is a topological space then S is called a *separating ring* provided that S is a subbase and that $S = \wedge.\vee.S$. In addition, S is called *normal* provided that for all $S_0,S_1 \in S$ with $S_0 \cap S_1 = \emptyset$ there are $S_0',S_1' \in S$ with $S_0 \subset S_0'\backslash S_1'$, $S_1 \subset S_1'\backslash S_0'$ and $S_0' \cup S_1' = X$. A *normal base* is a normal separating ring; a *normal subbase* is a subbase which moreover is normal.

0.2. LEMMA. *Let* X *be a compact topological space and let* S *be a subbase for* X. *Then for all disjoint closed sets* $A_0,A_1 \subset X$ *there are disjoint* $T_0,T_1 \in \wedge.\vee.S$ *such that* $A_i \subset T_i$ $(i \in \{0,1\})$.

PROOF. Let $F := \{T \in \wedge.\vee.S \mid A_0 \subset T\}$. Then, since F is closed under finite intersections, the compactness of X implies that some member $F_0 \in F$ does not intersect A_1. Similarly one can choose $F_1 \in \wedge.\vee.S$ such that $A_1 \subset F_1$ and $F_1 \cap F_0 = \emptyset$. □

0.3. COROLLARY. *Let* X *be a compact topological space and let* S *be a subbase for* X *which is closed under finite intersections. Then for all clopen subsets* $A \subset X$ *there is a finite* $F_A \subset S$ *such that* $A = \cup F_A$. □

A subbase S for a topological space X is called *binary* provided that for all $L \subset S$ with $\cap L = \emptyset$ there are $L_0,L_1 \in L$ with $L_0 \cap L_1 = \emptyset$. In addition, the subbase S is called a T_1*-subbase* if for all $x \in X$ and $S \in S$ with $x \notin S$ there is an $S_0 \in S$ with $x \in S_0$ and $S_0 \cap S = \emptyset$.

0.4. LEMMA. *A binary subbase is a* T_1*-subbase.*

PROOF. Let S be a binary subbase for X. Let $S \in S$ and let $x \in X$ such that $x \notin S$. Since X is a T_1-space, there is an $F \subset S$ such that $\{x\} = \cap F$. Then $\cap F \cap S = \emptyset$ and consequently, by binarity of S, there is an $F \in F$ such that $F \cap S = \emptyset$. □

A space which admits a binary subbase is called *supercompact*. The proof of the following simple lemma is left to the reader.

0.5. LEMMA.

(i) *Any product of supercompact spaces is supercompact;*

(ii) *a space* X *admits a binary (normal) subbase iff it admits a binary (normal) subbase closed under arbitrary intersections.* □

The following lemma is used frequently in the sequel.

0.6. LEMMA. *Let* $\mathcal{S}$ *be a normal* T_1*-subbase for* X. *Then for all distinct* $x_0, x_1 \in X$ *there are* $S_0, S_1 \in \mathcal{S}$ *such that* $x_0 \in S_0 \setminus S_1$, $x_1 \in S_1 \setminus S_0$ *and* $S_0 \cup S_1 = X$.

PROOF. Obvious. □

[B] Some conventions

A *cardinal number* is an initial ordinal number, and an *ordinal number* is the set of all smaller ordinal numbers; the symbol ω denotes the least infinite cardinal and c is 2^{ω}. If we want to index a set X of cardinality κ we usually write $X = \{x_\alpha \mid \alpha \in \kappa\}$ or $X = \{x_\alpha \mid \alpha < \kappa\}$. A countable set A is indexed as $A = \{a_\alpha \mid \alpha \in \omega\}$ or as $A = \{a_n \mid n \in \mathbb{N}\}$; here $\mathbb{N}$ denotes the set of natural numbers. The cardinality of a set X is denoted by $|X|$; its powerset by $\mathcal{P}(X)$.

The domain of a function f is dom(f). If A and B are sets, then ${}^{A}B$ is the set of all functions from A to B; recall that each $f \in {}^{A}B$ is a subset of $A \times B$. If $f \in {}^{A}B$ then if $C \subset A$ then $f \restriction C$ denotes the restriction of f to C. So if $f, g \in {}^{A}B$ then $f \subset g$ means $f = g \restriction \mathrm{dom}(f)$.

If X_α ($\alpha \in \kappa$) are sets then $\prod_{\alpha \in \kappa} X_\alpha$ denotes their cartesian product. In addition, X^∞ or X^ω is the product of countably many copies of X.

Let $\mathcal{S}$ be a collection of subsets of a set X; then for any $A \subset X$ we write $\mathcal{S} \cap A = \{S \cap A \mid S \in \mathcal{S}\}$.

[C] Some definitions

We recall some definitions.

(a) For any topological space X, let

$$C(X) := \{f \in {}^{X}\mathbb{R} \mid f \text{ is continuous}\};$$

$$C^*(X) := \{f \in C(X) \mid f \text{ is bounded}\};$$

$$C(X,I) := \{f \in C^*(X) \mid f[X] \subset I = [0,1]\}.$$

(b) If $Y \subset X$ then Y is called C^*-*embedded* in X provided that for any $f \in C^*(Y)$ there is a $g \in C^*(X)$ such that $g \restriction Y = f$.

(c) A *zeroset* in X is a set of the form $\{x \in X \mid f(x) = 0\}$ with $f \in C^*(X)$. A *cozeroset* is the complement of a zeroset.
Define $Z(X) := \{Z \subset X \mid Z \text{ is a zeroset}\}$. It is well known that $Z(X)$ is a normal base iff X is a Tychonoff space and that $Z(X)$ is closed under countable intersections.

(d) An *F-space* (cf. GILLMAN & JERISON [52]) is a space in which every cozeroset is C^*-embedded. It is known that $\beta X \backslash X$ is an F-space if X is a noncompact locally compact and σ-compact topological space (cf. GILLMAN & JERISON [52]).

(e) A *pseudocompact space* is a space for which every real valued continuous function is bounded.

(f) IF $A \subset X$ then ∂A denotes the boundary of A, i.e. $\partial A = cl_X(A) \backslash int_X(A)$.

(g) A *continuum* is a compact connected Hausdorff space.

(h) A *Peano continuum* is a compact connected and locally connected metrizable space. It is well known that the class of Peano continua coincides with the class of continuous images of the closed unit segment $[0,1]$.

(i) *The Hilbert cube* I^∞ is the topological product of countably many copies of the closed unit segment $I = [0,1]$.
A Hilbert cube is a topological space which is homeomorphic to the Hilbert cube.
Q denotes the countably infinite product of copies of $[-1,1]$. Clearly Q is a Hilbert cube. Sometimes we will call Q also *the* Hilbert cube.
The pseudo-boundary $B(Q)$ of the Hilbert cube Q is $\{x \in Q \mid \exists i \in \mathbb{N}: |x_i| = 1\}$.
A pseudo-boundary is a subset A of the Hilbert cube Q for which there is an autohomeomorphism $\phi: Q \to Q$ such that $\phi[A] = B(Q)$.
The pseudo-interior of Q is the complement of $B(Q)$.
A pseudo-interior is the complement of a pseudo-boundary. It is known that a pseudo-interior of Q is homeomorphic to ℓ_2, the space of all square summable sequences in $\mathbb{R}$ (cf. ANDERSON [3]).

(j) An AR (Absolute Retract) is a space which is homeomorphic to a retract of Q.

(k) If (Y,d) is a compact metric space and if $f,g: X \to Y$ are continuous, then the *distance between* f *and* g is defined by

$$d(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\}.$$

(l) Let X be a topological space. We denote by 2^X the collection of nonvoid closed subsets of X. For all nonvoid $A_i \subset X$ $(i \leq n)$ define $\langle A_0,A_1,\ldots,A_n\rangle \subset 2^X$ by

$$\langle A_0,A_1,\ldots,A_n\rangle := \{B \in 2^X \mid B \subset \bigcup_{i\leq n} A_i \text{ and } B \cap A_i \neq \emptyset \ (i \leq n)\}.$$

As a (closed) subbase for a topology on 2^X we take the collection

$$\{\langle B\rangle \mid B \in 2^X\} \cup \{\langle B,X\rangle \mid B \in 2^X\}.$$

With this topology 2^X is called the *hyperspace* of X. The space 2^X is compact iff X is compact (cf. MICHAEL [75]) and moreover 2^X contains a homeomorph of X; the mapping $i: X \to 2^X$ defined by $i(x) := \{x\}$ is easily seen to be an embedding. The spaces X and $i[X]$ are often identified.

If $f: X \to Y$ is a closed continuous mapping, then there is a natural extension $2^f: 2^X \longrightarrow 2^Y$ of f defined by

$$2^f(A) := f[A].$$

This mapping is easily seen to be continuous.

[D] Set theoretic axioms

In this treatise we assume the axiom of choice; the only exception is made in section 2.1.

The *Continuum Hypothesis* (CH) states that $2^\omega = \omega_1$ (the first uncountable cardinal); in section 2.8 only we have some results depending on CH.

Martin's axiom (MA) (cf. MARTIN & SOLOVAY [74]) states that no compact ccc Hausdorff space is the union of less than $\mathfrak{c}$ nowhere dense sets. Clearly CH implies MA; however MA is weaker than CH (cf. SOLOVAY & TENNENBAUM [108]) and in particular it is consistent to assume MA and the negation of the Continuum Hypothesis (MA + $\neg$CH). Results depending on MA are to be found in section 1.2 and section 2.8; MA + $\neg$CH is used in example 2.8.28 only.

CHAPTER I

SUPERCOMPACT SPACES

The class of supercompact spaces - first introduced by DE GROOT [54] - is easy to define, but in general it is hard to decide whether or not a certain space belongs to it. A topological space is called *supercompact* if it possesses a binary subbase for its closed subsets where a collection of subsets $\mathcal{S}$ of a set X is called *binary* if for each subsystem $\mathcal{M} \subset \mathcal{S}$ with $\cap\mathcal{M} = \emptyset$ there are $M_0, M_1 \in \mathcal{M}$ such that $M_0 \cap M_1 = \emptyset$. Equivalently a space X is supercompact if there is a subbase for its closed sets (a closed subbase) such that each *linked* subsystem (a subsystem any two members of which meet) has a nonvoid intersection. Supercompactness of course can also be defined in a dual form: a space X is supercompact iff there is a subbase $\mathcal{U}$ for its open sets such that each covering of X by elements of $\mathcal{U}$ contains a subcover consisting of at most two elements of $\mathcal{U}$.

Clearly, by the lemma of ALEXANDER, each supercompact space is compact. In addition the class of supercompact spaces is closed under products. However closed subspaces of supercompact spaces need not be supercompact (cf. BELL [14]) and it is unknown whether Hausdorff continuous images of supercompact Hausdorff spaces are supercompact (VERBEEK [119] has given a simple example of a nonsupercompact T_1 space which is the continuous image of a supercompact space).

Hausdorff continuous images of supercompact Hausdorff spaces are natural generalizations of dyadic spaces (Hausdorff continuous images of generalized Cantor discontinua). It is known that

every compact metric space is supercompact (cf. STROK & SZYMAŃSKI [116])

and

if βX *is the continuous image of a supercompact Hausdorff space then* X *is pseudocompact* (cf. cor.1.1.7).

There are supercompact spaces that are not dyadic but we do not have an example of a dyadic space that is not supercompact. As a consequence of our results a compact infinite Hausdorff space in which no sequence converges is not the continuous image of a supercompact Hausdorff space. Thus $\beta\mathbb{N}$ and $\beta\mathbb{N}\setminus\mathbb{N}$ are not supercompact. We also present a "small" nonsupercompact compact Hausdorff space: there is a separable first countable compact Hausdorff space that is not the continuous image of a supercompact Hausdorff space (cf. also VAN DOUWEN & VAN MILL [43]).

As noted before STROK & SZYMAŃSKI [116] have shown that every compact metric space is supercompact (a simpler proof of this fact was given recently by VAN DOUWEN [42]). This theorem implies that every separable metric space admits at least one supercompact compactification. It seems reasonable to try to generalize this corollary for a larger class of spaces, for example, for the class of all separable semi-stratifiable spaces. Unfortunately this is not possible: we will show that Martin's axiom implies that there exists a countable stratifiable space no compactification of which is supercompact. Our example also shows that not every countable space admits a supercompact compactification, a result which is of independent interest.

DE GROOT [55], [56] and DE GROOT & SCHNARE [60] demonstrated that certain classes of supercompact topological spaces can be characterized by means of a binary subbase of a special kind. These results now can be derived using a more general method. We also discuss other classes of topological spaces which can be characterized by means of special binary subbases. As an application, using a result of ANDERSON [2], we give a new internal characterization of the Hilbert cube Q (cf. also VAN MILL & SCHRIJVER [81]).

An interesting subclass of the class of supercompact spaces consists of those spaces which possess a binary subbase which also is *normal* (two disjoint subbase elements are separated by disjoint complements of subbase elements). Such spaces are surprisingly nice, for example in this class of spaces connectedness implies local connectedness (cf. VERBEEK [119]) and (generalized) arcwise connectedness (see section 1.5) and the fixed point property for continuous functions (cf. VAN DE VEL [118]), while metrizability and connectedness imply contractibility and local contractibility (see section 1.5). Moreover such a space is a retract of the hyperspace of its nonvoid closed subsets and a retract of its superextension.

1.1. Supercompact spaces

In this section we study "topological properties" of Hausdorff continuous images of supercompact Hausdorff spaces. Of course, being the continuous image of a supercompact Hausdorff space is itself such a topological property. However we want properties which are easier to recognize. As a consequence of our results it will follow that a compact Hausdorff space in which no sequence converges is not the continous image of a supercompact Hausdorff space. Several examples will be given. The results of this section were obtained in collaboration with E. VAN DOUWEN, cf. [43].

1.1.1. Let X be a supercompact Hausdorff space which admits a continuous mapping, say f, onto the topological space Y. Let $\mathcal{S}$ be a binary closed subbase for X. Without loss of generality assume that $\mathcal{S}$ is closed under arbitrary intersection. For $A \subset X$ define $I(A) \subset X$ by

$$I(A) := \bigcap\{S \in \mathcal{S} \mid A \subset S\}.$$

Notice that $cl_X(A) \subset I(A)$, since each element of $\mathcal{S}$ is closed, that $I(I(A)) = I(A)$ and that $I(A) \subset I(B)$ if $A \subset B$, for all $A,B \subset X$ (the operator I defined in this way will play an important role in our investigations; see sections 1.3, 1.5, 2.5, 2.6, 2.7, 2.10, 3.1, 3.2 and 3.4).

1.1.2. LEMMA. *Let $p \in X$. If U is a neighborhood of p and if A is a subset of X with $p \in cl_X(A)$, then there is a subset B of A with $p \in cl_X(B)$ and $I(B) \subset U$.*

PROOF. Since X is regular, p has a neighborhood V such that $cl_X(V) \subset U$. Choose a finite $\mathcal{F} \subset \mathcal{S}$ such that $cl_X(V) \subset \bigcup\mathcal{F} \subset U$ (lemma 0.2). Now $\mathcal{F}$ is finite, and $A \cap V \subset \bigcup\mathcal{F}$, and $p \in cl_X(A \cap V)$; hence there is an $S \in \mathcal{F}$ with $p \in cl_X(A \cap V \cap S)$. Let $B := A \cap V \cap S$. Then $p \in cl_X(B)$, and $B \subset A$, and $I(B) \subset S \subset \bigcup\mathcal{F} \subset U$. □

1.1:3. DEFINITION. If T is a subspace of Y, a family $\mathcal{A}$ of subsets of Y is called a *network for* T *in* Y, if for each $p \in T$ and each neighborhood U of p in Y there is an $A \in \mathcal{A}$ with $p \in A \subset U$ (if $T = Y$, then $\mathcal{A}$ simply is a network for Y).

1.1.4. LEMMA. *Let Y be a Hausdorff space which is a continuous image of a supercompact Hausdorff space. If K is any countable infinite subset of Y,* *then the subspace*

$$E := \{y \in Y \mid y \in \mathrm{cl}_Y(K\setminus\{y\}),\ \textit{and no nontrivial sequence in } Y \textit{ converges to } y\}$$

of Y *has a countable network in* Y.

<u>PROOF</u>. Let X be a supercompact Hausdorff space X with binary subbase $\mathcal{S}$; without loss of generality we may assume that $\mathcal{S}$ is closed under arbitrary intersection. Suppose there is a continuous surjection $f\colon X \to Y$. Choose any countable subset J of X such that $f[J] = K$. Since J has only countably many finite subsets, the family

$$\mathcal{A} := \{f[I(F)] \mid F \text{ is a finite subset of } J\}$$

is countable. We claim that it is a network for E in Y.

Let $y \in E$ be arbitrary, let U be any neighborhood of y in Y, and let $J^* := J\setminus f^{-1}[\{y\}]$.

Since f is a closed map (Y is Hausdorff), and $f[J^*] = K\setminus\{y\}$, and $y \in \mathrm{cl}_Y(K\setminus\{y\})$, there is an $x \in \mathrm{cl}_X(J^*)$ with $f(x) = y$. Then lemma 1.1.2 implies that there is a $B \subset J^*$ such that $x \in \mathrm{cl}_X(B)$ and $I(B) \subset f^{-1}[U]$. We will show that there is a finite $F \subset B$ such that $y = f(x) \in f[I(F)]$. Since y and U are arbitrary, and $f[I(F)] \subset f[I(B)] \subset U$, it would follow that $\mathcal{A}$ is a network for E in Y.

Enumerate B as $\{b_k \mid k \in \omega\}$, and for each $n \in \omega$ define Z_n and T_n by

$$Z_n := \bigcap_{k\le n} I(\{x,b_k\}) \cap I(\{b_k \mid k \le n\})$$

$$T_n := \bigcap_{k\le n} I(\{x,b_k\}) \cap I(B).$$

<u>CLAIM</u>. *There is an* $n_0 \in \omega$ *such that* $f[Z_n] = \{y\}$ *for all* $n \ge n_0$.

Indeed, first observe that $\bigcap_{b\in B} I(\{x,b\}) = \{x\}$. Evidently $x \in I(\{x,b\})$ for all $b \in B$. Let $t \in X\setminus\{x\}$ be arbitrary. By lemma 1.1.2 there is a $C \subset B$ such that $x \in \mathrm{cl}_X(C)$ and $I(C) \subset X\setminus\{t\}$. Choose any $b \in C$. Then $t \notin I(\{x,b\})$, since $\{x,b\} \subset \mathrm{cl}_X(C) \subset I(C)$, which implies that $I(\{x,b\}) \subset I(I(C)) = I(C)$.

To proceed with the proof of the claim, notice that, since $x \in \mathrm{cl}_X(B) \subset$ $\subset I(B)$, it follows from the fact that $\bigcap_{b\in B} I(\{b,x\}) = \{x\}$ that $\bigcap_{n\in\omega} T_n = \{x\}$.

But $Z_n \subset T_n$ for each $n \in \omega$, and $\{T_n \mid n \in \omega\}$ is a decreasing collection of closed sets in a compact space, hence

(*) if V is any neighborhood of x in X, then there is an $m_0 \in \omega$ such that $Z_k \subset V$ for all $k \geq m_0$.

Now assume the claim to be false. Then for each $k \in \omega$ there is a $z(k) \geq k$ with $f[Z_{z(k)}] \neq \{y\}$. But $Z_n \neq \emptyset$ for all $n \in \omega$ since $\mathcal{S}$ is binary (this is the only point in the proof where we use the fact that $\mathcal{S}$ is binary). Consequently, for each $k \in \omega$ we can choose a $y_k \in f[Z_{z(k)}]\setminus\{y\}$. Then the sequence $\langle y_k\rangle_{k\in\omega}$ converges to y. Indeed, let U be any neighborhood of $y = f(x)$. Then there is an $m_0 \in \omega$ such that $Z_k \subset f^{-1}[U]$ for all $k \geq m_0$. Since $z(k) \geq k$ for all $k \in \omega$, it follows that $y_k \in U$ for all $k \geq m_0$. Since $y_k \neq y$ for all $k \in \omega$, this contradicts $y \in E$.

Now define $F := \{b_k \mid k \leq n_0\}$, where n_0 is as in the claim. Then F is a finite subset of J such that $y \in f[I(F)] \subset U$. □

Now we can formulate the main result of this section.

1.1.5. THEOREM. *Let Y be a Hausdorff space which is a continuous image of a supercompact Hausdorff space, and let K be a countably infinite subset of Y. Then*

(a) *at least one cluster point in K is the limit of a nontrivial convergent sequence in Y (not necessarily in K), and*
(b) *at most countably many cluster points of K are not the limit of some nontrivial convergent sequence in Y.*

PROOF. Let Y and K be as in theorem 1.1.5 and let E be as in lemma 1.1.4. We will first show that E is countable. Let $\mathcal{A}$ be a countable network for E in Y. In order to show that E is countable it suffices to show that for each $p \in E$ there is a finite $\mathcal{F}_p \subset \mathcal{A}$ such that $\bigcap\mathcal{F}_p = \{p\}$, since $\mathcal{A}$ as only countably many finite subfamilies.

Let $p \in E$ be arbitrary. List $\{A \in \mathcal{A} \mid p \in A\}$ as $\{A_n \mid n \in \omega\}$. We claim that $\bigcap_{i\leq n} A_i = \{p\}$ for some $n \in \omega$. For assume not. Then we can pick for each $n \in \omega$ an $a_n \in (\bigcap_{i\leq n} A_i)\setminus\{p\}$. Since each neighborhood of p in Y contains some A_n, it follows that the sequence $\langle a_n\rangle_{n\in\omega}$ converges to p. Since $a_n \neq p$, for all $n \in \omega$, this contradicts $p \in E$.

We next show that (a) holds. Suppose not. Then $cl_Y(K) = K \cup E$, hence $cl_Y(K)$ is countable. But each compact countable Hausdorff space is metrizable, hence each cluster point of K is the limit of a nontrivial convergent sequence of points in K. Contradiction. □

1.1.6. COROLLARY. *$\beta\mathbb{N}$, and $\beta\mathbb{N}\setminus\mathbb{N}$ and $\beta\mathbb{R}\setminus\mathbb{R}$, or, more generally, any infinite compact Hausdorff F-space, or, yet more generally, any infinite compact Hausdorff space in which no sequence converges, cannot be a continuous image of a supercompact Hausdorff space.*

1.1.7. COROLLARY. *If βX is the continuous image of a supercompact Hausdorff space, then X is pseudocompact (cf. also* M. BELL [14]).

PROOF. If X is not pseudocompact, there is a continuous $f\colon \beta X \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in X$, while $f(x) = 0$ for some $x \in \beta X\setminus X$. Let $Y := f^{-1}[(0,\infty)]$ and for each $n \geq 1$ pick $p_n \in Y$ with $f(p_n) < 1/n$; let $P := \{p_n \mid n \geq 1\}$. Then Y is σ-compact, and P is a countably infinite subset of βX all cluster points of which are in $\beta X\setminus Y$. In view of theorem 1.1.5 it now suffices to observe that no point of $\beta X\setminus Y$ is the limit of a nontrivial convergent sequence in βX. For completeness sake, we give the (known) proof.

Suppose that $p \in \beta X\setminus Y$ is the limit of a nontrivial convergent sequence. Then there is a countably infinite $D \subset \beta X$ such that (*) every neighborhood of p contains all but finitely many points of D, while also $p \notin D$. Then D is closed and discrete in $D \cup Y$. But $D \cup Y$ is normal, being σ-compact, and $\beta(D \cup Y) = \beta X$ since $X \subset D \cup Y \subset \beta X$; hence D is C^*-embedded in βX. This contradicts (*). □

Theorem 1.1.5 suggests some questions we can not answer at the moment.

1.1.8. QUESTION. *Let Y be a Hausdorff continuous image of a supercompact Hausdorff space (or even a supercompact Hausdorff space). If K is a countable subset of Y, then is every cluster point of K the limit of a nontrivial convergent sequence in Y? Equivalently, is a point of Y the limit of a nontrivial convergent sequence iff it is a cluster point of a countable subset of Y?*

1.1.9. QUESTION. *Is there a nonsupercompact Hausdorff space which is a continuous image of some supercompact Haudorff space?*

We do not even know the answer for irreducible maps or for retractions. Indeed, we do not even know if $X \times Y$ supercompact implies that X and

Y are supercompact.

1.1.10. <u>QUESTION</u>. *Is there a nonsupercompact Hausdorff space* X *and a Hausdorff space* Y *such that* $X \times Y$ *is supercompact?*

We know that the answer to the above question is affirmative if we replace "supercompact" by "having a normal binary subbase". SZYMAŃSKI [117] recently has given an example of a (compact metric) AR which admits no binary normal subbase. However, by a recent result of EDWARDS [45], each AR is a Hilbert cube factor, that is a space whose product with the Hilbert cube is homeomorphic to the Hilbert cube. Hence SZYMAŃSKI's [117] example multiplied with the Hilbert cube admits a binary normal subbase.

With respect to question 1.1.9 we only have the information that VERBEEK's [119] example cited in the introduction of this chapter is the continuous image of a supercompact Hausdorff space.

Corollary 1.1.7 generalizes the fact that X is pseudocompact if βX is dyadic (recall that a *dyadic* space is a Hausdorff continuous image of some product of a family of two-point discrete spaces). Corollary 1.1.6 was also (essentially) known for dyadic spaces, cf. ENGELKING & PELCYNSKI [50], footnote 2; see also ENGELKING [47] theorem 1.5. This suggests which other theorems on dyadic spaces generalize. None of the theorems on dyadic spaces recorded in EFIMOV & ENGELKING [46], ENGELKING [47] or ENGELKING & PELCYNSKI [50] which are not related to corollary 1.1.6 or 1.1.7 can be generalized for Hausdorff continuous images of supercompact Hausdorff spaces, see the examples below, with the possible exception of the theorem that closed G_δ-subspaces of dyadic spaces are dyadic ([50], theorem 2). This leads to the following question.

1.1.11. <u>QUESTION</u>. *Is a closed* G_δ*-subspace of a supercompact Hausdorff space supercompact? a continuous image of a supercompact space?*

We now sketch some examples. Note that the first three of our examples are compact linearly orderable spaces, while all four are supercompact.

1.1.12. <u>EXAMPLES</u>. (a) *The Alexandroff double arrow line* A, *i.e.* $[0,1] \times \{0,1\} \setminus \{\langle 0,0\rangle, \langle 0,1\rangle\}$, *topologized by the lexicographic order.*

If π: $A \to [0,1]$ is the "projection", then π is a continuous surjection, yet there is no (closed) metrizable $M \subset A$ with $\pi[M] = [0,1]$, cf. [50], cor. on p.56. Also, A is a nonmetrizable supercompactification of a metrizable space (any countable dense subspace), cf. [50] appendix, and A is first coutanble but not second countable, cf. [46], theorem 4.

(b) ω_1+1, *the space of all ordinals less than or equal to* ω_1.

The point ω_1 is not the limit of a nontrivial convergent sequence in ω_1+1, cf. [47], cor. 2 to theorem 1.5. (Note however that theorem 1.1.5 is a partial generalization of the theorem that every non-isolated point of a dyadic space is the limit of a nontrivial convergent sequence.)

(c) *An Aronszajn line.*

An Aronszajn line, L, can be constructed from an Aronszajn tree in the same way one constructs a Souslin line from a Souslin tree, cf. RUDIN [97]. It is known that there is a collection $\{U_\alpha \mid \alpha<\omega_1\}$ of dense open sets in L such that $U_\alpha \supset U_\beta$ if $\alpha<\beta$, and $\bigcap_{\alpha<\omega_1} U_\alpha = \emptyset$. So [46] theorem 3 does not generalize.

(d) *The Alexandroff double* D *of the product* $P=\{0,1\}^c$ (see ENGELKING [49]).

The underlying set of D is $P\times\{0,1\}$. Points of $P\times\{0\}$ are isolated in D. A basic neighborhood of $\langle x,1\rangle$ has the form $U\times\{0,1\}\backslash\{\langle x,0\rangle\}$, where U is a neighborhood of x in P.

It is a straightforward exercise to show that D is supercompact. Let B be any closed subspace without isolated points of P which is not the continuous image of a supercompact Hausdorff space, e.g. a homeomorph of $\beta\mathbb{N}\backslash\mathbb{N}$. Then $B\times\{0,1\}$ is the closure of the open subset $B\times\{0\}$ of the supercompact space D, yet it is not supercompact, not even the continuous image of a supercompact Hausdorff space, since the "natural" map from $B\times\{0,1\}$ to B is continuous.

1.1.13. Examples of compact Hausdorff spaces which are not supercompact, obtained from theorem 1.1.5, are not first countable and have cardinality at least 2^c. This suggests two questions: are first countable compact Hausdorff spaces supercompact? and: are "small" compact Hausdorff spaces supercompact? These questions are answered in the negative by examples 1.1.17 and 1.1.18.

1.1.14. Let α be an ordinal less than or equal to ω. We are interested in ${}^\alpha 2$. An element of ${}^\alpha 2$ can be considered to be an α-sequence of 0's and 1's. As usual we denote $\bigcup_{n<\omega} {}^n 2$ the set of finite sequences of 0's and 1's, by ${}^{\underset{\sim}{\omega}}2$. For each $f\in {}^\omega 2$ we define

$$I(f) := \{g \in {}^{\underset{\sim}{\omega}}2 \mid g \subset f\},$$

the set of initial sequences of f; $I(f)$ can be seen as the set of finite approximations to f. It is clear that

(1) if $f,g \in {}^{\omega}2$ are distinct, then $I(f) \cap I(g)$ is finite.

In other words, $\{I(f) \mid f \in {}^{\omega}2\}$ is an *almost disjoint* collection of subsets of the countable set ${}^{\underline{\omega}}2$.

The set $T := {}^{\underline{\omega}}2 \cup {}^{\omega}2$, partially ordered by inclusion, is a tree (in the sense of JECH [66]), the so-called *Cantor tree*, cf. RUDIN [98]. We give T the usual tree topology by using the set of all open intervals as a base. To be specific: points of ${}^{\underline{\omega}}2$ are isolated, and a basic neighborhood of $f \in {}^{\omega}2$ contains f and all but finitely many points of $I(f)$. The topological space T is first countable, and every subspace is locally compact, by (1).

The set ${}^{\omega}2$ can be viewed as a product of countably many two-point discrete spaces. Under the product topology ${}^{\omega}2$ is nothing but the Cantor discontinuum, a basis for this topology is

$$\{\{f \in {}^{\omega}2 \mid f \supset g\} \mid g \in {}^{\underline{\omega}}2\},$$

as the reader can easily verify. We start with a simple but useful lemma on the almost disjoint family $\{I(f) \mid f \in {}^{\omega}2\}$.

1.1.15. LEMMA. *Let* G *be any uncountable subset of* ${}^{\omega}2$. *Then there are a* $g \in G$ *and an infinite* $H \subset G\backslash\{g\}$ *such that* $I(h) \cap I(h') \subset I(g)$ *for any two distinct* $h,h' \in H$ *(then also* $(I(h) \cup \{h\}) \cap (I(h') \cup \{h'\}) \subset I(g)$*).*

PROOF. In this proof we provide ${}^{\omega}2$ with the topology of the Cantor discontinuum. Then G is an uncountable separable metric space, hence we can find a nonisolated point g in G. Basic neighborhoods of g in G have the form

$$\{h \in G \mid \exists f \in I(g) \cap {}^{n}2 : f \subset h\}, \qquad n \in \omega$$

hence we can find $H = \{h_n \mid n \in \omega\} \subset G\backslash\{g\}$ such that

$$\min\{k \in \omega \mid g(k) \neq h_n(k)\} < \min\{k \in \omega \mid g(k) \neq h_{n+1}(k)\}$$

for all $n \in \omega$. Then g and H are as required. $\square$

This lemma implies the following

1.1.16. PROPOSITION. *Let* $L \subset {}^{\omega}2$ *be uncountable. Then no Hausdorff compactification of the subspace* ${}^{\omega}2 \cup L$ *of* T *is the continuous image of a supercompact Hausdorff space.*

PROOF. Denote the subspace ${}^{\omega}2 \cup L$ of T by Z. Let αZ be any Hausdorff compactification of Z. Let X be a supercompact Hausdorff space with binary subbase $\mathcal{S}$ and assume that there is a continuous surjection $\xi\colon X \to \alpha Z$. Also assume that $\mathcal{S}$ is closed under arbitrary intersection.

For each $g \in {}^{\omega}2$ choose an $a(g) \in \xi^{-1}[\{g\}]$. If $f \in L$ then the set $I(f) \cup \{f\}$ is open in Z and compact, hence it is clopen in αZ. Consequently $\xi^{-1}[I(f) \cup \{f\}]$ is clopen in X and hence it is the union of some finite subfamily of $\mathcal{S}$ (cf. lemma 0.). It follows that for each $f \in L$ we can choose an $S(f) \in \mathcal{S}$ such that

(2) $\qquad S(f) \subset \xi^{-1}[I(f) \cup \{f\}]$ and $S(f) \cap \{a(g) \mid g \in I(f)\}$ is infinite.

Since L is uncountable and ${}^{\omega}2$ is countable it follows that for some $p \in {}^{\omega}2$ the set

$$G = \{f \in L \mid a(p) \in S(f)\}$$

is uncountable. By lemma 1.1.15 there is a $g \in G$ and an infinite $H \subset G\backslash\{g\}$ such that

(3) $\qquad (I(h)\cup\{h\}) \cap (I(h')\cup\{h'\}) \subset I(g)$ for distinct $h,h' \in H$.

Since $(I(a)\cup\{a\}) \cap (I(b)\cup\{b\})$ is finite for distinct $a,b \in {}^{\omega}2$ it follows from (2) and (3) that

(4) $\qquad \{S(h)\backslash\xi^{-1}[I(g) \cup \{g\}] \mid h \in H\}$ is a disjoint collection of nonempty subsets of X.

Since $\xi^{-1}[I(g) \cup \{g\}]$ is a clopen subset of X, so is its complement in X. Hence $X\backslash(\xi^{-1}[I(g) \cup \{g\}])$ is the union of a finite subfamily of $\mathcal{S}$. It now follows from (4) that there is an $S \in \mathcal{S}$ with

(5) $\qquad S \cap (\xi^{-1}[I(g) \cup \{g\}]) = \emptyset$

such that there are distinct $h,h' \in H$ such that S intersects both $S(h)$ and $S(h')$. But $S(h)$ and $S(h')$ intersect, since $a(p) \in S(h) \cap S(h')$, consequently $\{S,S(h),S(h')\}$ is linked. However, it follows from (2), (3) and

(5) that

$$\begin{aligned} S \cap S(h) \cap S(h') &\subset S \cap (\xi^{-1}[I(h) \cup \{h\}]) \cap (\xi^{-1}[I(h') \cup \{h'\}]) \\ &= S \cap \xi^{-1}[(I(h) \cup \{h\}]) \cap (I(h') \cup \{h'\})] \\ &\subset S \cap \xi^{-1}[I(g)] \\ &= \emptyset. \end{aligned}$$

This is a contradiction, since S is binary. □

REMARK. This lemma is similar to the proof in BELL [14]. It was discovered independently, but only after learning about BELL's result (i.e. not every compact Hausdorff space is supercompact).

Now we can describe the examples promised in 1.1.13.

1.1.17. EXAMPLE. *A separable first countable compact Hausdorff space which is not the continuous image of a supercompact Hausdorff space.*

We will describe a first countable Hausdorff compactification of $T = {}^{\underset{\smile}{\omega}}2 \cup {}^{\omega}2$. Then proposition 1.1.16 implies that this compactification is the desired example since it is not the continuous image of a supercompact Hausdorff space. The basic idea is to identify the points of the subset ${}^{\omega}2$ of T with the isolated points of the Alexandroff double (cf. ENGELKING [49]) of the Cantor discontinuum, in the "natural way". It will be technically convenient to change the underlying set of T to $\{0\} \times {}^{\underset{\smile}{\omega}}2 \cup \{1\} \times {}^{\omega}2$, and the underlying set of the Cantor discontinuum to $\{2\} \times {}^{\omega}2$, if only to tell the two ${}^{\omega}2$'s apart.

Let K be $\{0\} \times {}^{\underset{\smile}{\omega}}2 \cup \{1,2\} \times {}^{\omega}2$. We topologize K by assigning to each $x \in K$ a neighborhood base $\{U(x,n) \mid n \in \omega\}$. For $\langle i,k\rangle \in K$ define

$$U(\langle i,f\rangle,n) = \begin{cases} \{\langle i,f\rangle\} & \text{if } i = 0; \\ \{\langle i,f\rangle\} \cup \{\langle 0,f{\restriction}k\rangle \mid k \geq n\} & \text{if } i = 1; \\ \{\langle j,g\rangle \in K \mid j \in 3,\ f{\restriction}n \subset g\} \setminus U(\langle 1,f\rangle),0) & \text{if } 1 = 2. \end{cases}$$

The straightforward check that this is a valid neighborhood assignment for a Hausdorff topology is left to the reader. Note that the subspace $\{1,2\} \times {}^{\omega}2$ of K is the Alexandroff double of the Cantor discontinuum, and that $\{0\} \times {}^{\underset{\smile}{\omega}}2 \cup \{1\} \times {}^{\omega}2$ is a dense subspace of K which is homeomorphic to T. Hence if K is compact proposition 1.1.16 will imply that K cannot be the continuous image of a supercompact Hausdorff space.

It remains to show that K indeed is compact. For $\langle i,f\rangle \in K$ let $n(i,f) \in \omega$ be arbitrary. We have to show that the open cover

$$\mathcal{U} = \{U(\langle i,f\rangle, n(i,f)) \mid \langle i,f\rangle \in K\}$$

of K has a finite subcover. Since the subspace $\{2\} \times {}^{\omega}2$ (which is homeomorphic to the Cantor discontinuum) is compact, there are for some $p \in \omega$ functions $f_0,\dots,f_p \in {}^{\omega}2$ such that

$$\mathcal{U}_0 = \{U(\langle 2,f_i\rangle, n(2,f_i)) \mid 0 \le i \le p\}$$

covers $\{2\} \times {}^{\omega}2$. Then $\mathcal{U}_0$ covers $\{1\} \times {}^{\omega}2$, with possible exception of the points $\langle 1,f_i\rangle$, $0 \le i \le p$. Let

$$\mathcal{U}_1 = \{U(\langle 1,f_i\rangle, n(1,f_i)) \mid 0 \le i \le p\}$$

and define m by

$$m := \max\{n(j,f_i) \mid j \in \{1,2\},\ 0 \le i \le p\}.$$

A straightforward check shows that $\mathcal{U}_0 \cup \mathcal{U}_1$ covers all points of K with possible exception of the points of the finite set $\bigcup_{k<m} {}^{k}2$. It follows that $\mathcal{U}$ has a finite subcover. □

1.1.18. EXAMPLE. *A separable compact Hausdorff space with ω_1 points which is not the continuous image of a supercompact Hausdorff space.*

Choose any subset L of ${}^{\omega}2$ with cardinality ω_1. Then the subspace $S = {}^{\underline{\omega}}2 \cup L$ of T is a locally compact space with ω_1 points, hence the one-point compactification of S has all properties required. □

1.1.19. We now show that examples 1.1.17 and 1.1.18 are close to being supercompact. Note that if X is compact, then any open *base* for X consisting of clopen sets is a closed subbase for X.

1.1.20. PROPOSITION. *Let* E *be either example* 1.1.17 *or example* 1.1.18, *and let* I *be the (countable) set of isolated points of* E. *Then*

(a) $E\backslash I$ *is supercompact;*

(b) E *has a base $\mathcal{B}$ consisting of clopen sets such that for any $\mathcal{A} \subset \mathcal{B}$ with $\bigcap\mathcal{A} = \emptyset$ there are $A_0,A_1,A_2 \in \mathcal{A}$ with $A_0 \cap A_1 \cap A_2 = \emptyset$.*

PROOF. We prove this for example 1.1.18 and leave the proof for example 1.1.17 to the reader. Notice that (a) is trivial since $E\setminus I$ is the one-point compactification $D \cup \{p\}$ of a discrete space D.

To prove (b), for $f \in L$ and $n \in \omega$ let

$$B(f,n) := \{f\} \cup f\restriction(\omega\setminus n)$$

and let

$$\mathcal{T} := \{B(f,n) \mid f \in L,\ n \in \omega\}.$$

Let

$$\mathcal{U} := \{E\setminus\bigcup\{B(f,0) \mid f \in F\} \mid F \subset L \text{ is finite}\}.$$

Evidently $\mathcal{U}$ is a neighborhood base for the point p at infinity. Consequently $\mathcal{B} := \mathcal{U} \cup \mathcal{T} \cup {}^{\omega}2$ is a base for E. Clearly the elements of $\mathcal{B}$ are clopen.

Let $\mathcal{A}$ be any subfamily of $\mathcal{B}$ such that $A_0 \cap A_1 \cap A_2 \neq \emptyset$ for all $A_1, A_2, A_3 \in \mathcal{A}$. Define F and $\mathcal{F}$ by:

$$F := \{f \in L \mid \exists n \in \omega : B(f,n) \in \mathcal{A}\}$$

$$\mathcal{F} := \mathcal{A} \cap \mathcal{T}.$$

CASE 1: $F = \emptyset$. Then $\mathcal{A}$ contains a singleton or $\mathcal{A} \subset \mathcal{U}$ which implies $p \in \bigcap\mathcal{A}$.

CASE 2: $|F| = 1$. Let $F = \{f\}$. Clearly, if $U \in \mathcal{U}$, $g \in L$ and $g \notin U$ then $B(g,n) \cap U = \emptyset$ for all $n \in \omega$. It follows that $f \in \bigcap\mathcal{A}$.

CASE 3: $|F| > 1$. We claim that

(*) there are $B(a,p)$ and $B(b,q)$ in $\mathcal{F}$ such that $B(a,p) \cap B(b,q) = \bigcap\mathcal{F}$.

For any $f,g \in {}^{\omega}2$ we can define $d(f,g) \leq \omega$ by

$$d(f,g) := \max\{\alpha \leq \omega \mid f\restriction\alpha = g\restriction\alpha\}.$$

Let $B(f,m)$ and $B(g,n)$ be any two members of $\mathcal{F}$ with $f \neq g$. Then for any $h \in {}^{\omega}2$, if $j \geq d(f,g)$ then $B(h,j)$ can not intersect both $B(f,m)$ and $B(g,n)$. Since any two members of $\mathcal{F}$ intersect, it follows that

$$p := \max\{n \in \omega \mid \exists h \in F : B(h,n) \in \mathcal{F}\}$$

exists. Choose any $a \in F$ such that $B(a,p) \in \mathcal{F}$. Let

$$s := \min\{n \in \omega \mid \exists h \in F : (h \neq a \text{ and } d(a,h) = n)\}$$

and choose any $B(b,q) \in \mathcal{F}$ such that $d(a,b) = s$. Since $q \leq p$ one easily verifies that $B(a,p) \cap B(b,q) \subset \bigcap\mathcal{F}$. This completes the proof of (*).

Let $j = d(a,b)$. Then $a{\restriction}j \in B(a,p) \cap B(b,q)$, and if $f \in B(a,p) \cap B(b,q)$, then $f = a{\restriction}i$ for some $i \leq j$. It is clear from the form of the members of $\mathcal{U}$ that $U \in \mathcal{U}$ and $a{\restriction}j \notin U$, then $a{\restriction}i \notin U$ for any $i \leq j$. Since $A_0 \cap A_1 \cap A_2 \neq \emptyset$ for any $A_0, A_1, A_2 \in \mathcal{A}$, it follows that $a{\restriction}j \in \bigcap\mathcal{A}$. □

1.2. A countable stratifiable space no compactification of which is supercompact *)

In section 1.1 we gave an example of a locally compact separable first countable space of cardinality ω_1 that admits no supercompact compactification (see proposition 1.1.16 and example 1.1.18). It now is natural to ask whether there is a *countable* space without supercompact Hausdorff compactification. Obviously such a space cannot be first countable, since a (regular) first countable countable space is metrizable and has an orderable compactification. By the same argument the example cannot be locally compact. Under MARTIN's axiom there exists a countable space with only one nonisolated point which admits no supercompact Hausdorff compactification. Hence this example is locally compact and first countable in all points but one.

The example also answers another natural question. As noted before the theorem of STROK & SZYMAŃSKI [116] implies that every separable metric space admits at least one supercompact compactification. It seems reasonable to try to generalize this corollary for a larger class of spaces, for example for the class of all separable stratifiable spaces or, more generally, for the class of all separable semi-stratifiable spaces. Unfortunately this is not possible since the space, constructed in this section, turns out to be stratifiable.

1.2.1. The example depends on the existence of P-points in $\beta\mathbb{N}\setminus\mathbb{N}$. A point p of a topological space X is called a *P-point* if the intersection of countably many neighborhoods of p is again a neighborhood of p. MARTIN's axiom (cf. 0.D) implies that there is a P-point in $\beta\mathbb{N}\setminus\mathbb{N}$ [18], see also [99] and [40]. It is conjectured that there exist P-points in $\beta\mathbb{N}\setminus\mathbb{N}$ without

*) This section will also be published separately in Bull. L'Acad. Pol. Sci.

set-theoretic assumptions; but this is as yet open.

1.2.2. THEOREM. *Let* p *be a* P-*point in* $\beta\mathbb{N}\setminus\mathbb{N}$. *Then the subspace* $\mathbb{N}\cup\{p\}$ *of* $\beta\mathbb{N}$ *has the property that no Hausdorff compactification of it is supercompact.*

PROOF. Define $X := \mathbb{N}\cup\{p\}$, where p is a P-point in $\beta\mathbb{N}\setminus\mathbb{N}$. Let αX be any Hausdorff compactification of X and let $f\colon \beta X = \beta\mathbb{N} \to \alpha X$ be the unique mapping which extends id_X. Notice that $f^{-1}[\{p\}] = \{p\}$.

Assume that $\mathcal{S}$ is a binary closed subbase for αX, closed under arbitrary intersection, and as in section 1.1 for $A \subset \alpha X$ let $I(A)$ be defined by

$$I(A) := \bigcap\{S \in \mathcal{S} \mid A \subset S\}.$$

Notice that $cl_{\alpha X}(A) \subset I(A)$, since each element of $\mathcal{S}$ is closed, that $I(I(A)) = I(A)$ and that $I(A) \subset I(B)$, for all $A \subset B \subset \alpha X$.

Let C be defined by

$$C := \{n \in \mathbb{N} \mid I(\{p,n\}) \cap (\alpha X\setminus X) \neq \emptyset\}.$$

For $n \in C$ choose an $x_n \in I(\{p,n\}) \cap (\alpha X\setminus X)$ and let $B := \{x_n \mid n \in C\}$.

CLAIM 1: $p \notin cl_{\alpha X}(B)$.

Indeed, as $f^{-1}[B]$ is a countable union of closed sets in $\beta\mathbb{N}\setminus\mathbb{N}$ which not contains p, it follows that, since p is a P-point,

$$p \notin cl_{\beta\mathbb{N}\setminus\mathbb{N}}(f^{-1}[B]) = cl_{\beta\mathbb{N}}(f^{-1}[B])$$

and consequently $p \notin f[cl_{\beta\mathbb{N}}(f^{-1}[B])]$ for otherwise $f^{-1}[\{p\}]$ would consist of more than one point. Now, as $B \subset f[cl_{\beta\mathbb{N}}(f^{-1}[B])]$ and as f is a closed mapping we conclude that $p \notin cl_{\alpha X}(B)$.

Choose open sets $U,V \subset \alpha X$ such that $p \in U \subset cl_{\alpha X}(U) \subset V$ and $V \cap cl_{\alpha X}(B) = \emptyset$. Let $T = \bigcup_{i\le n} S_i$ be an element of $\vee.\mathcal{S}$ ($S_i \in \mathcal{S}$, $i \le n$) such that $cl_{\alpha X}(U) \subset T \subset V$ (cf. lemma 0.2). Then

$$p \in cl_{\alpha X}(U) = cl_{\alpha X}(U\cap\mathbb{N}) = \bigcup_{i\le n} cl_{\alpha X}(U\cap\mathbb{N}\cap S_i),$$

and consequently there is an $i_0 \le n$ such that $p \in cl_{\alpha X}(U\cap\mathbb{N}\cap S_{i_0})$. Define $M := U \cap \mathbb{N} \cap S_{i_0}$. Then M is infinite and

$$p \in \mathrm{cl}_{\alpha X}(M) \subset I(M) \subset S_{i_0} \subset V$$

(this is the same technique as used in lemma 1.1.2).

<u>CLAIM 2</u>: *For each* $m \in M$ *the set* $I(\{p,m\})$ *is finite and does not intersect* $\alpha X \backslash X$.

The latter is trivial since $I(\{p,m\}) \cap B \subset I(M) \cap B \subset V \cap B = \emptyset$. To prove the former assume that $I(\{p,m\})$ were infinite. Then $I(\{p,m\}) \cap \mathbb{N}$ were infinite and as $I(\{p,m\}) \cap \mathbb{N}$ is C^*-embedded in X it does not converge to p; consequently

$$\emptyset \neq \mathrm{cl}_{\alpha X}(I(\{p,m\}) \cap \mathbb{N}) \cap (\alpha X \backslash X) \subset I(\{p,m\}) \cap (\alpha X \backslash X),$$

which is a contradiction.

Now for every ordinal $\kappa \leq \omega_1$ define a finite subset $A(\kappa)$ of M such that

(i) if $p \in \mathrm{cl}_{\alpha X}(\bigcup_{\mu<\kappa} A(\mu))$ then $A(\kappa) = \emptyset$;

(ii) if $p \notin \mathrm{cl}_{\alpha X}(\bigcup_{\mu<\kappa} A(\mu))$ then $A(\kappa) \neq \emptyset$ and $I(A(\kappa) \cup \{p\}) = A(\kappa) \cup \{p\}$ and $A(\kappa) \cap \bigcup_{\mu<\kappa} A(\mu) = \emptyset$.

Take a point $m \in M$ and define $A(o) := I(\{p,m\}) \cap \mathbb{N}$. Then $A(o)$ has all desired properties. Suppose that all $A(\mu)$ have been constructed for $\mu < \kappa \leq \omega_1$. Assume that $p \notin \mathrm{cl}_{\alpha X}(\bigcup_{\mu<\kappa} A(\mu))$. Using the same technique as above there exists an infinite $N_0 \subset M$ such that $p \in \mathrm{cl}_{\alpha X}(N_0) \subset I(N_0)$ and $I(N_0) \cap \mathrm{cl}_{\alpha X}(\bigcup_{\mu<\kappa} A(\mu)) = \emptyset$. Take $n \in N_0$ and define $A(\kappa) := I(\{p,n\}) \cap \mathbb{N}$. Then $A(\kappa)$ is as required.

As there are only countably many finite subsets of M there exists a $\kappa < \omega_1$ such that $p \in \mathrm{cl}_{\alpha X}(\bigcup_{\mu<\kappa} A(\mu))$. Then, since $\bigcup_{\mu<\kappa} A(\mu) \cup \{p\}$ is not a convergent sequence, there is a $q \in \mathrm{cl}_{\alpha X}(\bigcup_{\mu<\kappa} A(\mu)) \cap (\alpha X \backslash X)$. Take an infinite $L \subset \bigcup_{\mu<\kappa} A(\mu)$ such that

$$q \in \mathrm{cl}_{\alpha X}(L) \subset I(L) \subset \alpha X \backslash \{p\}.$$

As L is infinite there exist two different ordinals κ_0, κ_1 less than κ such that L intersects both $A(\kappa_0)$ and $A(\kappa_1)$. Then the subsystem

$$\{I(L), A(\kappa_0) \cup \{p\},\ A(\kappa_1) \cup \{p\}\}$$

of $\mathcal{S}$ is linked, but has a void intersection. This is a contradiction. □

1.2.3. A topological space X is called *stratifiable* (cf. BORGES [19]) if to each open subset U of X one can assign a sequence of open sets $\{U_n\}_{n=1}^{\infty}$ such that

(a) $\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \mathrm{cl}_X(U_n) = U$;

(b) $U_n \subset V_n$ whenever $U \subset V$ (where $\{V_n\}_{n=1}^{\infty}$ is the sequence assigned to V).

It is easy to see that each metrizable space is stratifiable while the converse need not be true.

If $p \in \beta\mathbb{N}\setminus\mathbb{N}$ then $\mathbb{N} \cup \{p\}$ clearly is stratifiable. Consequently MARTIN's axiom implies that there is a countable stratifiable space no Hausdorff compactification of which is supercompact. We do not have a metrizable space no Hausdorff compactification of which is supercompact. This suggests the following question.

1.2.4. QUESTION. *Is there a metrizable space no Hausdorff compactification of which is supercompact?*

1.3. Subbase characterizations of compact topological spaces

Often, an important class of topological spaces can be characterized by the fact that each element of the class possesses a subbase of a special kind. For example compact spaces (ALEXANDER's lemma), completely regular spaces (DE GROOT & AARTS [57]), second countable spaces (by definition), metrizable spaces (BING, cf. [86]), (products of) orderable spaces (VAN DALEN & WATTEL [39]; VAN DALEN [38]; DE GROOT & SCHNARE [60]). Such characterizations we shall call *subbase characterizations*.

DE GROOT has observed that certain classes of supercompact spaces can be characterized by means of special *binary* subbases; among the results obtained by him were the nice internal characterization of I^n and I^{∞} ([55]) and the characterization of products of compact orderable spaces ([60]). Also he discovered the duality between supercompact spaces and graphs ([56]). DE GROOT represented a supercompact space with binary subbase $\mathcal{S}$ by the intersection graph of $\mathcal{S}$, i.e. the graph with vertex set $\mathcal{S}$ and an edge between S_0 and S_1 in $\mathcal{S}$ if and only if $S_0 \cap S_1 \neq \emptyset$. DE GROOT proved that the space under consideration is completely determined by this graph.

We will derive DE GROOT's results using a slight modification:

a supercompact space with binary subbase $\mathcal{S}$ will be represented by the non-intersection graph of $\mathcal{S}$. This method, which of course is not essentially different, has some advantages; e.g. connectedness and bipartiteness of this latter graph imply interesting properties of the spaces under consideration; also product structures become trivialities. Moreover, our graph representation if often helpful to determine a subbase characterization.

The results of this section are taken from the joint paper VAN MILL & SCHRIJVER [81].

1.3.1. Here we define the notion of an interval structure, and use this concept to characterize supercompactness. Next we demonstrate a correspondence between graphs and supercompact spaces.

1.3.2. DEFINITION. Let X be a set and let I: $X \times X \to \mathcal{P}(X)$. Write $I(x,y) = I((x,y))$. Then I is called an *interval structure* on X if:

(i) $x,y \in I(x,y)$ $\qquad (x,y \in X)$,

(ii) $I(x,y) = I(y,x)$ $\qquad (x,y \in X)$,

(iii) if $u,v \in I(x,y)$ then $I(u,v) \subset I(x,y)$ $\qquad (u,v,x,y \in X)$,

(iv) $I(x,y) \cap I(x,z) \cap I(y,z) \neq \emptyset$ $\qquad (x,y,z \in X)$.

Axioms (i), (ii) and (iii) together can be replaced by the following axiom:

$$u,v \in I(x,y) \quad \text{iff} \quad I(u,v) \subset I(x,y) \qquad (u,v,x,y \in X).$$

A subset B of X is called I-*convex* if for all $x,y \in B$ we have $I(x,y) \subset B$.

1.3.3. THEOREM. *Let X be a topological space. Then X is supercompact if and only if X is compact and possesses a (closed) subbase $\mathcal{S}$ and an interval structure I such that each $S \in \mathcal{S}$ is I-convex.*

PROOF. Let X be a supercompact space and let $\mathcal{S}$ be a binary subbase for X. Define $I_{\mathcal{S}}$: $X \times X \to \mathcal{P}(X)$ by

$$I_{\mathcal{S}}((x,y)) := \bigcap\{S \in \mathcal{S} \mid x,y \in S\} \qquad (x,y \in X).$$

Then it is easy to show that $I_{\mathcal{S}}$ is an interval structure on X and that each element of $\mathcal{S}$ is $I_{\mathcal{S}}$-convex.

Conversely, let X be a compact space with a closed subbase $\mathcal{S}$ consisting of I-convex sets, where I is an interval structure on X. We will show that

$\mathcal{S}$ is binary.

Let $\mathcal{S}' \subset \mathcal{S}$ such that $\cap\mathcal{S}' = \emptyset$. Then, since X is compact, there exists a finite $\mathcal{S}_0' \subset \mathcal{S}$ such that $\cap\mathcal{S}_0' = \emptyset$. Hence it is enough to prove the following: if $S_1,\ldots,S_k \in \mathcal{S}$ and $S_1 \cap \ldots \cap S_k = \emptyset$ then there exist $i,j \le k$ such that $S_i \cap S_j = \emptyset$. We will prove this by induction with respect to k.

For $k=1$ or 2 there is nothing to prove. Therefore assume that $k \ge 3$ and that the statement is true for all $k' < k$. Define

$$\begin{aligned} T_1 &= \phantom{S_1 \cap{}} S_2 \cap S_3 \cap S_4 \cap \ldots \cap S_k, \\ T_2 &= S_1 \cap \phantom{S_2 \cap{}} S_3 \cap S_4 \cap \ldots \cap S_k, \\ T_3 &= S_1 \cap S_2 \cap \phantom{S_3 \cap{}} S_4 \cap \ldots \cap S_k. \end{aligned}$$

If one of the T_i's is empty, then the induction hypothesis applies. Therefore suppose neither is empty and take $x \in T_1$, $y \in T_2$ and $z \in T_3$. Then

$$\begin{aligned} x,y &\in S_3 \cap S_4 \cap \ldots \cap S_k, \\ x,z &\in S_2 \cap S_4 \cap \ldots \cap S_k, \\ y,z &\in S_1 \cap S_4 \cap \ldots \cap S_k, \end{aligned}$$

and thus

$$\begin{aligned} I(x,y) &\subset S_3 \cap S_4 \cap \ldots \cap S_k, \\ I(x,z) &\subset S_2 \cap S_4 \cap \ldots \cap S_k, \\ I(y,z) &\subset S_1 \cap S_4 \cap \ldots \cap S_k. \end{aligned}$$

But

$$\begin{aligned} \emptyset \neq I(x,y) \cap I(x,z) \cap I(y,z) &\subset (S_3 \cap S_4 \cap \ldots \cap S_k) \cap (S_2 \cap S_4 \cap \ldots \cap S_k) \cap \\ &\quad \cap (S_1 \cap S_4 \cap \ldots \cap S_k) \\ &= S_1 \cap S_2 \cap \ldots \cap S_k. \end{aligned}$$

This contradicts our hypothesis. □

For some related ideas see GILMORE [53].

1.3.4. REMARK. As noted in the introduction, the notion of an interval structure is used extensively in the theory of maximal linked systems and of supercompact spaces. It is simple but useful and often is helpful to prove local properties of supercompact spaces.

1.3.5. Now we turn our attention to the announced correspondence between graphs and supercompact spaces.

A *graph* G is a pair (V,E), in which V is a set, called the set of *vertices*, and E is a collection of unordered pairs of elements of V, that is $E \subset \{(v,w) \mid v,w \in V,\ v \neq w\}$. Pairs in E are called *edges*. Usually a graph is represented by a set of points in a space with lines between two points if these two points form an edge. A subset V' of V is called *independent* if for all $v,w \in V'$ we have $\{v,w\} \notin E$. A *maximal independent subset* of V is an independent subset not contained in any other independent subset. By Zorn's lemma each independent subset of V is contained in some maximal independent subset. We write

$$\mathcal{I}(G) := \{V' \subset V \mid V' \text{ is maximal independent}\}$$

and for each $v \in V$

$$B_v := \{V' \in \mathcal{I}(G) \mid v \in V'\}.$$

Finally let $\mathcal{B}(G)$ be defined by

$$\mathcal{B}(G) := \{B_v \mid v \in V\}.$$

The *graph space* T(G) of G is the topological space with $\mathcal{I}(G)$ as underlying point set and with $\mathcal{B}(G)$ as a (closed) subbase.

If $\mathcal{S}$ is a collection of sets then the *non-intersection graph* $G(\mathcal{S})$ of $\mathcal{S}$ if the graph with vertex-set $\mathcal{S}$ and with edges the collection of all pairs $\{S_1,S_2\}$ such that $S_1 \cap S_2 = \emptyset$. The following theorem follows from observations made by DE GROOT [56]:

1.3.6. <u>THEOREM</u>. *A topological space* X *is supercompact iff it is the graph space of a graph, in particular*

(i) *if* X *has a binary subbase* $\mathcal{S}$ *then* X *is homeomorphic to the graph space of* $G(\mathcal{S})$;

(ii) *For any graph* G, *the graph space* T(G) *is supercompact with* $\mathcal{B}(G)$ *as a binary subbase.*

Let G_j be a graph ($j \in J$); the *sum* $\sum_{j\in J} G_j$; of these graphs is the graph with vertex set a disjoint union of the vertex sets of the G_j ($j \in J$) and edge set the corresponding union of the edge sets. These sums of graphs and products of topological spaces are related by the following theorem:

1.3.7. THEOREM. *Let* J *be a set and for each* $j \in J$ *let* G_j *be a graph. Then* $T(\sum_{j\in J} G_j)$ *is homeomorphic to* $\Pi_{j\in J} T(G_j)$.

PROOF. Straightforward. □

1.3.8. DEFINITION. A collection S of subsets of a set X is called *weakly normal* if for each $S_0, S_1 \in S$ with $S_0 \cap S_1 = \emptyset$ there exists a finite covering M of X by elements of S such that each element of M meets at most one of S_0 and S_1.

Weakly normal closed subbases for topological spaces play an important role in characterizing complete regularity (cf. DE GROOT & AARTS [57]). They turn out to be the right natural generalizations to subbases of *normal bases* as defined by FRINK [51], STEINER [114] and many others. This will be discussed in chapter 4.

Clearly weak normality of a collection S of subsets of a subset X must imply properties of the corresponding non-intersection graph $G(S)$. We call a graph (V,E) *weakly normal* if for each $\{v,w\} \in E$ there are $v_1,\ldots,v_k,w_1,\ldots,w_\ell \in V$ $(k,\ell \geq 0)$ such that:

$$\{v,v_1\},\ldots,\{v,v_k\},\{w,w_1\},\ldots,\{w,w_\ell\} \in E$$

and in addition, whenever

$$v_1',\ldots,v_k',w_1',\ldots,w_\ell' \in V$$

with

$$\{v_1,v_1'\},\ldots,\{v_k,v_k'\},\{w_1,w_1'\},\ldots,\{w_\ell,w_\ell'\} \in E,$$

then

$$\{v_1',\ldots,v_k',w_1',\ldots,w_\ell'\}$$

is not independent.

1.3.9. THEOREM. *Let* X *be a supercompact space with binary subbase* S *and let* X *be the graph space of the graph* G. *The following assertions are equivalent*:

(i) X *is a Hausdorff space;*

(ii) $\mathcal{S}$ *is a weakly normal subbase;*

(iii) G *is a weakly normal graph.*

PROOF. (i) ⇒ (ii). Take $S_1,S_2 \in \mathcal{S}$ with $S_1 \cap S_2 = \emptyset$. As X is normal (compact and Hausdorff) there exist closed sets C and D in X with

$$C \cap S_1 = \emptyset = S_2 \cap D \quad \text{and} \quad C \cup D = X.$$

Since X is compact and C and D are intersections of finite unions of sets in $\mathcal{S}$, we can take C and D to be finite intersections of finite unions of sets in $\mathcal{S}$, or, what is the same, finite unions of finite intersections of sets in $\mathcal{S}$.

Since $C \cap S_1 = \emptyset$ each of the finite intersections composing C has an empty intersection with S_1. Now the binarity of $\mathcal{S}$ implies that we can replace these finite intersections by single sets of $\mathcal{S}$. Hence we may suppose that C is a finite union of elements of $\mathcal{S}$. Similarly we can take D as a finite union of elements of $\mathcal{S}$.

(ii) ⇒ (1). By lemma 0.4 $\mathcal{S}$ is a T_1-subbase. Now the result follows from a theorem due to DE GROOT & AARTS [57].

(i) ⟺ (iii). The simple proof is left to the reader. □

This theorem now implies the following remarkable fact, which was first observed by DE GROOT [56].

1.3.10. THEOREM. *The following assertions are equivalent:*

(i) X *is compact metric;*

(ii) X *has a countable weakly normal binary subbase;*

(iii) X *is homeomorphic to the graph space of a countable weakly normal graph.*

PROOF. Part (i) ⇒ (ii) follows from STROK & SZYMAŃSKI's [116] result and theorem 1.3.9. The other implications follow from URYSOHN's metrization theorem. □

From this theorem we can derive a, in our opinion, remarkable characterization of the Cantor discontinuum. We call a graph (V,E) *locally finite* if for all $v \in V$ the set $\{w \in V \mid \{v,w\} \in E\}$ is finite.

1.3.11. THEOREM. *The following assertions are equivalent:*

(i) X *is homeomorphic to the Cantor discontinuum;*

(ii) X *is homeomorphic to the graph space of a countable locally finite graph with infinitely many edges.*

PROOF. (i) ⇒ (ii). By theorem 1.3.7 X is homeomorphic to the graph space of the following graph (cf. DE GROOT [56]);

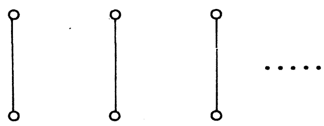


Figure 1.

ii) ⇒ (i). We shall show that X is a compact metric totally disconnected space without isolated points; hence it will follow that X is homeomorphic to the Cantor discontinuum.

Let G be a countable locally finite graph with infinitely many edges. We will first show that the closed subbase $\mathcal{B}(G)$ of T(G) consists of clopen sets. Take $v \in V$. Since G is locally finite, there are $w_1, w_2, \ldots, w_n \in V$ such that

$$\{w_1, \ldots, w_n\} = \{w \in V \mid \{v,w\} \in E\}.$$

Now for all $i \in \{1,2,\ldots,n\}$ the set B_{w_i} is closed and consequently $\bigcup_{i=1}^{n} B_{w_i}$ is closed too. It is obvious that

$$X \setminus \bigcup_{i=1}^{n} B_{w_i} = B_v,$$

and hence B_v is open.

It now follows that T(G) is Hausdorff, by lemma 0.4; moreover it is compact totally disconnected and second countable. Hence T(G) is a compact metric totally disconnected topological space.

Finally we show that T(G) has no isolated points. For suppose to the contrary there is a $V' \in \mathcal{I}(G)$ such that $\{V'\} = \bigcap_{i=1}^{m} B_{v_i}$. That is, if $V'' \in \mathcal{I}(G)$ and $\{v_1, v_2, \ldots, v_m\} \subset V''$ then $V' = V''$. Let W be the set

$$\{w \in V \mid \{v_i, w\} \in E \text{ for some } i \in \{1,2,\ldots,m\}\}.$$

Since G is locally finite, W is finite. Now the set

$$E' = \{\{v,w\} \mid w \in W,\ v \in V\}$$

also is finite. Since E is infinite there is an edge $\{a,b\} \in E\backslash E'$. It is easy to see that $a \notin W$ and $b \notin W$, hence $\{v_1,\ldots,v_m,a\}$ and $\{v_1,\ldots,v_m,b\}$ both are independent sets of vertices, and hence both contained in a maximal independent set, say in V''_a and V''_b. As $\{v_1,\ldots,v_m\} \subset V''_a$ and $\{v_1,\ldots,v_m\} \subset V''_b$ it follows that $V''_a = V''_b = V'$; hence $a,b \in V'$. But $\{a,b\} \in E$, hence V' is not independent; this is a contradiction. □

1.3.12. We will now give a correspondence between spaces induced by a lattice and graph spaces obtained from bipartite graphs. Let $(X,\leq)$ be a lattice with universal bounds 0 and 1. If a and b are elements of X then $[a,b]$ will denote the set

$$[a,b] := \{x \in X \mid a \leq x \leq b\}.$$

The *interval space* of X is the topological space with underlying set X and with (closed) subbase the collection

$$\mathcal{S} := \{[0,x] \mid x \in X\} \cup \{[x,1] \mid x \in X\}.$$

Spaces obtained in this way are called *lattice spaces*. According to a theorem of FRINK (cf. BIRKHOFF [17]) *the interval space of a lattice* $(X,\leq)$ *is compact iff* $(X,\leq)$ *is complete*.

1.3.13. THEOREM. *Every compact lattice space is supercompact.*

PROOF. Let $(X,\leq)$ be a complete lattice and define an interval structure (cf. definition 1.3.2) I on X by

$$I(x,y) := [x\wedge y, x\vee y].$$

This is easily seen to be an interval structure while moreover the subbase $\mathcal{S}$ for X defined in 1.3.12 consists of I-convex sets; consequently X is supercompact by theorem 1.3.3. □

1.3.14. A graph (V,E) is called *bipartite* if V can be partioned in two sets V_0 and V_1 such that each edge consists of an element in V_0 and an element of V_1. A well-known and easily proved theorem in graph theory, see e.g. WILSON [129], tells us that *a graph* (V,E) *is bipartite if and only if each circuit is even,* that is, whenever

$$\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{k-1},v_k\},\{v_k,v_1\}$$

are edges in E, then k is even (this characterization uses a weak form of the axiom of choice).

We call a collection S of subsets of a set X *bipartite* if the non-intersection graph $G(S)$ is bipartite.

1.3.15. THEOREM. *The following assertions are equivalent:*
(i) X *is homeomorphic to a compact lattice space;*
(ii) X *possesses a binary bipartite subbase;*
(iii) X *is homeomorphic to the graph space of a bipartite graph.*

PROOF. (i) $\Rightarrow$ (ii). Let $(X,\leq)$ be a complete lattice; the subbase

$$S = \{[0,x] \mid x \in X\} \cup \{[x,1] \mid x \in X\}$$

is binary and bipartite.
(ii) $\Rightarrow$ (i). Let X be a topological space with a binary bipartite subbase S; let $S = S_0 \cup S_1$, such that $S_0 \cap S_1 = \emptyset$ and $\bigcap S_0 \neq \emptyset$ and $\bigcap S_1 \neq \emptyset$ (this is possible since S is binary and bipartite). Define an order "$\leq$" on X by

$$x \leq y \text{ iff } y \in S \text{ whenever } x \in S \in S_1.$$

The relation "$\leq$" is reflexive and transitive; "$\leq$" is antisymmetric too. For suppose that $X \neq y$ and $x \leq y \leq x$. Since the subbase S is T_1 (lemma 0.4) there are $S,T \in S$ such that $x \in S$, $y \in T$ and $S \cap T = \emptyset$. From this it follows that either $S \in S_1$ or $T \in S_1$. If $S \in S_1$ then $y \in S$, since $x \leq y$. But this is a contradiction. On the other hand if $T \in S_1$ then $x \in T$, since $y \leq x$. This also is a contradiction.

We will show that "$\leq$" defines a complete lattice by proving that for each $X' \subset X$ there is a $z \in X$ such that $z = \sup X'$.

Let $X' \subset X$. Define

$$S_0' := \{S \in S_0 \mid X' \subset S_0\}$$

and

$$S_1' := \{T \in S_1 \mid T \cap S \neq \emptyset \text{ for all } S \in S_0'\}$$

respectively.

Now $\bigcap S_0' \cap \bigcap S_1' \neq \emptyset$, since $\bigcap S_0' \neq \emptyset \neq \bigcap S_1'$ and also $S \cap T \neq \emptyset$ for all $S \in S_0'$ and $T \in S_1'$ (notice that S is binary!). Choose $z \in \bigcap S_0' \cap \bigcap S_1'$. This point z is an upper bound for X', for let $x \in X'$ and let $x \in T \in \mathcal{S}_1$; then $T \in \mathcal{S}_1'$ and consequently $z \in T$. Therefore $z \leq x$ for all $x \in X'$.

Suppose now that $x \leq z'$ for all $x \in X'$ and that $z \not\leq z'$. Then there exists a $T \in \mathcal{S}_1$ with the properties $z \in T$ and $z' \notin T$. As $\mathcal{S}$ is binary and bipartite there is an $S \in \mathcal{S}_0$ such that $S \cap T = \emptyset$ and $z' \in S$. Now, $X' \subset S$, since otherwise there must be an $x_0 \in X'$ and a $T' \in \mathcal{S}_1$ with the properties $x_0 \in T'$ and $T' \cap S = \emptyset$. Then, since $x_0 \leq z'$ we have that $z' \in T'$, which contradicts the fact that $S \cap T' = \emptyset$. Therefore $X' \subset S$, which implies that $S \in \mathcal{S}_0'$. But $z \notin S$, which cannot be the case since $z \in \bigcap \mathcal{S}_0' \cap \mathcal{S}_1'$.

Finally the topology induced by the lattice-ordering $\leq$ coincides with the original topology of the space X. Indeed, for $x \in X$ we have that

$$[x,1] = \bigcap\{S \in \mathcal{S}_1 \mid x \in S\},$$

as can easily be seen.

Furthermore

$$[0,x] = \bigcap\{S \in \mathcal{S}_0 \mid x \in S\},$$

for suppose that $y \leq x$ and that $y \notin S$ for some $S \in \mathcal{S}_0$ with $x \in S$. Then there exists a $T \in \mathcal{S}_1$ such that $S \cap T = \emptyset$ and $y \in T$. Hence $x \in T$, contradicting the fact that $S \cap T = \emptyset$.

Also if $T \in \mathcal{S}_1$, let

$$\mathcal{S}_0' := \{S \in \mathcal{S}_0 \mid S \cap T \neq \emptyset\}.$$

Then $T \cap \bigcap\mathcal{S}_0' \neq \emptyset$, since $\mathcal{S}$ is binary. Choose $z \in T \cap \bigcap\mathcal{S}_0'$. We will show that

$$[z,1] = T.$$

If $z \leq y$, then $y \in T$ since $z \in T$. If $y \in T$ and $z \not\leq y$, then there exists an $S \in \mathcal{S}_0$ such that $y \in S$ and $z \notin S$. However, $S \cap T \neq \emptyset$ and consequently $S \in \mathcal{S}_0'$ and $z \in S$, which is a contradiction.

Conversely, if $S \in \mathcal{S}_0$ let

$$\mathcal{S}_1' = \{T \in \mathcal{S}_1 \mid S \cap T \neq \emptyset\}.$$

Then $S \cap \bigcap\mathcal{S}_1' \neq \emptyset$, since $\mathcal{S}$ is binary. Choose $z \in S \cap \bigcap\mathcal{S}_1'$. We will show that

$$[0,z] = S.$$

If $y \le z$ and $y \notin S$ then $y \in T$ for some $T \in \mathcal{S}$ with $S \cap T = \emptyset$. Hence $z \notin T$, which contradicts the fact that $y \le z$. If $y \in S$ and $y \not\le z$ then there is some $T \in \mathcal{S}_1$ such that $y \in T$ and $z \notin T$. Then $S \cap T \ne \emptyset$ and $T \in \mathcal{S}_1'$. Hence $z \in T$, contradicting the fact that $z \notin T$.

(ii) $\Rightarrow$ (iii). Let X be a space with a binary bipartite subbase $\mathcal{S}$. By definition $G(\mathcal{S})$ is bipartite and, by theorem 1.3.6 X is homeomorphic to the graph space of $G(\mathcal{S})$.

(iii) $\Rightarrow$ (ii). Let G be a bipartite graph. It is easy to see that the binary subbase $\mathcal{B}(G)$ for the graph space of G is bipartite. $\square$

1.3.16. We now turn our attention to compact tree-like spaces, which also will be characterized with the help of weakly comparable subbases and graphs.

A *tree-like space* is a connected space in which every two distinct points x and y can be separated by a third point z, i.e. x and y belong to different components of $X\setminus\{z\}$. Obviously every connected orderable space is tree-like; however, the class of tree-like spaces is much bigger; see e.g. KOK [70].

A collection $\mathcal{S}$ of subsets of a set X is called *normal* if for every $S_0,S_1 \in \mathcal{S}$ with $S_0 \cap S_1 = \emptyset$ there exist $T_0,T_1 \in \mathcal{S}$ with $S_0 \cap T_1 = \emptyset = T_0 \cap S_1$ and $T_0 \cup T_1 = X$. Clearly a normal collection is weakly normal, cf. definition 1.3.8. In addition $\mathcal{S}$ is called *weakly comparable* if for all $S_0,S_1,S_2 \in \mathcal{S}$ satisfying $S_0 \cap S_1 = \emptyset = S_0 \cap S_2$ it follows that either $S_1 \subset S_2$ or $S_2 \subset S_1$ or $S_1 \cap S_2 = \emptyset$ (the notion "comparable" will be defined in 1.3.26).

A collection $\mathcal{S}$ of subsets of a set X will be called *connected (strongly connected)* if there is no partition of X in two (finitely many) elements of $\mathcal{S}$.

1.3.17. PROPOSITION. *Let $\mathcal{S}$ be a weakly comparable collection of subsets of the set* X. *The following properties are equivalent:*

(i) $\mathcal{S}$ *is normal and connected;*

(ii) $\mathcal{S}$ *is weakly normal and strongly connected.*

PROOF. (i) $\Rightarrow$ (ii). Let $\mathcal{S}$ be weakly comparable, normal and connected. Clearly $\mathcal{S}$ is weakly normal. Suppose that $\mathcal{S}$ is not strongly connected and let k be the minimal number such that there are pairwise disjoint sets $S_1,\dots,S_k$ in $\mathcal{S}$ with union X. Since $\mathcal{S}$ is connected, $k \ge 3$. As $S_1 \cap S_2 = \emptyset$ there exist, by the normality of $\mathcal{S}$, T_1 and T_2 in $\mathcal{S}$ such that

$S_1 \cap T_2 = \emptyset = T_1 \cap S_2$ and $T_1 \cup T_2 = X$. Now S_3 intersects either T_1 or T_2. Without loss of generality we may suppose that $S_3 \cap T_1 \neq \emptyset$. Hence since $S_2 \cap T_1 = \emptyset = S_2 \cap S_3$, by the weak comparability of $\mathcal{S}$, $S_3 \cap T_1 = \emptyset$ or $T_1 \subset S_3$ or $S_3 \subset T_1$. Since the first two cases are impossible it follows that $S_3 \subset T_1$. In the same way one proves that for each $j = 4,\dots,k$ either $S_j \subset T_1$ or $S_j \cap T_1 = \emptyset$. Hence there exists a smaller number of pairwise disjoint sets in $\mathcal{S}$ covering X.

(ii) $\Rightarrow$ (i). Let $\mathcal{S}$ be a weakly normal, strongly connected, weakly comparable collection of subsets of X. We need only show that $\mathcal{S}$ is normal. To prove this, let $T_0, T_1 \in \mathcal{S}$ such that $T_0 \cap T_1 = \emptyset$. Let k be the minimal number such that there are $S_1,\dots,S_k$ in $\mathcal{S}$ covering X and such that each S_i meets at most one of T_0 and T_1. By the minimality of k we may suppose that no two of these subsets $S_1,\dots,S_k$ are contained in each other. If $k = 2$ then we are done.

Suppose therefore $k \geq 3$. We prove that the sets $S_1,\dots,S_k$ are pairwise disjoint. We only prove that $S_1 \cap S_2 = \emptyset$. To the contrary assume that $S_1 \cap S_2$ were nonvoid. By the weak comparability of $\mathcal{S}$ they are neither both disjoint from T_0 nor they are both disjoint from T_1. We may suppose therefore $S_1 \cap T_0 \neq \emptyset \neq S_2 \cap T_1$. Since now $S_1 \cap T_1 = \emptyset = T_1 \cap T_0$ it follows that either $S_1 \subset T_0$ or $T_0 \subset S_1$. If $S_1 \subset T_0$ then $T_0 \cap S_2 \supset S_1 \cap S_2 \neq \emptyset$, which cannot be the case. It follows that $T_0 \subset S_1$ and similarly $T_1 \subset S_2$. We may suppose that $S_3 \cap T_0 = \emptyset$. Since also $S_2 \cap T_0 = \emptyset$ we have $S_3 \cap S_2 = \emptyset$. From this it follows that $S_3 \cap T_1 = \emptyset$ and since also $S_1 \cap T_1 = \emptyset$, we have $S_3 \cap S_1 = \emptyset$. Now from the weak comparability of $\mathcal{S}$ it follows from $S_3 \cap S_2 = \emptyset = S_3 \cap S_1$ that $S_2 \cap S_1 = \emptyset$, which is a contradiction.

Since there are no pairwise disjoint sets $S_1,\dots,S_k$ in $\mathcal{S}$ with union X, it cannot be the case that $k \geq 3$. Hence $\mathcal{S}$ is normal. $\square$

1.3.18. A graph (V,E) is called *normal* if for each edge $\{v,w\} \in E$ there are edges $\{v,v'\}$ and $\{w,w'\}$ in E such that whenever $\{v',v''\}$ and $\{w',w''\}$ are edges then also $\{v'',w''\}$ is an edge (see figure 2).

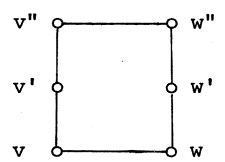


Figure 2.

Clearly each normal graph is a weakly normal graph (see 1.3.8).

A graph (V,E) is called *weakly comparable* if for each "path" $\{v_0,v_1\}$, $\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\}$ of edges either $\{v_1,v_3\} \in E$ or $\{v_0,v_3\} \in E$ or $\{v_1,v_4\} \in E$ (see figure 3).

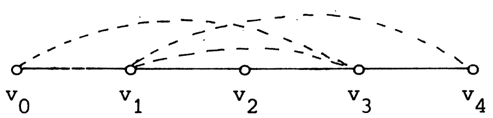


Figure 3.

A graph (V,E) is called *contiguous* (BRUIJNING [26]) if for each edge $\{v,w\} \in E$ there exist edges $\{v,v'\}$ and $\{w,w'\}$ such that $\{v',w'\} \notin E$.

A graph (V,E) is *connected* if for each two vertices $v,w \in V$ there is a path of edges $\{v,v_1\},\{v_1,v_2\},\ldots,\{v_k,w\}$ $(k \in \mathbb{N})$.

Finally, we call a collection $\mathcal{S}$ of subsets of a set X *graph-connected* if the corresponding non-intersection graph $G(\mathcal{S})$ is connected.

We need a simple lemma.

1.3.19. LEMMA. *Let $\mathcal{S}$ be a binary collection of subsets of the set X with non-intersection graph $G(\mathcal{S})$. Then*

(i) *$\mathcal{S}$ is normal iff $G(\mathcal{S})$ is normal;*

(ii) *$\mathcal{S}$ is weakly comparable iff $G(\mathcal{S})$ is weakly comparable;*

(iii) *$\mathcal{S}$ is connected iff $G(\mathcal{S})$ is contiguous.*

PROOF. Notice that $S_1 \cup \ldots \cup S_k = X$ $(S_i \in \mathcal{S},\ i \le k)$ if and only if the following holds in $G(\mathcal{S})$: for all $S_1',\ldots,S_k'$ such that $\{S_i,S_i'\}$ is an edge of $G(\mathcal{S})$ $(i \le k)$ the set $\{S_1',S_2',\ldots,S_k'\}$ is not independent. □

1.3.20. If X is a tree-like space then a subset A of X is called a *segment* if A is a component of $X\backslash\{x_0\}$ for certain $x_0 \in X$. KOK [70] has shown that *every segment in a tree-like space is open*. In particular any tree-like space is Hausdorff.

1.3.21. THEOREM. *Let X be a topological space. Then the following properties are equivalent:*

(i) X *is compact tree-like;*

(ii) X *possesses a binary normal connected (closed) subbase* T *such that for all* $T_0, T_1 \in T$ *we have that either* $T_0 \subset T_1$ *or* $T_1 \subset T_0$ *or* $T_0 \cap T_1 = \emptyset$ *or* $T_0 \cup T_1 = X$;

(iii) X *is homeomorphic to the graph space of a connected normal contiguous weakly comparable graph.*

PROOF. (i) ⇒ (ii). Let X be compact tree-like and let U denote the collection of segments of X. Since every two distinct points of X are contained in disjoint segments, the compactness of X implies that U is an open subbase for the topology on X. We will show that for all $U_0, U_1 \in U$ either $U_0 \cup U_1 = X$ or $U_0 \cap U_1 = \emptyset$ or $U_0 \subset U_1$ or $U_1 \subset U_0$. To prove this, take $U_0, U_1 \in U$ and suppose that U_i is a component of $X\backslash\{x_i\}$ $(i \in \{0,1\})$. Without loss of generality we may assume that $x_0 \neq x_1$. Suppose that $X\backslash\{x_i\} = U_i + U_i^*$ $(i \in \{0,1\})$ (this means $U_i \cap U_i^* = \emptyset$ and $X\backslash\{x_i\} = U_i \cup U_i^*$). We have to consider two cases:

(a) Suppose first that $x_1 \in U_0$. We again distinguish two subcases:

(a)$^{(i)}$ $x_0 \in U_1$. It then follows that $cl_X(U_0^*) = U_0^* \cup \{x_0\} \subset U_1$, since $cl_X(U_0^*)$ is connected. This implies $U_0 \cup U_1 = X$.

(a)$^{(ii)}$ $x_0 \in U_1^*$. Then $cl_X(U_1) \subset U_0$, since $cl_X(U_1)$ is connected. Therefore $U_1 \subset U_0$.

(b) Suppose that $x_1 \in U_0^*$. We distinguish two subcases:

(b)$^{(i)}$ $x_0 \in U_1$. This implies that $cl_X(U_0) \subset U_1$, since $cl_X(U_0)$ is connected. Hence $U_0 \subset U_1$.

(b)$^{(ii)}$ $x_0 \in U_1^*$. Now we have $cl_X(U_0) \subset U_1^*$, since $cl_X(U_0)$ is connected. Therefore $U_0 \subset U_1^*$ and consequently $U_0 \cap U_1 = \emptyset$.

Now define $T := \{X\backslash U \mid U \in U\}$, Then T is a closed subbase for X such that for all $T_0, T_1 \in T$ either $T_0 \cup T_1 = X$ or $T_0 \cap T_1 = \emptyset$ or $T_0 \subset T_1$ or $T_1 \subset T_0$. In particular T is weakly comparable. To show that T is binary it suffices to show that each covering of X by elements of U contains a subcover consisting of two elements of U. Indeed, let A be an open cover of X by elements of U. By the compactness of X the cover A has a finite subcover $\{U_1, \ldots\ldots, U_n\}$. In addition we may assume that $\emptyset \neq U_i \not\subset U_j$ for $i \neq j$. We claim that for each $U_i \in \{U_1, \ldots, U_n\}$ there exists a $U_j \in \{U_1, \ldots, U_n\}$ such that $U_i \cap U_j \neq \emptyset$; for assume to the contrary that for some fixed

$i \leq n$ it were true that $U_i \cap U_j = \emptyset$ for all $i \neq j \leq n$. As $\{U_1,\ldots,U_n\}$ is a covering of X it would follow that X is not connected, which is a contradiction. Therefore $U_i \cup U_j = X$. Consequently $\mathcal{T}$ is a binary subbase.

As X is Hausdorff, by theorem 1.3.9, $\mathcal{T}$ is weakly normal, which implies that $\mathcal{T}$ is normal by proposition 1.3.17, since trivially $\mathcal{T}$ is strongly connected (notice that $\mathcal{T}$ consists of closed sets).

(ii) $\Rightarrow$ (i). Since $\mathcal{T}$ is a binary subbase evidently X is compact. Therefore we must prove that X is tree-like. We will check the connectedness first.

Suppose that X is not connected. Then there are closed disjoint sets G and H such that $G \cup H = X$ and $G \neq \emptyset \neq H$. As G and H are intersections of finite unions of elements of $\mathcal{T}$ and as G and H are disjoint, the compactness of X implies that G and H both are finite intersections of finite unions of elements of $\mathcal{T}$, or, what is the same, finite unions of intersections. Let m be the minimal number such that there are $G_1,\ldots,G_m$ such that

(i) $G_1,\ldots,G_m$ are nonvoid and intersections of subbase elements;

(ii) $G_1 \cup \ldots \cup G_m = X$;

(iii) there is an $I \subset \{1,2,\ldots,m\}$ such that

$$\bigcup_{i\in I} G_i \neq \emptyset \neq \bigcup_{j\notin I} G_j \quad \text{and} \quad \bigcup_{i\in I} G_i \cap \bigcup_{j\notin I} G_j = \emptyset.$$

We first prove that $G_i \cap G_j = \emptyset$ if $i \neq j$. Suppose that $G_i \cap G_j \neq \emptyset$ for some $i \neq j$.

<u>CLAIM</u>: $G_i \cup G_j = \bigcap\{T \in \mathcal{T} \mid G_i \cup G_j \subset T\}$.

Indeed, take $x \notin G_i \cup G_j$. Then, since G_i and G_j are intersections of subbase elements there are T_0 and T_1 in $\mathcal{T}$ such that $G_i \subset T_0$, $G_j \subset T_1$ and $x \notin T_0 \cup T_1$. Now since $T_0 \cap T_1 \supset G_i \cap G_j \neq \emptyset$ and $T_0 \cup T_1 \neq X$ it follows that either $T_0 \subset T_1$ or $T_1 \subset T_0$. Therefore $x \notin T$ for some $T \in \mathcal{T}$ with $G_i \cup G_j \subset T$.

Now it follows that m is not the minimal number of sets with the above properties, which is a contradiction.

Second we prove that each G_i is an element of $\mathcal{T}$. Suppose that some $G_i \notin \mathcal{T}$. Let $j \neq i$. Then, since G_i is an intersection of subbase elements and $\mathcal{T}$ is binary, there is a $T \in \mathcal{T}$ such that $G_i \subset T$ and $T \cap G_j = \emptyset$. The sequence $G_1,\ldots,G_{i-1},T,G_{i+1},\ldots,G_m$ is also a sequence with the above properties (i), (ii) and (iii). So again $T \cap G_k = \emptyset$ if $k \neq i$, hence $G_i \subset T \subset X\backslash\bigcup_{k\neq i} G_k$, which implies that $G_i = T$ and therefore $G_i \in \mathcal{T}$.

Hence there is a collection $G_1,\ldots,G_m$ of pairwise disjoint subbase elements covering X and as $\mathcal{T}$ is weakly comparable, and hence by proposition 1.3.17 is strongly connected, this is a contradiction. This proves that X is connected.

We will now show that every two distinct points can be separated by a third. Let $x,y \in X$ such that $x \neq y$. As X is a T_1-space we have that $\{z\} = \bigcap\{T \in \mathcal{T} \mid z \in T\}$ for all $z \in X$ and consequently, since $\mathcal{T}$ is binary, there exist $T_0,T_1 \in \mathcal{T}$ such that $x \in T_0$, $y \in T_1$ and $T_0 \cap = \emptyset$ (cf. lemma 0.4). The normality of $\mathcal{T}$ implies the existence of $T_0',T_1' \in \mathcal{T}$ such that $T_0' \cup T_1' = X$ and $T_0 \cap T_1' = \emptyset = T_0' \cap T_1$. Define

$$\mathcal{A} := \{T \in \mathcal{T} \mid T \cup T_0' = X\}.$$

Since X is connected we have that $\mathcal{A} \cup \{T_0'\}$ is a linked system and consequently $T_0' \cap \bigcap\mathcal{A} \neq \emptyset$. We claim that this intersection consists of one point. We assume to the contrary that $z_0,z_1 \in T_0' \cap \bigcap\mathcal{A}$ with $z_0 \neq z_1$. In the same way as above there exist $S_0,S_1 \in \mathcal{T}$ such that $z_0 \in S_0\backslash S_1$ and $z_1 \in S_1\backslash S_0$ and $S_0 \cup S_1 = X$. Since $z_0 \notin S_1$ we have that $S_1 \notin \mathcal{A}$ and consequently $T_0' \cup S_1 \neq X$. Hence $T_0' \subset S_1$ or $S_1 \subset T_0'$; notice that $S_1 \cap T_0' \neq \emptyset$. However this implies that $S_1 \subset T_0'$, since $z_0 \notin S_1$. With the same technique one proves that $S_0 \subset T_0'$; but this is a contradiction since $T_0' \neq X$. Let $\{z_0\} := T_0' \cap \bigcap\mathcal{A}$. Then z_0 is a separation point of x and y, since T_0' and $\bigcap\mathcal{A}$ are closed subsets of X such that $T_0' \cup (\bigcap\mathcal{A}) = X$ and $x \in T_0'$ and $y \in \bigcap\mathcal{A}$. This proves that X is compact tree-like.

(ii) $\Rightarrow$ (iii). Let X be a space possessing a binary normal connected subbase $\mathcal{T}$ such that for all $T_0,T_1 \in \mathcal{T}$ we have that either $T_0 \subset T_1$ or $T_1 \subset T_0$ or $T_0 \cap T_1 = \emptyset$ or $T_0 \cup T_1 = X$. We may suppose that $\emptyset \notin \mathcal{T}$ and $X \notin \mathcal{T}$. Then the non-intersection graph $G(\mathcal{T})$ is normal by lemma 1.3.19. Also $G(\mathcal{T})$ is weakly comparable since $\mathcal{T}$ is weakly comparable, as is easy to show. $G(\mathcal{T})$ is contiguous since $\mathcal{T}$ is connected (lemma 1.3.19). So we only need to prove that $G(\mathcal{T})$ is connected. Let $T_0,T_1 \in \mathcal{T}$, then either

(a) $T_0 \cap T_1 = \emptyset$; hence there is an edge in $G(\mathcal{T})$ between T_0 and T_1; or,

(b) $T_0 \cup T_1 = X$; hence there are T_0' and T_1' in $\mathcal{T}$ such that $T_0 \cap T_0' = T_0' \cap T_1' = T_1' \cap T_1 = \emptyset$, forming a path in $G(\mathcal{T})$ connecting T_0 and T_1; or,

(c) $T_0 \subset T_1$; hence there is a $T_2 \in \mathcal{T}$ such that $T_0 \cap T_2 = \emptyset = T_2 \cap T_1$, giving a path connecting T_0 and T_1; or,

(d) $T_1 \subset T_0$; this case is similar to case (c).

(iii) ⇒ (ii). Let X be the graph space of a connected normal contiguous weakly comparable graph $G = (V,E)$. We will prove that the subbase $\mathcal{B}(G)$ for the graph space satisfies the conditions of (ii). $\mathcal{B}(G)$ clearly is binary, normal and connected. Suppose now that $B_v, B_w \in \mathcal{B}(G)$ (cf. 1.3.5), with $v,w \in V$. Let $\{v,v_1\},\ldots,\{v_{k-1},w\} \in E$ be a path connecting v and w with minimal number k of edges. We will prove that always $B_v \cap B_w = \emptyset$ or $B_v \cup B_w = X$ or $B_v \subset B_w$ or $B_w \subset B_v$. The proof will be by induction to k. If $k = 1$ then $\{v,w\} \in E$ and hence $B_v \cap B_w = \emptyset$. Suppose that $k > 1$. There is a path of (minimal) length k-1 between v_1 and w, hence by induction hypothesis either

(a) $B_{v_1} \cap B_w = \emptyset$; i.e. $\{v,v_1\},\{v_1,w\} \in E$. It now follows that $\{v,w\} \notin E$ (otherwise $k = 1$) and therefore $B_v \subset B_w$ or $B_w \subset B_v$, for if not, there would be an edge $\{v,v'\} \in E$ and an edge $\{w,w'\} \in E$ such that $\{v,w'\} \notin E$ and $\{w,v'\} \notin E$, contradicting the weak comparability of G; or,

(b) $B_{v_1} \cup B_w = X$; since $B_v \cap B_{v_1} = \emptyset$ it follows that $B_v \subset B_w$; or,

(c) $B_{v_1} \subset B_w$; now $B_v \cap B_{v_{k-1}} = \emptyset$ and hence as in case (a) $B_v \subset B_w$ or $B_w \subset B_v$;, or

(d) $B_w \subset B_{v_1}$; then $B_v \cap B_w = \emptyset$, which implies that $k = 1$ (contradiction).

Therefore always $B_v \cap B_w = \emptyset$ or $B_v \cup B_w = X$ or $B_v \subset B_w$ or $B_w \subset B_v$. □

1.3.22. COROLLARY. *Each compact tree-like space is supercompact.* □

1.3.23. COROLLARY. *Let X be a topological space. Then the following properties are equivalent:*

(i) *X is a product of compact tree-like spaces;*

(ii) *X possesses a binary normal connected weakly comparable (closed) subbase;*

(iii) *X is homeomorphic to the graph space of a normal contiguous weakly comparable graph.*

PROOF. Notice that each graph is the sum of its components. Then apply theorem 1.3.7 and theorem 1.3.21. □

1.3.24. An interesting application of corollary 1.3.23 is the following. In [55], DE GROOT obtained a topological characterization of the n-cell I^n

and of the Hilbert cube I^∞ by means of binary subbases of a special kind (cf. theorem 1.3.31). ANDERSON [2] has proved that *the product of a countably infinite number of dendra is homeomorphic to the Hilbert cube*, where a *dendron* is defined to be a uniquely arcwise connected Peano continuum. It is well known, however, that a dendron is simply a compact metric tree-like space (cf. WHYBBURN [128]). Since the dimension of a dendron is 1, using our characterization of products of compact tree-like spaces, we are able to give a new characterization of the Hilbert cube, thus generalizing the result of DE GROOT mentioned above for the case of the Hilbert cube.

1.3.25. THEOREM. *A topological space* X *is homeomorphic to the Hilbert cube* I^∞ *if and only if* X *has the following properties:*

(i) X *is infinite dimensional;*

(ii) X *possesses a countable binary connected normal weakly comparable subbase.*

PROOF. The necessity follows from corollary 1.3.23, since the Hilbert cube is a product of compact tree-like spaces. The sufficiency follows from the fact that by corollary 1.3.23 X is homeomorphic to a countable product of dendra. As X is infinite dimensional this must be a countably infinite product. Hence X is homeomorphic to the Hilbert cube by the result of ANDERSON [2]. □

1.3.26. Now we will treat the relations between ordered spaces and comparable subbases and graphs. Note that an ordered space is the interval space of a totally ordered set. Hence clearly every ordered space is a lattice space while moreover a connected ordered space is tree-like.

Let X be a set and let $\mathcal{S}$ be a collection of subsets of X. The collection $\mathcal{S}$ is called *comparable* (cf. DE GROOT [55]) if for all $S_0, S_1, S_2 \in \mathcal{S}$ with $S_0 \cap S_1 = \emptyset = S_0 \cap S_2$ it follows that either $S_1 \subset S_2$ or $S_2 \subset S_1$. A graph (V,E) is called *comparable* if for each path $\{v_0,v_1\},\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\}$ of edges it follows that either $\{v_0,v_3\} \in E$ or $\{v_1,v_4\} \in E$ (cf. figure 4).

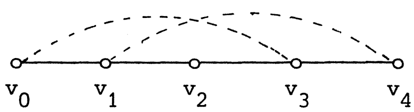


Figure 4.

1.3.27. LEMMA.

(i) *A graph* G *is comparable iff* G *is weakly comparable and bipartite.*

(ii) *Each comparable graph is normal.*

(iii) *A collection* S *of subsets of a set* X *is comparable iff it is weakly comparable and bipartite.*

(iv) *A comparable collection* S *of subsets of a set* X *is normal if it satisfies the following condition: for each* $x \in X$ *and each* $S \in S$ *with* $x \notin S$ *there exists an* $S_0 \in S$ *with* $x \in S_0$ *and* $S_0 \cap S = \emptyset$.

PROOF. The simple proof is left to the reader. □

1.3.28. THEOREM. *Let* X *be a topological space. Then the following properties are equivalent:*

(i) X *is compact orderable;*

(ii) X *possesses a binary graph-connected comparable subbase;*

(iii) X *is homeomorphic to the graph space of a connected comparable graph.*

PROOF. (i) ⇒ (ii). Let $(X,\leq)$ be an order-complete totally ordered set, with universal bounds 0 and 1. Clearly the subbase $S = \{[0,x] \mid x \in X, 0 \leq x < 1\} \cup \{[x,1] \mid x \in X, 0 < x \leq 1\}$ is binary, graph-connected and comparable.

(ii) ⇒ (i). Let X be a space with a binary graph-connected comparable subbase S. Since S is bipartite (lemma 1.3.27), S induces a lattice ordering "$\leq$" on X, such as in the proof of theorem 1.3.15 (ii) ⇒ (i). We only have to prove that this order is a total order. Suppose "$\leq$" is not total, that is suppose that for some $x,y \in X$ we have $x \not\leq y$ and $y \not\leq x$. Then there are $S,T \in S_1$ (see theorem 1.3.14) such that

$$x \in S, \quad y \notin S, \quad y \in T \text{ and } x \notin T.$$

Since S is graph-connected and bipartite there are $S_1,\ldots,S_k$ in S such that

$$S \cap S_1 = S_1 \cap S_2 = \ldots = S_{k-1} \cap S_k = S_k \cap T = \emptyset$$

with k odd (cf. 1.3.13 and 1.3.17). Suppose that k is the smallest number for which such a path in $G(S)$ exists. If $k \geq 3$ then $S_1 \cap S_2 = \emptyset = S_2 \cap S_3$ and hence $S_1 \subset S_3$ or $S_3 \subset S_1$. If $S_1 \subset S_3$ then

$$S \cap S_1 = S_1 \cap S_4 = S_4 \cap S_5 = \ldots = S_k \cap T = \emptyset,$$

which gives a shorter path from S to T.

The case $S_3 \subset S_1$ can be treated similarly.

Hence $k = 1$ and consequently $S \cap S_1 = \emptyset = S_1 \cap T$. Since $\mathcal{S}$ is comparable it now follows that $S \subset T$ or $T \subset S$. This means that either $x \in T$ or $y \in S$, both of which are contradictory.

(ii) ⇒ (iii). Let X be a space with a binary graph-connected comparable subbase $\mathcal{S}$. Then X is homeomorphic to the graph space of the graph $G(\mathcal{S})$, while moreover it is easy to see that $G(\mathcal{S})$ is connected and comparable.

(iii) ⇒ (ii). Let X be the graph space of a connected comparable graph $G = (V,E)$. The subbase $\mathcal{B}(G)$ is graph-connected since G is connected. Also $\mathcal{B}(G)$ is comparable, for suppose that $B_{v_1}, B_{v_2}, B_{v_3} \in \mathcal{I}(G)$ such that

$$B_{v_1} \cap B_{v_2} = \emptyset = B_{v_2} \cap B_{v_3}$$

and nevertheless $B_{v_1} \not\subset B_{v_3}$ and $B_{v_3} \not\subset B_{v_1}$.

Then $\{v_1,v_2\} \in E$ and $\{v_2,v_3\} \in E$; moreover there are V' and V'' in $\mathcal{I}(G)$ such that $V' \in B_{v_1} \backslash B_{v_3}$ and $V'' \in B_{v_3} \backslash B_{v_1}$.

As $v_3 \notin V'$ there is a $v_4 \in V'$ such that $\{v_3,v_4\} \in E$. As $v_1 \notin V''$ there is a $v_0 \in V''$ such that $\{v_0,v_1\} \in E$. Now

$$\{v_0,v_1\},\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\} \in E$$

and also $\{v_0,v_3\} \notin E$ (because $v_0,v_3 \in V''$) and $\{v_1,v_4\} \notin E$ (because $v_1,v_4 \in V'$). This contradicts the comparability of the graph G.

Hence the graph space $T(G)$ of G has a binary comparable graph connected subbase.

This completes the proof of the theorem. □

1.3.29. <u>COROLLARY</u>. (DE GROOT & SCHNARE [60].) *Let* X *be a topological space. Then the following statements are equivalent:*

(i) X *is a product of compact orderable spaces;*

(ii) X *possesses a binary comparable subbase;*

(iii) X *is homeomorphic to the graph space of a comparable graph.*

<u>PROOF</u>. Apply theorem 1.3.28 and theorem 1.3.7. □

1.3.30. <u>COROLLARY</u>. *Let* X *be a topological space. Then the following statements are equivalent:*

(i) X *is connected compact orderable;*

(ii) X *possesses a connected graph-connected comparable subbase;*

(iii) X *is homeomorphic to the graph space of a connected contiguous comparable graph.*

PROOF. Apply theorem 1.3.28 and theorem 1.3.21. □

1.3.31. COROLLARY. *Let* X *be a topological space. Then the following statements are equivalent:*

(i) X *is a product of connected compact orderable spaces;*

(ii) X *possesses a connected comparable subbase;*

(iii) X *is homeomorphic to the graph space of a contiguous comparable graph.*

PROOF. Combine corollary 1.3.30 and theorem 1.3.7. □

Adding countability conditions on the subbases and graphs one easily obtains characterizations of (products of) (connected) compact subsets of the real line (cf. DE GROOT [56], BRUIJNING [26]).

1.4. Regular supercompact spaces

STEINER [114] defined a compact space to be *regular Wallman* if it possesses a closed subbase $\mathcal{T}$ such that $\wedge.\vee.\mathcal{T}$ is a ring consisting of regular closed sets, i.e. each element of $\wedge.\vee.\mathcal{T}$ is the closure of its own interior. Regular Wallman spaces are Wallman compactification of each dense subspace (this will be discussed in chapter four) and many interesting classes of compact topological spaces turn out to be regular Wallman, for example the class of all compact metric spaces (AARTS [1], STEINER & STEINER [109]). Not all compact Hausdorff spaces are regular Wallman; SOLOMON [107] recently has given an example of a compact Hausdorff space that is not so.

It seems natural to define a topological space X to be *regular supercompact* provided that it possesses a (closed) *binary subbase* $\mathcal{T}$ such that $\wedge.\vee.\mathcal{T}$ is a ring consisting of regular closed sets. Obviously a regular supercompact space is (super)compact and regular Wallman. The space $\beta\mathbb{N}$ is a good example of a regular Wallman space (totally disconnected!) that is not regular supercompact. We do not have an example of a supercompact Hausdorff space that is not regular supercompact, or even of a supercompact

Hausdorff space that is not regular Wallman.

Regular supercompact spaces behave similar to regular Wallman spaces; for example products of regular supercompact spaces are again regular supercompact, closed subspaces of regular supercompact spaces need not be regular supercompact. But regular supercompact spaces have an additional property, they are not only a Wallman compactification of each dense subspace but they are also a *superextension of each dense subspace* (this will be proved in section 4.5).

Many interesting classes of regular Wallman spaces are regular supercompact. VAN DOUWEN [42] recently has shown that *every compact metric space is regular supercompact*. As a consequence of our results every compact orderable space is regular supercompact, every compact tree-like space of small weight is regular supercompact, and the superextension of a Lindelöf semi-stratifiable space is regular supercompact (section 4.5).

1.4.1. A topological space X is called *regular supercompact* provided that it possesses a binary subbase $\mathcal{T}$ such that $\wedge.\vee.\mathcal{T}$ is a ring consisting of regular closed sets.

The proof of theorem 1.4.2 will be postponed till section 4.5. For a precise definition and a discussion of superextensions, see chapter II.

1.4.2. <u>THEOREM</u>. *A regular supercompact space is a superextension of each dense subspace.*

This theorem is of interest since intuitively superextensions are "big"; however theorem 1.4.2 tells us that superextensions can be compactifications as well.

1.4.3. <u>THEOREM</u>. *The topological product of regular supercompact spaces is regular supercompact.*

<u>PROOF</u>. Let $X = \Pi_{\alpha\in I} X_\alpha$ be a product of regular supercompact spaces and let $\mathcal{T}_\alpha$ be a binary subbase for X_α such that $\wedge.\vee.\mathcal{T}_\alpha$ is a ring consisting of regular closed sets ($\alpha \in I$). A straightforward check shows that

$$\mathcal{T} := \{\pi_\alpha^{-1}[T] \mid T \in \mathcal{T}_\alpha \ (\alpha \in I)\}$$

is a binary subbase for X such that $\wedge.\vee.\mathcal{T}$ is a ring consisting of regular closed sets. □

We now give some classes of topological spaces that are regular supercompact.

1.4.4. THEOREM. *Each compact metric space is regular supercompact.*

PROOF. See VAN DOUWEN [42]. □

1.4.5. THEOREM. *A compact orderable space is regular supercompact.*

PROOF. Let X be a compact ordered space and let A denote the collection of isolated points of X. Then $X\backslash cl_X(A)$ is a locally compact topological space without isolated points and therefore has disjoint dense subspaces (cf. HEWITT [64], theorem 47). So X has dense subspaces D and E, such that $A = D \cap E$ and all points isolated from the left belong to D and all points isolated from the right belong to E. Let a be the smallest element of X and let b be the largest element of X. Then

$$\mathcal{T} := \{[a,d] \mid d \in D\} \cup \{[e,b] \mid e \in E\}$$

is a binary closed subbase such that $\wedge.\vee.\mathcal{T}$ is a ring consisting of regular closed sets. □

1.4.6. REMARK. HAMBURGER [62] has shown that a compact orderable space is regular Wallman. This theorem was generalized by MISRA [85] who showed that the Čech-Stone compactification of a locally compact ordered space is regular Wallman. MISRA's theorem cannot be generalized for regular supercompactness since $\beta\mathbb{N}$, the Čech-Stone compactification of the natural numbers, is not supercompact (cf. BELL [14] and corollary 1.1.7). Hence $\beta\mathbb{N}$ is an example of a regular Wallman space that is not (regular) supercompact.

1.4.7. In section 1.3 we showed that every compact tree-like space is supercompact (theorem 1.3.21). This result suggests the question whether every compact tree-like space is regular supercompact. Simple examples show that the structure of compact tree-like spaces is much more complicated than the structure of ordered compacta. Therefore the simple proof of theorem 1.4.5 cannot be generalized. However it is possible that a modification of the technique "works", since each compact tree-like space is the continuous image of an ordered compactum, by a result of CORNETTE [32]. We give a partial answer to the general question by showing that each compact tree-like space of weight at most $\mathfrak{c}$ is regular supercompact.

1.4.8. THEOREM. *A compact tree-like space of weight at most $\mathfrak{c}$ is regular supercompact.*

PROOF. Let X be a compact tree-like space. Recall that the collection of complements of segments forms a closed subbase for X (theorem 1.3.21).

Let $\mathcal{T}$ be a collection of complements of components which is a subbase and which in addition is of cardinality at most $\mathfrak{c}$. Define

$$\mathcal{A} := \{(S,T) \mid S,T \in \mathcal{T} \text{ and } S \cap T = \emptyset\}.$$

List $\mathcal{A}$ as $\{A_\alpha \mid \alpha \in \mathfrak{c}\}$. By transfinite induction choose for each $\alpha \in \mathfrak{c}$ a point $p_\alpha \in X$ such that

(i) if $A_\alpha = (S,T)$ then p_α separates S from T;

(ii) $p_\alpha \notin \{p_\beta \mid \beta < \alpha\}$.

To define p_0, note that each element of $\mathcal{T}$ is connected and hence that if $A_0 = (S,T)$ then there exists a separation point $b \in X$ which separates S from T. Define $p_0 := b$.

Suppose that all p_β have been constructed for $\beta < \alpha$. Notice that

$$|\{p_\beta \mid \beta < \alpha\}| < \mathfrak{c}.$$

Let $A_\alpha = (S,T)$ and take $c \in S$ and $d \in T$. Define

$$Z = \{x \in X \mid x \text{ separates } c \text{ from } d\}.$$

It is well-known, cf. PROIZVOLOV [92], KOK [70], that Z is a connected orderable subspace of X (Z is ordered by the usual cut point order). The connectedness of Z implies that $U = Z\setminus(S\cup T)$ is a nonvoid open subset of Z, hence contains a nonvoid open order interval and consequently is of cardinality at least $\mathfrak{c}$. Also each $x \in U$ separates S from T. As $|\{p_\beta \mid \beta < \alpha\}| < \mathfrak{c}$ there is an $e \in U$ such that $e \notin \{p_\beta \mid \beta < \alpha\}$. Define $p_\alpha := e$. This completes the inductive construction.

Now, if $A_\alpha = (S,T)$ let U_α be the component of $X\setminus\{p_\alpha\}$ that contains T. Define $V_\alpha := X\setminus U_\alpha$ $(\alpha \in \mathfrak{c})$. Then $V_\alpha \cap T = \emptyset$ and $\partial V_\alpha = \{p_\alpha\}$ $(\alpha \in \mathfrak{c})$. Clearly $\mathcal{V} := \{V_\alpha \mid \alpha \in \mathfrak{c}\}$ is a closed subbase for X. This subbase also is binary since it is a subcollection of the collection of complements of segments which is binary (theorem 1.3.21). Finally $\wedge.\vee.\mathcal{V}$ is a ring consisting of regular closed sets. For take $\alpha_0 < \alpha_1 < \ldots < \alpha_n$ ($\alpha_i \in C$, $i \leq n$). Then $V_{\alpha_0} \cap \ldots \cap V_{\alpha_n}$ is regular closed since $\partial V_{\alpha_i} \cap \partial V_{\alpha_j} = \emptyset$ for all $\alpha_i \neq \alpha_j$ and each V_{α_i} is regular closed. Each finite union of regular closed sets is regular closed and hence $\wedge.\vee.\mathcal{V}$ is a ring consisting of regular closed sets. □

Theorem 1.4.8 suggests the following question:

1.4.9. QUESTION. *Is every compact tree-like space regular supercompact?*

1.4.10. We will now describe how to construct regular supercompact compactifications of discrete spaces.

STEINER & STEINER [110] have shown the following theorem: *Let* X *be an infinite discrete space and let* K *be a compact space with a dense subset of cardinality less than or equal to that of* X. *Then* X *has a (Hausdorff) compactification* αX *with* K *as remainder, i.e.* $\alpha X \backslash X$ *is homeomorphic to* K.

The construction of this compactification is very simple. Express X as the union of disjoint subsets X_i ($i \in \omega$) each of cardinality $|X|$. Let D be a dense subset of K with cardinality less than or equal to $|X|$. Construct a function f of X into K which maps each X_i onto D. Let X^* be the Alexandroff one point compactification of X. The closure of the graph of f in $X^* \times K$ is a compactification αX of X with K as remainder. The restriction of the projection onto the second coordinate of the product $X^* \times K$ to αX clearly is a retraction of αX onto K.

1.4.11. THEOREM. *Let* X *be an infinite discrete space and let* K *be a Hausdorff regular supercompact space with a dense subset of cardinality less than or equal to that of* X. *Then* X *has a Hausdorff compactification* αX *with the following properties:*

(i) $K = \alpha X \backslash X$;

(ii) αX *is regular supercompact.*

PROOF. Let αX be the "graph-closure" compactification of STEINER & STEINER, described above, and let $r\colon \alpha X \to K$ be a retraction. Let $\mathcal{T}$ be a binary subbase for K such that $\wedge.\vee.\mathcal{T}$ is a ring consisting of regular closed sets. Clearly

$$\mathcal{S} := \{\{x\} \mid x \in X\} \cup \{\alpha X \backslash \{x\} \mid x \in X\} \cup \{r^{-1}[T] \mid T \in \mathcal{T}\}$$

is a closed subbase for the topology on αX.

CLAIM 1. *S is binary.*

Indeed, let $M \subset S$ be a linked system with an empty intersection. By the compactness of αX we may assume that M is finite. It is clear that M does not contain a singleton. Hence we may write

$$M = \{r^{-1}[T_0],\ldots,r^{-1}[T_n],\alpha X\setminus\{x_0\},\ldots,\alpha X\setminus\{x_m\}\}.$$

Since T is binary we have that $\bigcap_{i\le n} T_i \neq \emptyset$ and consequently

$$\emptyset \neq \bigcap_{i\le n} T_i \subset \bigcap_{i\le n} r^{-1}[T_i] \cap \bigcap_{j\le m} \alpha X\setminus\{x_j\},$$

since r is a retraction. This is a contradiction.

CLAIM 2. *$\wedge.\vee.S$ consists of regular closed sets.*

Since $\wedge.\vee.T$ is a ring consisting of regular closed sets it suffices to show that

$$r^{-1}[T] \cap (\alpha X\setminus\{x_0,\ldots,x_n\})$$

is regular closed in αX for all $T \in T$ and $x_i \in X$ ($i \le n$, $n \in \omega$). But this is a triviality since it is easy to see that

$$(r^{-1}[\mathrm{int}_K(T)] \cup (r^{-1}[T] \cap X)) \setminus \{x_0,\ldots,x_n\}$$

is a dense open set in $r^{-1}[T] \cap (\alpha X\setminus\{x_0,\ldots,x_n\})$ for all $T \in T$ and $x_i \in X$ ($i \le n$, $n \in \omega$). □

This theorem implies that there are many Hausdorff compactifications of $\mathbb{N}$ that are regular supercompact. Also it is easy to construct *nonmetrizable* regular supercompact Hausdorff compactifications of $\mathbb{N}$. For example, let K be a separable nonmetrizable compact orderable space. Then theorem 1.4.5 and theorem 1.4.11 imply that there is a Hausdorff compactification $\alpha\mathbb{N}$ of $\mathbb{N}$ with K as remainder and which is regular supercompact.

We finish this section with an open question:

1.4.12. QUESTION. *Is there a supercompact Hausdorff space that is not regular supercompact, or, more generally, is there a supercompact Hausdorff space that is not regular Wallman?*

1.5. Partial orderings on supercompact spaces

Supercompact spaces which possess a binary subbase which also is *normal* (cf. 1.3.16) behave surprisingly nice. In some sense these spaces have much in common with (products of) compact tree-like spaces (section 1.3). It is well-known that a compact tree-like space

(a) *can be partially ordered in a natural way* (cf. WARD [123]);

(b) *is locally connected* (cf. PROIZVOLOV [92]);

(c) *is (generalized) arcwise connected* (cf. PROIZVOLOV [92]);

(d) *has the fixed point property for continuous functions* (cf. WALLACE [120]).

We will show that a space with a binary normal subbase satisfies (a), (b) and (c) if it is connected. Property (b) for these spaces is originally due to VERBEEK [119] and property (d) was proved recently by VAN DE VEL [118]. Basic tools in the proofs will be partial orderings and nearest point mappings defined in 1.5.2. These mappings are fundamental and will from now on be applied everywhere in this treatise.

Finally we show that a space with a binary normal subbase is a retract of the hyperspace of its nonvoid closed subsets. As a corollary it follows, using a result of WOJDYSLAWSKI [130], that if in addition such a space is connected and metrizable it is an Absolute Retract.

1.5.1. Let X be a topological space and let $\mathcal{S}$ be a binary normal (cf. 1.3.16) subbase for X. Notice that the normality of $\mathcal{S}$ implies that X is Hausdorff since $\mathcal{S}$ is a T_1-subbase (lemma 0.4) and that each supercompact Hausdorff space possesses a binary weakly normal subbase (theorem 1.3.9). Without loss of generality we assume that $X \in \mathcal{S}$.

For each subset $A \subset X$ let $I_{\mathcal{S}}(A)$ be defined by

$$I_{\mathcal{S}}(A) := \bigcap\{S \in \mathcal{S} \mid A \subset S\}.$$

Notice that $cl_X(A) \subset I_{\mathcal{S}}(A)$, since $\mathcal{S}$ is a closed subbase, that $I_{\mathcal{S}}(I_{\mathcal{S}}(A)) = I_{\mathcal{S}}(A)$ and that $I_{\mathcal{S}}(A) \subset I_{\mathcal{S}}(B)$ if $A \subset B$, for all $A,B \subset X$. If A is a two point set, say $A = \{x,y\}$, then we usually write $I_{\mathcal{S}}(x,y)$ in stead of $I_{\mathcal{S}}(\{x,y\})$. The set $I_{\mathcal{S}}(x,y)$ is interpreted as a "segment" joining x and y. The function $I: X\times X \to \mathcal{P}(X)$ defined by $I((x,y)) := I_{\mathcal{S}}(x,y)$ is an interval structure (cf. 1.3.2 and 1.3.3).

A *partially ordered topological space* (in the sense of WARD [122]) is a topological space Y endowed with a partial order, $\leq$, which is continuous in the sense that the graph of $\leq$ is closed in $Y \times Y$. A partial order "$\leq$" is called *order dense* if $x < y$ implies that there is a $z \in Y$ such that $x < z < y$. A *chain* in a partial ordered set is a subset which is linear with respect to the partial order. A point is called *minimal (maximal)* if it has no proper predecessor (successor).

For a given point $p \in X$ define a binary relation $\leq_p$ on X by

$$x \leq_p y \quad \text{iff} \quad I_S(p,x) \subset I_S(p,y).$$

In theorem 1.5.13 we will show that $\leq_p$ is a continuous partial ordering for X. The notation $x \leq_p y$ is not such a good notation, since the ordering $\leq_p$ also depends on the choice of the subbase S, and a topological space can have many totally distinct binary normal subbases. For notational simplicity we suppress the subindex S in the ordering; from the context the meaning of $x \leq_p y$ will always be clear.

1.5.2. THEOREM. *Let X be a topological space and let S be a binary normal subbase for X. Let $A \subset X$.*

(i) *For every $x \in X$ the set*

$$\bigcap_{a \in A} I_S(x,a) \cap I_S(A)$$

is a singleton.

We denote the unique point of this intersection by $r(x)$.

(ii) *$r: X \to I_S(A)$ is a retraction.*

(iii) *For all $x \in X$, the point $r(x)$ is the greatest lower bound with respect to $\leq_x$ of A.*

PROOF. (i). Define B(A) by $B(A) := \bigcap_{a \in A} I_S(x,A) \cap I_S(A)$. Notice that the binarity of S implies that B(A) is nonvoid. Assume that p and q are two distinct elements of B(A). By normality of S there are $S_0, S_1 \in S$ such that $p \in S_0 \setminus S_1$, $q \in S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. If $A \cap S_0 = \emptyset$, then $A \subset S_1$ and consequently

$$B(A) \subset I_S(A) \subset S_1,$$

which is impossible. Therefore $A \cap S_0 \neq \emptyset$. In the same way also $A \cap S_1 \neq \emptyset$. Now, as $\{S_0, S_1\}$ is a covering of X there is an $i \in \{0,1\}$ such that $x \in S_i$; say $x \in S_0$. Take $a_0 \in A \cap S_0$. Then

$$q \in B(A) \subset I_{\mathcal{S}}(x,a_0) \subset S_0 ,$$

which is a contradiction.

(ii). To prove the continuity of r, let $S \in \mathcal{S}$ and take $x \notin r^{-1}[S]$. Then $r(x) \notin S$ and as $\{r(x)\} = \bigcap_{a\in A} I_{\mathcal{S}}(x,a) \cap I_{\mathcal{S}}(A)$ we conclude, by the binarity of $\mathcal{S}$, that either $I_{\mathcal{S}}(A) \cap S = \emptyset$ or $I_{\mathcal{S}}(x,a_0) \cap S = \emptyset$ for some $a_0 \in A$. In the first case $r^{-1}[S] = \emptyset$, hence is closed. In the second case, choose S_0 and S_1 in $\mathcal{S}$ such that $I_{\mathcal{S}}(x,a_0) \subset S_0 \cap (X\backslash S_1)$ and $S \subset S_1 \cap (X\backslash S_0)$ and $S_0 \cup S_1 = X$. Then $U = X\backslash S_1$ is a neighborhood of x which misses $r^{-1}[S]$. Hence once more $r^{-1}[S]$ is closed; consequently r is continuous. Clearly r is a retraction.

(iii). First of all, let us check that $r(x)$ is a lower bound for A. Take $a \in A$; then $r(x) \in I_{\mathcal{S}}(x,a)$, by construction, and consequently $I_{\mathcal{S}}(x,r(x)) \subset I_{\mathcal{S}}(x,a)$. Hence, by definition, $r(x) \le_x a$.

Now assume that $p \le_x a$ for all $a \in A$. Then $p \le_x r(x)$, for assume to the contrary that $p \not\le_x r(X)$. Then $p \notin I_{\mathcal{S}}(x,r(x))$ and by the normality of $\mathcal{S}$ there are $S_0,S_1 \in \mathcal{S}$ such that $p \in S_0\backslash S_1$, $I_{\mathcal{S}}(x,r(x)) \subset S_1\backslash S_0$ and $S_0 \cup S_1 = X$. The set A is not contained in S_1, for otherwise $p \notin I_{\mathcal{S}}(x,a)$ for all $a \in A$. Hence A intersects S_0 and, consequently, so does $I_{\mathcal{S}}(A)$. Moreover $I_{\mathcal{S}}(x,a)$ intersects S_0 for all $a \in A$ since $p \in I_{\mathcal{S}}(x,a) \cap S_0$. Therefore the system

$$\{S_0\} \cup \{S \in \mathcal{S} \mid A \subset S\} \cup \{S \in \mathcal{S} \mid \exists a \in A : I_{\mathcal{S}}(x,a) \subset S\}$$

is linked. By the binarity of $\mathcal{S}$ it has a nonvoid intersection; consequently

$$\emptyset \ne S_0 \cap I_{\mathcal{S}}(A) \cap \bigcap_{a\in A} I_{\mathcal{S}}(x,a) = S_0 \cap \{r(x)\},$$

which is a contradiction, since $r(x) \notin S_0$. □

1.5.3. <u>COROLLARY</u>. *For all* $x,y,z \in X$ *the set* $I_{\mathcal{S}}(x,y) \cap I_{\mathcal{S}}(y,z) \cap I_{\mathcal{S}}(x,z)$ *is a singleton.* □

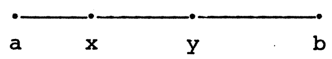
The greatest lowerbound of $A \subset X$ with respect to the binary relation $\le_x$ is denoted by $glb_x(A)$.

1.5.4. <u>COROLLARY</u>. *For all* $A \subset X$ *and* $x \in X$ *we have that* $glb_x(A) = glb_x(I_{\mathcal{S}}(A))$.

<u>PROOF</u>. $\{glb_x(A)\} = \bigcap_{a\in A} I_{\mathcal{S}}(x,a) \cap I_{\mathcal{S}}(A) \supset \bigcap_{a\in I_{\mathcal{S}}(A)} I_{\mathcal{S}}(x,a) \cap I_{\mathcal{S}}(A) = \bigcap_{a\in I_{\mathcal{S}}(A)} I_{\mathcal{S}}(x,a) \cap I_{\mathcal{S}}(I_{\mathcal{S}}(A)) = \{glb_x(I_{\mathcal{S}}(A))\}$. □

The following proposition indicates why we think of $I_S(x,y)$ as a segment joining x and y. It will be used in theorem 1.5.13 and theorem 1.5.14.

1.5.5. <u>PROPOSITION</u>. *If* $y \in I_S(a,b)$ *and* $x \in I_S(a,y)$ *then* $y \in I_S(x,b)$.



<u>PROOF</u>. Assume that $y \notin I_S(x,b)$. By the normality of S there are $S_0, S_1 \in S$ such that $y \in S_0 \setminus S_1$ and $I_S(x,b) \subset S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. Now if $a \in S_1$ then $I_S(a,b) \subset S_1$ and consequently $y \in S_1$ which is impossible. Therefore $a \in S_0$; but since $y \in S_0$ it follows that $x \in S_0$ since $x \in I_S(a,y) \subset S_0$. This is a contradiction. $\square$

1.5.6. <u>DEFINITION</u>. A subset $A \subset X$ is called *S-closed* if $A = I_S(A)$.

Recall that a subset $A \subset X$ is called *S-convex* if for all $x,y \in A$ we have that $I_S(x,y) \subset A$ (cf. definition 1.3.2). Clearly each S-closed set $A \subset X$ also is S-convex. Simple examples show that the converse need not be true. For example, an S-convex set need not even be a closed set. The two concepts coincide on the collection of closed subsets of X, as the following theorem shows.

1.5.7. <u>THEOREM</u>. *Let* X *be a topological space which possesses a binary normal subbase* S. *For a closed set* A *in* X *the following assertions are equivalent:*

(i) A *is S-closed;*

(ii) A *is S-convex.*

<u>PROOF</u>. We only need to check (ii) $\Rightarrow$ (i). Indeed, assume there is a closed set B in X which is S-convex and not S-closed. Choose $x \in I_S(B) \setminus B$. By theorem 1.5.2 (i) we have that $\{x\} = \bigcap_{b \in B} I_S(x,b) \cap I_S(B) \subset \bigcap_{b \in B} I_S(x,b)$. We claim that $\{x\} = \bigcap_{b \in B} I_S(x,b)$. Indeed, assume there is a $z \in \bigcap_{b \in B} I_S(x,b) \setminus \{x\}$. Then $z \leq_x b$ for all $b \in B$ and consequently $z \leq_x \mathrm{glb}_x(B) = \mathrm{glb}_x(I_S(B)) = \{x\}$, by theorem 1.5.2 (i), (ii) and corollary 1.5.4. Therefore $z \in I_S(x,x) = \{x\}$ which is a contradiction.

Define $T := \{I_S(x,b) \cap B \mid b \in B\}$. Then clearly T consists of subsets of B, closed in B and hence in X. We will show that T is a linked system of

$\mathcal{S}$-convex sets. Choose $b_0, b_1 \in B$. Then, as $\mathcal{S}$ is binary $I_{\mathcal{S}}(b_0,b_1) \cap I_{\mathcal{S}}(b_1,x) \cap I_{\mathcal{S}}(x,b_0) \neq \emptyset$ (cf. 1.3.2 and 1.3.3) and as $I_{\mathcal{S}}(b_0,b_1) \subset B$, by assumption, $I_{\mathcal{S}}(x,b_0) \cap B$ and $I_{\mathcal{S}}(x,b_1) \cap B$ must intersect. As B is $\mathcal{S}$-convex, it is easily seen that $\mathcal{T}$ consists of $\mathcal{S}$-convex sets too.

As in the proof of theorem 1.3.3 it can be shown that $\bigcap\mathcal{T} \neq \emptyset$. However, this is a contradiction since $\bigcap\mathcal{T} = \bigcap_{b\in B} I_{\mathcal{S}}(x,b) \cap B = \{x\} \cap B = \emptyset$. □

The following result follows from theorem 1.5.2.

1.5.8. COROLLARY. *Let* X *be a topological space and let* $\mathcal{S}$ *be a binary normal subbase for* X. *Then*

(i) *Each* $\mathcal{S}$*-closed set is a retract of* X.

(ii) *If* X *is connected, then each* $\mathcal{S}$*-closed set is connected; in particular each interval* $I_{\mathcal{S}}(x,y)$ *is connected* $(x,y \in X)$.

(iii) (cf. VERBEEK [119]). *If* X *is connected then* X *is locally connected.*

PROOF. (i) and (ii) follow from theorem 1.5.2.

To prove (iii), take $x \in X$ and let U be any neighborhood of x. Choose finitely many $S_0,S_1,\ldots,S_n \in \mathcal{S}$ such that $x \notin \bigcup_{i\leq n} S_i \supset X\setminus U$. For each $i \leq n$ choose $S_i' \in \mathcal{S}$ such that $x \in \operatorname{int}_X(S_i') \subset S_i'$ and $S_i' \cap S_i = \emptyset$. This is possible since $\mathcal{S}$ is normal and T_1. Then $V := \bigcap_{i\leq n} S_i'$ is a closed neighborhood of x, contained in U. Moreover it is clear that V is $\mathcal{S}$-closed, and hence connected ((ii)). □

1.5.9. Let X be a topological space. A *mean* m is a continuous map $m\colon X\times X \to X$ such that $m(x,x) = x$ for all $x \in X$ and $m(x,y) = m(y,x)$ for all $x,y \in X$. We will construct a mean on every supercompact space with a binary normal subbase. First we need a simple lemma.

1.5.10. LEMMA. *If* $\mathcal{S}$ *is a binary normal closed subbase for* X, *then the mapping* $f\colon X\times X\times X \to X$ *defined by*

$$\{f(x,y,z)\} = I_{\mathcal{S}}(x,y) \cap I_{\mathcal{S}}(x,z) \cap I_{\mathcal{S}}(y,z)$$

is a continuous surjection.

PROOF. Clearly f is well defined (cf. corollary 1.5.3). To prove the continuity of f let $S \in \mathcal{S}$ and take $(x,y,z) \notin f^{-1}[S]$. Then

$$I_{\mathcal{S}}(x,y) \cap I_{\mathcal{S}}(x,z) \cap I_{\mathcal{S}}(y,z) \cap S = \emptyset,$$

and hence, by binarity of S, without loss of generality $I_S(x,y) \cap S = \emptyset$. Choose $S_0 \in S$ such that $I_S(x,y) \subset \mathrm{int}_X(S_0) \subset S_0$ and $S_0 \cap S = \emptyset$. Then the neighborhood

$$U = \pi_0^{-1}[\mathrm{int}_X(S_0)] \cap \pi_1^{-1}[\mathrm{int}_X(S_0)]$$

of $(x,y,z) \in X\times X\times X$ does not intersect $f^{-1}[S]$, as can easily been seen. Hence $f^{-1}[S]$ is closed in $X\times X\times X$ and consequently f is continuous. Also f is surjective, since for an arbitrary $x \in X$ we have that

$$\{f(x,x,x)\} = I_S(x,x) \cap I_S(x,x) \cap I_S(x,x) = \{x\},$$

which completes the proof of the lemma. □

1.5.11. PROPOSITION. *Any topological space which possesses a binary normal closed subbase has a mean.*

PROOF. Let S be a binary normal closed subbase for the topological space X. Let f be defined as in lemma 1.5.10. Fix a point $p \in X$ and define $m: X\times X \to X$ by $m := f{\restriction}\{p\}\times X\times X$. Then m is a continuous map of $X \times X$ onto X. Furthermore $\{m(x,x)\} = I_S(x,x) \cap I_S(x,p) \cap I_S(p,x) = \{x\}$ for all $x \in X$ and $\{m(x,y)\} = I_S(x,y) \cap I_S(x,p) \cap I_S(p,y) = I_S(x,y) \cap I_S(y,p) \cap I_S(x,p) = \{m(x,y)\}$ for all $x,y \in X$. Therefore m is a mean. □

1.5.12. Proposition 1.5.11 gives us many easy examples of spaces which are supercompact but which do not possess a binary normal subbase (recall that each supercompact Hausdorff space possesses a binary weakly normal subbase, cf. 1.3.9). For example the supercompact space

$$Y = \{(0,y) \mid -1 \le y \le 1\} \cup \{(x,\sin \tfrac{1}{x}) \mid 0 < x \le 1\}$$

possesses no binary normal subbase, since this space has no mean (cf. BACON [13]).

That Y is supercompact is not trivial; it follows of course from the theorem of STROK & SZYMAŃSKI [116] (see also VAN DOUWEN [42]), but the binary subbase obtained from their theorem cannot be described well. Therefore we will indicate another binary subbase for Y. For each $n \in \omega$ define

$$x_n := \frac{2}{(2n+1)\pi}.$$

Notice that $\sin(\frac{1}{x_n}) = 1$ if n is even and that $\sin(\frac{1}{x_n}) = -1$ if n is odd. Let $r\colon Y \to [-1,1]$ be the projection onto the second coordinate. It can be shown that

$$\{(r^{-1}[x,1])\backslash C \mid -1 \le x \le 1 \text{ and } C \text{ is a component of } r^{-1}[x,1]\} \cup$$
$$\cup \{(r^{-1}[-1,x])\backslash C \mid -1 \le x \le 1 \text{ and } C \text{ is a component of } r^{-1}[-1,x]\} \cup$$
$$\cup \{(x,\sin\tfrac{1}{x}) \mid x \in [x_n,p] \text{ with } x_n \le p \le x_{n-1},\ n \in \omega\} \cup$$
$$\cup \{(x,\sin\tfrac{1}{x}) \mid x \in [p,x_n] \text{ with } x_{n+1} \le p \le x_n,\ n \in \omega\}$$

is a binary closed subbase for Y. Moreover it is obvious that this subbase is not normal.

That Y possesses no binary normal subbase also follows from corollary 1.5.8 (iii) since Y is not locally connected. However, this argument cannot be used in the class of connected and locally connected spaces. Then proposition 1.5.11 applies. For example, the n-spheres S_n are supercompact, but do not have a mean (cf. AUMANN [7]) and consequently they cannot possess a binary normal subbase.

15.13. THEOREM. *Let* X *be a topological space and let* $\mathcal{S}$ *be a binary normal subbase for* X. *Let* $p \in X$. *Then*

(i) $\le_p$ *is a continuous partial ordering for* X;

(ii) $\{y \in X \mid y \le_p x\} = I_{\mathcal{S}}(p,x)$ *for all* $x \in X$;

(iii) $\{y \in X \mid x \le_p y\}$ *is* $\mathcal{S}$*-closed for all* $x \in X$;

(iv) $\{y \in X \mid x \le_p y \le_p z\} = I_{\mathcal{S}}(x,z)$ *for all* $x,z \in X$ *with* $x \le_p z$;

(v) *if* X *is connected, then* $\le_p$ *is order dense.*

PROOF. (i) From the definition $I_{\mathcal{S}}$ it is clear that $\le_p$ is reflexive. It is symmetric too, for take $x,y \in X$ with $x \le_p y$ and $y \le_p x$. Then, by definition $x \in I_{\mathcal{S}}(p,y)$ and $y \in I_{\mathcal{S}}(p,x)$. But corollary 1.5.3 shows that

$$I_{\mathcal{S}}(p,x) \cap I_{\mathcal{S}}(p,y) \cap I_{\mathcal{S}}(x,y) \supset \{x,y\}$$

is a singleton. Finally transitivity of $\le_p$ is obvious.

To prove that $\le_p$ is continuous, let $(x,y) \in X\times X$ such that $x \not\le_p y$ and $y \not\le_p x$. Then $\{z\} = I_{\mathcal{S}}(p,x) \cap I_{\mathcal{S}}(p,y) \cap I_{\mathcal{S}}(x,y)$ is not an element of $\{x,y\}$. Let U be any neighborhood of z such that $cl_X(U) \cap \{x,y\} = \emptyset$. By lemma 1.5.10 there are disjoint neighborhoods V_0 and V_1 of x and y such that

(a) $(V_0 \cup V_1) \cap cl_X(U) = \emptyset$;

(b) for all $a \in V_0$ and $b \in V_1$ we have that $I_{\mathcal{S}}(p,a) \cap I_{\mathcal{S}}(p,b) \cap I_{\mathcal{S}}(a,b) \subset U$.

Then $V_0 \times V_1$ is a neighborhood of $(x,y) \in X \times X$ which has an empty intersection with the graph of $\leq_p$.

(ii) The simple proof is left to the reader.

(iii) Clearly $\{y \in X \mid x \leq_p y\}$ is closed in X (cf. WARD [124]). We will show that $\{y \in X \mid x \leq_p y\}$ is $\mathcal{S}$-convex. Then, by theorem 1.5.7 the set $\{y \in X \mid x \leq_p y\}$ is $\mathcal{S}$-closed.

Take $a,b \in \{y \in X \mid x \leq_p y\}$ and take $c \in I_{\mathcal{S}}(a,b)$. Assume that $x \notin I_{\mathcal{S}}(p,c)$. Then take $S_0, S_1 \in \mathcal{S}$ such that $I_{\mathcal{S}}(p,c) \subset S_0 \setminus S_1$ and $x \in S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. If a and b are both contained in S_1 then so is $I_{\mathcal{S}}(a,b)$, contradicting $c \notin S_1$. Therefore either $a \in S_0$ or $b \in S_0$. Assume that $a \in S_0$. Then p and a are both contained in S_0; consequently $I_{\mathcal{S}}(p,a) \subset S_0$. This is a contradiction since $x \in I_{\mathcal{S}}(p,a)$.

(iv) Notice that

$$\begin{aligned}\{y \in X \mid x \leq_p y \leq_p z\} &= \{y \in X \mid x \leq_p y\} \cap \{y \in X \mid y \leq_p z\} \\ &= \{y \in X \mid x \leq_p y\} \cap I_{\mathcal{S}}(p,z),\end{aligned}$$

which is an intersection of two $\mathcal{S}$-closed sets (by (iii)) and hence is $\mathcal{S}$-closed itself. Therefore $I_{\mathcal{S}}(x,z) \subset \{y \in X \mid x \leq_p y\} \cap I_{\mathcal{S}}(p,z)$. Now take

$$q \in \{y \in X \mid x \leq_p y\} \cap I_{\mathcal{S}}(p,z).$$

Then $x \in I_{\mathcal{S}}(p,q)$ and $q \in I_{\mathcal{S}}(p,z)$, hence $q \in I_{\mathcal{S}}(x,z)$ by proposition 1.5.5.

(v) Take $x,y \in X$ and assume that $x <_p y$. Define

$$A := I_{\mathcal{S}}(p,x)$$

and

$$B := \{z \in X \mid y \leq_p z\}$$

respectively; note that B is $\mathcal{S}$-closed by (iii).

Then A and B are two disjoint $\mathcal{S}$-closed sets, since $\leq_p$ is a partial ordering. By normality of $\mathcal{S}$ there exist $S_0, S_1 \in \mathcal{S}$ such that $A \subset S_0 \setminus S_1$ and $B \subset S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. Choose a point z_0 in $S_0 \cap S_1$ (X is connected!); by 1.5.3 we can define q by

$$\{q\} := I_S(x,z_0) \cap I_S(y,z_0) \cap I_S(x,y).$$

Then $q \in S_0 \cap S_1 \cap I_S(x,y)$ and consequently $q \notin A \cup B$; hence $q \neq x$ and $q \neq y$. But as $q \in I_S(x,y)$ it follows from (iv) that $x \leq_p q \leq_p y$. Therefore $x <_p q <_p y$. □

1.5.14. THEOREM. *Let* X *be a topological space and let* S *be a binary normal subbase for* X. *Choose* $p,q \in X$. *Then the ordering* $\leq_p$ *induces a lattice ordering on* $I_S(p,q)$. *Moreover*

(i) $x \leq_p y$ *iff* $y \leq_q x$ *for all* $x,y \in I_S(p,q)$;

(ii) $\{y \in I_S(p,q) \mid x \leq_p y\} = I_S(x,q)$ *for all* $x \in I_S(p,q)$;

(iii) *the family* $\{I_S(p,x) \mid x \in I_S(p,q)\} \cup \{I_S(x,q) \mid x \in I_S(p,q)\}$ *is a closed subbase for* $I_S(x,y)$; *hence* $I_S(x,y)$ *is a compact lattice space* (cf. 1.3.12);

(iv) *if* X *is connected, then* $\leq_p$ *is order dense on* $I_S(p,q)$.

PROOF. (i) Since $x \leq_p y$ iff $x \in I_S(p,y)$ and $y \leq_q x$ iff $y \in I_S(q,x)$, this follows from proposition 1.5.5.

(ii) Since $y \in I_S(x,q)$ iff $y \leq_q x$, this is a restatement of (i).

(iii) Indeed, choose $x,y \in I_S(p,q)$ such that $x \neq y$. The system

$$\{I_S(p,x), I_S(x,q), I_S(p,y), I_S(y,q)\}$$

is a system of S-closed sets with an empty intersection, for $I_S(p,x) \cap I_S(x,q) = \{x\}$, by corollary 1.5.3 ($x \in I_S(p,q)$!) and similarly $I_S(p,y) \cap I_S(y,q) = \{y\}$. Therefore, by the binarity of S, either $I_S(p,x) \cap I_S(y,q) = \emptyset$ or $I_S(p,y) \cap I_S(x,q) = \emptyset$. Without loss of generality we may assume that $I_S(p,x) \cap I_S(y,q) = \emptyset$. Choose S_0 and S_1 in S such that $I_S(p,x) \subset S_0 \setminus S_1$ and $I_S(y,q) \subset S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. We will show that $S_0 \cap I_S(p,q) = I_S(p, \mathrm{glb}_q(S_0))$.

Recall that $\mathrm{glb}_q(S_0) = \bigcap_{s \in S_0} I_S(s,q) \cap I_S(S_0) = \bigcap_{s \in S_0} I_S(s,q) \cap S_0$. Therefore, as $p \in S_0$, $\mathrm{glb}_q(S_0) \in I_S(p,q)$; moreover as $\mathrm{glb}_q(S_0) \in S_0$ we conclude that $\{p, \mathrm{glb}_q(S_0)\} \subset S_0 \cap I_S(x,p)$ and consequently

$$I_S(p, \mathrm{glb}_q(S_0)) \subset S_0 \cap I_S(p,q).$$

Now assume that there is a $y \in ((S_0 \cap I_S(p,q)) \setminus I_S(p, \mathrm{glb}_q(S_0)))$. Choose $T_0, T_1 \in S$ such that $y \in T_0 \setminus T_1$ and $I_S(p, \mathrm{glb}_q(S_0)) \subset T_1 \setminus T_0$ and $T_0 \cup T_1 = X$. Now, if $q \in T_1$, then $I_S(p,q) \subset T_1$, which is a contradiction, since $y \in I_S(p,q)$.

Therefore $q \in T_0$. This, however, also is a contradiction since then $glb_q(S_0) \in T_0$. We conclude that $S_0 \cap I_S(p,q) = I_S(p,glb_q(S_0))$. Similarly, using (i), we can derive $S_1 \cap I_S(p,q) = I_S(glb_p(S_1),q)$.

Now, by lemma 0.1, $\{I_S(p,x) \mid x \in I_S(p,q)\} \cup \{I_S(x,q) \mid x \in I_S(p,q)\}$ is a closed subbase for $I_S(p,q)$ (note that $I_S(p,q)$ is compact!)

It remains to establish (iv); this can be done using the same technique as in theorem 1.5.13 (v). □

1.5.15. A point x in a topological space X is called an *endpoint* if its complement $X\backslash\{x\}$ is connected. We call a topological space X (generalized) *arcwise connected* if for each two distinct x and y in X there is a totally ordered compact connected subspace of X containing both x and y. Then x and y are connected by an ordered continuum L such that $L\backslash\{x,y\}$ is connected; i.e. x and y are the only two endpoints of L.

1.5.16. THEOREM. *Let X be a connected topological space and let $\mathcal{S}$ be a binary normal subbase for X. Then X is (generalized) arcwise connected.*

PROOF. Choose $x,y \in X$ and consider the connected subspace $I_S(x,y)$ (corollary 1.5.8 (ii)). Then $I_S(x,y)$ is partially ordered by $\le_x$ and $\le_x$ is order dense (theorem 1.5.14 (iv)). An easy application of Zorn's lemma shows that there is a maximal chain L in $I_S(x,y)$. But as $\le_x$ is order dense so is the induced (total) order on L. Moreover by a theorem of WARD [124], L is closed and connected in $I_S(x,y)$ (this is very easy to show). Therefore, in virtue of theorem 1.5.14, L is an ordered compactum that clearly contains both x and y. □

1.5.17. For a topological space X, let 2^X be the space of all nonempty closed subsets of X topologized by the Vietoris topology, i.e. a basis for the open sets consists of all sets

$$\langle O_0,O_1,\dots,O_n\rangle = \{G \in 2^X \mid G \subset \bigcup_{i=0}^{n} O_i \text{ and } G \cap O_i \neq \emptyset \text{ for all } i \le n\}$$

where $O_0,O_1,\dots,O_n$ is an arbitrary finite collection of open subsets of X (cf. MICHAEL [75]). The space 2^X is called the *hyperspace* of X. For many strong results concerning hyperspaces, see WOJDYSLAWSKI [130], CURTIS & SCHORI [36],[37], SCHORI & WEST [102] and WEST [127].

Hyperspaces are widely used in general topology; for our purposes too they will turn out to be of great help.

1.5.18. THEOREM. *Let* X *be a topological space with binary normal subbase* $\mathcal{S}$. *Then the mapping* $\xi\colon 2^X \times X \to X$ *defined by* $\xi(A,x) := \mathrm{glb}_x(A)$ *is continuous.*

PROOF. Let $S \in \mathcal{S}$ and suppose that $(A,x) \notin \xi^{-1}[S]$. Then $\mathrm{glb}_x(A) \notin S$. By the normality of $\mathcal{S}$ there are S_0, S_1 in $\mathcal{S}$ such that $\mathrm{glb}_x(A) \in S_0\setminus S_1$, $S \subset S_1\setminus S_0$ and $S_0 \cup S_1 = X$. Clearly A intersects $X\setminus S_1$, for otherwise $I_{\mathcal{S}}(A) \subset S_1$ which would imply that $\mathrm{glb}_x(A) \in S_1$. If $A \not\subset X\setminus S_1$, then clearly $x \notin S_1$. Let

$$V := \langle X\setminus S_1\rangle \times X \qquad \text{if } A \subset X\setminus S_1$$

and

$$V := \langle X\setminus S_1, X\rangle \times (X\setminus S_1) \qquad \text{if } A \not\subset X\setminus S_1.$$

Then V is an open neighborhood of (A,x) which, in addition, does not intersect $\xi^{-1}[S]$. For take $(B,y) \in V$. In the first case, $B \subset X\setminus S_1$, whence $\xi(B,y) \in X\setminus S_1 \subset X\setminus S$. In the second case, choose $b \in B \cap (X\setminus S_1)$. Then $\{b,y\} \subset S_0$; consequently $\xi(B,y) \in S_0 \subset X\setminus S$. □

1.5.19. Recall that a topological space X can be embedded in 2^X by the mapping $i(x) := \{x\}$ (MICHAEL [75]). We will identify X and $i[X]$. A topological space X which possesses a binary normal subbase will be called, from now on, *normally supercompact*.

1.5.20. COROLLARY. *A normally supercompact space* X *is a retract of its hyperspace* 2^X. *If, in addition,* X *is connected and metrizable then* X *is an Absolute Retract.*

PROOF. Let $\mathcal{S}$ be a binary normal subbase for X. Fix a point $p \in X$ and define $r\colon 2^X \to X$ by $r(A) := \xi(A,p)$, where ξ is as defined in theorem 1.5.18. Then r is a continuous retraction. For take $x \in X$. Then

$$\{r(x)\} = I_{\mathcal{S}}(x,p) \cap I_{\mathcal{S}}(\{x\}) = \{x\}.$$

If in addition X is connected and metrizable, then X is a Peano continuum (corollary 1.5.8 (iii)). Hence 2^X is an Absolute Retract (WOJDYSLAWSKI [130]; even $2^X \approx Q$, the Hilbert cube, see CURTIS & SCHORI [36]). Therefore X is an Absolute Retract too. □

1.5.21. If X has a binary normal subbase $\mathcal{S}$ then the subspace $H(X,\mathcal{S}) := \{C \in 2^X \mid C \text{ is } \mathcal{S}\text{-closed}\}$ of 2^X is of particular interest. It will be discussed in section 2.10. From the results obtained there we mention the following:

(a) *the mapping* ϕ: $2^X \to H(X,S)$ *defined by* $\phi(A) := I_S(A)$ *is a retraction (hence* $H(X,S)$ *is compact!) (theorem* 2.10.5*)*;

(b) $H(X,S)$ *has a binary normal subbase (*corollary 2.10.12*)*.

$H(X,S)$ inherits a partial ordering (by inclusion) from 2^X, which is order dense if X is connected.

1.5.22. THEOREM. *Let* X *be a topological space which possesses a binary normal subbase* S. *Then* $H(X,S)$ *is a densely ordered (by inclusion) compact subset of* 2^X *if and only if* X *is connected.*

PROOF. $H(X,S)$ always is compact (cf. theorem 2.10.5). Assume that X is connected. Choose $A,B \in H(X,S)$ such that A is a proper subset of B. Take $x \in B\setminus A$ and let $S_0,S_1 \in S$ such that $A \subset S_0 \cap (X\setminus S)$, $x \in S_1\setminus S_0$ and $S_0 \cup S_1 = X$. This is possible since S is normal and since $A \in H(X,S)$. Then $\{S_0,S_1,B\}$ is a linked system consisting of S-closed sets, hence $S_0 \cap S_1 \cap B \neq \emptyset$ since S is binary. Take $b \in S_0 \cap S_1 \cap B$ and define $C := S_0 \cap B$. Then $A \subset C \subset B$ and $A \neq C$ since $b \in C\setminus A$ and $C \neq B$ since $x \in B\setminus C$. Clearly $C \in H(X,S)$.

Conversely, assume that $H(X,S)$ is a densely ordered (by inclusion) compact subset of 2^X. Take $A \in H(X,S)$ and let L_A be a maximal chain, in $H(X,S)$, that contains A. Notice that $X \in L_A$. Then, since $H(X,S)$ is compact and densely ordered by inclusion, L_A is compact and connected (WARD [124]). But then $H(X,S) = \bigcup\{L_A \mid A \in H(X,S)\}$ is connected too. As each singleton in X is S-closed, $X \subset H(X,S)$ and as X is a retract of $H(X,S)$ by corollary 1.5.20 we conclude that X is connected. □

1.5.23. COROLLARY. *Let* X *be a connected topological space which admits a binary normal closed subbase* S. *Then for each* $x \in X$ *there is a compact connected linearly ordered space* J, *with endpoints* a *and* b, and a continuous "*contraction*" $p: X\times J \to X$ *such that* $p\restriction X\times\{a\}$ *is constant with values on* x *and* $p\restriction X\times\{b\}$ *is the identity mapping. If, in addition,* X *is metrizable then so is* J *and consequently* p *becomes an ordinary contraction.*

PROOF. Choose $x \in X$ and let $L \subset H(X,S)$ be a maximal chain that contains $\{x\}$. Then L is densely ordered by inclusion (theorem 1.5.22) and consequently L is a compact connected ordered space. Also $\{\{x\},X\}$ are the only endpoints of L as can easily be seen. Now let $p: L\times X \to X$ be the restriction to $L \times X$ of the mapping ξ, described in theorem 1.5.18. Then it is easy to see that p satisfies the required properties.

If in addition X is metrizable, then 2^X is metrizable (cf. ENGELKING [48], problem P.4H) and consequently so is its subspace J. But then J is homeomorphic to the closed unit interval [0,1] (WARD [124]). □

The technique, used in the proof of the above corollary, is due to VAN DE VEL [118].

Finally, we present some questions which at the moment we cannot answer. In section 1.3 we showed that each compact tree-like space is supercompact. A compact tree-like space is rim finite (cf. PROIZVOLOV [92]), i.e. each point admits arbitrary small neighborhoods with finite boundaries. This suggests the question whether any rim finite continuum is supercompact.

1.5.24. QUESTION. *Are rim finite continua supercompact?*

It should be noticed that a rim finite continuum is the continuous image of a supercompact Hausdorff space; indeed, it is even the continuous image of an ordered continuum (cf. WARD [125]). Not all rim finite continua are normally supercompact, since the 1-sphere S_1 is rim finite but not contractible (cf. corollary 1.5.20).

1.5.25. QUESTION. *When is a normally supercompact space the continuous image of an ordered compactum?*

Not all connected spaces with a binary normal subbase are the continuous image of an ordered compactum. For example, I^c is not the continuous image of an ordered compactum, since it is not hereditarily normal.

1.6. Notes

DE GROOT [54],[55] conjectured that every compact metric space is supercompact (which was proved to be correct by STROK & SZYMAŃSKI [116]) and also that not every compact Hausdorff space is supercompact (which was proved by BELL [14]). Theorem 1.1.5 indicates why certain compact Hausdorff spaces are not supercompact, but there are still many questions left.

After learning that not every compact Hausdorff space is supercompact, VAN DOUWEN and the author together improved BELL's result. These results are included in the previous chapter; they fill section 1.1. They will also be published separately in a forthcoming paper (cf. VAN DOUWEN & VAN MILL [43]).

We also have some comments concerning section 1.3. As noted there, supercompact spaces can be characterized as being those spaces obtainable as the graph-space of a graph. This approach was developped by DE GROOT [56] and it turned out to be useful (cf. DE GROOT [56], BRUIJNING [26], SCHRIJVER [105]). BRUIJNING [26] used the graph-theoretical method's of DE GROOT by reproving an internal characterization of I^n and I^∞ (cf. DE GROOT [55]). SCHRIJVER [105] used non-intersection graphs instead of intersection graphs and considerably simplified and generalized the techniques; among others he reproved in a simple way all the results in DE GROOT & SCHNARE [60] and obtained some new subbase characterizations of certain classes of topological spaces. The author proved the subbase characterization of (products of) compact tree-like spaces (cf. VAN MILL [76]); in particular that every compact tree-like space is supercompact, which was proved independently by BROUWER & SCHRIJVER [24] (cf. also BROUWER [23]) using a different method. BROUWER & SCHRIJVER [24] used interval structures (which were first used by SCHRIJVER). Finally SCHRIJVER and the author jointly wrote a paper in which we included the interval structures, results from [105] and [76] and also some new techniques (cf. VAN MILL & SCHRIJVER [81]). This paper was the basis for section 1.3.

CHAPTER II

SUPEREXTENSIONS

In this chapter we will construct for each topological space X and for each suitable closed subbase S a supercompact superspace $\lambda(X,S)$ of X, in the same way as FRINK [51], SHANIN [106a], and others, constructed a compactification $\omega(X,S)$ of X. The underlying set of $\lambda(X,S)$ is the set of maximal linked systems in S; the topology is induced by a natural Wallman subbase for the closed subsets. The space $\lambda(X,S)$ is called *the superextension of* X *relative the subbase* S (cf. DE GROOT [54]), and in case S consists of all the closed subsets of X we usually write λX instead of $\lambda(X,S)$, calling λX *the superextension of* X.

The spaces $\lambda(X,S)$ are supercompact, in a very natural way: their canonical defining subbases are binary. It is not surprising that one has to use something like the axiom of choice to prove this (cf. FRINK [51], STEINER [114]). The first section in this chapter deals with the question what set theoretic assumptions we have to make in order to extend arbitrary linked systems to maximal linked systems. We do this in the setting of Boolean algebras. We will reprove SCHRIJVER's [106] theorem that the statement

(*) *each linked system in a Boolean algebra can be extended to at least one maximal linked system,*

is strictly weaker than Stone's representation theorem; also (*) is independent of the usual axioms of set theory since, as SCHRIJVER [106] has shown, (*) implies that each product of sets containing at most two elements is nonempty (that is to say: (*) implies C_2, the axiom of choice for two sets, cf. JECH [66]). We will show that (*) is equivalent to a weaker form of the representation theorem of Stone; for this we define near-subalgebras of Boolean algebras. Each subalgebra is a near-subalgebra; (*) is equivalent to the statement that each Boolean algebra is isomorphic to a near-subalgebra of a $\mathcal{P}(X)$.

The other sections in this chapter deal with topological properties of superextensions. Some properties are inherited from the underlying space, such as: *λX is connected if X is connected* (cf. VERBEEK [119]). But other properties are new and unexpected and they turn out to be fundamental, such as: *λX is locally connected if X is connected* (cf. VERBEEK [119]).

The superextension λX of a topological space X always is a "big" space, in case X is normal, *the dimension of λX either is zero (in case* Ind $X = 0$*) or infinite*. Also λX contains βX, the Čech-Stone compactification of X, as a subspace (again we only consider normal spaces) (cf. VERBEEK [119]). This is a consequence of the fact that X is C^*-embedded in λX and this can be shown using a result of JENSEN [59] (cf. also VERBEEK [119]). We will extend the result of JENSEN in such a way that it becomes applicable in more general situations. Here we apply ideas of STEINER & STEINER [111], [112].

Subspaces of superextensions often have rich structures. In section 2.8 a first attempt is made to describe some subspaces which appear to be interesting. For a normal space X we define a subspace $\Sigma(X)$ of $\lambda(X)$ which seems to behave as the "remainder" of the "extension" λX of X; as we will show $\Sigma(X)$ has much in common with $\beta X\backslash X$. In particular, as a consequence of our results $\Sigma(X)$ is compact iff X is locally compact iff $\Sigma(X)$ is homeomorphic to $\lambda(\beta X\backslash X)$. Of particular interest is the space $\Sigma(\mathbb{N})$. This is in fact the space of all uniform maximal linked system on $\mathbb{N}$. The space $\Sigma(\mathbb{N})$ can be characterized in about the same way as PAROVIČENKO [91] characterized $\beta\mathbb{N}\backslash\mathbb{N}$. This characterization is valid under CH, the Continuum Hypothesis. By an example of VAN DOUWEN [40] the Continuum Hypothesis is indeed essential here. There is a locally compact, separable, σ-compact topological space M for which $\beta M\backslash M$ and $\beta\mathbb{N}\backslash\mathbb{N}$ are homeomorphic under CH but not under MA + $\neg$CH. VAN DOUWEN's example also shows that CH is essential in our characterization of $\Sigma(\mathbb{N})$. The spaces $\Sigma(\mathbb{N})$ and $\Sigma(M)$ are homeomorphic under CH but not under MA + $\neg$CH.

In section 2.10 we try to define a general notion of convexity in topological spaces; convexity with respect to a certain closed subbase. This section has in fact little to do with superextensions; it is hyperspace theory. But to prove our theorems we use superextensions extensively. Some of the consequences of this section were used in 1.5.22 and the same results will also be used in section 2.7. There we show that the superextension of a normal space, with the property that each finite subset is contained in a metrizable continuum, is contractible. This is really a nice theorem. As a consequence it follows that $\lambda\mathbb{R}$, the superextension of the real line $\mathbb{R}$, is contractible, in contrast with $\beta\mathbb{R}$ (this space is not even path connected). The contractibility of $\lambda\mathbb{R}$ was claimed previously by VERBEEK [119]; his proof is incorrect however, since it relies on the contactibility of $\beta\mathbb{R}$. The results about convexity in topological spaces and about contractibility of superextensions were obtained in good cooperation with M. VAN DE VEL (cf. VAN MILL & VAN DE VEL [82], [83]).

2.1. Linked systems and the Stone representation theorem

This section deals with logical independency of some axioms in Boolean algebra's. Our main interest is in (maximal) linked systems, which are natural generalizations of filters. We refer to the book of HALMOS [61] for general concepts concerning Boolean algebras.

2.1.1. DEFINITION. Let $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ be a Boolean algebra. A subset $M \subset B$ is called a *linked system* if $m_0 \wedge m_1 \neq 0$ for all $m_0, m_1 \in M$. A *maximal linked system* is a linked system not properly contained in any other linked system.

It is easy to verify that the lemma of Zorn implies that each linked system in a Boolean algebra can be extended to at least one maximal linked system. However, much weaker axioms imply this fact, cf. SCHRIJVER [106]. We deal with the following axioms:

FA : *Each Boolean algebra contains an ultrafilter.*

FA': *Each filter in a Boolean algebra is contained in at least one ultrafilter.*

LA : *Each Boolean algebra contains a maximal linked system.*

LA': *Each linked system in a Boolean algebra is contained in at least one maximal linked system.*

Again it is easy to see that FA and FA' are equivalent, forming quotient algebra's (cf. JECH [66]). Also, LA and LA' are equivalent (SCHRIJVER [106]; cf. 2.1.7 below) but this is less trivial.

2.1.2. LEMMA (LA'). *Let* $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ *be a Boolean algebra. Then for* *all* $x,y \in B$ *there is a maximal linked system* $L \subset B$ *such that* $|L \cap \{x,y\}| = 1$.

PROOF. If x equals y, then the linked system $\{x\}$ is contained in at least one maximal linked system L, by LA'.

If x is not equal to y, then we may assume, without loss of generality, that $y \not\leq x$. Clearly, $x' \wedge y \neq 0$. Then the linked system $\{x',y\}$ is contained in at least one maximal linked system $L \subset B$. Then $L \cap \{x,y\} = \{y\}$, since $x' \in L$. □

Let $(X,\leq)$ be a partially ordered set; then each subset A of X will be partially ordered by the induced ordering $\leq_A$, defined by $a \leq_A b$ iff $a \leq b$ $(a,b \in A)$.

2.1.3. DEFINITION. Let $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ be a Boolean algebra. A subset $A \subset B$ is called a *near-subalgebra* of $\mathcal{B}$ provided that

(i) $(A,\leq_A)$ is a distributive lattice;

(ii) $0,1 \in A$;

(iii) $\forall a \in A$: $a' \in A$.

For any two elements a_0 and a_1 of the near-subalgebra A of $\mathcal{B}$ write

$$a_0 \wedge_A a_1 \qquad (a_0 \vee_A a_1)$$

for the greatest lower bound (least upper bound) of a_0 and a_1. We then have

2.1.4. LEMMA. *Let* A *be a near-subalgebra of the Boolean algebra* $\mathcal{B}$. *Then* $a \wedge_A b \leq a \wedge b$ *and* $a \vee b \leq a \vee_A b$ *for all* $a,b \in A$. □

2.1.5. PROPOSITION. *Let* A *be a near-subalgebra of Boolean algebra* $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$. *Then* $\mathcal{A} = \langle A,0,1,',\wedge_A,\vee_A\rangle$ *is a Boolean algebra. Moreover each subalgebra of a Boolean algebra is a near-subalgebra.* □

The proof of this proposition is straightforward.
Proposition 2.1.5 suggests the question whether each near-subalgebra of a Boolean algebra is a subalgebra (in the usual sense). The answer to this question is in the negative, as the following example shows.

2.1.6. EXAMPLE. *A near-subalgebra which is not a subalgebra.*
In $\mathcal{P}(\{1,2,3,4\})$ let $\mathcal{A} := \{\emptyset,\{1\},\{2\},\{3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$.

It is easy to see that $\mathcal{A}$ is a near-subalgebra, which is not a subalgebra of $\mathcal{P}(X)$. For example $\{1\} \in \mathcal{A}$ and $\{2\} \in \mathcal{A}$ while $\{1,2\} \notin \mathcal{A}$. □

2.1.7. THEOREM. *The following statements are equivalent:*

(i) LA;

(ii) LA';

(iii) *each Boolean algebra is isomorphic to a near-subalgebra of some* $\mathcal{P}(X)$.

PROOF. SCHRIJVER [106] has first shown that (i) is equivalent to (ii). We will present a different and simpler proof here. As obviously (ii) $\Rightarrow$ (i), we need only prove (i) $\Rightarrow$ (ii). Indeed, let $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ be a Boolean algebra; let $M \subset B$ be a maximal linked system. If $L \subset B$ is a linked system, then define

$$L' := \{m \in M \mid m \wedge \ell \neq 0 \ (\forall \ell \in L)\} \cup \{m' \mid m \in M \text{ and } \exists \ell \in L: m \wedge \ell = 0\}.$$

Then it is easily seen that L' is a maximal linked system that contains L.

(ii) $\Rightarrow$ (iii). Let $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ be a Boolean algebra. Define

$$X = \{L \subset B \mid L \text{ is a maximal linked system}\}.$$

Then X is nonvoid, because of LA'. For any $b \in B$ define

$$b^+ := \{L \in X \mid b \in L\}.$$

Define a function

$$\phi: B \to \mathcal{P}(X) \quad \text{by} \quad \phi(b) := b^+.$$

CLAIM. $\phi[B]$ *is a near-subalgebra of* $\mathcal{P}(X)$ *and* $\phi: B \to \phi[B]$ *is an isomorphism.*

Indeed, first notice that $\phi(0) = \emptyset$. Also $\phi(x') = \{L \in X \mid x' \in L\} = \{L \in X \mid x \notin L\} = X\backslash\{L \in X \mid x \in L\} = \phi(x)^c$, since each element $L \in X$ is a maximal linked system.

We will proceed to show that $\phi[B]$ is a near-subalgebra of $\mathcal{P}(X)$ and for this it only remains to be shown that $(\phi[B],\subset)$ is a lattice.

Choose $x^+,y^+ \in \phi[B]$. Let us show that $(x \wedge y)^+$ is the greatest lower bound of x^+ and y^+ in $[B]$. Trivially $(x \wedge y)^+ \subset x^+ \cap y^+$; therefore suppose that $z^+ \subset x^+ \cap y^+$. Now, $z^+ \subset x^+$ implies that $z \leq x$, for suppose to the contrary that $z \nleq x$. Then the linked system $\{x',z\}$ is contained in a maximal linked system $L \in X$. Hence $L \in z^+$ and $L \notin x^+$, since $x' \in L$. This is a contradiction. Hence $z \le x$ and in the same way also $z \le y$. Consequently $z \le x \wedge y$; thus $z^+ \subset (x \wedge y)^+$.

In the same way $(x \vee y)^+$ is the least upper bound of x^+ and y^+ in $\phi[B]$. Hence $\phi[B]$ is a near-subalgebra of $P(X)$. Also it is clear that $\phi\colon B \to \phi[B]$ is an homomorphism, since for example $\phi(x \wedge y) = (x \wedge y)^+ = x^+ \wedge_{\phi[B]} y^+ = \phi(x) \wedge_{\phi[B]} \phi(y)$. Finally, ϕ is injective. For take $x,y \in B$ such that $x \neq y$. By lemma 2.1.2 there is an $L \in X$ such that $|L \cap \{x,y\}| = 1$. This implies that $x^+ \neq y^+$ and consequently $\phi(x) \neq \phi(y)$. We conclude that $\phi\colon B \to \phi[B]$ is an isomorphism.

(iii) $\Rightarrow$ (i). Let $\mathcal{B}$ be a near-subalgebra of some $P(X)$. Choose $x_0 \in X$ and define

$$\mathcal{L} := \{L \in \mathcal{B} \mid x_0 \in L\}.$$

We will show that $\mathcal{L}$ is a maximal linked system.

First of all notice that $\mathcal{L} \neq \emptyset$ since $X \in \mathcal{L}$. Also $\mathcal{L}$ is a linked system. For suppose $L_0, L_1 \in \mathcal{L}$ such that $L_0 \wedge_{\mathcal{B}} L_1 = \emptyset$. Then $L_0 \le_{\mathcal{B}} (X \backslash L_1)$ and consequently $L_0 \subset (X \backslash L_1)$, since $\mathcal{B}$ is a near-subalgebra. This is a contradiction. Finally $\mathcal{L}$ is a maximal linked system, since for all $B \in \mathcal{B}$ either $B \in \mathcal{L}$ or $X \backslash B \in \mathcal{L}$.

This completes the proof of the theorem. □

2.1.8. In [106] SCHRIJVER showed that LA follows from OEP, the order extension principle, which can be formulated as follows:

OEP: *Each partial order on a set can be extended to a total order.*

He also proved that LA implies C_2, where

C_2 : *Each product of sets, each containing at most two elements, is nonempty.*

It is unlikely that LA is equivalent to OEP, although LA is equivalent to a statement which seems to be very close to OEP. We define

REP (*relation extension principle*): *For each Boolean algebra* $\mathcal{B} = \langle B,0,1,',\wedge,\vee \rangle$ *there is a binary relation* R *on* B *satisfying:*

(i)	$x \le y$ *implies* xRy	$(x,y \in B)$;
(ii)	xRy *or* yRx	$(x,y \in B)$;
(iii)	$\neg(xRx'$ *and* $x'Rx)$	$(x \in B)$;
(iv)	xRy *and* yRz *implies* xRz	$(x,y,z \in B)$.

(Notice that R *is a total pre-ordering.)*

2.1.8. THEOREM. LA *is equivalent to* REP.

PROOF. Let $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ be a Boolean algebra and let M be a maximal linked system in B. Then the relation R on B defined by xRy iff ($x' \in M$ or $y \in M$) satisfies all requirements.

On the other hand, let $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ be a Boolean algebra and let R be a binary relation on B satisfying (i)-(iv). Let $M := \{x \in B \mid x'Rx\}$. We will show that M is a maximal linked system. To prove that M is linked, take $a,b \in M$. Suppose to the contrary, that $a \wedge b = 0$. Then $a \leq b'$ and $b \leq a'$. Therefore

$$aRb'RbRa'Ra,$$

since $a,b \in M$. But then aRa' and a'Ra (by (iv)), which contradicts (iii). Finally M is a maximal linked system since for all $x \in B$ either x'Rx or xRx' and consequently $x \in M$ or $x' \in M$. $\square$

REMARK. The proof of the implication REP $\Rightarrow$ LA is the same as SCHRIJVER's [106] proof OEP $\Rightarrow$ LA.

As clearly OEP implies REP we conclude that OEP implies LA and hence, as OEP is weaker than FA (JECH [66]), that LA if weaker than FA.

2.2. Superextensions; some preliminaries

In this section we will describe how to construct superextensions of topological spaces; we give some simple lemma's which we frequently use without explicit reference. Moreover we will characterize the class of all superextensions of a given topological space.

2.2.1. Let X be a topological space and let $\mathcal{S}$ be a subbase for the closed subsets of X. Recall the following definitions; $\mathcal{S}$ is defined to be

(i) a T_1-*subbase* if for each $x_0 \in X$ and $S \in \mathcal{S}$ with $x_0 \notin S$ there exists a $T \in \mathcal{S}$ with $x_0 \in T$ and $T \cap S = \emptyset$ (cf. 0.A);

(ii) a *weakly normal subbase* if for each $S,T \in \mathcal{S}$ with $S \cap T = \emptyset$ there is a finite cover $\mathcal{M}$ of X by elements of $\mathcal{S}$ such that each element of $\mathcal{M}$ meets at most one of S and T (cf. 1.3.8);

(iii) *a normal subbase* if for each $S_0, T_0 \in \mathcal{S}$ with $S_0 \cap T_0 = \emptyset$ there exist $S_1, T_1 \in \mathcal{S}$ with $S_1 \cap T_0 = \emptyset = T_1 \cap S_0$ and $S_1 \cup T_1 = X$ (cf. 1.3.16).

Finally we define $\mathcal{S}$ to be

(iv) *a supernormal subbase* if $\mathcal{S}$ is normal while moreover for all $S \in \mathcal{S}$ and closed $G \subset X$ with $S \cap G = \emptyset$ there exists an $S_0 \in \mathcal{S}$ such that $G \subset S_0$ and $S \cap S_0 = \emptyset$.

A *maximal linked system,* or briefly *mls,* in $\mathcal{S}$ is a linked system of $\mathcal{S}$ not properly contained in any other linked system of $\mathcal{S}$. Usually we do not explicitly mention $\mathcal{S}$.

The simple propositions 2.2.2 and 2.2.3 and the proof of theorem 2.2.4 can be found in [119].

2.2.2. PROPOSITION. *Let M_0, M_1 be mls's in $\mathcal{S}$. Then*

(i) $\emptyset \notin M_0$;

(ii) *if* $S \in M_0$, $T \in \mathcal{S}$ *and* $S \subset T$ *then* $T \in M_0$;

(iii) *if* $S \in \mathcal{S} \backslash M_0$ *then* $\exists T \in M_0$: $S \cap T = \emptyset$;

(iv) $M_0 \neq M_1$ *iff* $\exists S \in M_0$, $\exists T \in M_1$: $S \cap T = \emptyset$;

(v) *if* $S, T \in \mathcal{S}$ *and* $S \cup T = X$ *then* $S \in M_0$ *or* $T \in M_0$. □

The above proposition shows that maximal linked systems in some respects behave like ultrafilters. Define

$$\lambda(X,\mathcal{S}) := \{M \subset \mathcal{S} \mid M \text{ is a maximal linked system in } \mathcal{S}\}.$$

If $\mathcal{S}$ is a T_1-subbase then for each $x \in X$ we have that $M_x := \{S \in \mathcal{S} \mid x \in S\}$ is an mls in $\mathcal{S}$; the function $\underline{i}: X \to \lambda(X,\mathcal{S})$ defined by $\underline{i}(x) := M_x$ is one to one.

For $A \subset X$ we define

$$A^+ := \{M \in \lambda(X,\mathcal{S}) \mid A \text{ contains a member of } M\}.$$

2.2.3. PROPOSITION. *For any* $A, B \subset X$ *we have*

(i) $A \subset B$ *implies* $A^+ \subset B^+$;

(ii) $A \cap B = \emptyset$ *implies* $A^+ \cap B^+ = \emptyset$;

(iii) *if* $A, B \in \mathcal{S}$ *then* $A \cap B = \emptyset$ *iff* $A^+ \cap B^+ = \emptyset$;

(iv) *if* $A, B \in \mathcal{S}$ *then* $A \cup B = X$ *iff* $A^+ \cup B^+ = \lambda(X,\mathcal{S})$;

(v) *if* $A \in \mathcal{S}$ *then* $A^+ \cup (X \backslash A)^+ = \lambda(X,\mathcal{S})$.

As a closed subbase for a topology on $\lambda(X,\mathcal{S})$ we take

$$\mathcal{S}^+ := \{S^+ \mid S \in \mathcal{S}\}.$$

With this topology $\lambda(X,\mathcal{S})$ is called *the superextension of* X *with respect to the subbase* $\mathcal{S}$. In case $\mathcal{S}$ consists of all the closed subsets of X, then $\lambda(X,\mathcal{S})$ is denoted λX and is called *the* superextension of X.

Zorn's lemma implies that each linked system $\mathcal{M} \subset \mathcal{S}$ is contained in at least one maximal linked system $\mathcal{M}' \subset \mathcal{S}$. This proves theorem 2.2.4 (iv).

2.2.4. THEOREM.

(i) *If* $\mathcal{S}$ *is a* T_1*-subbase then* $\underline{i}: X \to \lambda(X,\mathcal{S})$ *is an embedding;*

(ii) $\lambda(X,\mathcal{S})$ *is* T_1*;*

(iii) $\lambda(X,\mathcal{S})$ *is Hausdorff if* $\mathcal{S}$ *is normal, since* $\mathcal{S}^+$ *is normal if* $\mathcal{S}$ *is normal;*

(iv) $\lambda(X,\mathcal{S})$ *is supercompact; in fact* $\mathcal{S}^+$ *is binary;*

(v) *for all* $S \in \mathcal{S}$: $\underline{i}^{-1}[S^+] = S$.

In case $\underline{i}$ is a topological embedding we will always identify X and $\underline{i}[X]$. Because of theorem 2.2.4 (iv), if $\mathcal{S}$ is a T_1-subbase the closure of X in $\lambda(X,\mathcal{S})$ is a compactification of X, the so called GA (de Groot-Aarts) *compactification* $\beta(X,\mathcal{S})$ *of* X *with respect to the subbase* $\mathcal{S}$. These compactifications were introduced by DE GROOT and AARTS in [57]. They showed that *if* $\mathcal{S}$ *is weakly normal then* $\beta(X,\mathcal{S})$ *is a Hausdorff compactification of* X; consequently X is completely regular. The counterpart of this theorem is also true: *if* $\beta(X,\mathcal{S})$ *is Hausdorff then* $\mathcal{S}$ *is weakly normal* (cf. 4.6.2). The GA compactifications will be discussed in detail in chapter four.

The following theorem is simple but useful; it will be used frequently in chapter 3.

2.2.5. THEOREM. *Let* $\mathcal{S}$ *be a binary subbase for the topological space* X. *Let* Y *be a subspace of* X *such that for all* $S_0,S_1 \in \mathcal{S}$ *with* $S_0 \cap S_1 \neq \emptyset$ *also* $S_0 \cap S_1 \cap Y \neq \emptyset$. *Then* X *is homeomorphic to* $\lambda(Y,\mathcal{S}\cap Y)$.

PROOF. Define a function $\phi: X \to \lambda(Y,\mathcal{S}\cap Y)$ by $\phi(x) := \{S \cap Y \mid S \in \mathcal{S}$ and $x \in S\}$. We will show that ϕ is a homeomorphism.

To prove that ϕ is well defined, choose $x \in X$. Then clearly $\phi(x)$ is a linked system. Assume it were not maximally linked. Choose $S_0 \in \mathcal{S}$ such that $\phi(x) \cup \{S_0 \cap Y\}$ is linked but $S_0 \cap Y \notin \phi(x)$. Clearly $x \notin S_0$. Choose $T \in \mathcal{S}$ such that $x \in T$ and $T \cap S_0 = \emptyset$ (this is possible since $\mathcal{S}$ is a T_1-subbase). But then $T \cap Y \in \phi(x)$ and $(T \cap Y) \cap (S_0 \cap Y) = \emptyset$, which is a contradiction. Hence ϕ is well defined.

Also ϕ is one to one and surjective. For take $x,y \in X$ such that $x \neq y$. Choose S and T in $\mathcal{S}$ such that $x \in S$, $y \in T$ such that $S \cap T = \emptyset$. But then $S \cap Y \in \phi(x)$ and $T \cap Y \in \phi(y)$ and as $(S \cap Y) \cap (T \cap Y) = \emptyset$ it follows that $\phi(x) \neq \phi(y)$. To prove that ϕ is surjective, take $M \in \lambda(Y, \mathcal{S} \cap Y)$. Define $L = \{S \in \mathcal{S} \mid S \cap Y \in M\}$. Then L is a linked system (in $\mathcal{S}$) and consequently, since $\mathcal{S}$ is binary, there is an $x \in \bigcap L$. It now is not hard to see that $\phi(x) = M$.

Finally ϕ and ϕ^{-1} are continuous. This is trivial since for all $S \in \mathcal{S}$ we have $x \in \phi^{-1}[(S \cap Y)^+]$ iff $\phi(x) \in (S \cap Y)^+$ iff $S \cap Y \in \phi(x)$ iff $x \in S$. Therefore $\phi^{-1}[(S \cap Y)^+] = S$.

We conclude that ϕ is a homeomorphism. □

2.2.6. <u>COROLLARY</u> (VERBEEK [119]). *Every superextension of a topological space* X *can be regarded as a superextension of a compactification of* X, *viz.*

$$\lambda(X,\mathcal{S}) \approx \lambda(\beta(X,\mathcal{S}),\mathcal{S}'),$$

where

$$\mathcal{S}' = \{S^+ \cap \beta(X,\mathcal{S}) \mid S \in \mathcal{S}\}.$$

<u>PROOF</u>. Let $S_0, S_1 \in \mathcal{S}$. If $S_0^+ \cap S_1^+ \neq \emptyset$ then $S_0 \cap S_1 \neq \emptyset$ and consequently $(S_0^+ \cap \beta(X,\mathcal{S})) \cap (S_1^+ \cap \beta(X,\mathcal{S})) \neq \emptyset$, since $S_i \subset S_i^+ \cap \beta(X,\mathcal{S})$ $(i \in \{0,1\})$. Now apply theorem 2.2.5. □

Theorem 2.2.5 implies much more; it was the starting point for the author's proof that $\lambda I \approx I^\infty$. Also theorem 2.2.5 allows us to construct *nice* superextensions of topological spaces. Let us demonstrate this by an example. It is clear that the canonical subbase of right- and left-tails of a linearly ordered compact space is binary and also that if $\mathcal{T}$ is a binary subbase for X then $\lambda(X,\mathcal{T})$ is homeomorphic to X (in the obvious way). In particular the subbase $\mathcal{S} = \{[0,x] \mid x \in I\} \cup \{[x,1] \mid x \in I\}$ is a binary subbase for the unit segment $I = [0,1]$, and consequently $\lambda(I,\mathcal{S})$ is homeomorphic to I. Hence the unit segment is a superextension of the unit segment, VERBEEK ([119], p.136) gives a list of superextensions of the unit segment, but none of the examples is homeomorphic to

the unit square or to a higher dimensional hypercube. Theorem 2.2.5 gives us for each $n \in \mathbb{N}$ an easily described subbase S_n for which $\lambda(I,S_n)$ is homeomorphic to I^n. Let us describe S_2. To this end define an embedding of I into $[0,\frac{1}{3}]^2$ as suggested in the following figure.

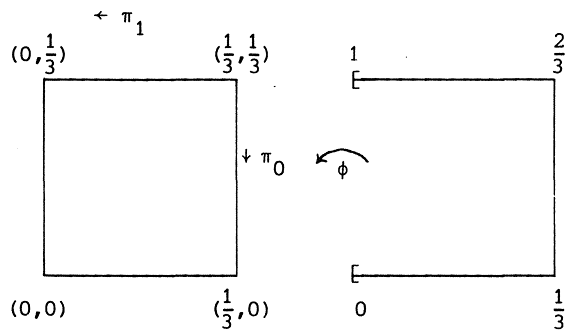


Figure 5.

Define a binary subbase T for $[0,\frac{1}{3}]^2$ by

$$T := \{\pi_i^{-1}[0,x] \mid 0 \le x \le \tfrac{1}{3},\ i \in \{0,1\}\} \cup \{\pi_i^{-1}[x,\tfrac{1}{3}] \mid 0 \le x \le \tfrac{1}{3},\ i \in \{0,1\}\}.$$

That T is a binary subbase is easily checked (of course this is also shown in lemma 0.5). Also it is clear that for all $T_0,T_1 \in T$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap \phi[I] \neq \emptyset$. Hence theorem 2.2.5 implies that $\lambda(\phi[I],T \cap \phi[I]) \approx [0,\frac{1}{3}]^2$. Therefore

$$S_2 := \{[0,x] \cup [1-x,1] \mid 0 \le x \le \tfrac{1}{3}\} \cup \{[\tfrac{1}{3}-x,\tfrac{2}{3}+x] \mid 0 \le x \le \tfrac{1}{3}\} \cup$$
$$\cup \{[\tfrac{1}{3},x] \mid \tfrac{1}{3} \le x \le \tfrac{2}{3}\} \cup \{[x,1] \mid \tfrac{1}{3} \le x \le \tfrac{2}{3}\}$$

is a subbase for I such that $\lambda(I,S_2) \approx I^2$.

To get I^3 as a superextension of I we must embed I in $[0,\frac{1}{7}]^3$ as suggested in figure 6.

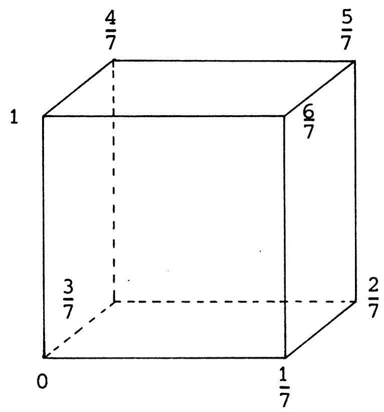


Figure 6.

Therefore

$$\begin{aligned} S_3 := &\{[0,x] \cup [\tfrac{3}{7}-x,\tfrac{4}{7}+x] \cup [1-x,1] \mid 0 \le x \le \tfrac{1}{7}\} \cup \\ &\cup \{[\tfrac{1}{7}-x,\tfrac{2}{7}+x] \cup [\tfrac{5}{7}-x,\tfrac{6}{7}+x] \mid 0 \le x \le \tfrac{1}{7}\} \cup \{[0,\tfrac{3}{7}+x] \mid 0 \le x \le \tfrac{1}{3}\} \cup \\ &\cup \{[\tfrac{4}{7}-x,1] \mid 0 \le x \le \tfrac{1}{7}\} \cup \{[0,\tfrac{1}{7}+x] \cup [\tfrac{6}{7}-x,1] \mid 0 \le x \le \tfrac{1}{7}\} \cup \\ &\cup \{[\tfrac{2}{7}-x,\tfrac{5}{7}+x] \mid 0 \le x \le \tfrac{1}{7}\} \end{aligned}$$

is a subbase for I such that $\lambda(I,S_3) \approx I^3$. It is clear that with a simple induction we now can construct the subbases S_n $(n \in \mathbb{N})$.

Using an embedding of I in I^∞ we can also construct a subbase S_∞ for I for which $\lambda(I,S_\infty) \approx I^\infty$. We will not describe the subbase S_∞ as there are much nicer subbases for I for which the corresponding superextension is homeomorphic to the Hilbert cube Q, cf. chapter 3. But it must be noticed that the first subbase for the closed unit segment with a superextension homeomorphic to the Hilbert cube was constructed in the indicate manner.

2.3. Extending continuous functions to superextensions

In this section we deal with the question under what conditions continuous functions can be extended over superextensions. This is of importance of course, since several properties of superextensions can be derived by considering the space to be a quotient of a superextension with a richter subbase (cf. VERBEEK [119]).

G.A. JENSEN [59] gives a solution of the extension problem but for some purposes her solution is not satisfactory. We will extend JENSEN's result, but our result still is not really satisfactory because we cannot give a necessary and sufficient condition for extension of continuous functions.

2.3.1. DEFINITION. Let S and T be two families of closed sets in the topological space X. We way that S *separates* T if for any $T_0, T_1 \in T$ with $T_0 \cap T_1 = \emptyset$ there exist $S_0, S_1 \in S$ such that $T_i \subset S_i$ $(i \in \{0,1\})$ and $S_0 \cap S_1 = \emptyset$.

Notation: $T \sqsubset S$.

2.3.2. DEFINITION (VERBEEK [119]). Let S be a T_1-subbase for the topological space X. Then a linked system $M \subset S$ is called a *pre-mls* if M is contained in precisely one mls $M' \in \lambda(X,S)$.

The following lemma will be used frequently without reference. It is straightforward to prove.

2.3.3. LEMMA (VERBEEK [119]). *Let S be a closed T_1-subbase for the topological space* X *and let $M \in \lambda(X,S)$. Then*

(i) *a linked system $P \subset S$ is a pre-mls iff $\forall S,S' \in S$: ($P \cup \{S\}$ and $P \cup \{S'\}$ are linked $\Rightarrow S \cap S' \neq \emptyset$);*

(ii) *if $P \subset S$ is a pre-mls, contained in M, then $M = \{S \in S \mid P \cup \{S\}$ is linked$\}$.*

The unique S-mls that contains a pre-mls $M \subset S$ is denoted by $\underline{M}$. We say that M *is a pre-mls for* $\underline{M}$.

We now can formulate the main result in this section.

2.3.4. THEOREM. *Let S be a T_1-subbase for* X, *let T be a normal T_1-subbase for* Y *and let* $f\colon X \to Y$ *be a continuous map satisfying*

$$\{f^{-1}[T] \mid T \in \mathcal{T}\} \sqsubset \mathcal{S}.$$

Then f *can be extended to a continuous map* $\bar{f}: \lambda(X,\mathcal{S}) \to \lambda(Y,\mathcal{T})$.
Moreover, if f *is onto, then* $\bar{f}$ *is onto.*
If f *is one to one and* $\{f[S] \mid S \in \mathcal{S}\} \sqsubset \mathcal{T}$ *then* $\bar{f}$ *is an embedding.*

PROOF. Define

$$\mathcal{A} := \{A \subset X \mid A \in \mathcal{S} \text{ or } \exists T \in \mathcal{T}: A = f^{-1}[T]\}.$$

Then $\mathcal{A}$ is a T_1-subbase for X. Choose $\mathcal{M} \in \lambda(X,\mathcal{S})$.

CLAIM 1. $\mathcal{M}$ *is a pre-mls in* $\mathcal{A}$.

Indeed, assume to the contrary that $\mathcal{M}$ were not a pre-mls in $\mathcal{A}$. Then there exist $A_0, A_1 \in \mathcal{A}$ with $A_0 \cap A_1 = \emptyset$ and $\mathcal{M} \cup \{A_i\}$ is linked ($i \in \{0,1\}$). Without loss of generality we may assume that $A_i \notin \mathcal{S}$ ($i \in \{0,1\}$) for if, say $A_0 \in \mathcal{S}$, it would follow that, since $\mathcal{M}$ is a maximal linked system, $A_0 \in \mathcal{M}$, which is a contradiction since $A_0 \cap A_1 = \emptyset$. Hence $A_i \in \{f^{-1}[T] \mid T \in \mathcal{T}\}$ ($i \in \{0,1\}$). Take $S_i \in \mathcal{S}$ such that $A_i \subset S_i$ ($i \in \{0,1\}$) and $S_0 \cap S_1 = \emptyset$. Now $\mathcal{M} \cup \{A_i\}$ is linked implies that $\mathcal{M} \cup \{S_i\}$ is linked and therefore $S_i \in \mathcal{M}$ ($i \in \{0,1\}$). This contradicts the linkedness of $\mathcal{M}$.

Now, let $\underline{\mathcal{M}}$ be the unique mls in $\mathcal{A}$ that contains $\mathcal{M}$.

CLAIM 2. $P\underline{\mathcal{M}} := \{T \in \mathcal{T} \mid f^{-1}[T] \in \underline{\mathcal{M}}\}$ *is a pre-mls in* $\mathcal{T}$.

Clearly $P\underline{\mathcal{M}}$ is linked. Suppose that $P\underline{\mathcal{M}}$ were not a pre-mls. Then there exist $T_0, T_1 \in \mathcal{T}$ such that $P\underline{\mathcal{M}} \cup \{T_i\}$ is linked ($i \in \{0,1\}$) but $T_0 \cap T_1 = \emptyset$. The normality of $\mathcal{T}$ implies the existence of $T_i' \in \mathcal{T}$ ($i \in \{0,1\}$) such that $T_0' \cup T_1' = Y$ and $T_0 \cap T_1' = \emptyset = T_0' \cap T_1$. Then $f^{-1}[T_0'] \cup f^{-1}[T_1'] = X$ and consequently, by proposition 2.2.2 (v), either $f^{-1}[T_0'] \in \underline{\mathcal{M}}$ or $f^{-1}[T_1'] \in \underline{\mathcal{M}}$. Without loss of generality assume that $f^{-1}[T_0'] \in \underline{\mathcal{M}}$. But then $T_0' \in P\underline{\mathcal{M}}$, which is a contradiction since $T_0' \cap T_1 = \emptyset$.

Now define

$$\bar{f}: \lambda(X,\mathcal{S}) \to \lambda(Y,\mathcal{T}) \text{ by } \bar{f}(\mathcal{M}) := \underline{P\underline{\mathcal{M}}}\ .$$

CLAIM 3. $\bar{f}$ *is continuous.*

It suffices to show that $\bar{f}^{-1}[T^+]$ is closed in $\lambda(X,\mathcal{S})$ for all $T \in \mathcal{T}$.

Therefore choose $T_1 \in \mathcal{T}$ arbitrarily and assume that $\mathcal{M} \notin \bar{f}^{-1}[T_1^+]$. Then $\bar{f}(\mathcal{M}) \notin T_1^+$ and consequently $P\underline{\mathcal{M}} \cup \{T_1\}$ is not linked, by claim 2. Choose $T_0 \in P\underline{\mathcal{M}}$ such that $T_0 \cap T_1 = \emptyset$. Also choose $T_i' \in \mathcal{T}$ $(i \in \{0,1\})$ such that $T_0' \cup T_1' = X$ and $T_0' \cap T_1 = \emptyset = T_0 \cap T_1'$. As $T_0 \in P\underline{\mathcal{M}}$ also $T_0' \in P\underline{\mathcal{M}}$ and consequently $\mathcal{M} \cup \{f^{-1}[T_0']\}$ is linked. Now as $\{f^{-1}[T] \mid T \in \mathcal{T}\} \sqsubset \mathcal{S}$ there are S_0 and S_1 in $\mathcal{S}$ satisfying $f^{-1}[T_0] \subset S_0$ and $f^{-1}[T_1'] \subset S_1$ and $S_0 \cap S_1 = \emptyset$. Define $U = X \backslash S_1$. We then have

$$f^{-1}[T_0] \subset S_0 \subset U \subset f^{-1}[T_0'].$$

Now, $T_0 \in P\underline{\mathcal{M}}$ implies that $\mathcal{M} \cup \{f^{-1}[T_0]\}$ is linked and therefore also $\mathcal{M} \cup \{S_0\}$ is linked. Hence $S_0 \in \mathcal{M}$ and consequently $\mathcal{M} \in U^+$. We claim that U^+ is a neighborhood of $\mathcal{M}$ which does not intersect $\bar{f}^{-1}[T_1^+]$. For take $\mathcal{L} \in U^+ \cap \bar{f}^{-1}[T_1^+]$. Then there is an $L \in \mathcal{L}$ such that $L \subset U$. Hence $\{f^{-1}[T_0']\} \cup \mathcal{L}$ is linked and therefore $T_0' \in \bar{f}(\mathcal{L})$. This is a contradiction, since $T_1 \cap T_0' = \emptyset$.

It now follows that $\bar{f}^{-1}[T^+]$ is closed and hence that $\bar{f}$ is continuous.

CLAIM 4. *The diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \underline{i}_X}\downarrow & & \downarrow{\scriptstyle \underline{i}_Y} \\ \lambda(X,\mathcal{S}) & \xrightarrow{\ \bar{f}\ } & \lambda(Y,\mathcal{T}) \end{array}$$

commutes.

Indeed, let $x \in X$. Then $\underline{i}(x)$ is the $\mathcal{S}$-mls $\{S \in \mathcal{S} \mid x \in S\}$ and $\bar{f}(\underline{i}(x))$ is the unique $\mathcal{T}$-mls containing the pre-mls

$$\{T \in \mathcal{T} \mid \{f^{-1}[T]\} \cup \{S \in \mathcal{S} \mid x \in S\} \text{ is linked}\}.$$

Let us show that $\underline{i}(f(x))$ contains this pre-mls. It then follows that $\bar{f}(\underline{i}(x)) = \underline{i}(f(x))$. Choose $T_1 \in \mathcal{T}$ such that $\{f^{-1}[T_1]\} \cup \{S \in \mathcal{S} \mid x \in S\}$ is linked, while moreover $f(x) \notin T_1$. Now, by the fact that $\mathcal{T}$ is a T_1-subbase, there is a $T_0 \in \mathcal{T}$ such that $f(x) \in T_0$ and $T_0 \cap T_1 = \emptyset$. Choose S_0 and S_1 in $\mathcal{S}$ satisfying $f^{-1}[T_i] \subset S_i$ $(i \in \{0,1\})$ and $S_0 \cap S_1 = \emptyset$. Then $x \in f^{-1}[T_0] \subset S_0$ which implies that $S_0 \cap f^{-1}[T_1] \neq \emptyset$. Contradiction.

CLAIM 5. *If* f *is onto then* $\bar{f}$ *is onto.*

Let $\mathcal{K} \in \lambda(Y,\mathcal{T})$ and define

$$\mathcal{L} := \{S \in \mathcal{S} \mid \exists T \in \mathcal{K}\colon f^{-1}[T] \subset S\}.$$

Since f is a surjection, L is a linked system. Choose $M \in \lambda(X,S)$ such that $L \subset M$. We assert that $\bar{f}(M) = K$. For this it suffices to prove that K contains the pre-mls $P\underline{M}$. Let us assume, to the contrary, that for some $T_0 \in P\underline{M}$ we have that $T_0 \notin K$. Then there is a $T_1 \in K$ such that $T_0 \cap T_1 = \emptyset$. Choose S_0 and S_1 in S such that $f^{-1}[T_i] \subset S_i$ $(i \in \{0,1\})$ and $S_0 \cap S_1 = \emptyset$. As $f^{-1}[T_0] \in \underline{M}$ also $S_0 \in \underline{M}$ and consequently $S_0 \in M$. But $T_1 \in K$ implies that $S_1 \in L \subset M$. This contradicts the linkedness of M.

<u>CLAIM</u> 6. *If* f *is one to one and* $\{f[S] \mid S \in S\} \sqsubset T$, *then* $\bar{f}$ *is an embedding.*

First notice that $\bar{f}\colon \lambda(X,S) \to \lambda(Y,T)$ is a closed mapping, since $\lambda(X,S)$ is compact and $\lambda(Y,T)$ is Hausdorff (theorem 2.2.4 (iii)).

It suffices to show that $\bar{f}$ is one to one. For this take $M_0, M_1 \in \lambda(X,S)$ such that $M_0 \neq M_1$. Choose S_0 and S_1 in S such that $S_i \in M_i$ $(i \in \{0,1\})$ and $S_0 \cap S_1 = \emptyset$. Clearly $f[S_0] \cap f[S_1] = \emptyset$ and hence there exist T_0 and T_1 in T such that $f[S_i] \subset T_i$ $(i \in \{0,1\})$ and $T_0 \cap T_1 = \emptyset$. As $S_i \subset f^{-1}[T_i]$ it follows that $T_i \in P\underline{M}_i$ $(i \in \{0,1\})$ and therefore $\bar{f}(M_0) \neq \bar{f}(M_1)$.

This completes the proof of the theorem. □

As noted in the introduction of this section theorem 2.3.4 does not give a necessary and sufficient condition for extension of continuous functions over superextensions. But if we, moreover, assume that the closed subbase S for X is a separating ring (cf. 0.A) and that f is a surjection, then the condition mentioned in the theorem is necessary and sufficient.

2.3.5. <u>COROLLARY</u>. *Let S be a separating ring of closed subsets of* X, *and let T be a normal T_1-subbase for* Y *and let* $f\colon X \to Y$ *be a continuous surjection. Then the following assertions are equivalent:*

(i) *there is a continuous surjection* $\bar{f}\colon \lambda(X,S) \to \lambda(Y,T)$ *such that* $\bar{f}\restriction X = f$.

(ii) $\{f^{-1}[T] \mid T \in T\} \sqsubset S$.

<u>PROOF</u>. We only need to show that (i) implies (ii).

Choose T_0 and T_1 in T and assume that $f^{-1}[T_0] \cap f^{-1}[T_1] = \emptyset$. Without loss of generality we may assume that both $f^{-1}[T_0]$ and $f^{-1}[T_1]$ are nonvoid. As $\wedge.\vee.S^+$ is a separating ring in $\lambda(X,S)$ there are $S_{ij} \in S$ $(i,j \le n)$ and $V_{k\ell} \in S$ $(k,\ell \le p)$ such that

$$\bar{f}^{-1}[T_0^+] \subset \bigcap_{i\leq n} \bigcup_{j\leq n} S_{ij}^+$$

and

$$\bar{f}^{-1}[T_1^+] \subset \bigcap_{k\leq p} \bigcup_{\ell\leq p} V_{k\ell}^+$$

and

$$\bigcap_{i\leq n} \bigcup_{j\leq n} S_{ij}^+ \cap \bigcap_{k\leq p} \bigcup_{\ell\leq p} V_{k\ell}^+ = \emptyset.$$

This is possible, since $\bar{f}^{-1}[T_0^+] \cap \bar{f}^{-1}[T_1^+] = \emptyset$. Now as $\bar{f}$ restricted to X is f it follows that

$$f^{-1}[T_0] = \bar{f}^{-1}[T_0^+] \cap X \subset \bigcap_{i\leq n} \bigcup_{j\leq n} S_{ij}^+ \cap X = \bigcap_{i\leq n} \bigcup_{j\leq n} S_{ij}$$

and

$$f^{-1}[T_1] = \bar{f}^{-1}[T_1^+] \cap X \subset \bigcap_{k\leq p} \bigcup_{\ell\leq p} V_{k\ell}^+ \cap X = \bigcap_{k\leq p} \bigcup_{\ell\leq p} V_{k\ell}.$$

Now, as $\mathcal{S} = \wedge.\vee.\mathcal{S}$, $f^{-1}[T_0]$ and $f^{-1}[T_1]$ are separated by elements of $\mathcal{S}$. □

In the light of theorem 2.3.4, the question arises whether the condition of normality of the subbase $\mathcal{T}$ for Y can be weaked in a natural way, say to weak normality. The following example shows that the answer to this question is in the negative.

2.3.6. EXAMPLE. Let $X = S_1$ be the boundary of the closed unit-square I^2. As in section 2.2, define

$$\mathcal{T} := \{A \subset I^2 \mid A = \pi_i^{-1}[0,x] \vee A = \pi_i^{-1}[x,1] \ (i \in \{0,1\}),\ x \in I\}.$$

Then $\mathcal{T}$ is a binary normal closed subbase for I^2 and also for all $T_0, T_1 \in \mathcal{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$. Hence we may apply theorem 2.2.5. To this end, define

$$\mathcal{T}^* := \{T \cap X \mid T \in \mathcal{T}\}.$$

Then $\mathcal{T}^*$ is a closed T_1-subbase for X and also $\lambda(X,\mathcal{T}^*) \approx I^2$ (theorem 2.2.5). Finally let

$$\mathcal{S} := \{A \subset X \mid A \text{ is an interval of length less than } 1\}.$$

Then $\mathcal{S}$ is a weakly normal binary subbase for X, which is not normal of course. Also $\mathcal{S} \sqsubset \mathcal{T}^*$.

Now assume that the identity mapping on X can be extended to a continuous f: $\lambda(X,\mathcal{T}^*) \to \lambda(X,\mathcal{S})$. By the binarity of $\mathcal{S}$ we have that $\lambda(X,\mathcal{S}) = X$ and hence it would follow that $X = S_1$ is a retract of the closed unit square I^2, which is a contradiction. □

The following corollary of theorem 2.3.4 was not stated explicitly in VERBEEK [119]; because of its importance we present it here, but we must acknowledge that it certainly was known to VERBEEK.

2.3.7. <u>COROLLARY</u>. *Let* X *be a topological space which admits a binary normal closed subbase* $\mathcal{S}$. *Then the mapping* r: $\lambda X \to X$ *defined by*

$$\{r(M)\} := \cap\{S \in \mathcal{S} \mid S \in M\}$$

is a retraction. □

The normality of the subbase $\mathcal{S}$ also is essential in this corollary: the 1-sphere S_1 admits a binary weakly normal subbase while it is not a retract of λS_1, since the latter space is an Absolute Retract (corollary 1.5.20) (recall that

(i) X normal implies that λX has a binary normal subbase (theorem 2.2.4 (iii)),

(ii) X connected implies that λX is connected (VERBEEK [119], cf. also section 2.5), and

(iii) X compact metric implies λX is compact metric (VERBEEK [119], cf. also corollary 2.4.10).)

2.3.8. Theorem 2.3.4 also implies that always X is C^*-embedded in λX. We argue as follows: let f: $X \to I$ be a continuous function; then, as the unit segment I has a binary normal subbase, there is a continuous extension $\bar{f}$: $\lambda X \to I$ (theorem 2.3.4).

This suggests the question of whether for any compact Hausdorff space Z and for any continuous function f: $X \to Z$ there is a continuous extension $\bar{f}$: $\lambda X \to Z$. This is a nontrivial question which has a nontrivial answer. The machinery developed in section 1.1 settles the question negatively. For let id: $\mathbb{N} \to \beta\mathbb{N}$ be the identity mapping on $\mathbb{N}$. Then there is a no continuous f: $\lambda\mathbb{N} \to \beta\mathbb{N}$ which extends id, since if there were such an f it would follow that $\beta\mathbb{N}$ would be the continuous image of a supercompact Hausdorff space, which is not the case (corollary 1.1.7).

2.3.9. <u>COROLLARY</u>. *Let* X *be a Tychonoff space. Then the closure of* X *in* $\lambda(X,Z(X))$ *is* βX.

<u>PROOF</u>. We show that X is C^*-embedded in $\lambda(X,Z(X))$. For let $f: X \to I$ be a continuous mapping. Then for each closed set $A \subset I$ the set $f^{-1}[A]$ is a zero-set in X. Consequently by theorem 2.3.4 there is a continuous extension $\bar{f}: \lambda(X,Z(X)) \to I$. Thus the closure of X in $\lambda(X,Z(X))$ is a Hausdorff compactification of X (recall that $Z(X)$ is a normal base, cf. 0.C) in which X is C^*-embedded. Now, by a well-known characterization of βX (cf. GILLMAN & JERISON [52]) we obtain the desired result. □

2.3.10. The concept of supernormality for subbases (cf. definition 2.2.1) seems to be pathological, since in compactification theory a closed subbase almost always fails to be supernormal. In our construction for λI however, cf. chapter 3, subbases which are supernormal appear in a natural way and therefore it is worthwile to derive some properties of superextensions relative supernormal subbases, using theorem 2.3.4.

Our main interest lies in the following problem: given two subbases $\mathcal{S}$ and $\mathcal{T}$ of a topological space X, what can be said about $\lambda(X,\mathcal{S}\cup\mathcal{T})$ in terms of $\lambda(X,\mathcal{S})$ and $\lambda(X,\mathcal{T})$? In general the answer is: nothing; but if we make the additional assumption that $\mathcal{S}$ and $\mathcal{T}$ are both supernormal then there turns out to exist a very nice and very important relation between $\lambda(X,\mathcal{S}\cup\mathcal{T})$ and $\lambda(X,\mathcal{S})$ and $\lambda(X,\mathcal{T})$. We will show that then $\lambda(X,\mathcal{S}\cup\mathcal{T})$ can be embedded, in a natural way, in $\lambda(X,\mathcal{S}) \times \lambda(X,\mathcal{T})$. First we need some simple lemma's.

2.3.11. <u>LEMMA</u>. *Let* $\mathcal{S}$ *be a closed supernormal* T_1*-subbase for* X *and let* $\mathcal{U}$ *be a closed* T_1*-subbase such that* $\mathcal{S} \subset \mathcal{U}$. *Then for all* $\mathcal{M} \in \lambda(X,\mathcal{U})$ *the collection* $\mathcal{M} \cap \mathcal{S}$ *is an mls in* $\mathcal{S}$.

<u>PROOF</u>. Let $\mathcal{M} \in \lambda(X,\mathcal{U})$ and define $P\mathcal{M} := \mathcal{M} \cap \mathcal{S}$. From the normality of $\mathcal{S}$ it follows that $P\mathcal{M} \neq \emptyset$, and therefore $P\mathcal{M}$ is a linked system. Suppose that $P\mathcal{M}$ is not maximally linked. Then there exists an $S_0 \in \mathcal{S}$ such that $P\mathcal{M} \cup \{S_0\}$ is linked and $S_0 \notin P\mathcal{M}$. Clearly $S_0 \notin \mathcal{M}$ and consequently there is an $M \in \mathcal{M}$ such that $M \cap S_0 = \emptyset$. Since $\mathcal{S}$ is supernormal there is an $S^* \in \mathcal{S}$ with $M \subset S^*$ and $S^* \cap S_0 = \emptyset$. This is a contradiction, however, since $M \in \mathcal{M}$ implies that $S^* \in \mathcal{M}$ and therefore $S^* \in P\mathcal{M}$. □

2.3.12. <u>COROLLARY</u>. *Let* $\mathcal{S}$ *be a supernormal* T_1*-subbase for* X *and let* $\mathcal{U}$ *be a closed* T_1*-subbase for* X *such that* $\mathcal{S} \subset \mathcal{U}$. *Then* $\lambda(X,\mathcal{S})$ *is a Hausdorff* *quotiënt of* $\lambda(X,\mathcal{U})$ *under the mapping* f *defined by*

$$f(M) := M \cap S.$$

Moreover, f *is the identity of* X.

PROOF. This immediately follows from lemma 2.3.11 and from the proof of theorem 2.3.4. □

We now can formulate the announced embedding property of superextensions with respect to supernormal subbases.

2.3.13. THEOREM. *Let* $\{S_\alpha \mid \alpha \in I\}$ *be a collection of supernormal* T_1*-subbases for the topological space* X. *Then* $\bigcup_{\alpha\in I} S_\alpha$ *is a supernormal subbase for* X. *Moreover the mapping* $e\colon \lambda(X,\bigcup_{\alpha\in I} S_\alpha) \to \Pi'_{\alpha\in I}\, \lambda(X,S_\alpha)$ *defined by*

$$(e(M))_\alpha := M \cap S_\alpha$$

is an embedding.

PROOF. The statement that $\bigcup_{\alpha\in I} S_\alpha$ is a supernormal subbase can easily be checked using the fact that all the S_α's are supernormal ($\alpha \in I$).

Let $f_\alpha\colon \lambda(X,\bigcup_{\alpha\in I} S_\alpha) \to \lambda(X,S_\alpha)$ be the mapping described in corollary 2.3.12, i.e. $f_\alpha(M) = S_\alpha \cap M$. Then the evaluation mapping

$$e\colon \lambda(X,\bigcup_{\alpha\in I} S_\alpha) \to \Pi_{\alpha\in I}\, \lambda(X,S_\alpha)$$

defined by $(e(M))_\alpha = f_\alpha(M)$ is continuous. Also it is a closed mapping, since $\lambda(X,\bigcup_{\alpha\in I} S_\alpha)$ and $\Pi_{\alpha\in I}\, \lambda(X,S_\alpha)$ both are compact Hausdorff spaces (cf. theorem 2.2.4 (iii)). We will proceed to show that e is one to one.

To this end, choose two distinct elements M_0 and M_1 in $\lambda(X,\bigcup_{\alpha\in I} S_\alpha)$. In addition take $M_i \in M_i$ ($i \in \{0,1\}$) such that $M_0 \cap M_1 = \emptyset$. Choose $\alpha_0 \in I$ such that $M_0 \in S_{\alpha_0}$. Then, since S_{α_0} is supernormal and M_1 is an mls in $\bigcup_{\alpha\in I} S_\alpha$, we may assume that also $M_1 \in S_{\alpha_0}$. But then $M_i \in f_{\alpha_0}(M_i)$ ($i \in \{0,1\}$) by corollary 2.3.12, and as $M_0 \cap M_1 = \emptyset$ we conclude that $f_{\alpha_0}(M_0) \neq f_{\alpha_0}(M_1)$. □

If $\{S_\alpha \mid \alpha \in I\}$ is a collection of supernormal subbases for X then we will often study $\lambda(X,\bigcup_{\alpha\in I} S_\alpha)$ as a subspace of $\Pi_{\alpha\in I}\, \lambda(X,S_\alpha)$. Hence let us identify $\lambda(X,\bigcup_{\alpha\in I} S_\alpha)$ and $e[\lambda(X,\bigcup_{\alpha\in I} S_\alpha)]$. It then is useful to know what points of $\Pi_{\alpha\in I}\, \lambda(X,S_\alpha)$ belong to $\lambda(X,\bigcup_{\alpha\in I} S_\alpha)$. There is a simple characterization for these points, as the following lemma shows.

Notice that a point $x = (x_\alpha)_{\alpha\in I}$ of $\Pi_{\alpha\in I}\ \lambda(X,S_\alpha)$ is a point of which the coordinates are maximal linked systems, so that we can speak of $\bigcup_{\alpha\in I} x_\alpha$.

2.3.14. LEMMA. *Let* $\{S_\alpha \mid \alpha \in I\}$ *be a collection of supernormal subbases for* X. *Then* $x \in \Pi_{\alpha\in I}\ \lambda(X,S_\alpha)$ *belongs to* $\lambda(X,\bigcup_{\alpha\in I} S_\alpha)$ *if and only if* $\bigcup_{\alpha\in I} x_\alpha$ *is a linked system.*

PROOF. Let $S := \bigcup_{\alpha\in I} S_\alpha$. If $x \in \lambda(X,S)$ then $x = \bigcup_{\alpha\in I} x_\alpha$, so $\bigcup_{\alpha\in I} x_\alpha$ is linked. Conversely, let $\bigcup_{\alpha\in I} x_\alpha$ be linked. Then $\bigcup_{\alpha\in I} x_\alpha$ is an mls in S. Indeed, suppose $\bigcup_{\alpha\in I} x_\alpha \cup \{S\}$ is linked for some $S \in S_{\alpha_0}$, with $\alpha_0 \in I$. Then $x_{\alpha_0} \cup \{S\}$ is linked, hence $S \in x_{\alpha_0}$ since x_{α_0} is an mls in S_{α_0}. Therefore $S \in \bigcup_{\alpha\in I} x_\alpha$. It is easy to see that $e[\bigcup_{\alpha\in I} x_\alpha] = x$. □

The importance of theorem 2.3.13 and lemma 2.3.14 is that one can study the behaviour of a superextension relative the union of certain subbases in a product of superextensions. We will demonstrate this by two examples. The examples are both superextensions of the closed unit interval; they are constructed in a similar way as in section 2.2. Hence we have to use theorem 2.2.5. The examples are both homeomorphic to I^3 and hence they are homeomorphic. This demonstrates that a topological space can have many quite distinct binary (normal) subbases.

2.3.15. EXAMPLES.

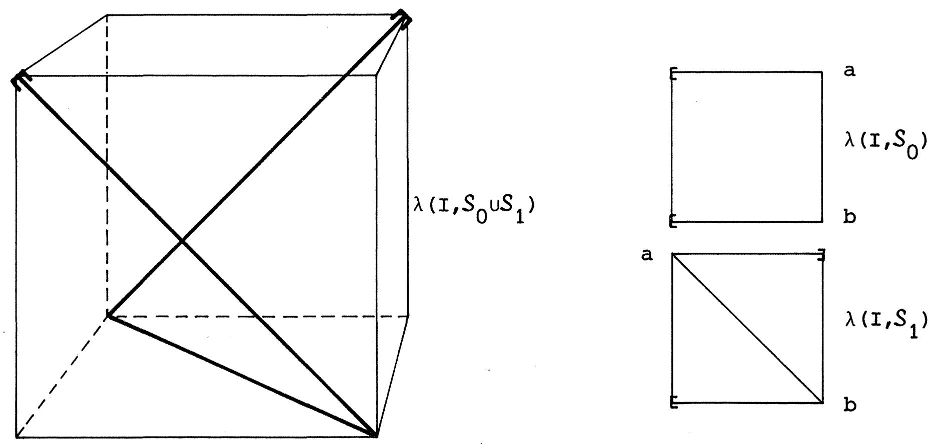


Figure 7.

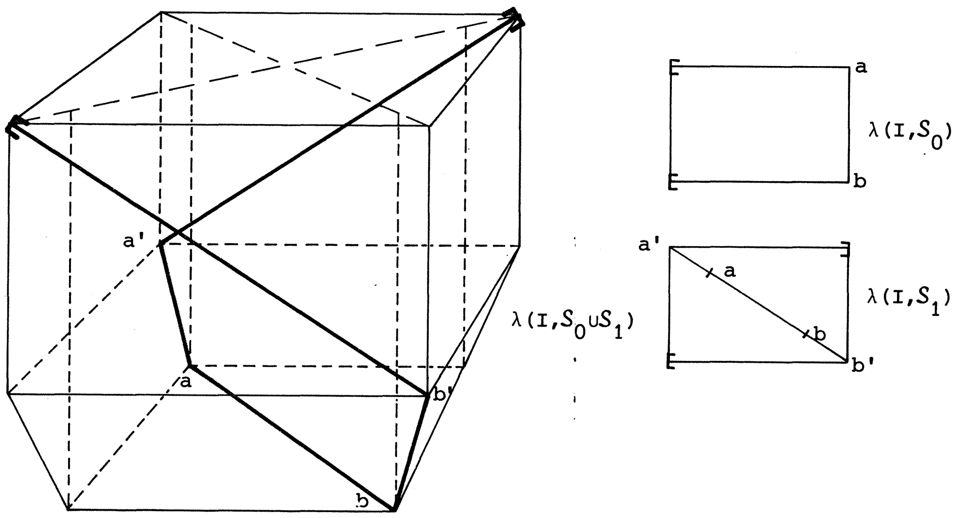


Figure 8.

2.4. A partial ordering on the set of all superextensions of a fixed space

It is natural to ask whether the set of all superextensions of a fixed topological space X can be partially ordered in a natural way, analogous to the usual ordering of Hausdorff compactifications (cf. DUGUNDJI [44]). This turns out to be the case. There also is a relation between the partial ordering of Hausdorff compactifications, mentioned above, and the partial ordering of superextensions.

2.4.1. DEFINITION. Two superextensions of a topological space X are defined to be *equivalent*, when there exists a homeomorphism between them which on X is the identity.

As a first step we derive a sufficient condition for equivalence of superextensions in terms of their generating subbases. This result was suggested by a theorem of STEINER [114].

2.4.2. THEOREM. *Let S and T be two T_1-subbases for* X *such that $S \sqsubset T$ and $T \sqsubset S$ (see definition* 2.3.1*). Then* $\lambda(X,S)$ *and* $\lambda(X,T)$ *are equivalent.*

<u>PROOF</u>. For $\mathcal{A} \subset \mathcal{S}$ define $P\mathcal{A} \subset \mathcal{T}$ by

$$P\mathcal{A} := \{T \in \mathcal{A} \mid \exists A \in \mathcal{A}\colon A \subset T\}.$$

For $\mathcal{B} \subset \mathcal{T}$ define $Q\mathcal{B} \subset \mathcal{S}$ by

$$Q\mathcal{B} := \{S \in \mathcal{S} \mid \exists B \in \mathcal{B}\colon B \subset S\}.$$

<u>CLAIM</u> 1. *If $\mathcal{M} \subset \mathcal{S}$ is a pre-mls in $\mathcal{S}$, then $P\mathcal{M}$ is a pre-mls in $\mathcal{T}$. If $\mathcal{N} \subset \mathcal{T}$ is a pre-mls in $\mathcal{T}$ then $Q\mathcal{N}$ is a pre-mls in $\mathcal{S}$.*

By symmetry it suffices to prove the first statement. Let $\mathcal{M} \subset \mathcal{S}$ be a pre-mls in $\mathcal{S}$. It is clear that $P\mathcal{M}$ is linked. Suppose $P\mathcal{M}$ were not a pre-mls in $\mathcal{T}$. Then there are disjoint T_0, T_1 in $\mathcal{T}$ such that $P\mathcal{M} \cup \{T_i\}$ is linked ($i \in \{0,1\}$). Since $\mathcal{T} \sqsubset \mathcal{S}$ there are disjoint S_0, S_1 in $\mathcal{S}$ with $T_i \subset S_i$ ($i \in \{0,1\}$). Clearly $P\mathcal{M} \cup \{S_i\}$ is linked ($i \in \{0,1\}$), hence $\mathcal{M} \cup \{S_i\}$ is linked ($i \in \{0,1\}$). For suppose there is an $M \in \mathcal{M}$ not intersecting S_0. Then $\mathcal{S} \sqsubset \mathcal{T}$ implies that there is a $T' \in \mathcal{T}$ such that $M \subset T'$ and $T' \cap S_0 = \emptyset$. Then $T' \in P\mathcal{M}$ which contradicts the linkedness of $P\mathcal{M} \cup \{T_0\}$. Therefore $\mathcal{M} \cup \{S_i\}$ is linked ($i \in \{0,1\}$) which contradicts the fact that $\mathcal{M}$ is a pre-mls.

Now define

$$\phi\colon \lambda(X,\mathcal{S}) \to \lambda(X,\mathcal{T}) \quad \text{and} \quad \psi\colon \lambda(X,\mathcal{T}) \to \lambda(X,\mathcal{S})$$

by

$$\phi(\mathcal{M}) := \underline{P\mathcal{M}} \qquad \text{and} \qquad \psi(\mathcal{N}) := \underline{Q\mathcal{N}}\ .$$

<u>CLAIM</u> 2. $\psi^{-1} = \phi$; *consequently ϕ is a bijection.*

By symmetry it suffices to prove that $\psi(\phi(\mathcal{M})) = \mathcal{M}$ for all $\mathcal{M} \in \lambda(X,\mathcal{S})$. Let $\mathcal{M} \in \lambda(X,\mathcal{S})$ be arbitrary. Then $QP\mathcal{M} \subset \psi(\phi(\mathcal{M}))$. But $QP\mathcal{M}$ is a pre-mls in $\mathcal{S}$ by claim 1, and it is easy to see that $QP\mathcal{M} \subset \mathcal{M}$. Hence $\psi(\phi(\mathcal{M})) = \mathcal{M}$.

<u>CLAIM</u> 3. *The diagram*

$$\begin{tikzcd} & \lambda(X,\mathcal{S}) \arrow[d, "\phi"] \\ X \arrow[ur, "\underline{i}_{\mathcal{T}}"] \arrow[r, "\underline{i}_{\mathcal{T}}"'] & \lambda(X,\mathcal{T}) \end{tikzcd}$$

commutes.

Indeed, let $x \in X$. Then $\underline{i}_{\mathcal{T}}(x)$ is the $\mathcal{T}$-mls $\{T \in \mathcal{T} \mid x \in T\}$, while $\phi(\underline{i}_{\mathcal{S}}(x))$ is the unique $\mathcal{T}$-mls containing the pre-mls

$$P\underline{i}_{\mathcal{S}}(x) = \{T \in \mathcal{T} \mid \exists S \in \mathcal{S}\colon x \in S \subset T\}.$$

However, if $T \in \underline{i}_{\mathcal{T}}(x)$, then clearly $P\underline{i}_{\mathcal{S}}(x) \cup \{T\}$ is linked, and so $T \in P\underline{i}_{\mathcal{S}}(x)$. It follows that $\underline{i}_{\mathcal{T}}(x) = \phi(\underline{i}_{\mathcal{S}}(x))$.

CLAIM 4. *ϕ is a homeomorphism.*

It suffices to show that ϕ is continuous, because for symmetry reasons it then follows that $\phi^{-1} = \psi$ is continuous too.

So take any $T \in \mathcal{T}$; we must prove that $\phi^{-1}[T^+]$ is closed in $\lambda(X,\mathcal{S})$. Now

$$\phi^{-1}[T^+] = \psi[T^+] = \{\psi(\mathcal{N}) \mid \mathcal{N} \in T^+\}$$

$$= \{\underline{Q\mathcal{N}} \mid \mathcal{N} \in \lambda(X,\mathcal{T}) \text{ and } T \in \mathcal{N}\}.$$

If $S \in \mathcal{S}$ and $T \subset S$ then $S \in Q\mathcal{N}$ for every $\mathcal{N} \in T^+$, hence $\underline{Q\mathcal{N}} \in S^+$ for any $\mathcal{N} \in T^+$; thus

$$\phi^{-1}[T^+] \subset \bigcap\{S^+ \mid S \in \mathcal{S} \text{ and } T \subset S\}.$$

Conversely, if $\mathcal{M} \notin \phi^{-1}[T^+]$, then $T \notin \phi(\mathcal{M})$ and consequently $P\mathcal{M} \cup \{T\}$ is not linked, so $T_0 \cap T = \emptyset$ for some $T_0 \in P\mathcal{M}$. As $\mathcal{T} \sqsubset \mathcal{S}$ there are $S_0, S \in \mathcal{S}$ such that $T_0 \subset S_0$, $T \subset S$ and $S_0 \cap S = \emptyset$. Exactly as in the proof of claim 1 we derive that $S_0 \in \mathcal{M}$; therefore $S \notin \mathcal{M}$, or $\mathcal{M} \notin S^+$. It now follows that

$$\phi^{-1}[T^+] = \bigcap\{S^+ \mid S \in \mathcal{S} \text{ and } T \subset S\}$$

and hence that $\phi^{-1}[T^+]$ is closed. $\square$

Theorem 2.4.2 leads us to the announced partial ordering on the class of all superextensions of a fixed topological space X.

2.4.3. DEFINITION. Let X be a topological space and let $\mathcal{K} := \{\lambda(X,\mathcal{S}) \mid \mathcal{S}$ is a T_1-subbase for X$\}$. Define an order "$\le$" on $\mathcal{K}$ by

$$\lambda(X,\mathcal{S}) \le \lambda(X,\mathcal{T}) \quad \text{iff} \quad \mathcal{S} \sqsubset \mathcal{T}.$$

2.4.4. COROLLARY. *If we identify equivalent superextions, "$\le$" is a partial order.*

PROOF. It suffices to prove that "$\le$" is an antisymmetric and this a corollary of theorem 2.4.2. □

2.4.5. Let F be a family of nonempty closed subsets of the topological space X. Then we put

$$\omega(X,F) := \{A \subset F \mid A \text{ is maximal with respect to the finite intersection property}\}.$$

For each $F \in F$ we define $F^* := \{A \in \omega(X,F) \mid F \in A\}$. As a closed subbase for a topology on $\omega(X,F)$ we take the collection

$$F^* := \{F^* \mid F \in F\}.$$

With this topology $\omega(X,F)$ is called a *Wallman space*. In case F is a T_1-subbase $\omega(X,F)$ is a compactification of X and is called the *Wallman compactification of* X *with respect to* F (cf. chapter 4). STEINER [114] showed

(a) *every Wallman space is compact, and* $\omega(X,F)$ *is homeomorphic to* $\omega(X,\wedge.\vee.F)$;

(b) *if* S *and* T *are separating rings of closed sets in* X, *then* $\omega(X,S)$ *and* $\omega(X,T)$ *are equivalent compactifications iff* $S \sqsubset T$ *and* $T \sqsubset S$.

The first part of (a) is also true for superextensions; every superextension is (super) compact. The second part unfortunately does not hold for superextensions.

2.4.6. <u>EXAMPLE</u>. Let $X = \{x_1,x_2,x_3\}$ be a space with 3 points with discrete topology. Define

$$S := \{\{x_1\},\{x_2\},\{x_3\}\}.$$

Then S is a closed binary subbase for X. Hence $\lambda(X,S) = X$. Let $T := \wedge.\vee.S$. Then there is precisely one free mls M in T (i.e. an mls with an empty intersection);

$$M = \{\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_2\},X\}.$$

Hence $\lambda(X,T)$ is a space of 4 points and hence is not homeomorphic to $\lambda(X,S)$. □

However (b) is true for superextensions; this is a direct consequence of theorem 2.4.2.

2.4.7. THEOREM. *Let S and T be two separating rings of closed subsets of* X. *Then* $\lambda(X,S)$ *and* $\lambda(X,S)$ *are equivalent iff* $S \sqsubset T$ *and* $T \sqsubset S$.

PROOF. This is a consequence of theorem 2.4.2 and of the proof of corollary 2.3.5. □

Finally, the partial ordering, constructed in this section, has much in common with the usual ordering or compactifications if we restrict ourselves to superextensions with respect to normal subbases.

2.4.8. COROLLARY. *Let S be a normal* T_1*-subbase for* X *and let T be a* T_1*-subbase for* X. *Then* $\lambda(X,S) \le \lambda(X,T)$ *implies that there is a continuous surjection* $f\colon \lambda(X,T) \to \lambda(X,S)$ *which on* X *is the identity.*

PROOF. This is a consequence of theorem 2.3.4. □

2.4.9. COROLLARY. *Let S be a separating ring of closed subsets of* X *and let T be a normal* T_1*-subbase for* X. *Then the following assertions are equivalent:*

(i) $\lambda(X,T) \le \lambda(X,S)$;

(ii) *there is a continuous surjection* $f\colon \lambda(X,S) \to \lambda(X,T)$ *such that* f *restricted to* X *is the identity.* □

The following important corollary of theorem 2.4.2. is due to VERBEEK [119].

2.4.10. COROLLARY. λX *is metrizable if and only if* X *is compact metrizable.*

PROOF. Assume that λX is metrizable; then X is normal and consequently λX is equivalent to $\lambda(X,Z(X))$ (cf. 2.4.2). Hence βX is a subspace of λX (cf. 2.3.9). But then βX is metrizable and hence X is compact.

On the other hand, assume that X is compact and metrizable. Let S be a countable closed base for X. Then $\wedge.\vee.S$ is a countable closed subbase for X which separates the closed subsets of X (cf. 0.2). Then $\lambda(X,\wedge.\vee.S)$ and λX are equivalent (theorem 2.4.2) and consequently $\lambda(X,\wedge.\vee.S)$ is a compact Hausdorff space with a countable closed subbase. Hence, by URYSOHN's metrization theorem (cf. DUGUNDJI [44]) $\lambda(X,\wedge.\vee.S)$ is metrizable and therefore λX is metrizable too. □

2.5. Connectedness in superextensions

We now turn our attention to connectedness in superextensions. Superextensions behave surprisingly nice with respect to connectedness. VERBEEK [119] showed that X *is connected if and only if* λX *is connected and locally connected*. From this, he derived that a superextension $\lambda(X,\mathcal{S})$ of a connected space X with respect to a normal T_1-subbase $\mathcal{S}$ is both connected and locally connected. Also the superextension $\lambda(X,\mathcal{S})$ of a connected space X with respect to a subbase $\mathcal{S}$ that contains all finite subsets of X is both connected and locally connected.

Since the Hilbert cube Q has a dense subset homeomorphic to the rationals it follows from theorem 1.4.5, theorem 1.4.3 and theorem 1.4.2 that the space of the rationals has a superextension homeomorphic to the Hilbert cube. In view of this example VERBEEK's results on connectedness of superextensions do not cover all situations (this he also noticed himself, see [119] p.143). We will show the following: *let* X *be a topological space and let* $\mathcal{S}$ *be a* T_1*-subbase for* X *that satisfies one of the following conditions:*

(i) $\mathcal{S}$ *is closed under finite unions;*

(ii) $\mathcal{S}$ *is normal.*

Then $\lambda(X,\mathcal{S})$ *is connected and locally connected if and only if for all nonvoid* $S_0,S_1 \in \mathcal{S}$: $(S_0 \cap S_1 = \emptyset \Rightarrow S_0 \cup S_1 \neq X)$. This proves once again, and at the same time generalizes some of the results of VERBEEK [119] mentioned above.

Our method of proof is not a generalization of VERBEEK's proof. We work with partial orderings while VERBEEK [119] used very technical results concerning types of maximal linked system.

2.5.1. THEOREM. *Let* $\mathcal{S}$ *be a normal* T_1*-subbase for the topological space* X. *Then the following assertions are equivalent:*

(i) $\lambda(X,\mathcal{S})$ *is connected;*

(ii) $\lambda(X,\mathcal{S})$ *is connected and locally connected;*

(iii) *for all nonvoid* $S_0,S_1 \in \mathcal{S}$: $(S_0 \cap S_1 = \emptyset \Rightarrow S_0 \cup S_1 \neq X)$.

PROOF. The implications (ii) $\Rightarrow$ (i), (i) $\Rightarrow$ (iii) are trivial. In addition (i) $\Rightarrow$ (ii) follows from corollary 1.5.8 (iii). Therefore we only prove (iii) $\Rightarrow$ (i).

In view of theorem 1.5.22 we need only show that $H(\lambda(X,\mathcal{S}),\mathcal{S}^+)$ is densely ordered by inclusion ($H(\lambda(X,\mathcal{S}),\mathcal{S}^+)$ is compact, cf. section 2.10). Therefore let A and B be elements of $H(\lambda(X,\mathcal{S}),\mathcal{S}^+)$ such that A is properly contained in B. Choose $\mathcal{M} \in B\backslash A$. As A is $\mathcal{S}^+$-closed, there are $M_i \in \mathcal{S}$ ($i \in \{0,1\}$) such that $\mathcal{M} \in M_0^+$, $A \subset M_1^+$ and $M_0^+ \cap M_1^+ = \emptyset$. Then $M_0 \cap M_1 = \emptyset$ and by the normality of $\mathcal{S}$ there are $T_i \in \mathcal{S}$ ($i \in \{0,1\}$) such that $M_0 \cap T_1 = \emptyset = T_0 \cap M_1$ and $T_0 \cup T_1 = X$. Then $T_0 \cap T_1 \neq \emptyset$, by our assumptions. Define $C := B \cap T_1^+$. Then $A \subset C \subset B$. We first note that A is a proper subset of C. Indeed, since $\{T_0^+, T_1^+, B\}$ is linked we have that $T_0^+ \cap T_1^+ \cap B \neq \emptyset$. Hence $\emptyset \neq T_0^+ \cap T_1^+ \cap C \subset C\backslash A$. Next we note that C is a proper subset of B, since $\mathcal{M} \in B\backslash C$. This completes the proof of the theorem. □

We now prove connectedness of superextensions with respect to subbases closed under finite unions.

From now on, let X be a topological space and let $\mathcal{S}$ be a T_1-subbase for X closed under finite unions. As in section 1.5 for all $\mathcal{M},\mathcal{N} \in \lambda(X,\mathcal{S})$ define $I(\mathcal{M},\mathcal{N}) \subset \lambda(X,\mathcal{S})$ by

$$I(\mathcal{M},\mathcal{N}) := \bigcap\{S^+ \mid S \in \mathcal{M} \cap \mathcal{N}\}.$$

We need a simple lemma, which is strongly related to theorem 1.5.13.

2.5.2. LEMMA.

(i) *For all $\mathcal{M},\mathcal{N},\mathcal{P} \in \lambda(X,\mathcal{S})$ the intersection $I(\mathcal{M},\mathcal{N}) \cap I(\mathcal{N},\mathcal{P}) \cap I(\mathcal{M},\mathcal{P})$ consists of one point;*

(ii) *for all $M \in \mathcal{M} \in I(\mathcal{N},\mathcal{P})$ we have that $M \in \mathcal{N}$ or $M \in \mathcal{P}$;*

(iii) *for all $\mathcal{M},\mathcal{N} \in \lambda(X,\mathcal{S})$ the relation $\leq_{\mathcal{M}}$ defined on $I(\mathcal{M},\mathcal{N})$ by $\mathcal{L} \leq_{\mathcal{M}} \mathcal{H}$ iff $\mathcal{L} \in I(\mathcal{M},\mathcal{H})$ is a partial ordering;*

(iv) *for all $\mathcal{M},\mathcal{N} \in \lambda(X,\mathcal{S})$ and all $\mathcal{L}_0,\mathcal{L}_1 \in I(\mathcal{M},\mathcal{N})$ such that $\mathcal{L}_0 \leq_{\mathcal{M}} \mathcal{L}_1$, the following holds: $I(\mathcal{L}_0,\mathcal{L}_1) = \{\mathcal{P} \in I(\mathcal{M},\mathcal{N}) \mid \mathcal{L}_0 \leq_{\mathcal{M}} \mathcal{P} \leq_{\mathcal{M}} \mathcal{L}_1\}$.*

PROOF. We will first prove (ii). To this end, take $M \in \mathcal{M} \in I(\mathcal{N},\mathcal{P})$ such that $M \notin \mathcal{N}$ and $M \notin \mathcal{P}$. Then there are $N \in \mathcal{N}$ and $P \in \mathcal{P}$ such that $M \cap N = \emptyset = M \cap P$. But then $M \cap (N \cup P) = \emptyset$ and as $I(\mathcal{N},\mathcal{P}) \subset (N \cup P)^+$ this is a contradiction (for $\mathcal{M} \in I(\mathcal{N},\mathcal{P})$ implies that $N \cup P \in \mathcal{M}$, contradicting the linkedness of $\mathcal{M}$).

To prove (i), take distinct $\mathcal{L},\mathcal{H} \in I(\mathcal{M},\mathcal{N}) \cap I(\mathcal{M},\mathcal{P}) \cap I(\mathcal{N},\mathcal{P})$. Also choose $L \in \mathcal{L}$ and $H \in \mathcal{H}$ such that $L \cap H = \emptyset$. By (ii) there are at least two distinct elements of $\{\mathcal{M},\mathcal{N},\mathcal{P}\}$ containing L. By the same reasons there are at least two distinct elements of $\{\mathcal{M},\mathcal{N},\mathcal{P}\}$ both containing H. Hence there is at least one element of $\{M,N,P\}$ containing both L and H, which is a contradiction.

To prove (iii), we only need to check that $\leq_M$ is anti-symmetric. Let $L_0,L_1 \in I(M,N)$ such that $L_0 \leq_M L_1$ and $L_1 \leq_M L_0$. Then $L_0 \in I(M,L_1)$ and consequently, by (i), $\{L_0\} = I(M,L_1) \cap I(M,L_0) \cap I(L_0,L_1)$. In the same way, as $L_1 \in I(M,L_0)$ we also have that $\{L_1\} = I(M,L_0) \cap I(M,L_1) \cap I(L_0,L_1)$. Hence L_0 equals L_1.

To prove (iv), take $L_0,L_1 \in I(M,N)$ such that $L_0 \leq_M L_1$. Choose $P \in I(L_0,L_1)$. Assume that $L_0 \not\leq P$. Then $L_0 \notin I(M,P)$ and consequently there is an $L \in L_0$ such that $L \notin M$ and $L \notin P$. Now, since $L_0 \in I(M,L_1)$, by (ii) it follows that $L \in L_1$. This is a contradiction since $L \in L_0 \cap L_1$ implies that

$$P \in I(L_0,L_1) \subset L^+.$$

This shows that $L_0 \leq_M P$. To prove that also $P \leq_M L_1$, notice that $L_0 \leq_M L_1$ implies that $P \in I(L_0,L_1) \subset I(M,L_1)$. Therefore $P \leq_M L_1$. This proves that $I(L_0,L_1) \subset \{P \in I(M,N) \mid L \leq_M P \leq_M L_1\}$. Now take $P \in I(M,N)$ such that $L_0 \leq_M P \leq_M L_1$ and assume that $P \notin I(L_0,L_1)$. Then there is a $P \in P$ such that $P \notin L_0$ and $P \notin L_1$. Since $P \in I(M,L_1)$ and since $P \notin L_1$, by (ii) it follows that $P \in M$. But then $P \in M \cap P$ which implies that $P \in L_0$ since $L_0 \in I(M,P)$. This is a contradiction. This completes the proof of the equality $I(L_0,L_1) = \{P \in I(M,N) \mid L_0 \leq_M P \leq_M L_1\}$. □

2.5.3. <u>THEOREM</u>. *Let* X *be a topological space and let* S *be a closed* T_1-*subbase for* X *which is closed under finite unions. Then the following assertions are equivalent:*

(i) $\lambda(X,S)$ *is connected;*

(ii) $\lambda(X,S)$ *is connected and locally connected;*

(iii) *for all nonvoid* $S_0,S_1 \in S$ $(S_0 \cap S_1 = \emptyset \Rightarrow S_0 \cup S_1 \neq X)$.

<u>PROOF</u>. The implications (ii) ⇒ (i) and (i) ⇒ (iii) are trivial. We will only establish the implication (iii) ⇒ (ii).

For this, take $M,N \in \lambda(X,S)$ and consider $I(M,N)$. By lemma 2.5.2 this set is partially ordered, by $\leq_M$. For simplicity of notation we from now on suppress the index M in the ordering.

<u>CLAIM</u> 1. $\leq$ *is order dense.*

Indeed, take distinct L_0 and L_1 in $I(M,N)$ such that $L_0 \leq L_1$. We assert that $|I(L_0,L_1)| > 2$. For assume to the contrary that $I(L_0,L_1) = \{L_0,L_1\}$. Choose $L_i \in L_i$ $(i \in \{0,1\})$ such that $L_0 \cap L_1 = \emptyset$. We will show that $L_0 \cup L_1 = X$. For choose $x \in X$. Then $I(x,L_0) \cap I(x,L_1) \cap I(L_0,L_1)$ is a singleton (cf. lemma 2.5.2 (i)). Hence, without loss of generality $\{L_0\} = I(x,L_0) \cap I(x,L_1) \cap I(L_0,L_1)$. Hence $L_0 \in I(x,L_1)$, which implies that $x \in L_0$, since $L_0 \notin L_1$ (cf. lemma 2.5.2 (ii)).

Therefore $L_0 \cup L_1 = X$; but this contradicts (iii).

We conclude that there is a $P \in I(L_0,L_1)$ such that $P \neq L_i$ $(i \in \{0,1\})$. However, it is clear that $L_0 \le P \le L_1$, which implies that $\le$ is order dense.

CLAIM 2. *There is an ordered continuum in* $I(M,N)$ *connecting* M *and* N.

Let L be a maximal chain in $I(M,N)$ (the existence of such a chain easily follows from Zorn's lemma). Clearly L contains both M and N. We will show that the subspace topology on L coincides with the order topology on L (notice that in general $\lambda(X,S)$ is not Hausdorff so that L need not be closed in $I(M,N)$). Then, by claim 1, L is densely ordered by $\le$ and consequently is connected (cf. WARD [124]). Also, L has two endpoints (M and N) which implies that L is compact.

To prove that the order topology on L coincides with the subspace topology on L, first notice that the order topology on L is weaker than the subspace topology on L because of lemma 2.5.2 (iv). Take $S_0 \in S$ such that $S_0^+ \cap I(M,N) \neq \emptyset$. We claim that $S_0^+ \cap L$ is an order interval in L, which will establish the claim. By lemma 2.5.2 (ii) either $S_0 \in M$ or $S_0 \in N$. Without loss of generality we may assume that $S_0 \in M$ and that $S_0 \notin N$, for if $S_0 \in M \cap N$ then $S_0^+ \cap L = L$. Choose a point H from

$$S_0^+ \cap \bigcap_{L \in L \cap S_0^+} \bigcap_{P \in L \setminus S_0^+} I(L,P).$$

This intersection is nonvoid since $\{S_0^+\} \cup \{I(L,P) \mid L \in L \cap S_0^+,\ P \in L \setminus S_0^+\}$ is a linked system. To prove this, choose $L_0, L_1 \in L \cap S_0^+$ and $P_0, P_1 \in L \setminus S_0^+$. We claim that $I(L_0,P_0) \cap I(L_1,P_1) \neq \emptyset$. Notice that $S_0^+ \cap L$ is order-convex in L, because of lemma 2.5.2 (iv). This implies that $\max\{L_0,L_1\} < \min\{P_0,P_1\}$ and consequently $I(L_0,P_0) \cap I(L_1,P_1) \neq \emptyset$. Therefore

$$S_0^+ \cap \bigcap_{L \in L \cap S_0^+} \bigcap_{P \in L \setminus S_0^+} I(L,P) \neq \emptyset.$$

We claim that $H \in L$. By the fact that L is a maximal chain we need only prove that any member of L and the point H are comparable. Assume that

$L_0 \in$ L and H are incomparable. As $H \in I(M,P)$ for all $P \in L\backslash S_0^+$ it follows that $H \leq P$ for all $P \in L\backslash S_0^+$. On the other hand $H \in I(L,N)$ for all $L \in L \cap S_0^+$ so that $L \leq H$ for all $L \in L \cap S_0^+$. This is a contradiction. We claim that $S_0^+ \cap L = \{L \in L \mid L \leq H\}$, which will complete the proof. Indeed, take $L \in S_0^+ \cap L$. Then $L \leq H$, as was proved above. On the other hand, take $P \in \{L \in L \mid L \leq H\}$ and assume that $P \notin S_0^+ \cap L$. Then $P \in L\backslash S_0^+$ and consequently $H \leq P$. Therefore $H = P$, which is a contradiction.

CLAIM 3. $\lambda(X,S)$ *is connected and locally connected.*

Indeed, by claim 2, $\lambda(X,S)$ is connected. The superextension $\lambda(X,S)$ is also locally connected. In order to prove this, let $M \in \lambda(X,S)$ and let U be an open neighborhood of M. Without loss of generality, U equals

$$\cap_{i\leq n} V_i^+,$$

where $X\backslash V_i \in S$ $(i \leq n)$. Take $L_0, L_1 \in \cap_{i\leq n} V_i^+$; then

$$I(L_0,L_1) \subset \cap_{i\leq n} V_i^+.$$

To prove this, fix $i_0 \leq n$; then $\{L_0,L_1\} \subset V_{i_0}^+$ and hence there are $L_i \in L_i$ $(i \in \{0,1\})$ such that $L_i \subset V_{i_0}$ $(i \in \{0,1\})$. But then $L_0 \cup L_1 \subset V_{i_0}$ and consequently

$$I(L_0,L_1) \subset (L_0 \cup L_1)^+ \subset V_{i_0}^+.$$

Hence, by claim 2, $\cap_{i\leq n} V_i^+$ is connected. $\square$

2.5.4. COROLLARY (VERBEEK [119]). *Let* X *be a topological space. Then the following assertions are equivalent:*

(i) X *is connected;*

(ii) λX *is connected;*

(iii) λX *is connected and locally connected.*

2.6. The dimension of λX

VERBEEK [119] proved the following results on the dimension of λX.

(a) λX *is zero-dimensional iff* X *is strongly zero-dimensional and normal;*

(b) λX *is infinite dimensional if* X *is normal and contains a subspace homeomorphic to* [0,1];

(c) *if* X *is compact metrizable then* λX *either is zero-dimensional (if* X *is) or is infinite dimensional.*

We will extend these results by showing that *for any normal space* X *we have:* $\dim(\lambda X) = \infty$ *iff* X *is not strongly zero-dimensional.*

2.6.1. Recall that a Tychonoff space X is called *strongly zero-dimensional* if its Čech-Stone compactification βX is zero-dimensional. Also recall that for any Tychonoff space X the superextension $\lambda(X,Z(X))$ is homeomorphic to $\lambda(\beta X)$ (cf. 2.2.6).

2.6.2. THEOREM. *Let* X *be a Tychonoff space. Then the following assertions are equivalent:*

(i) X *is not strongly zero-dimensional;*

(ii) $\lambda(X,Z(X))$ *is infinite dimensional.*

PROOF. (ii) ⇒ (i) follows from VERBEEK's [119] result, mentioned in the introduction of this section.

To prove (i) ⇒ (ii) assume that X is not strongly zero-dimensional and that $\lambda(X,Z(X))$ is not infinite dimensional, say $\dim \lambda(X,Z(X)) \le n$ ($n \in \omega$). Then βX is not zero-dimensional, in other words, βX is not totally disconnected. Choose a nontrivial closed connected set A in βX. As A is an infinite Hausdorff space, its cellularity is at least ω; choose open (in A) sets U_i ($i \in \omega$) such that

$$\mathrm{cl}_A(U_i) \cap \mathrm{cl}_A(U_j) = \emptyset \quad \text{iff} \quad i \neq j.$$

Now if $\mathrm{cl}_A(U_i)$ is totally disconnected, it admits a base of open and closed sets; hence there is an open and closed (in $\mathrm{cl}_A(U_i)$) set $C \subset U_i$, which is nonvoid. But then C is clopen in A, which contradicts A being connected.

Therefore we may assume that there is a collection K_i ($i \in \omega$) of connected closed sets in A satisfying

$$K_i \cap K_j = \emptyset \quad \text{iff} \quad i \neq j.$$

Now fix $p \in K_{n+1}$. We will show that $\lambda(X,Z(X))$ ($\approx \lambda(\beta X)$) contains a homeomorph of $\Pi_{i\le n} \lambda K_i$ which contradicts $\dim \lambda(X,Z(X)) \le n$ (cf. LIFANOV [73]) (notice that $\Pi_{i\le n} \lambda K_i$ is a product of $n+1$ compact (generalized) arcwise connected Hausdorff spaces (cf. theorem 2.5.3 and theorem 1.5.16) so that $\Pi_{i\le n} \lambda K_i$ contains a product of $n+1$ ordered compact connected spaces).

Define a mapping $\phi\colon \Pi_{i\le n} \lambda K_i \longrightarrow \lambda(\beta X)$ in the following manner:

$$\phi((M_0,\ldots,M_n)) := \{A \subset \beta X \mid A \text{ is closed and either } (A\cap K_i \in M_i \text{ for all } i \le n) \text{ or } (\exists i \le n\colon p \in A \text{ and } A \cap K_i \in M_i)\}.$$

It is easy to see that ϕ is well-defined, that is: $\phi((M_0,\ldots,M_n))$ is a maximal linked system for all $(M_0,\ldots,M_n) \in \Pi_{i\le n} \lambda K_i$.

CLAIM. *ϕ is injective and continuous.*

Indeed, choose $(M_i)_i, (N_i)_i \in \Pi_{i\le n} \lambda K_i$ such that $(M_i)_i \ne (N_i)_i$. Assume that $M_j \ne N_j$ for some $j \le n$. Then take $M \in M_j$ and $N \in N_j$ such that $M \cap N = \emptyset$. Notice that M and N are both contained in K_j. Then $M \cup \{p\} \in \phi((M_i)_i)$ and $N \cup \bigcup_{i\ne j} K_i \in \phi((N_i)_i)$ which proves that $\phi((M_i)_i) \ne \phi((N_i)_i)$ since $(M\cup\{p\}) \cap (N\cup\bigcup_{i\ne j} K_i) = \emptyset$.

Let D be a closed subset of βX and assume that $(M_i)_i \notin \phi^{-1}[D^+]$. Then $\phi((M_i)_i) \notin D^+$, or, equivalently $D \notin \phi((M_i)_i)$. We have to consider two cases:

CASE 1. $D \cap K_i \notin M_i$ for all $i \le n$.

Then $\bigcap_{i\le n} \pi_i^{-1}[(K_i\backslash D)^+]$ is a neighborhood of $(M_i)_i$ which misses $\phi^{-1}[D^+]$.

CASE 2. There is a $j \le n$ and an $M \in M_j$ such that $(\{p\}\cup M) \cap D = \emptyset$.

Then $\pi_j^{-1}[K_j\backslash D]$ is a neighborhood of $(M_i)_i$ which misses $\phi^{-1}[D^+]$.

It now follows that ϕ is an embedding, since $\Pi_{i\le n} \lambda K_i$ and $\lambda(\beta X)$ are both compact Hausdorff spaces. □

2.6.3. COROLLARY. *Let X be a normal space. Then the following assertions are equivalent:*

(i) X *is not strongly zero-dimensional;*

(ii) λX *is infinite dimensional.*

PROOF. If X is normal, then λX is homeomorphic to $\lambda(X,Z(X))$ (cf. theorem 2.4.2). Then apply theorem 2.6.2. □

2.6.4. COROLLARY. *Let* X *be a normal space. Then* λX *either is zero-dimensional or is infinite dimensional.* □

2.7. Path connectedness and contractibility of λX

The following results have been proved:

(i) *if* X *is compact Hausdorff, and either contractible or a suspension, then its superextension* λX *is contractible* (VERBEEK [119]);

(ii) *if* X *is a metric continuum, then* λX *is an* AR *(compact metric)* (VAN MILL [79], *also* 2.5.1, 2.4.21 *and* 1.5.20). *In particular* λX *is contractible;*

(iii) *if* X *is connected and normal, then* λX *is acyclic and has the fixed point property for continuous functions* (VAN DE VEL [118]).

In this section we make a first attempt to fill up the gaps which obviously exist between the above results. Among other things, we show that λX *is contractible if* X *is a continuum of finite category or if* X *is path connected, separable and normal*. We also show that *if* X *is seperable and normal then* λX *is contractible if and only if it is path connected.* The results in this section are taken from VAN MILL & VAN DE VEL [83].

2.7.1. For the remainder of this section, let X be a Tychonoff space; let $\mathcal{S}$ be a normal T_1-subbase for X. An $\mathcal{S}^+$-closed set in $\lambda(X,\mathcal{S})$ will be called *convex* for short (notice that each $\mathcal{S}^+$-closed set also is $\mathcal{S}^+$-convex and that conversely each closed $\mathcal{S}^+$-convex set is $\mathcal{S}^+$-closed (cf. theorem 1.5.7); this motivates our terminology). Also the subspace $H(\lambda(X,\mathcal{S}),\mathcal{S}^+)$ of $2^{\lambda(X,\mathcal{S})}$ (cf. 1.5.22 and section 2.10) will be denoted by $K(\lambda(X,\mathcal{S}))$.

In the following we need two results:

(a) *the map* $I_{\mathcal{S}^+}\colon 2^{\lambda(X,\mathcal{S})} \to K(\lambda(X,\mathcal{S}))$ *is a continuous retraction of* $2^{\lambda(X,\mathcal{S})}$ *onto* $K(\lambda(X,\mathcal{S}))$;

(b) *the map* $p\colon \lambda(X,\mathcal{S}) \times K(\lambda(X,\mathcal{S})) \to \lambda(X,\mathcal{S})$ *defined by* $p(\mathcal{M},A) := \mathrm{glb}_{\mathcal{M}}(A)$ *is continuous.*

Statement (b) is a direct consequence of theorem 1.5.2 (i) and theorem 1.5.18. We will refer to the map p described in (b) as the "*nearest point map of* $\lambda(X,\mathcal{S})$"; cf. also VAN DE VEL [118] and VAN MILL & VAN DE VEL [82].

The following general result will be our main tool for deriving contractibility results on $\lambda(X,\mathcal{S})$.

2.7.2. PROPOSITION. *Let $\mathcal{S}$ be a normal T_1-subbase for X and assume that there exists a continuous mapping ϕ: $[0,1] \to 2^X$ such that $\phi(0)$ is a singleton and $\phi(1) = X$. Then there is a contraction of $\lambda(X,\mathcal{S})$ onto $\phi(0)$ keeping $\phi(0)$ fixed.*

PROOF. Define a mapping ψ: $2^X \to 2^{\lambda(X,\mathcal{S})}$ by $\psi(A) := cl_{\lambda(X,\mathcal{S})}(A)$. This mapping is easily seen to be continuous, since $\lambda(X,\mathcal{S})$ is compact Hausdorff (cf. theorem 2.2.4 (iii)). Define

$$\phi': [0,1] \to 2^{\lambda(X,\mathcal{S})}$$

by $\phi'(t) := \cup\{\psi\phi(t') \mid t' \le t\}$. Then $\phi'(t)$ is compact, being the union of a compact family of compact sets, and ϕ' is easily seen to be continuous again. Notice that $\phi'(0) = \phi(0)$, that $\phi'(1) = \phi(1)$ and that ϕ' is increasing.

We now use the mapping $I_{\mathcal{S}^+}$: $2^{\lambda(X,\mathcal{S})} \longrightarrow K(\lambda(X,\mathcal{S}))$. It is easy to verify that $I_{\mathcal{S}^+}$ preserves singletons, and that $I_{\mathcal{S}^+}(S) = S^+$ for each $S \in \mathcal{S}$.

Let x_0 be the unique point in $I_{\mathcal{S}^+}(\phi'(0))$ and define a map F: $\lambda(X,\mathcal{S}) \times [0,1] \to \lambda(X,\mathcal{S})$ by

$$F(M,t) := p(M, I_{\mathcal{S}^+}(\phi'(t))),$$

where p is the nearest point mapping of $\lambda(X,\mathcal{S})$. Then, clearly F is continuous, and by the construction of the map p

$$F(M,0) = p(M,\{x_0\}) = x_0;$$

$$F(M,1) = p(M,\lambda X) \quad = M.$$

Moreover, $x_0 \in I_{\mathcal{S}^+}(\phi'(t))$ for each $t \in [0,1]$, whence

$$F(x_0,t) = p(x_0, I_{\mathcal{S}^+}(\phi'(t))) = x_0,$$

proving that F is a contraction of $\lambda(X,\mathcal{S})$ onto x_0 keeping x_0 fixed. □

Recall that a space X is said to be of *category* $\le n$ $(n < \omega)$ if X is the union of n closed subspaces $\{X_i\}_{i \le n}$, each deformable onto a point of X (cf. WILLARD [129]). A space X is of *finite category* if it is of category less than or equal to n, for some $n < \omega$.

2.7.3. COROLLARY. *Let* X *be a continuum of finite category and let* S *be a normal* T_1*-subbase for* X. *Then* $\lambda(X,S)$ *is contractible.*

PROOF. Let $X = \bigcup_{i=1}^{n} X_i$, where each X_i is a closed subspace of X which admits a mapping

$$F_i\colon X_i \times [0,1] \to X$$

with the properties: $F_i(-,0)$ is a constant map onto, say x_i, and $F_i(-,1)$ equals the inclusion $X_i \subset X$. It is easy to see, using the connectedness of X, that the space X is path connected. For each $i > 1$ we fix a path

$$\xi_i\colon [0,1] \to X$$

with $\xi_i(0) = x_1$ and $\xi_i(1) = x_i$ $(i \le n)$. Define

$$\phi_i\colon [0,1] \to 2^X$$

by $\phi_i(t) := F_i(X_i \times [0,t])$. It is easy to see that each ϕ_i is continuous (cf. VAN DE VEL [118], lemma 1.3). Let

$$\phi\colon [0,1] \to 2^X$$

be defined as follows

$$\phi(t) := \bigcup_{i=2}^{n} \xi_i[0,2t] \qquad \text{if } 0 \le t \le \tfrac{1}{2};$$

$$\phi(t) := \phi(\tfrac{1}{2}) \cup \bigcup_{i=1}^{n} \phi_i(2t-1) \qquad \text{if } \tfrac{1}{2} \le t \le 1.$$

Then ϕ is easily seen to be a continuous map with $\phi(0)$ a singleton and $\phi(1) = X$. Applying proposition 2.7.2 we find that $\lambda(X,S)$ is contractible. □

This corollary includes, as a particular case, the contractibility results of VERBEEK, mentioned in the introduction of this section. In fact, a contractible (compact Hausdorff) space is of category 1, and a (compact Hausdorff) suspension is of category 2.

The main result in this section is the following:

2.7.4. THEOREM. *Let* X *be a separable space such that each finite subset of* X *is contained in a metric continuum and let* S *be a normal* T_1*-subbase for* X. *Then* $\lambda(X,S)$ *is contractible.*

PROOF. We need two auxiliary results:

CLAIM 1. *There is an increasing sequence* $(K_n)_{n<\omega}$ *of metrizable subcontinua of* X*, such that* K_0 *is a singleton and* $(K_n)_{n<\omega}$ *converges to* X *in* 2^X.

Indeed, let $\{x_n \mid n < \omega\}$ be a countable dense subspace of X. For each $n < \omega$ let L_n be a metric subcontinuum of X containing $\{x_0,\ldots,x_n\}$. We choose $L_0 := \{x_0\}$. Then put

$$K_n = L_0 \cup \ldots \cup L_n,$$

for each $n < \omega$, so that $(K_n)_{n<\omega}$ is an increasing sequence of metrizable subcontinua of X whose union is dense in X.

The sets of the type

$$\langle O_0,\ldots,O_p,X\rangle$$

where each O_i is open $(i \le p)$, form a neighborhood base at $X \in 2^X$. Fix open sets $O_0,\ldots,O_p$ in X. For each $i \le p$ we can find $n(i) < \omega$ such that $K_n \cap O_i \neq \emptyset$ for all $n \ge n(i)$ (the sequence $(K_n)_{n<\omega}$ is increasing!). Hence, if $n_0 = \max\{n(i) \mid i \le p\}$ we have that $K_n \in \langle O_0,\ldots,O_p,X\rangle$ for each $n \ge n_0$. Therefore $(K_n)_{n<\omega}$ converges to X in 2^X.

CLAIM 2. *If* K *and* L *are metric subcontinua of* X*, with* $K \subset L$*, then there is a continuous increasing mapping* $\phi\colon [0,1] \to 2^X$ *with* $\phi(0) = K$ *and* $\phi(1) = L$.

Using the fact that 2^L is a subspace of 2^X, this statement is a direct consequence of a result in KURATOWSKI [72], vol. II.

We now combine the two statements. For each $n > 0$ we have a continuous increasing map (with rearranged domain)

$$\phi_n\colon [1-\tfrac{1}{n},\ 1-\tfrac{1}{n+1}] \longrightarrow 2^X,$$

such that $\phi_n(1-\frac{1}{n}) = K_{n-1}$ and $\phi_n(1-\frac{1}{n+1}) = K_n$. Since each ϕ_n is monotonic, and since $(K_n)_{n<\omega}$ converges to X, the map $\phi\colon [0,1] \to 2^X$, defined by

$$\phi := \bigcup_{n<\omega} \phi_n \cup (\{1\} \times X)$$

is also continuous. Applying proposition 2.7.2 yields the desired result. $\square$

2.7.5. Several classes of topological spaces are in the scope of theorem 2.7.4. For example the class of all separable path connected spaces. The class of spaces, described in theorem 2.7.4, is countably productive.

2.7.6. As a particular consequence of theorem 2.7.4, it follows that $\lambda\mathbb{R}$, the superextension of real line is contractible, in contrast with the fact that the Čech-Stone compactification $\beta\mathbb{R} \subset \lambda\mathbb{R}$ is not contractible (it is not even path connected).

By the above remark on productivity, a countable product of real lines also has a contractible superextension. Recall that $\mathbb{R}^{\infty}$ is homeomorphic to the separable Hilbert space ℓ_2 by a result of ANDERSON [3] (cf. also ANDERSON & BING [6]).

2.7.7. We now turn our attention to path connectedness of superextensions. It is rather surprising that the existence of dense path connected subspaces is easy to prove under fairly general circumstances. In contrast to this, it seems to be rather difficult to find an improvement of theorem 2.7.4 in the direction of path connectedness of λX. A partial explanation is provided by theorem 2.7.8 below, which shows that path connectedness and contractibility are equivalent on separable superextensions.

2.7.8. THEOREM. *Let* X *be a separable space and let* S *be a normal* T_1*-subbase for* X. *Then* $\lambda(X,S)$ *is contractible if and only if it is path connected.*

PROOF. By a result of VERBEEK ([119], p.96), $\lambda(X,S)$ is separable. Let $\{M_n \mid n \in \mathbb{N}\}$ be a countable dense subspace of $\lambda(X,S)$. For each $n \geq 1$ we fix a path

$$\alpha_n\colon [1-\frac{1}{n}, 1-\frac{1}{n+1}] \longrightarrow \lambda(X,S),$$

with $\alpha_n(1-\frac{1}{n}) = M_n$ and $\alpha_n(1-\frac{1}{n+1}) = M_{n+1}$. Lateral composition yields a continuous map on the half open interval $[0,1)$,

$$\alpha\colon [0,1) \longrightarrow \lambda(X,S),$$

the image of which contains the above dense subspace.

Define a mapping

$$\phi\colon [0,1] \longrightarrow 2^{\lambda(X,S)}$$

by $\phi(t) := \alpha[0,t]$ if $t < 1$ and $\phi(1) := \lambda(X,S)$. The continuity of ϕ follows from a rather obvious type of argument similar to the one in the proof of theorem 2.7.4.

Now, by proposition 2.7.2, $\lambda(\lambda(X,S),S^+)$ is contractible. But $\lambda(\lambda(X,S),S^+)$ is homeomorphic to $\lambda(X,S)$ (cf. theorem 2.2.5) and consequently $\lambda(X,S)$ is contractible. $\square$

2.7.9. THEOREM. *Let X be a topological space and let S be a normal T_1subbase for X. If X contains a dense path connected subspace, then so does $\lambda(X,S)$.*

PROOF. We need the following three auxiliary results:

CLAIM 1. *Let $M,N \in \lambda(X,S)$. If M and N can be joined by some path in $\lambda(X,S)$, then the interval $I_{S^+}(M,N)$ is path connected.*

Indeed, let $f\colon [0,1] \longrightarrow \lambda(X,S)$ be a path joining M and N. As $I_{S^+}(M,N)$ is a retract of $\lambda(X,S)$ (cf. theorem 1.5.2), hence we may assume that $f[0,1] \subset I_{S^+}(M,N)$. Let $P \in I_{S^+}(M,N)$. Then

$$I_{S^+}(M,P) \subset I_{S^+}(M,N) \quad \text{and} \quad I_{S^+}(M,P) \cap I_{S^+}(P,N) = \{P\},$$

cf. 1.3.2 and 1.5.3. Let

$$r\colon I_{S^+}(M,N) \longrightarrow I_{S^+}(M,P)$$

be the restriction of the retraction of $\lambda(X,S)$ onto $I_{S^+}(M,P)$ described in theorem 1.5.2. Then $r(M) = M$ and $r(N) = P$ and hence it follows that the path f "retracts" onto a path $r \circ f$ of $I(M,P)$ joining M and P. It now easily follows that $I_{S^+}(M,N)$ is path connected.

CLAIM 2. *If $A \subset \lambda(X,S)$ is path connected, then so is the space*

$$I_{S^+}(A\times A) = \bigcup\{I_{S^+}(x,y) \mid x,y \in A\}.$$

By claim 1, each interval $I_{S^+}(x,y)$ with $x,y \in A$, is path connected. Moreover A is a path connected subspace of $I_{S^+}(A\times A)$; therefore the desired result follows.

CLAIM 3. *Let $B \subset \lambda(X,S)$ be such that for all $x,y \in B$ the set $I_{S^+}(x,y) \subset B$. Then the closure $\mathrm{cl}_{\lambda(X,S)}(B)$ of B in $\lambda(X,S)$ is S^+-closed.*

Choose $x,y \in \mathrm{cl}_{\lambda(X,S)}(B)$ such that $I_{S^+}(x,y) \not\subset \mathrm{cl}_{\lambda(X,S)}(B)$. By the continuity of the mapping f, described in lemma 1.5.10, there are disjoint neighborhoods U and V of x and y such that

$$I_{S^+}(p,q) \not\subset cl_{\lambda(X,S)}(B)$$

for all $p \in U$ and $q \in V$. Choose $z_0 \in U \cap B$ and $z_1 \in V \cap B$. Then

$$I_{S^+}(z_0,z_1) \not\subset B,$$

which is a contradiction. Now, by theorem 1.5.7, $cl_{\lambda(X,S)}(B)$ is S^+-closed.

To prove the theorem, let $Y_0 \subset X$ be a dense path connected subspace. For each $n \in \omega$ we define, inductively

$$Y_{n+1} := I_{S^+}(Y_n \times Y_n).$$

Using claim 2, each Y_n is path connected. Since $Y_n \subset Y_{n+1}$ for all $n \in \omega$, we find that $Y := \bigcup_{n\in\omega} Y_n$ is path connected too. This subspace of $\lambda(X,S)$ obviously satisfies the conditions of claim 3, whence $cl_{\lambda(X,S)}(Y)$ is S^+-closed. But

$$X \subset cl_{\lambda(X,S)}(Y_0) \subset cl_{\lambda(X,S)}(Y),$$

and the only S^+-closed subsets of $\lambda(X,S)$ containing X is $\lambda(X,S)$ itself. This shows that Y is dense in $\lambda(X,S)$. □

2.7.10. Our final results in this section involve some particular dense subspaces of superextensions introduced in VERBEEK [119]. An mls $M \in \lambda X$ is said *to be defined on a closed set* $A \subset X$ if $M \cap A \in M$ for all $M \in M$. For any space X, let

$$\lambda_f(X) := \{M \in \lambda X \mid M \text{ is defined on some finite subset of } X\}$$

and

$$\lambda_{comp}(X) := \{M \in \lambda X \mid M \text{ is defined on some compact closed subset of } X\}.$$

2.7.11. THEOREM. *Let X be a normal space such that each finite subset of X is contained in a metrizable continuum. Then $\lambda_f(X)$ is path connected within $\lambda_{comp}(X)$ (notice that the space is not assumed to be separable).*

PROOF. Let $M,N \in \lambda_f(X)$; say, M is defined on $F \subset X$ and N is defined on $G \subset X$, where F and G are finite. By assumption there is a metrizable continuum $K \subset X$ containing $F \cup G$. The inclusion mapping $K \subset X$ induces an embedding $\lambda K \subset \lambda X$ (cf. theorem 2.3.4). Clearly $\lambda K \subset \lambda_{comp}(X)$. But λK is contractible (theorem 2.7.4) and hence M and N are joined by a path in $\lambda K \subset \lambda_{comp}(X)$. □

The results derived in this section suggest some questions:

2.7.12. QUESTION. *Find necessary and sufficient conditions on a continuum* X *in order for* λX *to be path connected/contractible.*

We found the following "controversial" examples:

2.7.13. EXAMPLES.

(i) *Let* X *be a compact tree-like space which is not path connected. Then* λX *is not path connected.*

(ii) *Let* $X = \beta\mathbb{R}$, *the Čech-Stone compactification of the real line* $\mathbb{R}$. *Then* X *is not path connected, but* λX *is contractible.*

The proofs are simple:

(i) a compact tree-like space admits a binary normal subbase (cf. theorem 1.3.21) and hence it is a retract of its superextension (cf. corollary 2.3.7).

(ii) $\lambda(\beta\mathbb{R})$ is homeomorphic to $\lambda\mathbb{R}$ (cf. VERBEEK [119]; also corollary 2.2.6 and theorem 2.4.2).

It is well known that AR's in the category of compact Hausdorff spaces are contractible and locally contractible: see e.g. SAALFRANK [101]. The two properties are not equivalent in general. However, in view of our result that λX is an AR (compact metric) iff X is a metrizable continuum, and in view of nice convexity structure of superextensions, one is lead to the following:

2.7.14. QUESTION. *Find conditions on a continuum* X *in order that* λX *be an* AR *(in the category of compact Hausdorff spaces).*

Concerning the superextensions of non-compact spaces we have no information on the necessity of the separability condition appearing in our present results.

2.8. Subspaces of superextensions; the spaces $\sigma(X)$ and $\Sigma(X)$ *)

In this section we will describe some subspaces of superextensions which seem interesting. This is only a first attempt; many questions are unsolved. We are particularly interested in subspaces of $\lambda\mathbb{N}$, the superextension of the natural numbers. It is clear, due to the definition of $\lambda\mathbb{N}$, that $\lambda\mathbb{N}$ and $\beta\mathbb{N}$, the Čech-Stone compactification of $\mathbb{N}$, must be related, but it is not clear in what way. It was noticed by VERBEEK [119] that $\lambda\mathbb{N}$ and $\beta\mathbb{N}$ are not homeomorphic, since $\lambda\mathbb{N}$ contains nontrivial convergent sequences. But $\lambda\mathbb{N}$ contains a dense set of isolated points (VERBEEK [119]) and hence can considered to be a compactification of $\mathbb{N}$; consequently $\lambda\mathbb{N}$ is a continuous image of $\beta\mathbb{N}$, however $\beta\mathbb{N}$ is not a continuous image of $\lambda\mathbb{N}$ (cf. corollary 1.1.6).

Proposition 2.2.3 implies that $\lambda\mathbb{N}$ is totally disconnected and has weight $\mathfrak{c}$. The isolated points in $\lambda\mathbb{N}$ are just the points with a finite defining set (VERBEEK [119]; recall that an mls $\mathcal{M} \in \lambda X$ is said to be defined on a closed set $A \subset X$ provided that $M \cap A \in \mathcal{M}$ for all $M \in \mathcal{M}$, cf. section 2.7). The space $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ is compact and possesses points with a countable neighborhood basis and points without a countable neighborhood basis. For example

$$\mathcal{M} = \{M \subset \mathbb{N} \mid \exists i > 1:\ \{1,i\} \subset M \text{ or } \{2,3,\ldots\} \subset M\}$$

can easily be seen to be an mls in $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ with a countable neighborhood basis.

An ultrafilter $\mathcal{F} \in \beta\mathbb{N}\setminus\mathbb{N} \subset \lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ is an example of a point without countable neighborhood basis (notice that each ultrafilter is a maximal linked system and hence that $\beta\mathbb{N}$ is a subset of $\lambda\mathbb{N}$; $\beta\mathbb{N}$ also is a subspace of $\lambda\mathbb{N}$; indeed it equals the closure of $\mathbb{N}$ in $\lambda\mathbb{N}$, cf. corollary 2.3.9). We see that $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ differs essentially from $\beta\mathbb{N}\setminus\mathbb{N}$. The following subspace of $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ at first glance seems to be closer to $\beta\mathbb{N}\setminus\mathbb{N}$ than $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$:

$$\sigma(\mathbb{N}) := \{\mathcal{M} \in \lambda\mathbb{N} \mid \mathcal{M} \text{ contains no finite set}\}.$$

Unfortunately, however, $\sigma(\mathbb{N})$ is separable, because of the following lemma, while $\beta\mathbb{N}\setminus\mathbb{N}$ is not.

2.8.1. LEMMA. *$\sigma(\mathbb{N})$ is a retract of $\lambda\mathbb{N}$.*

*) This section will also be published separately in Math. Z.

PROOF. Let $A = \{A \subset \mathbb{N} \mid |\mathbb{N}\setminus A| < \omega\}$. Then $\sigma(\mathbb{N}) = \cap\{A^+ \mid A \in A\}$ and hence by theorem 1.5.2, $\sigma(\mathbb{N})$ is a retract of $\lambda\mathbb{N}$. As $\lambda\mathbb{N}$ is separable (VERBEEK [119]), so is $\sigma(\mathbb{N})$. □

[The definition of $\sigma(\mathbb{N})$ suggests a more general definition. For any topological space X let $\sigma(X)$ be defined by

$$\sigma(X) := \{M \in \lambda X \mid M \text{ contains no compact set}\}.$$

We did not yet study the spaces $\sigma(X)$ in detail.]

The subspace $\Sigma(\mathbb{N}) := \{M \in \lambda\mathbb{N} \mid \text{for all } M_0, M_1 \in M\colon |M_0 \cap M_1| = \omega\}$ of $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ is a better candidate for an analogue of $\beta\mathbb{N}\setminus\mathbb{N}$. One can look at $\Sigma(\mathbb{N})$ as the set of all uniform maximal linked systems. This appears to be the most interesting subspace. More generally, for any topological space X, define

$$\Sigma(X) := \{M \in \lambda X \mid \text{for all } M_0, M_1 \in M\colon M_0 \cap M_1 \text{ is not compact}\}.$$

Notice that $\Sigma(X) = \emptyset$ if X is compact Hausdorff and that $\Sigma(X) \subset \sigma(X)$.

2.8.2. THEOREM. *Let X be a normal topological space. Then*

(i) $\Sigma(X) \subset \lambda X\setminus\lambda_f(X)$;

(ii) $\Sigma(X)$ *is compact iff X is locally compact;*

(iii) *if X is locally compact then* $\Sigma(X)$ *is homeomorphic to* $\lambda(\beta X\setminus X)$.

PROOF. (i) is trivial. To prove (ii), assume that $\Sigma(X)$ is compact. Notice that βX is closed in λX and consequently $\beta X\setminus X$ is closed in $\lambda X\setminus\lambda_f(X)$. Therefore, as $\beta X\setminus X \subset \Sigma(X)$, $\beta X\setminus X$ is closed in $\Sigma(X)$ too. It follows that $\beta X\setminus X$ is compact and consequently X is locally compact. The converse of (ii) follows from (iii), since $\lambda(\beta X\setminus X)$ is compact.

To prove (iii), assume that X is locally compact. For each closed subset $M \subset X$ define $M^* := \mathrm{cl}_{\beta X}(M)\setminus M$. Then $\{M^* \mid M \text{ is closed in } X\}$ is a closed base for the topology of $\beta X\setminus X$, closed under finite intersections and finite unions. Define a mapping $\phi\colon \lambda(\beta X\setminus X) \to \Sigma(X)$ by

$$\phi(M) := \{M \subset X \mid M^* \in M\}.$$

First we will show that ϕ is well-defined. Clearly $\phi(M)$ is a linked system for all $M \in \lambda(\beta X\setminus X)$. Suppose that $\phi(M)$ is not a maximal linked system for some $M \in \lambda(\beta X\setminus X)$. Then there exists a closed set $A \subset X$ such that $\phi(M) \cup \{A\}$ is linked, while $A \notin \phi(M)$. Then $A^* \notin M$ and consequently there exists an $M \in M$ such that $A^* \cap M = \emptyset$. By the compactness of $\beta X\backslash X$ there is a closed subset $B \subset X$ such that $M \subset B^*$ and $B^* \cap A^* = \emptyset$. As $M \in M$ it follows that $B^* \in M$ and consequently $B \in \phi(M)$. Therefore $B \cap A \neq \emptyset$. But $B^* \cap A^* = \emptyset$ implies that $B \cap A$ is compact. Choose a relatively compact neighborhood U of $A \cap B$ and define $C := B\backslash U$. Then $C^* = B^*$ and consequently also $C \in \phi(M)$. This is a contradiction, since $C \cap A = \emptyset$. Also it is clear that $\phi(M) \in \Sigma(X)$; for take $M,N \in \phi(M)$ such that $M \cap N$ is compact. Then $M^* \cap N^* = \emptyset$ and consequently M is not linked. Contradiction.

Let B be a closed subset of X. Then

$$\begin{aligned} M \in \phi^{-1}[B^+ \cap \Sigma(X)] &\text{ iff } \phi(M) \in B^+ \cap \Sigma(X) \\ &\text{ iff } \phi(M) \in B^+ \\ &\text{ iff } B^* \in M \\ &\text{ iff } M \in (B^*)^+. \end{aligned}$$

Therefore $\phi^{-1}[B^+ \cap \Sigma(X)] = (B^*)^+$ (the first "plus" is taken in λX, the second in $\lambda(\beta X\backslash X)$!) showing that ϕ is continuous. Also it is not difficult to show that ϕ is one to one and surjective. As $\lambda(\beta X\backslash X)$ and $\Sigma(X)$ both are compact Hausdorff spaces, it follows that ϕ is a homeomorphism. □

2.8.3. REMARKS.

(i) The present proof of theorem 2.8.2 (ii) is due to E. VAN DOUWEN; he discovered a mistake in our original proof.

(ii) Theorem 2.8.2 shows that $\Sigma(\mathbb{N})$ is a homeomorph of $\lambda(\beta\mathbb{N}\backslash\mathbb{N})$ and hence that $\Sigma(\mathbb{N})$ is supercompact. The proof of theorem 2.8.2 shows that the subbase $\{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$ for $\Sigma(\mathbb{N})$ is binary. For this fact there is also an elementary proof. For take $M,N,P \in \Sigma(\mathbb{N})$. Then

$$I_{P(\mathbb{N})^+}(M,N) \cap I_{P(\mathbb{N})^+}(M,P) \cap I_{P(\mathbb{N})^+}(N,P)$$

consists of one point, say L (cf. corollary 1.5.3). Take $L_0,L_1 \in L$ and assume that $L_0 \cap L_1$ is finite. Then, as in the proof of lemma 2.5.2 L_0 and L_1 both belong to an element of $\{M,N,P\}$, which is a contradiction, since $\{M,N,P\} \subset \Sigma(\mathbb{N})$.

Now, theorem 1.3.3 implies that $\{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$ is a binary subbase for $\Sigma(\mathbb{N})$.

(iii) The supercompactness of $\Sigma(\mathbb{N})$ implies that $\beta\mathbb{N}\backslash\mathbb{N}$ and $\Sigma(\mathbb{N})$ are not homeomorphic after all, since $\beta\mathbb{N}\backslash\mathbb{N}$ is an F-space (cf. 0.C), and no infinite compact F-space is supercompact (cf. corollary 1.1.6).

We will now derive some properties of $\Sigma(\mathbb{N})$ (and hence of $\lambda(\beta\mathbb{N}\setminus\mathbb{N})$).

2.8.4. LEMMA. *The cellularity of* $\Sigma(\mathbb{N})$ *is* $\mathfrak{c}$.

PROOF. Let $\mathcal{A}$ be an almost disjoint collection of infinite subset of $\mathbb{N}$ of cardinality $\mathfrak{c}$; i.e. for all $A \in \mathcal{A}$ we have $|A| = \omega$ while $|A \cap B| < \omega$ for all distinct $A, B \in \mathcal{A}$ (there is such a collection, cf. GILLMAN & JERISON [52]). Then $\{A^+ \cap \Sigma(\mathbb{N}) \mid A \in \mathcal{A}\}$ is a collection of $\mathfrak{c}$ pairwise disjoint open subsets of $\Sigma(\mathbb{N})$. For take distinct $A, B \in \mathcal{A}$ and $\mathcal{M} \in A^+ \cap B^+ \cap \Sigma(\mathbb{N})$. Then $|A \cap B| = \omega$ since $\mathcal{M} \in \Sigma(\mathbb{N})$. Contradiction.

Since weight $(\lambda\mathbb{N}) = \mathfrak{c}$, the weight of $\Sigma(\mathbb{N})$ also equals $\mathfrak{c}$ (recall that $\beta\mathbb{N}\setminus\mathbb{N} \subset \Sigma(\mathbb{N})$). $\square$

2.8.5. Let κ be any cardinal. The following principle is called $P(\kappa)$.

Let $\mathcal{A}$ *be a collection of fewer than* κ *subsets of* $\mathbb{N}$ *such that each finite subcollection of* $\mathcal{A}$ *has infinite intersection. Then there is an infinite* $F \subset \mathbb{N}$ *such that* $F\setminus A$ *is finite for all* $A \in \mathcal{A}$.

It is easy to show that $P(\omega_1)$ holds in ZFC and moreover that Martin's axiom (MA) implies $P(\mathfrak{c})$ (BOOTH [18]). Also $P(\kappa)$ implies that $2^\lambda = \mathfrak{c}$ for each infinite $\lambda < \kappa$ (ROTHBERGER [96]). Clearly $P(\omega_2)$ implies the negation of the Continuum Hypothesis.

It is easy to show that $P(\kappa)$ is equivalent to the statement that each nonvoid intersection of fewer than κ open subsets of $\beta\mathbb{N}\setminus\mathbb{N}$ has nonempty interior. Unfortunately $P(\kappa)$ does not imply the same property for $\Sigma(\mathbb{N})$. In fact we will show that there is a nonvoid countable intersection of clopen subsets of $\Sigma(\mathbb{N})$ with a void interior. The following lemma shows that $P(\kappa)$ works for intersections of open sets in $\Sigma(\mathbb{N})$ containing an ultrafilter.

2.8.6. LEMMA $[P(\kappa)]$. *Let* A *be an intersection of fewer than* κ *open subsets of* $\Sigma(\mathbb{N})$. *If* $A \cap (\beta\mathbb{N}\setminus\mathbb{N}) \neq \emptyset$ *then there is an infinite* $B \subset \mathbb{N}$ *such that* $B^+ \cap \Sigma(\mathbb{N}) \subset A$. *In particular,* A *has a nonvoid interior.*

PROOF. Let $A = \bigcap\{O_\alpha \mid \alpha \in \beta\}$, where $\beta < \kappa$ and each O_α is open in $\Sigma(\mathbb{N})$. Take a point $\mathcal{F} \in A \cap (\beta\mathbb{N}\setminus\mathbb{N})$. For each $\alpha \in \beta$ choose an $F_\alpha \in \mathcal{F}$ such that $F_\alpha^+ \cap \Sigma(\mathbb{N}) \subset O_\alpha$. This is possible since it is easy to see that $\{F^+ \cap \Sigma(\mathbb{N}) \mid F \in \mathcal{F}\}$ is a neighborhood *basis* for $\mathcal{F}$ in $\Sigma(\mathbb{N})$. Then $\{F_\alpha \mid \alpha \in \beta\}$ is a collection of fewer than κ subsets of $\mathbb{N}$ each finite subcollection of which has infinite intersection. Choose an infinite $B \subset \mathbb{N}$

such that $|B\backslash F_\alpha| < \omega$ for all $\alpha \in \beta$. We will show that

$$B^+ \cap \Sigma(\mathbb{N}) \subset \bigcap\{F_\alpha^+ \cap \Sigma(\mathbb{N}) \mid \alpha \in \beta\}.$$

Indeed, choose a point $M \in (B^+ \cap \Sigma(\mathbb{N}))\backslash(F_\alpha^+ \cap \Sigma(\mathbb{N}))$ for some $\alpha \in \beta$. Then $F_\alpha \notin M$ and consequently $\mathbb{N}\backslash F_\alpha \in M$. Hence $|B \cap (\mathbb{N}\backslash F_\alpha)| = \omega$, since $M \in \Sigma(\mathbb{N})$. Contradiction. Therefore $B^+ \cap \Sigma(\mathbb{N}) \subset F_\alpha^+ \cap \Sigma(\mathbb{N})$ $(\alpha \in \beta)$ and as B is infinite, $B^+ \cap \Sigma(\mathbb{N})$ is a nonvoid open set in $\Sigma(\mathbb{N})$. □

2.8.7. <u>REMARK</u>. In the proof of lemma 2.8.6 we showed that $A^+ \cap \Sigma(\mathbb{N}) \subset B^+ \cap \Sigma(\mathbb{N})$ iff $|A\backslash B| < \omega$. This is a property of the binary subbase $\{A^+ \cap \Sigma(\mathbb{N}) \mid A \subset \mathbb{N}\}$. The binary subbase $\{A^+ \mid A \subset \mathbb{N}\}$ does not have this property. For example let $A = \{1\}$ and $B = \{1,2\}$. Define an mls $M \in \lambda\mathbb{N}$ by

$$M := \{C \subset \mathbb{N} \mid \{1,2\} \subset C \text{ or } \{1,3\} \subset C \text{ or } \{2,3\} \subset C\}.$$

It is easy to see that M is an mls. Moreover $M \in B^+\backslash A^+$ and yet $|B\backslash A| < \omega$.

We will now give an example showing that lemma 2.8.6 cannot be sharpened.

2.8.8. <u>EXAMPLE</u>. *A countable nonvoid intersection of clopen subsets of $\Sigma(\mathbb{N})$ with a void interior.*

Inductively we construct a collection $\{A_n \mid n \in \omega\}$ of infinite subsets of $\mathbb{N}$ such that for all $i \in \omega$

(i) $k \le \ell \le i \Rightarrow |A_k \cap A_\ell| = \omega$;

(ii) $k \le i \Rightarrow |A_k \backslash \bigcup_{\substack{j\le i\\ j\ne k}} A_j| = \omega$;

(iii) $|\mathbb{N}\backslash \bigcup_{j\le i} A_j| = \omega$;

(iv) $k < \ell < m \le i \Rightarrow A_k \cap A_\ell \cap A_m = \emptyset$.

To define A_0 just pick an infinite subset of $\mathbb{N}$ with an infinite complement. Suppose that $\{A_j \mid 0 \le j \le i\}$ are defined satisfying (i) - (iv). For each $k \le i$ choose an infinite

$$C_k \subset A_k \backslash \bigcup_{\substack{j\le i\\ j\ne k}} A_j$$

such that also

$$(A_k \backslash \bigcup_{\substack{j\le i\\ j\ne k}} A_j) \backslash C_k$$

is infinite. Choose an infinite $D \subset \mathbb{N}\backslash\bigcup_{j\le i} A_j$ such that also $(\mathbb{N}\backslash\bigcup_{j\le i} A_j)\backslash D$ is infinite. Define $A_{i+1} := \bigcup_{j=0}^{i} C_j \cup D$. Then clearly (i), (ii) and (iii) are satisfied. Take $k, \ell \le i$ such that $k < \ell$. Then

$$A_k \cap A_\ell \cap A_{i+1} = A_k \cap A_\ell \cap \bigcup_{j=0}^{i} C_j = C_k \cap C_\ell = \emptyset;$$

hence (iv) is also satisfied.

We will show that the nonvoid set $\bigcap\{A_n^+ \mid n \in \omega\} \cap \Sigma(\mathbb{N})$ has a void interior (that $\bigcap\{A_n^+ \mid n \in \omega\} \cap \Sigma(\mathbb{N})$ is nonvoid is trivial since $|A_i \cap A_j| = \omega$ for all $i,j \in \omega$). First we prove one more simple lemma.

2.8.9. <u>LEMMA</u>. *Let* $M_\alpha \subset \mathbb{N}$ $(\alpha \in \beta)$ *such that* $\bigcap_{\alpha\in\beta} M_\alpha^+ \cap \Sigma(\mathbb{N}) \neq \emptyset$. *Then for all* $B \subset \mathbb{N}$ *we have* $\bigcap_{\alpha\in\beta} M_\alpha^+ \cap \Sigma(\mathbb{N}) \subset B^+ \cap \Sigma(\mathbb{N})$ *iff* $|M_{\alpha_0} \setminus B| < \omega$ *for some* $\alpha_0 \in \beta$.

<u>PROOF</u>. If $|M_\alpha \setminus B| < \omega$ for some $\alpha \in \beta$ then $M_\alpha^+ \cap \Sigma(\mathbb{N}) \subset B^+ \cap \Sigma(\mathbb{N})$ (cf. the proof of lemma 2.8.6) and consequently $\bigcap_{\alpha\in\beta} M_\alpha^+ \cap \Sigma(\mathbb{N}) \subset B^+ \cap \Sigma(\mathbb{N})$.

On the other hand, if $|M_\alpha \setminus B| = \omega$ for all $\alpha \in \beta$, then the linked system $\{M_\alpha \mid \alpha \in \beta\} \cup \{\mathbb{N} \setminus B\}$ is a linked system any two members of which meet in an infinite set. Hence this linked system can be extended to a maximal linked system

$$\mathcal{M} \in \bigcap_{\alpha\in\beta} M_\alpha^+ \cap (\mathbb{N} \setminus B)^+ \cap \Sigma(\mathbb{N}).$$

Contradiction. □

Now suppose there exists a nonvoid open (in $\Sigma(\mathbb{N})$) set $U \subset \bigcap\{A_n^+ \mid n \in \omega\} \cap \Sigma(\mathbb{N})$. Without loss of generality $U = \bigcap_{i\le n} M_i^+ \cap \Sigma(\mathbb{N})$ for some infinite $M_i \subset \mathbb{N}$ $(i \le n)$. Now lemma 2.8.9 implies that for each $m \in \omega$ there is a $k(m) \le n$ such that

$$|M_{k(m)} \setminus A_m| < \omega.$$

Hence there must be a $i \le n$ such that $B = \{m \in \omega \mid k(m) = i\}$ is infinite. Choose three elements $m_1, m_2, m_3 \in B$ such that $m_1 < m_2 < m_3$. Then clearly M_i is finite since $A_{m_1} \cap A_{m_2} \cap A_{m_3} = \emptyset$, which is a contradiction. □

2.8.10. <u>REMARK</u>. E. VAN DOUWEN has pointed out to me that lemma 2.8.6 and example 2.8.8 imply that $\Sigma(N)$ is not homogeneous. Indeed, let $\mathcal{F} \in \beta\mathbb{N} \setminus \mathbb{N}$, let L be a nonempty countable intersection of open sets in $\Sigma(\mathbb{N})$ with a void interior and let $\mathcal{L} \in L$. Then lemma 2.8.6 implies that there is no autohomeomorphism ϕ of $\Sigma(\mathbb{N})$ which maps $\mathcal{F}$ onto $\mathcal{L}$.

The above example shows that nonvoid countable intersections of open sets in $\Sigma(\mathbb{N})$ need not have nonvoid interiors in $\Sigma(\mathbb{N})$. The following theorem in any case implies that such intersections have cardinality $2^{\mathfrak{c}}$.

2.8.11. THEOREM. *Let* A *be a nonvoid countable intersection of open sets in* $\Sigma(\mathbb{N})$. *Then* A *contains a homeomorph of* $\beta\mathbb{N}\setminus\mathbb{N}$.

PROOF. Since $\{M^+ \mid M \subset \mathbb{N}\}$ is an open subbase for $\lambda\mathbb{N}$ there are $B_i \subset \mathbb{N}$ $(i \in \omega)$ such that

$$\emptyset \neq \bigcap_{i\in\omega} B_i^+ \cap \Sigma(\mathbb{N}) \subset A.$$

Then $\mathcal{B} = \{B_i \mid i \in \omega\}$ is a countable collection of subsets of $\mathbb{N}$, any two members of which meet in an infinite set. If $|\mathbb{N}\setminus B| < \omega$ for all $B \in \mathcal{B}$ then $\Sigma(\mathbb{N}) = \bigcap\{B^+ \cap \Sigma(\mathbb{N}) \mid B \in \mathcal{B}\} \subset A$ and hence clearly A contains a homeomorph of $\beta\mathbb{N}\setminus\mathbb{N}$. Therefore we may assume that there is a $B_0 \in \mathcal{B}$ such that $|\mathbb{N}\setminus B_0| = \omega$. Define

$$\mathcal{C} := \{B \cap B_0 \mid B \in \mathcal{B}\}.$$

Then $\mathcal{C}$ consists of countably many infinite subsets of B_0. List $\mathcal{C}$ as $\{C_i \mid i \in \omega\}$ such that each $C \in \mathcal{C}$ is listed countably many times.

Now, by induction, for each $i \in \omega$ pick $p_i, q_i \in C_i$ such that

(i) $p_i \neq q_i$;

(ii) $\{p_i,q_i\} \cap \{p_0,\ldots,p_{i-1},q_0,\ldots,q_{i-1}\} = \emptyset$.

Define $P = \{p_i \mid i \in \omega\}$ and $Q = \{q_i \mid i \in \omega\}$. Then P and Q are two disjoint infinite subsets of B such that $|P\cap C_i| = |Q\cap C_i| = \omega$ for all $i \in \omega$. Let $r\colon \Sigma(\mathbb{N}) \longrightarrow \bigcap\{B^+ \mid B \in \mathcal{B}\} \cap \Sigma(\mathbb{N})$ be a retraction defined by

$$r(\mathcal{N}) := \bigcap\{N^+\cap\Sigma(\mathbb{N}) \mid N \in \mathcal{N} \text{ and } |N\cap B| = \omega \text{ for all } B \in \mathcal{B}\} \cap \bigcap\{B^+\cap\Sigma(\mathbb{N}) \mid B \in \mathcal{B}\}$$

(cf. theorem 1.5.2).

Let $D := \mathbb{N}\setminus B_0$. We will show that $r\restriction\beta D\setminus D$ is a homeomorphism (notice that $\beta D\setminus D \subset \beta\mathbb{N}\setminus\mathbb{N} \subset \Sigma(\mathbb{N})$). Take two ultrafilters $\mathcal{F}_0, \mathcal{F}_1 \in \beta D\setminus D$ such that $\mathcal{F}_0 \neq \mathcal{F}_1$. Then there are $F_i \in \mathcal{F}_i$ such that $F_i \subset D$ $(i \in \{0,1\})$ and $F_0 \cap F_1 = \emptyset$. Clearly $F_0 \cup P \in \mathcal{F}_0$, $F_1 \cup Q \in \mathcal{F}_1$ and $(F_0\cup P) \cap (F_1\cup Q) = \emptyset$. Also $|(F_0\cup P) \cap B| = \omega = |(F_1\cup Q) \cap B|$ for all $B \in \mathcal{B}$. Hence $r(\mathcal{F}_0) \in (F_0\cup P)^+$ and $r(\mathcal{F}_1) \in (F_1\cup Q)^+$. But $(F_0\cup P)^+ \cap (F_1\cup Q)^+ = \emptyset$ and consequently $r(\mathcal{F}_0) \neq r(\mathcal{F}_1)$. Hence $r\restriction\beta D\setminus D$ is one to one and consequently $r\restriction\beta D\setminus D$ is a homeomorphism. □

2.8.12. COROLLARY. *No* $p \in \Sigma(\mathbb{N})$ *admits a countable neighborhood basis.*

A well-known property of $\beta\mathbb{N}\setminus\mathbb{N}$, under P(c), is that each nonvoid open set contains 2^{c} P_c-points (see e.g. VAN DOUWEN [40]). Recall that a point p of a topological space is called a P_c-*point* if the intersection of less than c neighborhoods of p is again a neighborhood of p. We will show that each nonvoid open set in $\Sigma(\mathbb{N})$ also contains 2^{c} P_c-points.

2.8.13. THEOREM [P(c)]. *Each nonvoid open set in* $\Sigma(\mathbb{N})$ *contains* 2^{c} P_c-*points.*

PROOF. Let $A := \{F \in \beta\mathbb{N}\setminus\mathbb{N} \mid F \text{ is a } P_c\text{-point}\}$. Define

$$B := \{M \in \Sigma(\mathbb{N}) \mid \exists F_i \in A\ (i \le n,\ n \in \omega)\ \exists L \in \lambda\{0,1,2,\dots,n\} :$$

$$M = \{F \subset \mathbb{N} \mid \exists L \in L\colon F \in F_i\ (i \in L)\}\}.$$

We will show that B consists of P_c-points of $\Sigma(\mathbb{N})$ and that each nonvoid open set contains 2^{c} elements of B. Indeed, take $M \in B$ and let $\{O_\alpha \mid \alpha \in \beta\}$ be a collection of less than c neighborhoods of M. Without loss of generality we may assume that each O_α is of the form M_α^+ with $M_\alpha \in M$ $(\alpha \in \beta)$. Choose $F_i \in A$ $(i \le n,\ n \in \omega)$ and $L \in \lambda\{0,1,2,\dots,n\}$ such that $M = \{F \subset \mathbb{N} \mid \exists L \in L\colon F \in F_i\ (i \in L)\}$. For each M_α choose $L_\alpha \in L$ such that $M_\alpha \in F_i$ for all $i \in L_\alpha$. For each $L \in L$ define $A(L) := \{\alpha \in \beta \mid L = L_\alpha\}$.

Fix $L \in L$. For each $i \in L$ choose $F_i(L) \in F_i$ such that $|F_i(L)\setminus M_\alpha| < \omega$ for all $\alpha \in A(L)$. This is possible since F_i is a P_c-point of $\beta\mathbb{N}\setminus\mathbb{N}$. Moreover for each $i \in \{0,1,2,\dots,n\}$ define $L_i := \{L \in L \mid i \in L\}$. Then let

$$F_i := \bigcap_{L \in L_i} F_i(L).$$

Clearly $F_i \in F_i$ $(i \le n)$. Finally define

$$U := \bigcap_{L \in L} \Big(\bigcup_{i \in L} F_i\Big)^+ \cap \Sigma(\mathbb{N}).$$

It is easy to see that U is a neighborhood of M such that $U \subset \bigcap_{\alpha\in\beta} O_\alpha^+$. This shows that B consists of P_c-points.

Now, let U be a nonvoid open set in $\Sigma(\mathbb{N})$. Take $M \in U$ and let $M_i \in M$ $(i \le n)$ such that $\bigcap_{i\le n} M_i^+ \cap \Sigma(\mathbb{N}) \subset U$. For each $i,j \in \{0,1,2,\dots,n\}$ take a P_c-point $F_{ij} = F_{ji} \in A$ such that $M_i \cap M_j \in F_{ij}$. This is possible since $|M_i \cap M_j| = \omega$. Take a maximal linked system $L \in \lambda(\{0,1,\dots,n\})$ such that

that for all $i \leq n$ the set $L_i = \{(i,j) \mid j \leq n\}$ is an element of L. Notice that $\{L_i \mid i \leq n\}$ is a linked system. Now define

$$N := \{F \subset \mathbb{N} \mid \exists L \in L\colon F \in F_{ij} \ ((i,j) \in L)\}.$$

We will show that N is a maximal linked system. Clearly N is linked. Now suppose that N is not maximally linked. Take $M \subset \mathbb{N}$ such that $N \cup \{M\}$ is linked while $M \notin N$. Define $E := \{(i,j) \mid M \in F_{ij}\}$. Clearly $E \neq \emptyset$ and also $\{E\} \cup L$ is linked. Hence, as L is a maximal linked system $E \in L$ and consequently $M \in N$. Contradiction.

Since each F_{ij} is an ultrafilter, N is a maximal linked system any two members of which meet in an infinite set and hence $N \in \Sigma(\mathbb{N})$. Also it is clear that $N \in U$ and that there are 2^{c} different choices for N. □

<u>REMARK</u>. The technique used in the proof of the previous theorem is due to VERBEEK [119].

2.8.14. It is well-known that $\beta\mathbb{N}\setminus\mathbb{N}$ is an F-space. In particular, a countable union of clopen subsets of $\beta\mathbb{N}\setminus\mathbb{N}$ is always C^*-embedded. The space $\Sigma(\mathbb{N})$ cannot be an F-space, since no infinite compact F-space is supercompact (cf. corollary 1.1.6). We give an example of a countable union of clopen subsets of $\Sigma(\mathbb{N})$ that is not C^*-embedded.

NEGREPONTIS [90] has shown that the closure of a countable union of clopen sets in $\beta\mathbb{N}\setminus\mathbb{N}$ is a retract of $\beta\mathbb{N}\setminus\mathbb{N}$. The following theorem shows that a similar assertion holds in $\Sigma(\mathbb{N})$ for suitable countable unions of clopen sets, taken from the "canonical" closed subbase $\{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$.

For the remainder of this section, let $S = \{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$. This subbase is binary and for all $S \in S$ the set $\Sigma(\mathbb{N})\setminus S$ is also in S. In particular, S is normal.

2.8.15. <u>THEOREM</u>. *Let* $\{A_\alpha \mid \alpha \in \beta\}$ *be a collection of* S*-closed sets such that* $A_\alpha \subset A_\gamma$ *iff* $\alpha < \gamma$. *Then* $\mathrm{cl}_{\Sigma(\mathbb{N})}(\bigcup_{\alpha\in\beta} A_\alpha)$ *equals* $I_S(\bigcup_{\alpha\in\beta} A_\alpha)$. *In particular* $\mathrm{cl}_{\Sigma(\mathbb{N})}(\bigcup_{\alpha\in\beta} A_\alpha)$ *is supercompact and is a retract of* $\Sigma(\mathbb{N})$.

<u>PROOF</u>. Clearly $\mathrm{cl}_{\Sigma(\mathbb{N})}(\bigcup_{\alpha\in\beta} A_\alpha) \subset I_S(\bigcup_{\alpha\in\beta} A_\alpha)$. Take two distinct points $M_0, M_1 \in \mathrm{cl}_{\Sigma(\mathbb{N})}(\bigcup_{\alpha\in\beta} A_\alpha)$ and assume that there exists a point $P \in \Sigma(\mathbb{N})$ such that

$$P \in I_S(M_0,M_1) \setminus \mathrm{cl}_{\Sigma(\mathbb{N})}\left(\bigcup_{\alpha\in\beta} A_\alpha\right).$$

Take finitely many $P_i \in \mathcal{P}$ $(i \le n,\ n \in \omega)$ such that $\bigcap_{i\le n} P_i^+ \cap \bigcup_{\alpha\in\beta} A_\alpha = \emptyset$. Now suppose that for some $\ell \le n$ we have that $P_\ell \notin \mathcal{M}_0$ and $P_\ell \notin \mathcal{M}_1$. Take $M_i \in \mathcal{M}_i$ such that $M_i \cap P_\ell = \emptyset$ $(i \in \{0,1\})$. Clearly $P_\ell \cap (M_0 \cup M_1) = \emptyset$ and also

$$I_{\mathcal{S}}(\mathcal{M}_0,\mathcal{M}_1) \subset (M_0 \cup M_1)^+.$$

However $P_\ell^+ \cap (M_0 \cup M_1)^+ = \emptyset$, which is a contradiction since $\mathcal{P} \in I_{\mathcal{S}}(\mathcal{M}_0,\mathcal{M}_1)$.

Therefore each P_ℓ either belongs to $\mathcal{M}_0$ or belongs to $\mathcal{M}_1$. Define

$$C_i = \{\ell \le n \mid P_\ell \in \mathcal{M}_i\} \qquad (i \in \{0,1\}).$$

Then $\bigcap_{\ell\in C_i} P_\ell^+$ is a neighborhood of $\mathcal{M}_i$ and hence intersects $\bigcup_{\alpha\in\beta} A_\alpha$ $(i \in \{0,1\})$.

Choose $\alpha_i \in \beta$ such that $\bigcap_{\ell\in C_i} P_\ell^+ \cap A_{\alpha_i} \neq \emptyset$ $(i \in \{0,1\})$. Without loss of generality assume that $\alpha_0 \le \alpha_1$. Then

$$\left\{\bigcap_{\ell\in C_0} P_\ell^+,\ \bigcap_{\ell\in C_1} P_\ell^+,\ A_{\alpha_1}\right\}$$

is a linked system of $\mathcal{S}$-convex sets; consequently, by the fact that $\mathcal{S}$ is binary

$$\emptyset \neq \bigcap_{\ell\in C_0} P_\ell^+ \cap \bigcap_{\ell\in C_1} P_\ell^+ \cap A_{\alpha_1} = \bigcap_{i\le n} P_i^+ \cap A_{\alpha_1} = \emptyset\ ,$$

which is a contradiction.

It now follows that $\mathrm{cl}_{\Sigma(\mathbb{N})}(\bigcup_{\alpha\in\beta} A_\alpha)$ is $\mathcal{S}$-convex and hence $\mathcal{S}$-closed, by theorem 1.5.7. Therefore $\mathrm{cl}_{\Sigma(\mathbb{N})}(\bigcup_{\alpha\in\beta} A_\alpha) = I_{\mathcal{S}}(\bigcup_{\alpha\in\beta} A_\alpha)$. Hence $\mathrm{cl}_{\Sigma(\mathbb{N})}(\bigcup_{\alpha\in\beta} A_\alpha)$ is supercompact (lemma 0.5) and is a retract of $\Sigma(\mathbb{N})$ (theorem 1.5.2). □

2.8.16. <u>COROLLARY</u>. *Let* $S_i \in \mathcal{S}$ *such that* $S_i \subset S_{i+1}$ *and* $S_{i+1}\setminus S_i \neq \emptyset$ $(i \in \omega)$. *Then* $\bigcup_{i\in\omega} S_i$ *is not* C^**-embedded in* $\Sigma(\mathbb{N})$.

<u>PROOF</u>. Notice that $A = \bigcup_{i\in\omega} S_i$ is not pseudocompact, since A is σ-compact, hence normal, and not countable compact. Now suppose that A is C^*-embedded in $\Sigma(\mathbb{N})$. Then $\mathrm{cl}_{\Sigma(\mathbb{N})}(A)$ is equivalent to the Čech-Stone compactification βA of A. Hence, by theorem 2.8.15, βA is supercompact and consequently A is pseudocompact (cf. corollary 1.1.7). Contradiction. □

2.8.17. There are still many questions to be asked concerning $\Sigma(\mathbb{N})$. For example theorem 2.8.11 says that each nonvoid countable intersection of open sets in $\Sigma(\mathbb{N})$ contains a homeomorph of $\beta\mathbb{N}\setminus\mathbb{N}$. Hence such an intersection contains many countable subspaces that are C^*-embedded. On the other hand $\Sigma(\mathbb{N})$ is supercompact and hence for each countable subspace K it follows that at least one cluster point of K is the limit of a nontrivial convergent sequence in $\Sigma(\mathbb{N})$ (cf. theorem 1.1.5). Hence there are also many countable subspaces of $\Sigma(\mathbb{N})$ that are not C^*-embedded. This suggests the following question:

2.8.18. QUESTION. *When is a countable* $A \subset \Sigma(\mathbb{N})$ C^**-embedded?*

Also we have said nothing about normality in $\Sigma(\mathbb{N})$. It is well-known that CH implies that $\beta\mathbb{N}\setminus\mathbb{N}\setminus\{p\}$ is not normal for all $p \in \beta\mathbb{N}\setminus\mathbb{N}$ (cf. COMFORT & NEGREPONTIS [31], RAJAGOPALAN [95], WARREN [126]). Hence if for each $p \in \Sigma(\mathbb{N})$ there is a copy of $\beta\mathbb{N}\setminus\mathbb{N}$ in $\Sigma(\mathbb{N})$ containing p, then CH also implies that $\Sigma(\mathbb{N})\setminus\{p\}$ is not normal. Theorem 2.8.11 suggests that such may well be the case.

2.8.19. QUESTION. *Is there for each* $p \in \Sigma(\mathbb{N})$ *a homeomorph of* $\beta\mathbb{N}\setminus\mathbb{N}$ *containing* p?

2.8.20. QUESTION. *Is it true that* $\Sigma(\mathbb{N})\setminus\{p\}$ *is not normal for all* $p \in \Sigma(\mathbb{N})$?

2.8.21. In [91], PAROVIČENKO characterized $\beta\mathbb{N}\setminus\mathbb{N}$ in terms of its Boolean algebra of clopen subsets. We will show that PAROVIČENKO's result allows us to give a characterization of $\Sigma(\mathbb{N})$, not in terms of its Boolean algebra of clopen subsets but in terms of the Boolean algebra $\{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$. Clearly $S = \{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$ is not a Boolean subalgebra of the Boolean algebra of clopen subsets of $\Sigma(\mathbb{N})$. Therefore we define for S new Boolean operations and show that, under the Continuum Hypothesis, the Boolean algebra thus obtained characterizes $\Sigma(\mathbb{N})$ and hence $\lambda(\beta\mathbb{N}\setminus\mathbb{N})$. PAROVIČENKO also uses the Continuum Hypothesis and from an example given by VAN DOUWEN [40] it follows that the Continuum Hypothesis is essential in this characterization: there is a locally compact, σ-compact and separable space M for which $\beta\mathbb{N}\setminus\mathbb{N}$ and $\beta M\setminus M$ are homeomorphic under CH but not under $P(\mathfrak{c}) + \neg CH$. This same example can be used for showing that in our characterization CH is essential. The spaces $\Sigma(M)$ and $\Sigma(\mathbb{N})$ are homeomorphic under CH, but not under $P(\mathfrak{c}) + \neg CH$. One might think that this immediately follows from VAN DOUWEN's result, using the equalities

$\Sigma(\mathbb{N}) \approx \lambda(\beta\mathbb{N}\setminus\mathbb{N})$ and $\Sigma(M) \approx \lambda(\beta M\setminus M)$ (cf. theorem 2.8.2). Such is not true, however. We will present an example of two compact metric spaces X and Y which are not homeomorphic while nevertheless λX and λY are homeomorphic.

PAROVIČENKO [91] has also, without using the Continuum Hypothesis, shown that each compact Hausdorff space of weight at most ω_1 is a continuous image of $\beta\mathbb{N}\setminus\mathbb{N}$. We will show that for $\Sigma(\mathbb{N})$ this is not true, since there is a compact Hausdorff space with ω_1 points which is not the continuous image of $\Sigma(\mathbb{N})$.

2.8.22. Let $\mathcal{B} = \langle B,0,1,',\wedge,\vee\rangle$ be a Boolean algebra. $\mathcal{B}$ is called *Cantor separable* if no strictly increasing sequence has a least upper bound, i.e. if

$$a_0 < a_1 < \dots < a_n < \dots < b,$$

then there exists an element $c < b$ such that $a_n < c$ for all $n \in \omega$. In addition $\mathcal{B}$ is called *Du Bois-Reymond separable* if a strictly increasing sequence can be separated from a strictly decreasing sequence dominating the increasing one, i.e. if

$$a_0 < a_1 < \dots < a_n < \dots < b_n < \dots < b_1 < b_0,$$

then there exists an element $c \in B$ such that $a_n < c < b_n$ for all $n \in \omega$. Finally $\mathcal{B}$ is called *dense in itself* if for all $a,c \in B$ with $a < c$ there is a $b \in B$ such that $a < b < c$.

2.8.23. PAROVIČENKO [91] has shown that, under CH, a compact totally disconnected Hausdorff space of weight $\mathfrak{c}$ which possesses no isolated points *is homeomorphic to* $\beta\mathbb{N}\setminus\mathbb{N}$ *if the Boolean algebra of clopen subsets of* X *is both Cantor and Du Bois-Reymond separable*. If fact he showed that all Boolean algebras of cardinality $\mathfrak{c}$ which are dense in themselves and which are both Cantor and Du Bois-Reymond separable are isomorphic under CH. We will use PAROVIČENKO's result in this form.

2.8.24. If X is a set and if S is a collection of subsets of X for any $A \subset X$ let $W_S(A) \subset X$ be defined by

$$W_S(A) := X \setminus I_S(X\setminus A).$$

The set $W_S(A)$ is sometimes called the *S-interior* of A, just as

$I_S(A) = \bigcap\{S \in S \mid A \subset S\}$ is called the *S-closure* of A.

For technical reasons we will assume for the remainder of this section that each closed subbase S for a topological space contains $\emptyset$ and X.

2.8.25. THEOREM [CH]. *Let X be a compact Hausdorff space of weight* $\mathfrak{c}$ *which possesses no isolated points. Then X is homeomorphic to* $\Sigma(\mathbb{N})$ *(and hence to* $\lambda(\beta(\mathbb{N})\setminus\mathbb{N})$*) iff X possesses a binary closed subbase* S *satisfying:*

(i) *for all* $S \in S$ *also* $X\setminus S \in S$;

(ii) *for all* $S_0, S_1 \in S$ *also* $I_S(S_0 \cup S_1) \in S$;

(iii) *for all* $S_0, S_1 \in S$: $I_S(S_0 \cup S_1) = X \Longleftrightarrow S_0 \cup S_1 = X$;

(iv) *for all* $S_0, S_1, S_2 \in S$: $W_S(S_0 \cap I_S(S_1 \cup S_2)) = I_S(W_S(S_0 \cap S_1) \cap W_S(S_0 \cap S_2))$;

(v) *if* $S_n \in S$, $S_n \supset S_{n+1}$ $(n \in \omega)$ *then* $\bigcap_{n\in\omega} S_n$ *contains a nonvoid element of* S;

(vi) *disjoint countable unions of elements of* S *have disjoint* S*-closures.*

PROOF. " $\Rightarrow$ ".

First we remark that $\Sigma(\mathbb{N})$ indeed is a compact Hausdorff space of weight $\mathfrak{c}$ without isolated points; this follows from proposition 2.2.3 and theorem 2.8.11. Also, S is a binary subbase for $\Sigma(\mathbb{N})$ which satisfies (i). In order to show that S also satisfies (ii), (iii) and (iv) we use the equalities

(1) $$I_S((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N}))) = (M_0 \cup M_1)^+ \cap \Sigma(\mathbb{N}).$$

(2) $$W_S((M_0^+ \cap \Sigma(\mathbb{N})) \cap (M_1^+ \cap \Sigma(\mathbb{N}))) = (M_0 \cap M_1)^+ \cap \Sigma(\mathbb{N}).$$

Let us prove (1) only.

Clearly $I_S((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N}))) \subset (M_0 \cup M_1)^+ \cap \Sigma(\mathbb{N})$. Suppose that there exists an $M \in ((M_0 \cup M_1)^+ \cap \Sigma(\mathbb{N}))\setminus I_S((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N})))$. Choose $L \subset \mathbb{N}$ such that $I_S((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N}))) \subset L^+ \cap \Sigma(\mathbb{N})$ and $M \notin L^+ \cap \Sigma(\mathbb{N})$. Then $M_i^+ \cap \Sigma(\mathbb{N}) \subset L^+ \cap \Sigma(\mathbb{N})$ implies that $|M_i\setminus L| < \omega$ $(i \in \{0,1\})$ (lemma 2.8.9) and hence that $|(M_0 \cup M_1)\setminus L| < \omega$, which is a contradiction since $M \in (M_0 \cup M_1)^+\setminus L^+$.

This shows that S satisfies (ii), and also (iii); for take $S_0, S_1 \in S$ such that $I_S(S_0 \cup S_1) = \Sigma(\mathbb{N})$. Let $S_i = M_i^+ \cap \Sigma(\mathbb{N})$ $(i \in \{0,1\})$. Then $\Sigma(\mathbb{N}) = (M_0 \cup M_1)^+ \cap \Sigma(\mathbb{N})$ by (1). Hence $|\mathbb{N}\setminus(M_0 \cup M_1)| < \omega$ and consequently

$(M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N})) = \Sigma(\mathbb{N})$ (notice that in general $|\mathbb{N}\setminus(M_0 \cup M_1)| < \omega$ need not imply $M_0^+ \cup M_1^+ = \lambda\mathbb{N}$!)

Using (1) and (2) it is easy to see that $\mathcal{S}$ satisfies (iv).

$\mathcal{S}$ also satisfies (v), because of lemma 2.8.6 (recall that $P(\omega_1)$ is true in ZFC and hence that we do not use CH or $P(\mathfrak{c})$ here).

Finally $\mathcal{S}$ satisfies (vi). Let $A = \bigcup_{i\in\omega} (M_i^+ \cap \Sigma(\mathbb{N}))$ and $B = \bigcup_{i\in\omega} (L_i^+ \cap \Sigma(\mathbb{N}))$ such that $A \cap B = \emptyset$. It now follows that $\bigcup_{i\in\omega} M_i^*$ and $\bigcup_{i\in\omega} L_i^*$ are disjoint subsets of $\beta\mathbb{N}\setminus\mathbb{N}$. As $\beta\mathbb{N}\setminus\mathbb{N}$ is an F-space (cf. 0.C) these two sets have disjoint closures. Therefore we can choose two disjoint sets E and F in $\mathbb{N}$ such that $\bigcup_{i\in\omega} M_i^* \subset E^*$ and $\bigcup_{i\in\omega} L_i^* \subset F^*$. Then $\bigcup_{i\in\omega} (M_i^+ \cap \Sigma(\mathbb{N})) \subset C^+ \cap \Sigma(\mathbb{N})$ and $\bigcup_{i\in\omega} (L_i^+ \cap \Sigma(\mathbb{N})) \subset F^+ \cap \Sigma(\mathbb{N})$, which establishes (v).

"$\Leftarrow$" Define operations $\wedge, \vee, '$ on $\mathcal{S}$ in the following manner:

$$A \wedge B = W_{\mathcal{S}}(A \cap B);$$

$$A \vee B = I_{\mathcal{S}}(A \cup B);$$

$$A' = X\setminus A.$$

We will show that $\langle\mathcal{S},\wedge,\vee,',0,1\rangle$ is a Boolean algebra, where $0 = \emptyset$ and $1 = X$. Notice that for all $A,B \in \mathcal{S}$ we have that $A \wedge B \subset A \cap B$ and $A \cup B \subset A \vee B$. Because of (ii) $A \vee B \in \mathcal{S}$ for all $A,B \in \mathcal{S}$. Also $A \wedge B \in \mathcal{S}$ for all $A,B \in \mathcal{S}$, because of the equality

$$A \wedge B = (A' \vee B')'.$$

To prove this, notice that $A \wedge B = \bigcup\{X\setminus S \mid S \in \mathcal{S} \text{ and } X\setminus S \subset A \cap B\} = \bigcup\{S \in \mathcal{S} \mid S \subset A \cap B\}$ by (i). Now take $S \in \mathcal{S}$ such that $S \subset A \cap B$. Then $A' \cup B' \subset S'$ and consequently $I_{\mathcal{S}}(A' \cup B') \subset S'$. Therefore $S \subset X\setminus I_{\mathcal{S}}(A' \cup B') = (A' \wedge B')'$. Since $(A' \vee B')' \in \mathcal{S}$, by (i) and (ii) it follows that $A \wedge B = (A' \vee B')'$.

Define a relation "$\leq$" on $\mathcal{S}$ by putting $A \leq B$ iff $A \wedge B = A$. Let us prove that $A \leq B$ iff $A \subset B$, for all $A,B \in \mathcal{S}$. Indeed, assume that $A \subset B$. Then $A \wedge B = (A' \vee B')' = (A')' = A$ and therefore $A \leq B$. Next, suppose that $A \leq B$ and that there exists an $x \in A\setminus B$. Then $x \notin A \wedge B$ and consequently $A \wedge B \neq A$. Contradiction.

It now follows that the relation "$\leq$" is a partial ordering. Also it is clear that for all $A,B \in \mathcal{S}$ the set $A \wedge B$ is the greatest lower bound of A and B with respect to this ordering and in the same way $A \vee B$ is the least upper bound for A and B. Hence $(\mathcal{S},\leq)$ is a lattice. Also $(\mathcal{S},\leq)$ is distributive because of (iv) and clearly it is complemented. Hence $\langle S,\wedge,\vee,',0,1\rangle$ is a Boolean algebra.

Let us show that this Boolean algebra is Cantor separable. Take $A_n \in S$ $(n \in \omega)$ and $B \in S$ such that $A_0 < \ldots < A_n < \ldots < B$. Define $S_n := B \wedge A_n'$ $(n \in \omega)$. We will show that $S_n \neq 0$ $(n \in \omega)$. For suppose to the contrary that for some $n_0 \in \omega$ we have $S_{n_0} = 0$. Then $1 = S_{n_0}' =$ $= (B \wedge A_{n_0}')' = B' \vee A_{n_0}$ and hence, by (iii), $B' \cup A_{n_0} = X$. This is a contradiction, since $A_{n_0} < B$ (notice that in fact we have shown that for all $A,B \in S$: $A \cap B \neq \emptyset$ iff $A \wedge B \neq 0$). Also $A_n < A_{n+1}$ implies that $B \wedge A_{n+1}' \subset$ $\subset B \wedge A_n$ $(n \in \omega)$. By (v) there is a nonvoid $C \in S$ such that $C \subset \bigcap_{n\in\omega} S_n$. Then $A_0 < A_1 < \ldots < A_n < \ldots < C' < B$.

Let us prove that $\langle S,\wedge,\vee,',0,1\rangle$ is dense in itself. Indeed, take $A,C \in S$ such that $A < C$. If $A = 0$, then $C \neq \emptyset$ implies that there are two distinct points x and y in C since X contains no isolated points. By the fact that S is binary there is an $S \in S$ such that $x \in S$ and $y \notin S$. Then define $B := C \wedge S$. Now $A < B < C$. If $A \neq 0$ define $D := C \wedge A'$. Then $D \neq 0$, since $C \cap A' \neq \emptyset$; define $B := D' \wedge C$. Clearly $A < B < C$.

Let us prove that $\langle S,\wedge,\vee,',0,1\rangle$ is Du Bois-Reymond separable. Suppose that $A_0 < \ldots < A_n < \ldots < B_n < \ldots < B_0$ for some $A_n, B_n \in S$ $(n \in \omega)$. Then $\bigcup_{n\in\omega} A_n$ and $\bigcup_{n\in\omega} B_n'$ are disjoint countable unions of elements of S and hence, by (vi), have disjoint S-closures. Let $C_0 := I_S(\bigcup_{n\in\omega} A_n)$ and $C_1 := I_S(\bigcup_{n\in\omega} B_n')$. By the binarity of S there now is a $D \in S$ such that $C_0 \subset D$ and $D \cap C_1 = \emptyset$. Then clearly $A_n < D$ and $B_n' < D'$ for all $n \in \omega$. It now follows that

$$A_0 < \ldots < A_n < \ldots < D < \ldots < B_n < \ldots < B_0.$$

The cardinality of S equals $\mathfrak{c}$ since X has weight $\mathfrak{c}$ and since S is a subbase. Now, by PAROVIČENKO's result the Boolean algebra $\langle S,\wedge,\vee,',0,1\rangle$ is isomorphic to the Boolean algebra of clopen subsets $CO(\beta\mathbb{N}\setminus\mathbb{N})$ of $\beta\mathbb{N}\setminus\mathbb{N}$. Let $\sigma\colon S \longrightarrow CO(\beta\mathbb{N}\setminus\mathbb{N})$ be an isomorphism. Define a function $\phi\colon X \longrightarrow \Sigma(\mathbb{N})$ by

$$\phi(x) := \{M \subset \mathbb{N} \mid M^* \in \{\sigma(S) \mid x \in S\}\}.$$

Recall that $M^* = cl_{\beta\mathbb{N}}(M)\setminus M$ for all $M \subset \mathbb{N}$. We will show that ϕ is a homeomorphism.

CLAIM 1. *Take $x \in X$; then $S_x := \{S \in S \mid x \in S\}$ is a maximal linked system in the Boolean algebra $\langle S,\wedge,\vee,',0,1\rangle$.*

Indeed, take $S_0, S_1 \in \mathcal{S}_x$. Then $S_0 \cap S_1 \neq \emptyset$ implies that $S_0 \wedge S_1 \neq \emptyset$, which shows that $\mathcal{S}_x$ is a linked system. Also $\mathcal{S}_x$ is maximally linked, for suppose that there is an $A \in \mathcal{S}$ such that $\mathcal{S}_x \cup \{A\}$ is linked but $A \notin \mathcal{S}_x$. Then $x \notin A$ and consequently $x \in A'$. But $A \cap A' = \emptyset$ implies that $A \wedge A' = 0$. Contradiction.

The Boolean isomorphism σ maps $\mathcal{S}_x$ onto a maximal linked system in $CO(\beta\mathbb{N}\setminus\mathbb{N})$. Now it follows that

$$\{M \subset \mathbb{N} \mid M^* \in \{\sigma(S) \mid x \in S\}\}$$

is a maximal linked system in $\mathcal{P}(\mathbb{N})$ and that it is an element of $\Sigma(\mathbb{N})$. Also, the fact that σ is an isomorphism implies that ϕ is one to one and surjective. Moreover ϕ is continuous, since $\phi^{-1}[M^+ \cap \Sigma(\mathbb{N})] = \sigma^{-1}[M^*]$ for all $M \subset \mathbb{N}$. Therefore ϕ is a homeomorphism. □

2.8.26. <u>COROLLARY</u> [CH]. *If* X *is a zero-dimensional noncompact* σ*-compact and locally compact space with* $|C(X)| = \mathfrak{c}$, *then* $\Sigma(X)$ *and* $\Sigma(\mathbb{N})$ *are homeomorphic.*

<u>PROOF</u>. It is easy to see that $\{M^+ \cap \Sigma(X) \mid M \text{ is open and closed in } X\}$ satisfies all conditions of theorem 2.8.25 (notice that X Lindelöf being σ-compact implies that for closed sets $A,B \subset X$ with $A \cap B = \emptyset$ there is an open and closed $U \subset X$ such that $A \subset U$ and $B \subset X\setminus U$). □

2.8.27. <u>REMARK</u>. Corollary 2.8.26 also follows directly from PAROVIČENKO's result. For if X is a zero-dimensional noncompact σ-compact and locally compact space with $|C(X)| = \mathfrak{c}$ then there is a homeomorphism $\phi\colon \beta X\setminus X \to \beta\mathbb{N}\setminus\mathbb{N}$ by PAROVIČENKO's characterization of $\beta\mathbb{N}\setminus\mathbb{N}$. This homeomorphism can be extended to a homeomorphism $\lambda(\phi)\colon \lambda(\beta X\setminus X) \longrightarrow \lambda(\beta\mathbb{N}\setminus\mathbb{N})$ (theorem 2.3.4). Now theorem 2.8.2 implies that $\Sigma(X)$ is homeomorphic to $\Sigma(\mathbb{N})$.

2.8.28. <u>EXAMPLE</u>. *A locally compact and* σ*-compact separable space* M *for which* $\Sigma(M)$ *and* $\Sigma(\mathbb{N})$ *are homeomorphic under* CH *but not under* $P(\mathfrak{c}) + \neg CH$.

As noted in the introduction of this chapter this example is based on an example of VAN DOUWEN [40].

Let $M = \mathbb{N} \times \{0,1\}^{\mathfrak{c}}$. Then clearly $\Sigma(M)$ and $\Sigma(\mathbb{N})$ are homeomorphic under CH (cf. corollary 2.8.26). Assume that $\omega_1 < \mathfrak{c}$ and let $K = \{0,1\}^{\mathfrak{c}}$. Let $\mathcal{K} := \{\pi_\alpha^{-1}[\{i\}] \mid \alpha \in \omega_1,\ i \in \{0,1\}\}$. Then $\{\mathbb{N} \times K \mid K \in \mathcal{K}\}$ is a collection of ω_1 clopen subsets of M each infinite subcollection of which has a void interior. As for each $\alpha \in \omega_1$ we have

$$(\mathbb{N} \times \pi_\alpha^{-1}[\{0\}]) \cup (\mathbb{N} \times \pi_\alpha^{-1}[\{1\}]) = M$$

for each $M \in \lambda M$ there is an $i \in \{0,1\}$ such that $\mathbb{N} \times \pi_\alpha^{-1}[\{i\}] \in M$. For each $M \in \Sigma(M)$ let $K(M) := \{K \in K \mid \mathbb{N} \times K \in M\}$. It follows that $K(M)$ is uncountable for each $M \in \Sigma(\mathbb{N})$ and also that $\{K(M) \mid M \in \Sigma(\mathbb{N})\}$ has cardinality 2^{ω_1}. Also

$$A := \{\cap\{(\mathbb{N}\times K)^+ \mid K \in K(M)\} \cap \Sigma(M) \mid M \in \Sigma(M)\}$$

covers $\Sigma(M)$. The collection A has cardinality 2^{ω_1} and consists of pairwise disjoint sets each an intersection of ω_1 clopen subsets of $\Sigma(M)$.

Let us prove that each $A \in A$ has a void interior. Assume there exist open and closed $C_0,\ldots,C_n \subset M$ such that

$$\emptyset \neq \bigcap_{i\leq n} C_i^+ \cap \Sigma(M) \subset A_0$$

for some $A_0 \in A$. Let $M_0 \in \Sigma(M)$ such that $A_0 = \cap\{(\mathbb{N}\times K)^+ \mid K \in K(M_0)\} \cap \Sigma(M)$. Now the fact that

$$\bigcap_{i\leq n} C_i^+ \cap \Sigma(M) \subset \cap\{(\mathbb{N}\times K)^+ \mid K \in K(M_0)\} \cap \Sigma(M)$$

implies that for all $K \in K(M_0)$ there is an $i_K \leq n$ such that $C_i\backslash(\mathbb{N}\times K)$ is compact; for otherwise $\bigcap_{i\leq n} C_i^+ \cap \Sigma(M) \not\subset (\mathbb{N}\times K)^+ \cap \Sigma(M)$.

Hence there is an $i_0 \leq n$ such that $L = \{K \in K(M_0) \mid i_K = i_0\}$ is uncountable. Also, clearly, C_{i_0} is not compact. Choose for every $L \in L$ an integer $i(L)$ such that $\emptyset \neq C_{i_0} \cap (\{i(L)\}\times K) \subset \{i(L)\} \times L$ (this is possible since $C_{i_0}\backslash(\mathbb{N}\times L)$ is compact!). There is an integer i such that the collection

$$B = \{L \in L \mid i(L) = i\}$$

is infinite, since L is uncountable. But then $\cap B$ has a nonvoid interior in K, since $\emptyset \neq C_{i_0} \cap (\{i\}\times K) \subset \{i\} \times \cap B$, which is a contradiction.

Now suppose that there is a homeomorphism $\phi\colon \Sigma(\mathbb{N}) \longrightarrow \Sigma(M)$. Take $F \in \beta\mathbb{N}\backslash\mathbb{N}$ and take $A \in A$ such that $F \in \phi^{-1}(A)$. As A is an intersection of ω_1 clopen sets, so is $\phi^{-1}(A)$. Also $\phi^{-1}(A)$ has a void interior. However P(C) + $\neg$CH implies that this intersection has a nonvoid interior (lemma 2.8.6). Contradiction. $\square$

2.8.29. EXAMPLE. *Two compact metric spaces* X *and* Y *which are not homeomorphic while yet* λX *and* λY *are homeomorphic.*

Let $X = I$, the closed unit interval and let $Y = \{(0,y) \mid -1 \le y \le 1\} \cup \cup \{x, \sin \frac{1}{x}) \mid 0 \le x \le 1\}$. In chapter 3 (sections 3.4 and 3.2) we will show that λX and λY both are homeomorphic to the Hilbert cube Q. □

2.8.30. EXAMPLE. *A separable compact Hausdorff space with* ω_1 *points which is not the continuous image of* $\Sigma(\mathbb{N})$.

In section 1.1 we gave an example of a separable compact Hausdorff space with ω_1 points which is not the continuous image of a supercompact Hausdorff space (cf. example 1.1.18). Hence this space is not the continuous image of $\Sigma(\mathbb{N})$. □

2.8.31. QUESTION. *Is there a separable supercompact first countable Hausdorff space which is not the continuous image of* $\Sigma(\mathbb{N})$?

2.9. Another nonsupercompact compact Hausdorff space

In section 1.1 we gave an example of a compact Hausdorff space which is not supercompact but which admits a closed subbase $\mathcal{T}$ such that for all $\mathcal{M} \subset \mathcal{T}$ with $\cap\mathcal{M} = \emptyset$ there are $M_0, M_1, M_2 \in \mathcal{M}$ such that $M_0 \cap M_1 \cap M_2 = \emptyset$. In this section we will present another space with this property. The space is a subspace of $\lambda\mathbb{N}$ and the subbase with the above property is just the restriction of the canonical binary subbase of $\lambda\mathbb{N}$ to the space under consideration. This makes the example of independent interest.

2.9.1. Let $\mathcal{S}$ denote the canonical binary subbase of $\lambda\mathbb{N}$ and for each $A \subset \lambda\mathbb{N}$, let $I(A)$ (as usual) be defined by $I(A) := \cap\{S \in \mathcal{S} \mid A \subset S\}$ (cf. section 1.1). We start with a simple but useful lemma.

2.9.2. LEMMA. *Let* $A \subset \lambda\mathbb{N}$. *Then for all* $M \in \mathcal{M} \in I(A)$ *there is an* $\mathcal{A} \in A$ *such that* $M \in \mathcal{A}$.

PROOF. Suppose that $M \notin \mathcal{A}$ for all $\mathcal{A} \in A$. Then $\mathbb{N}\setminus M$ belongs to each $\mathcal{A}$ in A and consequently $A \subset (\mathbb{N}\setminus M)^+$. But then $I(A) \subset (\mathbb{N}\setminus M)^+$, which is a contradiction since $\mathcal{M} \in I(A)$. □

2.9.3. EXAMPLE. *There is a subspace* X *of* $\lambda\mathbb{N}$ *with the following properties:*

(a) X *is not supercompact;*

(b) *for all* $\mathcal{M} \subset \{S \cap X \mid S \in \mathcal{S}\}$ *with* $\bigcap\mathcal{M} = \emptyset$ *there are* $M_0, M_1, M_2 \in \mathcal{M}$ *such that* $M_0 \cap M_1 \cap M_2 = \emptyset$.

Indeed, define

$$X := \{\mathcal{M} \in \lambda\mathbb{N} \mid \forall M_0, M_1, M_2 \in \mathcal{M}: [M_0 \cap M_1 \cap M_2 = \emptyset \Rightarrow \exists i \in \{0,1,2\}: 1 \in M_i]\}.$$

Notice that $\mathbb{N} \subset X$ and therefore, as we will show that X is closed in $\lambda\mathbb{N}$, also $\beta\mathbb{N} \subset X$.

CLAIM 1. X *is compact.*

Indeed, assume that $\mathcal{M} \notin X$. Then there exist $M_0, M_1, M_2 \in \mathcal{M}$ with $M_0 \cap M_1 \cap M_2 = \emptyset$ and $1 \notin M_i$ ($i \in \{0,1,2\}$). Then $M_0^+ \cap M_1^+ \cap M_2^+$ is an open neighborhood of $\mathcal{M}$ which obviously misses X. Hence X is closed in the compact space $\lambda\mathbb{N}$.

CLAIM 2. *The closed subbase* $\mathcal{T} = \{M^+ \cap X \mid M \subset \mathbb{N}\}$ *has the property that for each* $\mathcal{M} \subset \mathcal{T}$ *with* $\bigcap\mathcal{M} = \emptyset$ *there are* $M_0, M_1, M_2 \in \mathcal{M}$ *such that* $M_0 \cap M_1 \cap M_2 = \emptyset$.

Let $\mathcal{M} \subset \mathcal{T}$ be a subsystem any three members of which meet. We will show that $\bigcap\mathcal{M} \neq \emptyset$. This suffices to prove the claim.

We will show, by induction, that $\mathcal{M}$ has the finite intersection property; then, by claim 1, $\bigcap\mathcal{M} \neq \emptyset$. Assume that any $n-1$ members of $\mathcal{M}$ meet. If $n=2$ or $n=3$ then obviously any n members of $\mathcal{M}$ meet. Therefore we may assume that $n>3$. Let $M_i^+ \cap X \in \mathcal{M}$ ($i \in \{1,2,\dots,n\}$ and take, for each $i \in \{1,2,3,4\}$

$$L_i \in \bigcap_{j \neq i} M_j^+ \cap X.$$

Now define

$$\begin{aligned} Z := {} & I(\{L_1,L_2,L_3\}) \cap I(\{L_1,L_3,L_4\}) \cap I(\{L_1,L_2,L_4\}) \cap I(\{L_2,L_3,L_4\}) \\ & \cap I(\{1,L_1,L_2\}) \cap I(\{1,L_1,L_3\}) \cap I(\{1,L_1,L_4\}) \cap I(\{1,L_2,L_3\}) \\ & \cap I(\{1,L_2,L_4\}) \cap I(\{1,L_3,L_4\}). \end{aligned}$$

Notice that, as $\{M^+ \mid M \subset \mathbb{N}\}$ is binary the set Z is nonvoid. We claim that

$$Z \subset \bigcap_{i=1}^{n} M_i^+ \cap X.$$

That $Z \subset \bigcap_{i=1}^{n} M_i^+$ is trivial. We proceed to show that $Z \subset X$. Suppose there were an $N \in Z$ with $N_i \in N$ $(i \in \{1,2,3\})$ such that $N_1 \cap N_2 \cap N_3 = \emptyset$ and $1 \notin N_i$ $(i \in \{1,2,3\})$. We will derive a contradiction.

Fix $i \in \{1,2,3\}$ and let $A_i := \{j \in \{1,2,3,4\} \mid N_i \in L_j\}$. Then $|A_i| \geq 3$. Suppose that $|A_i| < 3$; then there exist distinct $\ell, m \in \{1,2,3,4\}\backslash A_i$. Then, as $N \in I(\{1,L_\ell,L_m\})$ and as $1 \notin N_i$, by lemma 2.9.2, we must have that either $N_i \in L_\ell$ or $N_i \in L_m$. Contradiction.

Now, $|A_i| \geq 3$ for all $i \in \{1,2,3\}$; therefore

$$A_1 \cap A_2 \cap A_3 \neq \emptyset.$$

Take $m \in A_1 \cap A_2 \cap A_3$. Then $N_i \in L_m$ for all $i \in \{1,2,3\}$ and as $L_m \in X$ this is a contradiction.

CLAIM 3. X *is not supercompact; it is not even the continuous image of a supercompact Hausdorff space.*

Assume that T is a binary closed subbase for X. We assume that T is closed under arbitrary intersection (cf. lemma 0.5). Let $A \subset P(\mathbb{N})\backslash\{1\})$ be an uncountable almost disjoint family of infinite sets which satisfies:

(*) For each uncountable $B \subset A$ there is a $B \in B$ and an infinite $C \subset B\backslash\{B\}$ such that $C \cap C' \subset B$ for all distinct $C, C' \in C$.

There is such an almost disjoint family, cf. 1.1.14 and lemma 1.1.15.

For each infinite $B \subset N$, the set $B^+ \cap X$ is clopen in X and consequently, since T is closed under arbitrary intersection, there exists a finite $F \subset T$ such that $B^+ \cap X = \bigcup F$ (cf. 0.3). Therefore there exists an $T(B) \in T$ such that $T(B) \subset B^+ \cap X$ and $T(B) \cap B$ is infinite.

As $\{T(A) \cap A \mid A \in A\}$ is an uncountable collection of subsets of $\mathbb{N}\backslash\{1\}$ there is an $n_0 \in \mathbb{N}\backslash\{1\}$ such that $A_1 = \{A \in A \mid n_0 \in T(A) \cap A\}$ is uncountable. Take an $A_0 \in A_1$ and an infinite $C \subset A_1$ such that

$$C \cap C' \subset A_0$$

for all distinct $C, C' \in C$. Then

$$\{T(C) \cap ((\mathbb{N}\setminus\{1\}) \setminus A_0)^+ \cap X \mid C \in \mathcal{C}\}$$

is an infinite disjoint collection nonvoid subsets of $((\mathbb{N}\setminus\{1\})\setminus A_0)^+ \cap X$. As this latter set is clopen in X, there is a finite $\mathcal{F} \subset \mathcal{T}$ such that $\cup\mathcal{F} = ((\mathbb{N}\setminus\{1\}\setminus A_0)^+ \cap X$. Choose a $T \in \mathcal{F}$ such that T intersects both T(C) and T(C') for certain $C,C' \in \mathcal{C}$ $(C \neq C')$. Then

$$\mathcal{L} = \{T,T(C),T(C')\}$$

is a linked system with a void intersection. That $\mathcal{L}$ is indeed linked is trivial since $n_0 \in T(C) \cap T(C')$. But

$$\begin{aligned}\cap\mathcal{L} &= T \cap T(C) \cap T(C') \\ &\quad \cap ((\mathbb{N}\setminus\{1\})\setminus A_0)^+ \cap C^+ \cap C'^+ \cap X \\ &= \emptyset,\end{aligned}$$

since $((\mathbb{N}\setminus\{1\})\setminus A_0) \cap C \cap C' \subset ((\mathbb{N}\setminus\{1\})\setminus A_0) \cap A_0 \subset (\mathbb{N}\setminus A_0) \cap A_0 = \emptyset$ and neither contains 1. Contradiction.

The assertion that X is not the continuous image of a supercompact space can be shown using the same technique, cf. proposition 1.1.16. □

REMARK. The proof of claim 3 of the above example is a simple modification of the technique used in the proof of proposition 1.1.16.

2.9.4. In section 2.8 we defined two subspaces $\sigma(\mathbb{N})$ and $\Sigma(\mathbb{N})$ of $\lambda\mathbb{N}$ which are, in some sense, related to the space X constructed in example 2.9.3. The spaces $\sigma(\mathbb{N})$ and $\Sigma(\mathbb{N})$ both have a void intersection with $\mathbb{N}$, but both contain $\beta\mathbb{N}\setminus\mathbb{N}$. Therefore $\sigma(\mathbb{N}) \cup \mathbb{N}$ and $\Sigma(\mathbb{N}) \cup \mathbb{N}$ are closed in $\lambda\mathbb{N}$. This suggests the question whether the spaces $\sigma(\mathbb{N}) \cup \mathbb{N}$ and $\Sigma(\mathbb{N}) \cup \mathbb{N}$ have the same properties as example 2.9.3 (recall that $\mathbb{N} \subset \beta\mathbb{N} \subset X$!). For $\sigma(\mathbb{N}) \cup \mathbb{N}$ this is disproved in the next proposition; $\Sigma(\mathbb{N}) \cup \mathbb{N}$ is more difficult to treat, however, it can also be shown that it differs in compactness type from X.

2.9.5. PROPOSITION.

(i) $\sigma(\mathbb{N})$ *and* $\Sigma(\mathbb{N})$ *are supercompact;*

(ii) $\sigma(\mathbb{N}) \cup \mathbb{N}$ *is supercompact; in fact* $\sigma(\mathbb{N}) \cup \mathbb{N} \approx \lambda(\mathbb{N}),\mathcal{H})$ *where* $\mathcal{H} = \{M \subset \mathbb{N} \mid |M| = 1 \vee |M| = \omega\}$;

(iii) $\Sigma(\mathbb{N}) \cup \mathbb{N}$ *is not supercompact;*

(iv) *the subbase* $\mathcal{T} := \{M^+ \cap (\sigma(\mathbb{N}) \cup \mathbb{N}) \mid M \subset \mathbb{N}\}$ *for* $\sigma(\mathbb{N}) \cup \mathbb{N}$ *has the property that for each* $n \geq 3$ *there is an* $\mathcal{F} \subset \mathcal{T}$ *with* $|\mathcal{F}| = n$ *and* $\bigcap\mathcal{F} = \emptyset$ *while* $\bigcap(\mathcal{F}\setminus\{F\}) \neq \emptyset$ *for all* $F \in \mathcal{F}$;

(v) *the subbase* $\mathcal{V} := \{M^+ \cap (\Sigma(\mathbb{N}) \cup \mathbb{N}) \mid M \subset \mathbb{N}\}$ *for* $\Sigma(\mathbb{N}) \cup \mathbb{N}$ *has the same property as* $\mathcal{T}$.

PROOF. (i) The supercompactness of $\sigma(\mathbb{N})$ follows from (ii). That $\Sigma(\mathbb{N})$ is supercompact was shown in theorem 2.8.2 (iii).

(ii) Define a mapping $\phi\colon \lambda(\mathbb{N},\mathcal{H}) \to \lambda\mathbb{N}$ by $\phi(\mathcal{M}) := \mathcal{M}$ (it is easy to see that an mls $\mathcal{M} \subset \mathcal{H}$ is also an mls in $\mathcal{P}(\mathbb{N})$). The simple proof that ϕ is an embedding and that $\phi[\lambda(\mathbb{N},\mathcal{H})] = \sigma(\mathbb{N}) \cup \mathbb{N}$ is left to the reader.

(iii) This can be proved using the same technique as in example 2.9.3 claim 3. Under P(c), we will give another proof, which uses theorem 1.1.5. Assume that $\Sigma(\mathbb{N}) \cup \mathbb{N}$ were supercompact. Then, by theorem 1.1.5, at most countably many points of $\beta\mathbb{N}\setminus\mathbb{N}$ are not the limit of a nontrivial convergent sequence in $\Sigma(\mathbb{N}) \cup \mathbb{N}$. As no sequence in $\mathbb{N}$ converges it follows that at most countably many points of $\beta\mathbb{N}\setminus\mathbb{N}$ are not the limit of a nontrivial convergent sequence in $\Sigma(\mathbb{N})$. Under P(c), there are $2^{\mathfrak{c}}$ $P_{\mathfrak{c}}$-points in $\beta\mathbb{N}\setminus\mathbb{N}$ (VAN DOUWEN [40]). It is easy to see that a $P_{\mathfrak{c}}$-point in $\beta\mathbb{N}\setminus\mathbb{N}$ is also a $P_{\mathfrak{c}}$-point in $\Sigma(\mathbb{N})$. But a $P_{\mathfrak{c}}$-point is not the limit of a nontrivial convergent sequence. Hence there are $2^{\mathfrak{c}}$ points in $\beta\mathbb{N}\setminus\mathbb{N}$ which are not the limit of a nontrivial convergent sequence in $\Sigma(\mathbb{N})$. Contradiction.

(iv) Fix $n \geq 3$ and define $\mathcal{F} := \{(\{1,2,\ldots,n\}\setminus\{i\})^+ \mid i \leq n\}$. Then $|\mathcal{F}| = n$ and $\bigcap\mathcal{F} \cap (\sigma(\mathbb{N}) \cup \mathbb{N}) = \emptyset$ while $\bigcap(\mathcal{F}\setminus\{F\}) \cap (\sigma(\mathbb{N})\setminus\mathbb{N}) \neq \emptyset$ for all $F \in \mathcal{F}$, as can easily be seen.

(v) This can be proved in the same way as in (iv). □

2.10. Subbases, convex sets and hyperspaces

In this section we will study the operator $I_{\mathcal{S}}$, defined in 1.5.1. We do not restrict ourselves to binary normal subbases. For any topological space X and for any closed subbase $\mathcal{S}$ for X we define

$$I_{\mathcal{S}}(A) := \bigcap\{S \in \mathcal{S} \mid A \subset S\}$$

for all $A \subset X$ (an empty intersection will represent, by convention, the whole space X). The set $I_{\mathcal{S}}(A)$ is called the *$\mathcal{S}$-closure* of A, or, the *$\mathcal{S}$-convex closure* of A. By definition, $H(X,\mathcal{S})$ will denote the space of all nonvoid S-closed sets, endowed with the subspace topology of 2^X.

We are interested in compactness properties for the spaces $H(X,S)$. Our main result in this section is that if X is a compact space and if S is a normal T_1-subbase which is closed under arbitrary intersection, then $H(X,S)$ *is compact if and only if* $H(X,S)$ *is a retract of* 2^X*, and also if and only if the map* $I_S\colon 2^X \longrightarrow H(X,S)$ (*which sends each closed set* $A \subset X$ *onto its* S*-closure*) *is a retraction*.

We first prove that if S is a binary normal subbase for X then $H(X,S)$ is compact though establishing that the closure operator I_S is a retraction. This fact then is used to obtain the general compactness result cited above.

The results in this section are taken from VAN MILL & VAN DE VEL [82].

We start with the following remarkable result:

2.10.1. THEOREM. *Let* S *be a binary normal subbase for* X. *Then the operation of intersecting two* S*-closed sets is continuous.*

PROOF. First notice that X is normal, being compact and Hausdorff (cf. 2.2.4 (iii)). Let

$$\Lambda \subset H(X,S) \times H(X,S)$$

be the subspace of all pairs (A,B) such that $A \cap B \neq \emptyset$. We have to show that the mapping

$$\cap\colon \Lambda \longrightarrow H(X,S)$$

assigning to $(A,B) \in \Lambda$ the S-closed set $A \cap B$, is continuous. We shall use the open subbase of 2^X, consisting of all sets of type $\langle O\rangle$ or $\langle O,X\rangle$, where $O \subset X$ is open.

Assume first that $(A,B) \in \Lambda$ and that $O \subset X$ is an open set such that $A \cap B \subset O$. A straightforward argument, using the normality of X, then shows that there exists a neighborhood V_0 of A and a neighborhood V, of B, in 2^X, such that $(V_0 \times V_1) \cap \Lambda$ is mapped into $\langle O\rangle$ by the intersection operator.

Assume next that $A \cap B \cap O \neq \emptyset$ for some pair $(A,B) \in \Lambda$ and for some open set $O \subset X$. Let $x \in A \cap B \cap O$. Since S is a normal T_1-subbase (cf. 0.4) there are $S_1,\ldots,S_n \in S$ such that

$$x \in \operatorname{int}_X \left(\bigcap_{i=1}^{n} S_i\right) \subset \bigcap_{i=1}^{n} S_i \subset O.$$

Hence $(\langle \cap_{i=1}^n S_i, X\rangle \times \langle \cap_{i=1}^n S_i, X\rangle) \cap \Lambda$ is a neighborhood of (A,B) and for each pair (A',B') in this neighborhood the system $\{A',B',S_1,\ldots,S_n\}$ is linked. Hence, by binarity of $\mathcal{S}$, also

$$\bigcap_{i=1}^{n} S_i \cap A' \cap B' \neq \emptyset.$$

It follows that $A' \cap B' \cap O = \emptyset$ for all $(A',B') \in (\langle \cap_{i=1}^n S_i, X\rangle \times \langle \cap_{i=1}^n S_i, X\rangle) \cap \Lambda$. □

2.10.2. It can easily been deduced from theorem 2.10.1 (or proved directly with the above method) that n-fold intersection is also continuous on the hyperspace of $\mathcal{S}$-closed sets, associated to a normal binary subbase $\mathcal{S}$.

The continuity of the 2-fold intersection operator - even in one variable at the time - seems to be fairly exceptional in hyperspaces, as can be seen from the next example.

2.10.3. EXAMPLE. Let X be the unit 2-cell. For each $t \in [0,\pi]$ we let $F_t \in 2^X$ be the line segment joining 0 and e^{it} (regarding X as a subspace of the complex numbers). The mapping

$$F\colon [0,\pi] \longrightarrow 2^X,$$

sending t onto F_t obviously is continuous. The map

$$G\colon F[[0,\pi]] \longrightarrow 2^X$$

assigning to F_t the set $F_\pi \cap F_t$, is not continuous, since the image of $G \circ F$ consists of the two points $\{0\}$ and F_π of 2^X. □

Before passing to general normal subbases, we need one other theorem dealing with binary normal subbases. We begin with the following auxiliary result (compare lemma 1.5.10).

2.10.4. LEMMA. *Let $\mathcal{S}$ be a binary normal subbase for the topological space* X. *For each* $n \geq 2$ *the mapping* $f\colon X^{n+1} \longrightarrow X$, *which sends an* $(n+1)$-*tuple* $(x,x_1,\ldots,x_n) \in X^{n+1}$ *onto the unique point in* $\cap_{i=1}^n I_{\mathcal{S}}(x,x_i) \cap \cap I_{\mathcal{S}}(\{x_1,\ldots,x_n\})$, *is continuous.*

PROOF. The uniqueness of $f(x,x_1,\ldots,x_n)$ is a consequence of theorem 1.5.2.

To prove the continuity, let $S \in \mathcal{S}$ and let $(x,x_1,\ldots,x_n) \in X^{n+1} \setminus f^{-1}[S]$. Then

$$\bigcap_{i=1}^{n} I_S(x,x_i) \cap I(\{x_1,\dots,x_n\}) \cap S = \emptyset,$$

and S being binary, we have that either $I_S(x,x_i) \cap S = \emptyset$ for some $i \le n$, or that $I_S(\{x_1,\dots,x_n\}) \cap S = \emptyset$.

In the first case, using the normality of S, there is an $S_0 \in S$ such that

$$I_S(x,x_i) \subset \mathrm{int}_X(S_0) \subset S_0 \subset X\setminus S.$$

Let $\pi_j\colon X^{n+1} \to X$ denote the projection mapping onto the j^{th} coordinate. Then

$$U = \Pi_0^{-1}[\mathrm{int}_X(S_0)] \cap \Pi_i^{-1}[\mathrm{int}_X(S_0)]$$

is a neighborhood of $(x,x_1,\dots,x_n)$ which does not meet $f^{-1}[S]$. For, if $(y,y_1,\dots,y_n) \in U$, then $\{y,y_i\} \subset \mathrm{int}_X(S_0) \subset S_0$, whence

$$f(y,y_1,\dots,y_n) \in I_S(y,y_i) \subset S_0 \subset X\setminus S.$$

In the second case one can proceed in the same way. First, choose $S_0 \in S$ such that

$$I_S(\{x,x_1,\dots,x_n\}) \subset \mathrm{int}_X(S_0) \subset S_0 \subset X\setminus S.$$

Then, let $U := \bigcap_{i=1}^{n} \Pi_i^{-1}[\mathrm{int}_X(S_0)]$. This set is a neighborhood of $(x,x_1,\dots,x_n)$ not meeting $f^{-1}[S]$. □

2.10.5. THEOREM. *Let S be a normal binary subbase for the topological space X. Then the map $I_S\colon 2^X \to H(X,S)$ is a continuous retraction of 2^X onto $H(X,S)$ (in particular $H(X,S)$ is compact).*

PROOF. For simplfication of notation, write $r = I_S$. Let us prove that r is continuous. Fix an open set $O \subset X$ and assume first that $A \in r^{-1}[\langle O\rangle]$. Then $I_S(A) \subset O$. Since X is compact and since S is a closed subbase, there exists $S_{ij} \in S$ ($i,j \le n$, $n \in \omega$) such that

$$X\setminus O \subset \bigcup_{i\le n} \bigcap_{j\le n} S_{ij} \subset X\setminus I_S(A).$$

Since S is normal and binary, we have that the collection of S-closed also is normal (cf. 0.5). For each $i \le n$, we therefore can choose $T_i \in S$ such that

$$I_{\mathcal{S}}(A) \subset \mathrm{int}_X(T_i) \subset T_i \subset X \setminus \bigcap_{j\le n} S_{ij}.$$

Define $Z := \bigcap_{i\le n} T_i$. Then Z is $\mathcal{S}$-closed and

$$I_{\mathcal{S}}(A) \subset \mathrm{int}_X(Z) \subset Z \subset O.$$

For each $A' \in \langle Z\rangle$ we have that $I_{\mathcal{S}}(A') \subset Z \subset O$, proving that $\langle Z\rangle$ is a neighborhood of A which is mapped by r into $\langle O\rangle$.

Assume next that $A \in r^{-1}[\langle O,X\rangle]$. Choose $p \in I_{\mathcal{S}}(A) \cap O$.

<u>CLAIM</u> 1. $\{p\} = \bigcap_{a\in A} I_{\mathcal{S}}(p,a)$.

Indeed, choose $z \in \bigcap_{a\in A} I_{\mathcal{S}}(p,a)$ such that $z \neq p$. By the fact that $\mathcal{S}$ is a normal T_1-subbase (cf. 0.4), there are $S_0,S_1 \in \mathcal{S}$ such that $z \in S_0\setminus S_1$, $p \in S_1\setminus S_0$ and $S_0 \cup S_1 = X$. Now if $A \cap S_1 = \emptyset$ it would follow that

$$A \subset I_{\mathcal{S}}(A) \subset S_0,$$

which is a contradiction since $p \in I_{\mathcal{S}}(A)$. Therefore, there is an $a_0 \in A \cap S_1$. But then

$$\bigcap_{a\in A} I_{\mathcal{S}}(p,a) \subset I_{\mathcal{S}}(p,a_0) \subset S_1,$$

which also is a contradiction since $z \notin S_1$.

By claim 1, and by the compactness of X there exist finitely many $a_i \in A$ $(i \le n,\ n \in \omega)$ such that

$$\bigcap_{i\le n} I_{\mathcal{S}}(p,a_i) \subset O.$$

Consequently, using the notation of lemma 2.10.4,

$$\{f(p,a_0,a_1,\ldots,a_n)\} = \bigcap_{i\le n} I_{\mathcal{S}}(a_i,p) \cap I_{\mathcal{S}}(\{a_0,a_1,\ldots,a_n\}) \subset O.$$

By the continuity of f, cf. lemma 2.10.4, there exist open neighborhoods V_i of a_i $(i \le n)$ such that $f(p,a_0',a_1',\ldots,a_n') \in O$ for all $n+1$-tuples $(a_0',a_1',\ldots,a_n') \in \prod_{i\le n} V_i$. Hence, the set $\langle V_0,V_1,\ldots,V_n,X\rangle$ is a neighborhood of $A \in 2^X$, which is mapped by r into O. For take $B \in \langle V_0,V_1,\ldots,V_n,X\rangle$ and choose $b_i \in B \cap V_i$ $(i \le n)$. Then

$$\emptyset \neq \bigcap_{i\le n} I_{\mathcal{S}}(p,b_i) \cap I_{\mathcal{S}}(\{b_0,b_1,\ldots,b_n\}) \subset O \cap I_{\mathcal{S}}(B),$$

since $\{b_0,b_1,\dots,b_n\} \subset B$. In particular, $O \cap I_S(B) \neq \emptyset$, or, equivalently, $r(B) \in \langle O,X\rangle$.

Finally, clearly $r(C) = C$ for each S-closed set C, proving that r is a retraction. □

2.10.6. CURTIS & SCHORI [36] have shown that $C(X)$, the space of all subcontinua of X, is a Hilbert cube factor (that is, a space of which the product with the Hilbert cube is homeomorphic to the Hilbert cube) if and only if X is a Peano continuum. In particular, this implies that $C(X)$ is a retract of 2^X. Theorem 2.10.5 implies that for the class of dendra, a subclass of the class of all Peano continua, such a retraction can be explicitly described. For, the collection of subcontinua of a dendron X is a binary normal closed subbase for X (in theorem 1.3.21 it was shown that the collection of complements of segments of a compact tree-like space is a binary normal subbase. As each connected closed subset A of a compact tree-like space X is the intersection of all complements of segments containing A, it follows that the collection of subcontinua of X is also a binary normal subbase).

We now can prove the following compactness theorem for normal subbases.

2.10.7. <u>THEOREM</u>. *Let X be a compact space and let S be a normal T_1-subbase for X which is closed under arbitrary intersection. Then the following assertions are equivalent:*

(i) $H(X,S)$ *is compact;*

(ii) *the map* I_S *is a retraction of* 2^X *onto* $H(X,S)$*;*

(iii) $H(X,S)$ *is a retract of* 2^X*;*

(iv) *regarding X as a subspace of its superextension $\lambda(X,S)$, the operation of intersection with X yields a continuous mapping*

$$H(\lambda(X,S),S^+) \supset S^+\setminus\{\emptyset\} \longrightarrow H(X,S);$$

(v) $H(X,S)$ *has a closed normal T_1-subbase consisting of all sets of type*

$$\langle S\rangle \cap H(X,S), \quad or, \quad \langle S,X\rangle \cap H(X,S) \qquad (S \in S);$$

(vi) I_S *is continuous on the space of all finite subsets of X, and in addition a nonempty closed set $A \subset X$ is S-closed iff for each finite $F \subset A$ also $I_S(F) \subset A$.*

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious, using the fact that 2^X is compact (cf. MICHAEL [75]). We shall prove the following statements: (i) $\Rightarrow$ (ii) and (iv); (iv) $\Rightarrow$ (i) $\Rightarrow$ (v) $\Rightarrow$ (iv); (i) $\Longleftrightarrow$ (vi). We assume throughout that $\emptyset \notin \mathcal{S}$ (and hence that $\emptyset \notin \mathcal{S}^+$), allowing us to identify $\mathcal{S}$ with $H(X,\mathcal{S})$, since $\mathcal{S}$ is closed under intersection.

PROOF. (i) $\Rightarrow$ (ii) and (iv). Let g denote the composed mapping

$$2^X \xrightarrow{\;i\;} 2^{\lambda(X,\mathcal{S})} \xrightarrow{I_{\mathcal{S}^+}} H(\lambda(X,\mathcal{S}),\mathcal{S}^+),$$

where i sends $A \subset X$ onto $A \subset \lambda(X,\mathcal{S})$; let h be the restriction of g to $\mathcal{S} = H(X,\mathcal{S})$. It is easy to see that $h[\mathcal{S}] \subset \mathcal{S}^+$ and that h has a two-sided inverse, which is the mapping

$$\cdot \cap X\colon \mathcal{S}^+ \longrightarrow \mathcal{S}$$

which sends $S^+ \in \mathcal{S}^+$ onto $S = S^+ \cap X \in \mathcal{S}$. By theorem 2.10.5, the map g (and hence h) is continuous. Since $\mathcal{S}$ is compact and Hausdorff, h is a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}^+$, showing that $\cdot\cap X$ is continuous.

For each $A \in 2^X$ we have that

$$I_{\mathcal{S}^+}(A) = \bigcap\{S^+ \mid S \in \mathcal{S},\ A \subset S^+\} = \bigcap\{S^+ \mid A \subset S \in \mathcal{S}\},$$

and therefore

$$I_{\mathcal{S}^+}(A) = \left(I_{\mathcal{S}}(A)\right)^+.$$

This shows that $g[2^X] = h[H(X,\mathcal{S})]$, and hence that

$$h^{-1}g\colon 2^X \longrightarrow H(X,\mathcal{S})$$

is a well-defined continuous map; clearly $h^{-1}g = I_{\mathcal{S}}$.

(iv) $\Rightarrow$ (i). Assume that the map

$$\cdot\cap X\colon \mathcal{S}^+ \longrightarrow \mathcal{S}$$

is continuous. We first prove that $\mathcal{S}^+$ is a closed (and hence compact) subspace of $H(\lambda(X,\mathcal{S}),\mathcal{S}^+)$. Let $C \in H(\lambda(X,\mathcal{S}),\mathcal{S}^+)\setminus\mathcal{S}^+$. If $C \cap X = \emptyset$, then $\langle\lambda(X,\mathcal{S})\setminus X\rangle$ is a neighborhood of C which misses $\mathcal{S}^+$ (since each $S^+ \in \mathcal{S}^+$ satisfies $S^+ \cap S = S \neq \emptyset$).

Assume next that $C \cap X \neq \emptyset$, and let $\mathcal{C} \subset \mathcal{S}^+$ be such that $C = \bigcap\mathcal{C}$. Then

$$C \cap X = \cap\{S^+ \mid S^+ \in \mathcal{C}\} \cap X = \cap\{S \mid S^+ \in \mathcal{C}\} \in \mathcal{S}.$$

Also, $(C\cap X)^+ \subset C$. In fact, if $\mathcal{M} \in (C\cap X)^+\setminus C$, then $C \cap X \in \mathcal{M}$ and some $M \in \mathcal{M}$ satisfies $M^+ \cap C = \emptyset$. Hence

$$M^+ \cap C \cap X = M \cap (C\cap X) = \emptyset,$$

contradicting that $\mathcal{M}$ is linked. Since $C \notin \mathcal{S}^+$, we have that $(C\cap X)^+ \neq C$, and using the above inclusion, there must be some maximal linked system $\mathcal{N} \in \lambda(X,\mathcal{S})$ such that $\mathcal{N} \in C\setminus(C\cap X)^+$. Let $N \in \mathcal{N}$ be such that $N \cap (C\cap X) = \emptyset$. By the normality of $\mathcal{S}$ there exist $S_0, S_1 \in \mathcal{S}$ so that

$$N \subset X\setminus S_0, \quad C \cap X \cap X\setminus S_1 \quad \text{and} \quad (X\setminus S_0) \cap (X\setminus S_1) = \emptyset.$$

Observe that $N^+ \cap C \neq \emptyset$ and that $N^+ \subset \lambda(X,\mathcal{S})\setminus S_0^+$. Then the collection

$$\langle\lambda(X,\mathcal{S})\setminus S_0^+, \lambda(X,\mathcal{S})\rangle \cap \langle\lambda(X,\mathcal{S})\setminus S_1^+, \lambda(X,\mathcal{S})\setminus X\rangle$$

is a neighborhood of C which misses $\mathcal{S}^+$. In fact, if $D \in \mathcal{S}$ is such that D^+ is in the above neighborhood, then

$$D = D^+ \cap X \subset ((\lambda(X,\mathcal{S})\setminus S_1^+) \cup (\lambda(X,\mathcal{S})\setminus X)) \cap X = X\setminus S;$$

$$\emptyset \neq D^+ \cap (\lambda(X,\mathcal{S})\setminus S_0^+)$$

and consequently

$$\emptyset \neq D \cap (X\setminus S_0),$$

which is a contradiction.

(i) $\Rightarrow$ (v). First, notice that for each $S \in \mathcal{S}$,

$$2^X\setminus\langle S\rangle = \langle X\setminus S, X\rangle;$$

$$2^X\setminus\langle S,X\rangle = \langle X\setminus S\rangle,$$

and hence that the sets of the form $\langle S\rangle$, $\langle S,X\rangle$, with $S \in \mathcal{S}$, are closed. Assume that $H(X,\mathcal{S})$ is compact, let $\mathcal{B} \subset H(X,\mathcal{S})$ be a closed subset, and let $S \in H(X,\mathcal{S})\setminus\mathcal{B}$. Then for each $B \in \mathcal{B}$ we have either $B \not\subset S$ or that $S \not\subset B$.

If $B \not\subset S$, then choose $x \in B\setminus S$. By the normality of $\mathcal{S}$ there exist $S_B, S_C \in \mathcal{S}$ such that

$$x \in S_B\setminus S_C, \quad S \subset S_C\setminus S_B \quad \text{and} \quad S_B \cup S_C = X.$$

In particular, $x \in B \cap \mathrm{int}_X(S_B)$, and hence it follows that $\langle S_B, X\rangle$ is a neighborhood of B which does not contain S.

If $S \not\subset B$, then choose $y \in S\backslash B$. Again, there exist $S_B, S_C \in \mathcal{S}$ such that

$$y \in S_C\backslash S_B, \quad B \subset S_B\backslash S_C \quad \text{and} \quad S_B \cup S_C = X.$$

In particular, $\langle S_B\rangle$ is a neighborhood of B that does not contain S.

Since $\mathcal{B}$ is compact, a finite number of the selected neighborhoods of type $\langle S_B\rangle$ or $\langle S_B, X\rangle$ suffices to cover $\mathcal{B}$. Hence it follows that the sets of type $\langle A\rangle$ or $\langle A, X\rangle$, $A \in H(X,\mathcal{S})$, form a closed subbase for $H(X,\mathcal{S})$.

This subbase is T_1: assume that $A, B \in H(X,\mathcal{S})$ and that $A \notin \langle B\rangle$. Choose $x \in A\backslash B$. Since $\mathcal{S}$ is a T_1-subbase, there is an $S \in \mathcal{S}$ such that $x \in S$ and $S \cap B = \emptyset$. Hence, $A \in \langle S, X\rangle$ and $\langle S, X\rangle \cap \langle B\rangle = \emptyset$. If $A \notin \langle B, X\rangle$, then $A \cap B = \emptyset$. It follows that $A \in \langle A\rangle$ and $\langle A\rangle \cap \langle B, X\rangle = \emptyset$.

Finally we prove that this subbase is normal. Notice that for each pair of $\mathcal{S}$-closed sets D_1 and D_2,

$$X \in \langle D_1, X\rangle \cap \langle D_2, X\rangle \cap H(X,\mathcal{S}).$$

Hence we are only concerned with the following two cases ($C_1, C_2 \in H(X,\mathcal{S})$).

(a) $\langle C_1\rangle \cap \langle C_2\rangle \cap H(X,\mathcal{S}) = \emptyset$. Then $C_1 \cap C_2 = \emptyset$. By the normality of $\mathcal{S}$, there exist $S_1, S_2 \in \mathcal{S}$ such that $C_1 \cap S_2 = \emptyset = S_1 \cap C_2$ and $S_1 \cup S_2 = X$. It easily follows that

$$\langle C_1\rangle \subset \langle S_1, X\rangle\backslash\langle S_2, X\rangle;$$
$$\langle C_2\rangle \subset \langle S_2, X\rangle\backslash\langle S_1, X\rangle;$$
$$\langle S_1, X\rangle \cup \langle S_2, X\rangle = 2^X,$$

yielding the desired result (after intersecting with $H(X,\mathcal{S})$).

(b) $\langle C_1\rangle \cap \langle C_2, X\rangle \cap H(X,\mathcal{S}) = \emptyset$. Then $C_1 \cap C_2 = \emptyset$. Choosing $S_1, S_2 \in \mathcal{S}$ as above, it can easily be seen that

$$\langle C_1\rangle \subset \langle S_1\rangle\backslash\langle S_2, X\rangle;$$
$$\langle C_2, X\rangle \subset \langle S_2, X\rangle\backslash\langle S_2\rangle;$$
$$\langle S_1\rangle \cup \langle S_2, X\rangle = 2^X.$$

(v) $\Rightarrow$ (iv). Let $f = \cdot\cap X\colon \mathcal{S}^+ \longrightarrow \mathcal{S}$. For each $\mathcal{S}$-closed set C it is easy to see that

$$f^{-1}[\langle C\rangle \cap \mathcal{S}] = \langle C^+\rangle \cap \mathcal{S}^+;$$

$$f^{-1}[\langle C,X\rangle \cap \mathcal{S}] = \langle C^+,\lambda(X,\mathcal{S})\rangle \cap \mathcal{S}^+.$$

Using the fact that the sets of type $\langle S\rangle \cap H(X,\mathcal{S})$ or $\langle S,X\rangle \cap H(X,\mathcal{S})$, where $S \in H(X,\mathcal{S})$, form a closed subbase for $H(X,\mathcal{S})$, it follows that f is continuous.

(i) $\Rightarrow$ (vi). The continuity of the map $I_{\mathcal{S}}$ on finite subsets of X follows from (i) $\Rightarrow$ (ii). Let $A \in 2^X$. If A is $\mathcal{S}$-closed, then $I_{\mathcal{S}}(F) \subset A$ for each finite $F \subset A$. If the latter is true, then $A \in H(X,\mathcal{S})$. In fact, let $\langle O_1,\ldots,O_n\rangle$ be a basic neighborhood of A, where $O_1,\ldots,O_n \subset X$ are open. For each $i \le n$ fix an $a_i \in A \cap O_i$, and let $F = \{a_1,\ldots,a_n\}$. Then

$$I_{\mathcal{S}}(F) \subset A \subset \bigcup_{i=1}^{n} O_i \quad \text{and} \quad I_{\mathcal{S}}(F) \cap O_i \neq \emptyset \text{ for all } i \le n,$$

and hence $\langle O_1,\ldots,O_n\rangle$ meets $H(X,\mathcal{S})$. It follows that A is in the closure of $H(X,\mathcal{S})$, which equals $H(X,\mathcal{S})$ by compactness.

(vi) $\Rightarrow$ (i). Let $A \in 2^X\backslash H(X,\mathcal{S})$. Then there is a finite $F = \{a_1,\ldots,a_n\} \subset A$ such that $I_{\mathcal{S}}(F) \not\subset A$. Fix $x \in I_{\mathcal{S}}(F)\backslash A$. By the regularity of X there exist disjoint open sets $O,P \subset X$ such that $x \in P$ and $A \subset O$. Since $I_{\mathcal{S}}$ is continuous on finite sets, there exist open sets $O_i \subset O$ with $a_i \in O_i$ $(i \le n)$ and such that

$$I_{\mathcal{S}}(\{a_1',\ldots,a_n'\}) \cap P \neq \emptyset$$

for all $(a_1',\ldots,a_n') \in \prod_{i\le n} O_i$.

Then $\langle O,O_1,\ldots,O_n\rangle$ is a neighborhood of A which does not meet $H(X,\mathcal{S})$. In fact, if $B \in \langle O,O_1,\ldots,O_n\rangle$, then there exist $b_1,\ldots,b_n \in B$ such that $b_i \in B \cap O_i$ for each $i \le n$, and hence $I_{\mathcal{S}}(\{b_1,\ldots,b_n\}) \cap P \neq \emptyset$. Also $B \subset O$, and hence $I_{\mathcal{S}}(\{b_1,\ldots,b_n\}) \not\subset B$, proving that B is not $\mathcal{S}$-closed.

This completes the proof of the theorem. $\square$

2.10.8. Theorem 2.10.7 shows that a closed subbase $\mathcal{S}$ which (a) is normal and T_1; (b) is closed under arbitrary intersections; and (c) yields a compact hyperspace of $\mathcal{S}$-closed sets, must have quite strong properties. The most interesting types of examples are the normal binary subbases, and the ones described below. It is possible, however, to find other non-trivial (i.e. different from $H(X)$) examples of such compact subbases.

2.10.9. EXAMPLE. Let X be a compact convex subspace of a locally convex vectorspace, and let S be the collection of all closed (linearly) convex subsets of X. Then S is easily seen to be a T_1-subbase for X, which is closed under arbitrary intersection. By the HAHN-BANACH theorem (cf. RUDIN [100]), S is also normal. This subbase is compact, as can be derived from an obvious argument on line segments and continuity of the algebraic operations in the vectorspace. Hence theorem 2.10.7 implies that the hyperspace of all closed convex subsets of X is a retract of 2^X.

Our next examples illustrate the interference of the conditions (a), (b) and (c) listed in 2.10.8.

2.10.10. EXAMPLE. Let X be a locally connected continuum. Then C(X) (cf. 2.10.6) is a closed T_1-subbase of X which is compact. C(X) is closed under arbitrary intersections iff X is hereditarily unicoherent, in which case X is a compact tree-like space and C(X) is a normal binary subbase (cf. 2.10.6 and theorem 1.3.21). □

2.10.11. EXAMPLE. Let S^1 denote the unit circle, metrized by arc distance. The following sets are easily seen to be closed subbases for S^1, for each real number r with $0 < r \leq 2\pi$:

$$S_r := \{C \in C(S^1) \mid \text{diameter of } C \leq r\};$$

$$S'_r := \{C \in C(S^1) \mid \text{diameter of } C < r\}.$$

Let E^2 denote the unit 2-cell. There is a wellknown homeomorphism (cf. CURTIS & SCHORI [37])

$$h: C(S^1) \longrightarrow E^2,$$

constructed as follows: $h(S^1) = 0$, and for $C \in C(S^1)$, $C \neq S^1$ the image h(C) of C is the point of E^2 on the line segment joining 0 with the middle point of the arc C on a distance

$$1 - \frac{1}{2\pi} \text{ (diameter of C)}$$

to the origin.

Applying this map to the subspaces S_r, S'_r of $C(S^1)$, it is easy to see that S_r is compact for each r and that S'_r is non-compact for each r. The subbase S_r (resp. S'_r) is closed under arbitrary intersections iff $r < \pi$

(resp. iff $r \le \pi$). The subbase $\mathcal{S}_r$ is non-normal for each $r < \pi$, and $\mathcal{S}'_r$ is normal iff $r > \pi$.

None of the above subbases therefore satisfies (a), (b) and (c) simultaneously. Notice that, if $r < \frac{2\pi}{3}$, then $\mathcal{S}_r$ and $\mathcal{S}'_r$ even are binary (but not normal). □

We now present some corollaries of theorem 2.10.5 and of theorem 2.10.7.

2.10.12. <u>COROLLARY</u>. *Let $\mathcal{S}$ be a binary normal subbase for* X. *Then* $H(X,\mathcal{S})$ *has a binary normal subbase.*

<u>PROOF</u>. Applying theorem 2.10.5 and theorem 2.10.7, we conclude that $H(X,\mathcal{S})$ admits a closed normal T_1-subbase consisting of all sets of type

$$\langle C\rangle \cap H(X,\mathcal{S}), \quad \text{or} \quad \langle C,X\rangle \cap H(X,\mathcal{S}),$$

where $C \in H(X,\mathcal{S})$. We claim that this subbase is binary.

Assume that the collection

$$\{\langle C_i\rangle \cap H(X,\mathcal{S}) \mid i \in I\} \cup \{\langle D_j,X\rangle \cap H(X,\mathcal{S}) \mid j \in J\}$$

is linked, where $C_i, D_j \in H(X,\mathcal{S})$ for each $i \in I$ and $j \in J$. Then there exist $\mathcal{S}$-closed sets

$$C_{ii'} \in \langle C_i\rangle \cap \langle C_{i'}\rangle \cap H(X,\mathcal{S}), \qquad i,i' \in I;$$

$$D_{ij} \in \langle C_i\rangle \cap \langle D_j,X\rangle \cap H(X,\mathcal{S}), \qquad i \in I,\ j \in J.$$

Hence,

$$\emptyset \ne C_{ii'} \subset C_i \subset C_{i'};$$

$$D_{ij} \subset C_i; \quad D_{ij} \cap D_j \ne \emptyset,$$

implying that for each $j \in J$ the collection

$$\{C_i \mid i \in I\} \cup \{D_j\}$$

is linked. Choose

$$x_j \in \bigcap_{i\in I} C_i \cap D_j, \qquad j \in J,$$

and let $A := I_S(\{x_j \mid j \in J\})$. Then $A \subset \bigcap_{i\in I} C_i$ and $A \cap D_j \neq \emptyset$ for all $j \in J$, proving that

$$A \in \bigcap_{i\in I} \langle C_i\rangle \cap \bigcap_{j\in J} \langle D_j, X\rangle \cap H(X,S).$$

This completes the proof of the corollary. $\square$

2.10.13. COROLLARY. *Let* X *be a continuum with a binary normal subbase* S. *Then*

(i) $H(X,S)$ *is an acyclic Lefschetz space (cf.* WILLARD [129]*), and it consequently has the fixed point property for continuous mappings;*

(ii) *if* X *is metrizable moreover, then* $H(X,S)$ *is a metric* AR.

PROOF. The space 2^X is connected (cf. MICHAEL [75]) and so is its retract $H(X,S)$. A connected space carrying a normal binary subbase is an acyclic Lefschetz space (cf. VAN DE VEL [118]).

If moreover X is metrizable, then 2^X is metrizable too, since X is compact and metrizable. Hence $H(X,S)$ is connected and metrizable, therefore an AR by corollary 1.5.2. $\square$

2.10.14. By a result of WOJDYSLAWSKI [130], the hyperspace of a Peano continuum is an AR (the hyperspace of a nondegenerate Peano continuum is even homeomorphic to the Hilbert cube, cf. CURTIS & SCHORI [36]). In case a metric compactum is not locally connected, the techniques discussed in the present section provide a way to construct hyperspaces which are AR's and which are rather close to the original space. Let S be a normal T_1-subbase for the compact metric connected space X. Then $\lambda(X,S)$ is metrizable, since it is a quotient of the compact metric space λX (cf. theorem 2.3.4 and corollary 2.4.21). Moreover $\lambda(X,S)$ is connected, by theorem 2.5.1. Therefore $\lambda(X,S)$ is an AR and consequently $H(\lambda(X,S),S^+)$ is an AR too, being a retract of an AR (theorem 2.10.5).

By a recent result of EDWARDS [45], every (compact metric) AR is a Hilbert cube factor. Consequently all hyperspaces, constructed above, are Hilbert cube factors. It is desirable to find conditions on the subbase S such that $H(\lambda(X,S),S^+)$ is not only a Hilbert cube factor but is homeomorphic to the Hilbert cube itself. Also one could ask whether the spaces $H(\lambda X,(2^X)^+)$ are homeomorphic to the Hilbert cube in case X is a nondegenerate metrizable continuum.

2.11. Notes

In the present chapter we have dealt with some topological properties of superextensions and of some of their subspaces. We expect that this treatment is only a first step. There remain many questions unsolved, for example the following ones: when is a superextension λX first countable?, or, when is a superextension λX hereditarily separable and hereditarily Lindelöf?, or, when is a superextension λX perfectly normal?, or, when is a superextension λX hereditarily normal? At the moment we are not able to solve these questions; we can only point out the following information:

(a) VERBEEK [119], p.135, has given an example of a first countable compact Hausdorff space X such that λX is not first countable;

(b) $\lambda\mathbb{N}$ is not first countable, not hereditarily separable, not hereditarily Lindelöf, not perfectly normal and not hereditarily normal.

Superextensions behave surprisingly nice with respect to connectedness, cf. 2.5; whenever a superextension is connected, it is not far from being locally connected. Our proof of the connectedness of certain superextensions is elementary, but not trivial. It is desirable to find a simple proof of our connectedness results.

The results in sections 2.7 and 2.10 are joint results of M. VAN DE VEL and the author, cf. VAN MILL & VAN DE VEL [82], [83].

Added: some of the above questions are answered by VAN DOUWEN, see section 5.2.

CHAPTER III

INFINITE DIMENSIONAL TOPOLOGY

In this chapter we concentrate on metrizable superextensions. Our main interest lies in infinite dimensional problems such as: *is the superextension of the closed unit interval homeomorphic to the Hilbert cube*? In section 3.4 we give an affirmative answer to this question, thus proving a conjecture of DE GROOT [59]. Recent developments in infinite dimensional topology, such as $2^X \approx Q$ iff X is a nondegenerate Peano continuum (cf. SCHORI & WEST [102],[103],[104] and CURTIS & SCHORI [36]) suggest that the above question should be attacked using methods from infinite dimensional topology. Indeed, such methods turn out to be very useful in our situation. We use near-homeomorphism techniques (cf. BROWN [25], SCHORI & WEST [102], [103],[104], CURTIS & SCHORI [36]) and inverse limits of Hilbert cubes. The bonding maps in the inverse sequences turn out to be near-homeomorphisms by results of CHAPMAN [28],[29].

In section 3.1 we derive some preliminary results concerning metrizability and superextensions. Among other things, we prove that each separable metric space which is not totally disconnected, admits a superextension homeomorphic to the Hilbert cube Q. As a consequence, the closed unit interval $I = [0,1]$ has a closed subbase $\mathcal{S}$ for which $\lambda(I,\mathcal{S}) \approx Q$. Unfortunately the subbase $\mathcal{S}$ obtained in this manner cannot be described well. Therefore, we describe in section 3.3 another subbase $\mathcal{S}$ for which $\lambda(I,\mathcal{S}) \approx Q$. This particular superextension is used in section 3.4 as the first step in an inverse limit representation of λI. There we show that λI can be approximated by superextensions $\lambda(I,\mathcal{S}_n) \approx Q$ $(n \in \mathbb{N})$ of I with cellular bonding maps. Combining several results in the literature it then follows that λI itself is homeomorphic to the Hilbert cube. The construction of the superextensions $\lambda(I,\mathcal{S}_n)$ $(n \in \mathbb{N})$ uses much geometry in the plane.

The final sections in this chapter are devoted to the construction of capsets in λI and to the study of some subspaces of superextensions. As a consequence of our results we show that the subspace $\lambda_{comp}(\mathbb{R})$ of $\lambda\mathbb{R}$ is homeomorphic to $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N}: |x_i| = 1\}$, thus disproving a conjecture of VERBEEK [119].

3.1. Metrizability and superextensions

This section contains some preliminary results concerning metrizability of superextensions. Of great importance is VERBEEK's [119] metric for λX. This metric allows us to recognize Z-sets in λX, and it reflects the nice geometric structure of λX.

3.1.1. One of the most important results in the theory of superextensions is VERBEEK's [119] theorem: *λX is metrizable if and only if X is compact and metrizable* (cf. also corollary 2.4.21). If (X,d) is compact metric then there is a metric $\bar{d}$ for λX such that $\underline{i}$: $(X,d) \hookrightarrow (\lambda X,\bar{d})$ is an isometry (VERBEEK [119]). We will study this metric in detail. Let us start with some definitions and some preliminary results.

If (X,d) is a metric space then for all $A \subset X$ and $\varepsilon \geq 0$ define

$$B_\varepsilon(A) := \{x \in X \mid d(x,A) \leq \varepsilon\}$$

$$U_\varepsilon(A) := \{x \in X \mid d(x,A) < \varepsilon\}.$$

For any $A,B \in 2^X$ the *Hausdorff distance* $d_H(A,B)$ is defined by

$$d_H(A,B) := \inf\{\varepsilon \geq 0 \mid A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A)\}.$$

If X is compact then d_H is a metric for 2^X (cf. ENGELKING [48]).

One might wonder whether one has to use the axiom of choice to extend a linked system $L \subset 2^X$ to a maximal linked system $L' \subset 2^X$ in case X is a compact metric space. The following lemma shows that this is possible using induction only.

3.1.2. LEMMA. *Let X be a compact metric space. Then each linked system $L \subset 2^X$ can be extended to at least one maximal linked system $L' \subset 2^X$.*

PROOF. Let $\{U_n \mid n \in \mathbb{N}\}$ be a countable open basis for X. It is easy to see that

$$T = \wedge.\vee.\{cl_X(U_n) \mid n \in \mathbb{N}\}$$

is a countable closed basis for X which is closed under finite intersections and finite unions. Suppose that $\mathcal{L} \subset 2^X$ is a linked system. Define

$$\mathcal{M} := \{T \in \mathcal{T} \mid \exists L \in \mathcal{L}: L \subset T\}.$$

Enumerate $\mathcal{T}$ as $\{T_n \mid n \in \mathbb{N}\}$. By induction, for each $n \in \mathbb{N}$ define a subcollection $\mathcal{M}_n$ of $\mathcal{T}$ in the following way:

(i) $\mathcal{M}_1 := \mathcal{M}$;

(ii) $\mathcal{M}_n := \mathcal{M}_{n-1}$ if $\mathcal{M}_{n-1} \cup \{T_n\}$ is not linked;

(iii) $\mathcal{M}_n := \mathcal{M}_{n-1} \cup \{T_n\}$ if $\mathcal{M}_{n-1} \cup \{T_n\}$ is linked.

Define $\mathcal{S} := \bigcup_{n=1}^{\infty} \mathcal{M}_n$. Then it is easy to see that

$$\mathcal{L}' := \{A \in 2^X \mid \forall S \in \mathcal{S}: A \cap S \neq \emptyset\}$$

is a maximal linked system that contains $\mathcal{L}$. □

3.1.3. In the proof of the above lemma we showed that each mls $\mathcal{M} \in \lambda X$, for compact metric X, contains a countable pre-mls (recall that a *pre-mls* $\mathcal{L} \subset 2^X$ is a linked system contained in at most one mls $\mathcal{L}' \subset 2^X$, cf. definition 2.3.2). (In general, this is not the case, cf. section 2.8.) The following lemma gives another proof of this fact.

3.1.4. LEMMA. *Let X be a topological space and let $\mathcal{M} \in \lambda X$. Then each dense subset $\mathcal{L} \subset \mathcal{M}$ (dense in $\cup\mathcal{M}$ as subspace of 2^X) is a pre-mls for $\mathcal{M}$. In particular, if X is compact metric, then any countable dense subset of $\mathcal{M}$ is a pre-mls for $\mathcal{M}$.*

PROOF. Suppose that $\mathcal{L} \subset \mathcal{M} \in \lambda X$ is dense in $\mathcal{M}$. Suppose that $\mathcal{L}$ is also contained in an mls $\mathcal{M}_0 \in \lambda X$ distinct from $\mathcal{M}$. Choose $M \in \mathcal{M}$, $M_0 \in \mathcal{M}_0$ such that $M \cap M_0 = \emptyset$. Then $\langle X \setminus M_0 \rangle$ is an open neighborhood of $M \in \mathcal{M}$; consequently there is an $L \in \mathcal{L}$ such that $L \in \langle X \setminus M_0 \rangle$. But then $L \cap M_0 = \emptyset$, which is a contradiction. □

3.1.5. REMARK. The converse of lemma 3.1.4 in general is not true. For example, define an mls $\mathcal{M} \in \lambda I$ by

$$\mathcal{M} := \{M \in 2^I \mid \{0,\tfrac{1}{2}\} \subset M \text{ or } \{\tfrac{1}{2},1\} \subset M \text{ or } \{0,1\} \subset M\}.$$

It is easily seen that M is an mls and also that $\{\{0,\frac{1}{2}\},\{\frac{1}{2},1\},\{0,1\}\}$ is a pre-mls for M. As M has continuously many points it cannot contain a dense subset consisting of three points.

3.1.6. A metric d for a space X is called *convex* provided that

$$B_{\delta_0}\big(B_{\delta_1}(A)\big) = B_{\delta_0+\delta_1}(A)$$

for any $A \in 2^X$ and $\delta_0,\delta_1 \geq 0$. It is well known that any Peano continuum admits a convex metric. The following lemma is also well known; for completeness sake we include it. The proof was suggested to me by M. VAN DE VEL.

3.1.7. LEMMA. *Let* $d\colon X\times X \to [0,\infty)$ *be a convex metric for the compact space* X. *Then the mapping* $e\colon 2^X\times[0,\infty) \to 2^X$ *defined by* $e(A,t) := B_t(A)$ *is continuous (*e *is sometimes called an expansion homotopy,* cf. CURTIS & SCHORI [37]*).*

PROOF. Let $O \subset X$ be open. We claim that $e^{-1}[\langle O\rangle]$ and $e^{-1}[\langle O,X\rangle]$ are open. To prove this, first assume that $(A,t) \in e^{-1}[\langle O,X\rangle]$. Then choose $x \in B_t(A) \cap O$ and choose $\varepsilon > 0$ such that $B_\varepsilon(x) \subset O$. Also choose $a \in A$ such that $d(x,a) \leq t$. We claim that

$$e[\langle B_{\varepsilon/2}(a),X\rangle \times (t-\tfrac{\varepsilon}{2}, t+\tfrac{\varepsilon}{2})] \subset \langle O,X\rangle.$$

Indeed, choose $(A',t') \in \langle B_{\varepsilon/2}(a),X\rangle \times (t-\frac{\varepsilon}{2}, t+\frac{\varepsilon}{2})$. Fix $a' \in B_{\varepsilon/2}(a) \cap A'$. Then

$$d(a',x) \leq d(a',a) + d(a,x) \leq d(a,x) + \frac{\varepsilon}{2}$$

and

$$d(a,x) \leq d(a',a) + d(a',x)$$

and therefore

$$d(a',x) \geq d(a,x) - d(a',a) \geq d(a,x) - \frac{\varepsilon}{2}.$$

We conclude that

$$d(a',x) \in [d(a,x)-\tfrac{\varepsilon}{2}, d(a,x)+\tfrac{\varepsilon}{2}].$$

As d is a convex metric, there is an $x' \in X$ such that $d(a',x') = \max\{d(a,x)-\frac{\varepsilon}{2},0\}$. Then $d(x',x) \leq \varepsilon$ and consequently $x' \in B_{d(a',x')}(A') \cap \cap O \subset B_{t'}(A') \cap O$.

To prove that $e^{-1}[\langle O\rangle]$ is open, assume that $(A,t) \in e^{-1}[\langle O\rangle]$. Then

$B_t(A) \subset O$. As X is compact there is an $\varepsilon > 0$ such that $B_\varepsilon(B_t(A)) \subset O$. Hence $B_{\varepsilon+t}(A) \subset O$. Therefore

$$e[\langle B_{t+\frac{\varepsilon}{2}}(A)\rangle \times [0,\tfrac{\varepsilon}{2}]] \subset \langle O\rangle.$$

This completes the proof of the lemma. □

3.1.8. <u>THEOREM</u>. *Let* X *be a topological space and let* $\mathcal{M} \in \lambda X$. *Then* $\mathcal{M}$ *is closed as subspace of* 2^X. *If in addition* X *is a Peano continuum then there is a retraction* $r\colon 2^X \to \mathcal{M}$.

<u>PROOF</u>. Choose $A \in 2^X$ such that $A \notin \mathcal{M}$. Then there is an $M \in \mathcal{M}$ such that $A \cap M = \emptyset$. Then $\langle X\backslash M\rangle$ is an open neighborhood of A which misses $\mathcal{M}$. For take $B \in \langle X\backslash M\rangle$. Then $B \cap M = \emptyset$ and consequently $B \notin \mathcal{M}$ since $\mathcal{M}$ is a linked system.

Assume that X is a Peano continuum. Let $d\colon X \times X \to [0,\infty)$ be a convex metric for X. Choose $A \in 2^X$.

<u>CLAIM</u> 1. *The set* $\{\varepsilon \geq 0 \mid B_\varepsilon(A) \in \mathcal{M}\}$ *has a minimum, denoted by* $t(A)$.

Indeed, let $\delta := \inf\{\varepsilon \geq 0 \mid B_\varepsilon(A) \in \mathcal{M}\}$ and assume that $B_\delta(A) \notin \mathcal{M}$. Take $M \in \mathcal{M}$ such that $B_\delta(A) \cap M = \emptyset$. Choose $\varepsilon > 0$ such that

$$B_\varepsilon\bigl(B_\delta(A)\bigr) \cap M = \emptyset.$$

Then $B_{\varepsilon+\delta}(A) \cap M = \emptyset$ and as $\delta = \inf\{\varepsilon \geq 0 \mid B_\varepsilon(A) \in \mathcal{M}\}$ it follows that there is a $\rho \in \{\varepsilon \geq 0 \mid B_\varepsilon(A) \in \mathcal{M}\}$ such that $\delta < \rho < \varepsilon + \delta$. Then

$$B_\rho(A) \subset B_{\varepsilon+\delta}(A)$$

implies that $B_\rho(A) \cap M = \emptyset$ and consequently $B_\rho(A) \notin \mathcal{M}$. Contradiction.

<u>CLAIM</u> 2. *If* $\lim_{n\to\infty} A_n = A$ *(in* 2^X *!) then* $\lim_{n\to\infty} t(A_n) = t(A)$.

Choose $\varepsilon > 0$. Then there is an $n_0 \in \mathbb{N}$ such that $d_H(A_n,A) < \varepsilon$ for all $n \geq n_0$. Fix arbitrary $m \geq n$. Now $B_\varepsilon(A) \supset A_m$ implies that

$$B_{t(A_m)}(A_m) \subset B_{t(A_m)}\bigl(B_\varepsilon(A)\bigr) = B_{t(A_m)+\varepsilon}(A);$$

consequently $t(A) \leq t(A_m) + \varepsilon$, since $B_{t(A_m)}(A_m) \in \mathcal{M}$.

On the other hand, $A \subset B_\varepsilon(A_m)$ and therefore

$$B_{t(A)}(A) \subset B_{t(A)}\big(B_\varepsilon(A_m)\big) = B_{t(A)+\varepsilon}(A_m)$$

which shows that $t(A_m) \le t(A) + \varepsilon$, since $B_{t(A)}(A) \in M$.

We conclude that $\lim_{n\to\infty} t(A_m) = t(A)$.

CLAIM 3. *The mapping* $r\colon 2^X \to M$ defined by $r(A) := B_{t(A)}(A)$ *is a retraction.*

The continuity follows from claim 2 and lemma 3.1.7. The fact that r is a retraction is trivial. □

3.1.9. COROLLARY. *Let X be a Peano continuum. Then each mls* $M \in \lambda X$ *is an AR, and consequently is a Q-factor.*

PROOF. Since 2^X is an AR (cf. WOJDYSLAWSKI [130]) the result follows from theorem 3.1.8 and the observation that each AR is a Q-factor (cf. EDWARDS [45]). □

3.1.10. If (X,d) is a compact metric space then there is a natural metric $\bar{d}$ for λX such that $\underline{i}\colon (X,d) \hookrightarrow (\lambda X,\bar{d})$ is an isometry. VERBEEK [119] has given the following expressions for $\bar{d}$;

(1) $\bar{d}(M,N) = \sup_{S\in M} \min_{T\in N} d_H(S,T)$

(2) $\quad = \min\{\varepsilon \ge 0 \mid \forall M \in M\colon B_\varepsilon(M) \in N \text{ and } \forall N \in N\colon B_\varepsilon(N) \in M\}$

(3) $\quad = \min\{\varepsilon \ge 0 \mid \forall M \in M\colon B_\varepsilon(M) \in N\}$

(4) $\quad = \min\{\varepsilon \ge 0 \mid \forall N \in N\colon B_\varepsilon(N) \in M\}$.

We need a simple generalization of this result.

3.1.11. LEMMA. *Let* (X,d) *be a compact metric space and let* M *be a pre-mls for* $\underline{M} \in \lambda X$. *Then for each* $N \in \lambda X$ *we have that* $d(\underline{M},N) = \min\{\varepsilon \ge 0 \mid \forall M \in M\colon B_\varepsilon(M) \in N\}$.

PROOF. Let $\delta := \inf\{\varepsilon \ge 0 \mid \forall M \in M\colon B_\varepsilon(M) \in N\}$. Assume that $B_\delta(M) \notin N$ for some $M \in M$. Take $\varepsilon > 0$ and $N \in N$ such that

$$B_{\delta+\varepsilon}(M) \cap N = \emptyset.$$

This is a contradiction, since $\delta+\varepsilon \in \{\varepsilon \geq 0 \mid \forall M \in \mathcal{M}: B_\varepsilon(M) \in \mathcal{N}\}$. We conclude that the set $\{\varepsilon \geq 0 \mid \forall M \in \mathcal{M}: B_\varepsilon(M) \in \mathcal{N}\}$ has a minimum, denoted by δ.

Obviously $\delta \leq \bar{d}(\underline{\mathcal{M}},\mathcal{N})$ (cf. 3.1.10 expression 3). Let us assume that $\delta < \bar{d}(\underline{\mathcal{M}},\mathcal{N})$. We will derive a contradiction. It follows that $B_\delta(M) \in \mathcal{N}$ for all $M \in \mathcal{M}$ and that $B_\delta(N) \notin \underline{\mathcal{M}}$ for some $N \in \mathcal{N}$. As $\mathcal{M}$ is a pre-mls for $\underline{\mathcal{M}}$ there is an $M \in \mathcal{M}$ such that

$$B_\delta(N) \cap M = \emptyset.$$

Since $B_\delta(M) \in \mathcal{N}$ there is a point $x \in N \cap B_\delta(M)$. Choose $y \in M$ such that $d(x,y) \leq \delta$. Then $y \in B_\delta(N) \cap M$, which is a contradiction. $\square$

3.1.12. The distance between two maps f and g: $X \to Y$, where (Y,d) is compact metric is defined by $d(f,g) = \sup_{x\in X} d(f(x),g(x))$. The identity mapping on X is denoted by id_X. A mapping $f: (X,d) \longrightarrow (Y,\rho)$ is called a *contraction* provided that $\rho(f(x),f(y)) \leq d(x,y)$ for all $x,y \in X$.

3.1.13. THEOREM. *Let* (X,d) *be a compact metric space and let* $\mathcal{M} \subset 2^X$ *be a linked system. Then there is a retraction* $r: \lambda X \longrightarrow \bigcap\{M^+ \mid M \in \mathcal{M}\}$ *satisfying:*

(i) r *is a contraction;*

(ii) $\bar{d}(\mathcal{N},r(\mathcal{N})) = \bar{d}(\mathcal{N},\bigcap\{M^+ \mid M \in \mathcal{M}\})$ *for all* $\mathcal{N} \in \lambda X$;

(iii) $\bar{d}(r,\mathrm{id}_{\lambda X}) \leq \sup_{M\in\mathcal{M}} d_H(X,M)$.

PROOF. Define r as in theorem 1.5.2. It follows from the definition of r that for all $\mathcal{N} \in \lambda X$ the collection

$$P(\mathcal{N}) = \{N \in \mathcal{N} \mid N \cap M \neq \emptyset \ (\forall M \in \mathcal{M})\} \cup \mathcal{M}$$

is a pre-mls for $r(\mathcal{N})$.

CLAIM 1. *r is a contraction.*

Indeed, choose $\mathcal{L},\mathcal{P} \in \lambda X$ and let $\varepsilon := \bar{d}(\mathcal{L},\mathcal{P})$. Choose $A \in P(\mathcal{L})$. If $A \in \mathcal{M}$ then clearly $B_\varepsilon(A) \in r(\mathcal{P})$. On the other hand if $A \in \mathcal{L}$ then $B_\varepsilon(A) \in \mathcal{P}$ (cf. 3.1.10 expression 3) and consequently $B_\varepsilon(A) \in P(\mathcal{P}) \subset r(\mathcal{P})$ since $B_\varepsilon(A)$ intersects all members from $\mathcal{M}$. From lemma 3.1.11 it now follows that $\bar{d}(r(\mathcal{L}),r(\mathcal{P})) \leq \varepsilon = \bar{d}(\mathcal{L},\mathcal{P})$.

CLAIM 2. $\bar{d}(\mathcal{N},r(\mathcal{N})) = \bar{d}(\mathcal{N},\bigcap\{M^+ \mid M \in \mathcal{M}\})$ *for all* $\mathcal{N} \in \lambda X$.

Choose $\mathcal{N} \in \lambda X$ and take $\mathcal{L} \in \bigcap\{\mathcal{M}^+ \mid M \in \mathcal{M}\}$ such that

$$\bar{d}(\mathcal{N},\mathcal{L}) < \bar{d}(\mathcal{N},r(\mathcal{N})).$$

Let $\varepsilon := \bar{d}(\mathcal{N},\mathcal{L})$. It then follows that $B_\varepsilon(N) \in \mathcal{L}$ for all $N \in \mathcal{N}$. But $\mathcal{L} \in \bigcap\{M^+ \mid M \in \mathcal{M}\}$ implies that each element $L \in \mathcal{L}$ intersects all members from $\mathcal{M}$. Consequently $B_\varepsilon(N) \in P(\mathcal{N}) \subset r(\mathcal{N})$ for all $N \in \mathcal{N}$. From lemma 3.1.11 it now follows that

$$\bar{d}(\mathcal{N},r(\mathcal{N})) \leq \varepsilon$$

which is a contradiction.

<u>CLAIM</u> 3. $\bar{d}(r,id_{\lambda X}) \leq \sup_{M\in\mathcal{M}} d_H(X,M)$.

Choose $\mathcal{N} \in \lambda X$ and consider $P(\mathcal{N})$. By lemma 3.1.11 we have

$$\bar{d}(\mathcal{N},r(\mathcal{N})) = \min\{\varepsilon \geq 0 \mid \forall A \in P(\mathcal{N}): B_\varepsilon(A) \in \mathcal{N}\}.$$

Let $\delta := \sup_{M\in\mathcal{M}} d_H(X,M)$. Notice that $\delta < +\infty$. Choose $A \in P(\mathcal{N})$. If $A \in \mathcal{M}$ then $B_\delta(A) = X \in \mathcal{N}$, since $\mathcal{N}$ is a maximal linked system. On the other hand if $A \notin \mathcal{M}$ then $A \in \mathcal{N}$ and then also $B_\delta(A) \in \mathcal{N}$. It now follows that $\bar{d}(\mathcal{N},r(\mathcal{N})) \leq \delta = \sup_{M\in\mathcal{M}} d_H(X,M)$. □

3.1.14. If Y is a closed subset of the normal space X then there is a natural embedding j_{YX} of λY in λX (cf. VERBEEK [119]) defined by

$$j_{YX}(\mathcal{M}) := \underline{\mathcal{M}}(= \{G \subset X \mid G \in 2^X \text{ and } G \cap Y \in \mathcal{M}\})$$

(that j_{YX} is an embedding also follows from theorem 2.3.4). We will always identify λY and $j_{YX}[\lambda Y]$.

3.1.15. <u>LEMMA</u>. *Let* Y *be a closed subset of the normal space* X. *Then* $\mathcal{M} \in \lambda X$ *is an element of* λY *if and only if* $\{M \cap Y \mid M \in \mathcal{M}\}$ *is linked.*

<u>PROOF</u>. Choose $\mathcal{M} \in \lambda X$. If $\mathcal{M} \in \lambda Y$ then $\{M \cap Y \mid M \in \mathcal{M}\}$ is a maximal linked system in Y and if $\{M \cap Y \mid M \in \mathcal{M}\}$ is linked, then it is easy to see that it is also maximal linked (in Y) and that $j_{YX}(\{M \cap Y \mid M \in \mathcal{M}\}) = \mathcal{M}$. □

3.1.16. A closed subset B of a metric space (X,d) is called a *Z-set* (cf. ANDERSON [4]) provided that for each $\varepsilon > 0$ there is a continuous $f_\varepsilon: X \to X\setminus B$ such that $d(f_\varepsilon,id_X) < \varepsilon$. Z-sets are very important in infinite dimensional topology and for later use we will give some classes of Z-sets in λX. The following result is an application of theorem 3.1.13.

3.1.17. THEOREM. *Let* (X,d) *be a metric continuum and let* $A \in 2^X$. *Then*
(i) A^+ *is a Z-set in* λX *iff* A *has a void interior in* X;
(ii) *if* $A \neq X$ *then* λA *is a Z-set in* λX.

PROOF. (i) If A has not a void interior in X then A^+ also has a nonvoid interior in λX. Consequently A^+ is not a Z-set.

Assume that A has a void interior in X. Choose $\varepsilon > 0$ and choose a finite subset $F \subset X$, disjoint from A, such that $d_H(F,X) < \varepsilon$. Let $f_\varepsilon \colon \lambda X \to F^+$ be the retraction of theorem 3.1.13. Then $\bar{d}(f_\varepsilon, id_{\lambda X}) < \varepsilon$ and as $F^+ \cap A^+ = \emptyset$, we have that $f_\varepsilon[\lambda X] \subset \lambda X \backslash A^+$.

(ii) Choose $\varepsilon > 0$ and choose two disjoint finite sets G_0 and G_1 in X such that $d_H(G_i,X) < \varepsilon$ $(i \in \{0,1\})$. Let $p \in X\backslash A$ and define

$$F_i := G_i \cup \{p\} \qquad (i \in \{0,1\}).$$

Let $f_\varepsilon \colon \lambda X \to F_0^+ \cap F_1^+$ be the retraction of theorem 3.1.13. Then

$$\bar{d}(f_\varepsilon, id_{\lambda X}) \leq \max\{d_H(F_0,X), d_H(F_1,X)\} < \varepsilon$$

and moreover $f_\varepsilon[\lambda X] \cap \lambda A = \emptyset$. For take $N \in f_\varepsilon[\lambda X] = F_0^+ \cap F_1^+$. Then $F_i \in N$ $(i \in \{0,1\})$ and $(F_0 \cap Y) \cap (F_1 \cap Y) = \emptyset$. Consequently, by lemma 3.1.5, $N \notin \lambda Y$. $\square$

3.1.18. Examples of Z-sets in the Hilbert cube Q are compact subsets of $(-1,1)^\infty$ and also closed subsets of Q which project onto a point in infinitely many coordinates (cf. ANDERSON [4]). In fact we have the following characterization: a closed subset B of Q is a Z-set iff there is an autohomeomorphism of Q which maps B onto a set which projects onto a point in infinitely many coordinates (cf. ANDERSON [4]). Also, a closed countable union of Z-sets is again a Z-set. Combining these two results it follows that in any case each convergent sequence in Q is a Z-set. This observation will be used in the proof of the following theorem.

We will also use ANDERSON's [4] homeomorphism extension theorem: *any homeomorphism between two Z-sets in Q can be extended to an autohomeomorphism of Q*. In particular, the Hilbert cube Q is homogeneous.

3.1.19. THEOREM. *For every separable metric not totally disconnected topological space X, there exists a normal closed T_1-subbase S such that $\lambda(X,S)$ is homeomorphic to the Hilbert cube Q.*

PROOF. Assume that X is embedded in Q and let C be a nontrivial component of X. Choose a convergent sequence B in C. Furthermore, define a sequence $\{y_n\}_{n=0}^{\infty}$ in Q by

$$(y_n)_i = \begin{cases} 1 & \text{if } i \neq n \\ -1 & \text{if } i = n, \end{cases}$$

for i = 1,2,...,.

It is clear that

$$\lim_{n\to\infty} y_n = y_0.$$

Moreover define $z \in Q$ by $z_i = 0$ (i = 1,2,...). Then

$$E = \{y_n \mid n \in \mathbb{N}\} \cup \{z\}$$

is a convergent sequence and therefore is homeomorphic to B. Since B and E are both Z-sets in Q (cf. remark 3.1.18) there is an autohomeomorphism of Q which maps B onto E (cf. remark 3.1.18). This procedure shows that we may assume that X is embedded in Q in such a way that $E \subset C$.

Let $\mathcal{T} = \{A \subset Q \mid \exists x \in [-1,1]\colon A = \pi_n^{-1}[-1,x] \vee A = \pi_n^{-1}[x,1]\ (n \in \mathbb{N})\}$ be the canonical binary normal subbase for Q. We claim that for all $T_0, T_1 \in \mathcal{T}$ with $T_0 \cap T_1 \neq \emptyset$ also $T_0 \cap T_1 \cap X \neq \emptyset$. To show this, choose $T_0, T_1 \in \mathcal{T}$ with $T_0 \cap T_1 \neq \emptyset$. We need only consider the following 4 cases:

CASE 1. $T_0 = \pi_{n_0}^{-1}[-1,x]$; $T_1 = \pi_{n_0}^{-1}[y,1]$ $(x \geq y;\ n_0 \in \mathbb{N})$.

Since $z \in T_0$ and $y_0 \in T_1$ and C is connected, it follows that $\emptyset \neq T_0 \cap T_1 \cap C \subset T_0 \cap T_1 \cap X$.

CASE 2. $T_0 = \pi_{n_0}^{-1}[-1,x]$; $T_1 = \pi_{n_1}^{-1}[y,1]$ $(n_0 \neq n_1)$.

Then $y_{n_0} \in T_0 \cap T_1 \cap X$.

CASE 3. $T_0 = \pi_{n_0}^{-1}[-1,x]$; $T_1 = \pi_{n_1}^{-1}[-1,y]$.

Then $z \in T_0 \cap T_1 \cap X$.

CASE 4. $T_0 = \Pi_{n_0}^{-1}[x,1]$; $T_1 = \Pi_{n_1}^{-1}[y,1]$.

Then $y_0 \in T_0 \cap T_1 \cap X$.

Theorem 2.2.5 now implies that $\lambda(X,\mathcal{T}\cap X)$ is homeomorphic to Q. That $\mathcal{T} \cap X$ is a normal T_1-subbase is straightforward and is left to the reader. □

3.1.20. Since the proof of theorem 3.1.19 uses the homeomorphism extension theorem the subbases derived from it are difficult to describe. For simple spaces however, such as the closed unit interval I or the n-spheres S_n there are subbases of easy description for which the corresponding superextensions are homeomorphic to the Hilbert cube; cf. VAN MILL & SCHRIJVER [80].

3.1.21. The final results in this section are devoted to mapping theorems. First let us give some definitions. A continuous surjection $f: (X,d) \twoheadrightarrow (X,d)$ is called a *near-homeomorphism* (cf. BROWN [25]) if for each $\varepsilon > 0$ there is an autohomeomorphism $\phi: X \to X$ such that $d(\phi,f) < \varepsilon$. Near-homeomorphisms are very useful in infinite dimensional topology. Let (X,d) and (Y,ρ) be metric spaces. A collection of functions $\mathcal{F} \subset C(X,Y)$ is called *equi-uniformly continuous* provided that for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x,y \in X$ with $d(x,y) < \delta$ we have that $\rho(f(x),f(y)) < \varepsilon$ for all $f \in \mathcal{F}$.

We need a simple lemma.

3.1.22. LEMMA. *Let* Y *be a normal space and let* $f: X \to Y$ *be a continuous closed surjection. Then there is a continuous surjection* $\lambda(f): \lambda X \twoheadrightarrow \lambda Y$, *defined by* $\lambda(f)(\mathcal{M}) := \{f[M] \mid M \in \mathcal{M}\}$, *which is an extension of* f.

PROOF. $\lambda(f)$ is just the mapping described in theorem 2.3.4. It is clear that, by the fact that f is closed, $\{f[M] \mid M \in \mathcal{M}\}$ is a pre-mls for $\bar{f}(\mathcal{M})$ ($\bar{f}$ defined as in the proof of theorem 2.3.4) for all $\mathcal{M} \in \lambda X$. Hence we need only show that $\lambda(f)(\mathcal{M})$ is an mls. Indeed, assume that for some $\mathcal{M} \in \lambda X$ we have that $\lambda(f)(\mathcal{M})$ were not an mls. Choose $A \in 2^Y$ such that $\lambda(f)(\mathcal{M}) \cup \{A\}$ is linked but $A \notin \lambda(f)(\mathcal{M})$. Then $f^{-1}[A] \notin \mathcal{M}$, since $f[f^{-1}[A]] = A$, and consequently there is an $M \in \mathcal{M}$ such that $f^{-1}[A] \cap M = \emptyset$. But this is a contradiction since $f[M] \in \lambda(f)(\mathcal{M})$ and $A \cap f[M] = \emptyset$. □

We now have the following theorem.

3.1.23. THEOREM. *Let* X *and* Y *be compact metric spaces and let* $\mathcal{F} \subset C(X,Y)$ *be* *a collection of surjections of* X *onto* Y. *Then*

(i) *if* $f \in \mathcal{F}$ *is a near-homeomorphism, then so is* $\lambda(f)$;

(ii) *if* $\mathcal{F}$ *is equi-uniformly continuous, then so is* $\{\lambda(f) \mid f \in \mathcal{F}\}$.

PROOF. (i) Identify X and Y and let d be a metric for X. Choose $\varepsilon > 0$ and choose a homeomorphism $\phi: X \to X$ such that $d(\phi, f) \leq \varepsilon$. From lemma 3.1.22 and theorem 2.3.4 it follows that $\lambda(\phi): \lambda X \longrightarrow \lambda X$ is a homeomorphism too. We will show that $\bar{d}(\lambda(\phi), \lambda(f)) \leq \varepsilon$.

For this, take $\mathcal{M} \in \lambda X$ and let

$$\delta := \bar{d}\big(\lambda(\phi)(\mathcal{M}), \lambda(f)(\mathcal{M})\big) =$$

$$= \min\{\varepsilon \geq 0 \mid \forall M \in \lambda(\phi)(\mathcal{M}): B_\varepsilon(M) \in \lambda(f)(\mathcal{M})\}.$$

Assume that there is an $M \in \lambda(\phi)(\mathcal{M})$ such that $B_\varepsilon(M) \notin \lambda(f)(\mathcal{M})$. Let $M = \phi[A]$, with $A \in \mathcal{M}$ (lemma 3.1.22). Choose $N \in \lambda(f)(\mathcal{M})$ such that $N \cap B_\varepsilon(M) = \emptyset$. Assume that $N = f[B]$, with $B \in \mathcal{M}$ (lemma 3.1.22). As $\mathcal{M}$ is a linked system, there is an $x \in A \cap B$. It now follows that

$$f(x) \in N$$

and

$$\phi(x) \in M \subset B_\varepsilon(M)$$

and $B_\varepsilon(M) \cap N = \emptyset$. But then $d(\phi(x), f(x)) > \varepsilon$, which is a contradiction.

(ii) This can be proved in the same way. □

3.1.24. REMARK. In theorem 3.1.23 (i) we showed that each near-homeomorphism $f: X \to X$ extends to a near-homeomorphism $\lambda(f): \lambda X \longrightarrow \lambda X$. The fact that f is a near-homeomorphism is not a necessary condition for $\lambda(f)$ to be a near-homeomorphism. From results derived in 3.2 and 3.4 it follows that *each* continuous surjection $f: I \to I$ extends to a near-homeomorphism $\lambda(f): \lambda I \longrightarrow \lambda I$.

3.2. Cell-like mappings and inverse limits

This section contains an approximation theorem for inverse limits of superextensions. We use corollary 1.5.20 to show that each continuous surjection f from a metrizable continuum X onto a metrizable continuum Y extends to a cell-like mapping $\lambda(f): \lambda X \longrightarrow \lambda Y$. Then, applying results of

CHAPMAN [28],[29] and BROWN [25] we get an approximation theorem for inverse limits of superextensions.

We first give an important consequence of corollary 1.5.21.

3.2.1. THEOREM. *Let* X *be a metrizable continuum and let* $\mathcal{S}$ *be a normal* T_1-*subbase for* X. *Then* $\lambda(X,\mathcal{S})$ *is an* AR. *In particular,* λX *is an* AR *if and only iff* X *is a metrizable continuum.*

PROOF. As λX is metrizable, so is $\lambda(X,\mathcal{S})$, being a Hausdorff quotient of a compact metric space (cf. VERBEEK [119]; also theorem 2.3.4). Moreover $\lambda(X,\mathcal{S})$ is connected (cf. VERBEEK [119]; also theorem 2.5.1). The result now follows from corollary 1.5.21 since the subbase $\{S^+ \mid S \in \mathcal{S}\}$ for $\lambda(X,\mathcal{S})$ is both binary and normal.

The second part of the present theorem follows from theorem 2.5.1. $\square$

3.2.2. The above theorem answers a question of VERBEEK [119] affirmatively. The second part of the above theorem was proved in [79]. There we asked whether every AR admits a binary normal subbase. This question was answered negatively by SZYMAŃSKI [117] who showed that BORSUK's two dimensional AR having the singularity of MAZURKIEWICZ (cf. BORSUK [20]) is a counterexample.

If X and Y are locally compact, then a map $f\colon X \to Y$ is called *proper* if the inverses of compact subsets of Y are compact in X. A proper map f is called *cell-like* or *cellular* (CE), if f is onto and point inverses have trivial shape (for the notion "*shape of a compactum*" see BORSUK [21],[22]).

We now can prove the following result, which is fundamental and important in the theory of superextensions.

3.2.3. THEOREM. *Let* $\mathcal{S}$ *be a normal* T_1-*subbase for the metrizable continuum* X, *let* $\mathcal{T}$ *be a normal* T_1-*subbase for* Y *and let* $f\colon X \to Y$ *be a continuous surjection. If* $\{f^{-1}[T] \mid T \in \mathcal{T}\} \subset \mathcal{S}$ *then the extension* $\bar{f}\colon \lambda(X,\mathcal{S}) \longrightarrow \lambda(Y,\mathcal{T})$ *of* f *described in theorem* 2.3.4 *has the property that each point inverse is an* AR. *In particular,* $\bar{f}$ *is cellular.*

PROOF. Let us use the notation of the proof of theorem 2.3.4. Take $\mathcal{M} \in \lambda(Y,\mathcal{T})$.

CLAIM. $\bar{f}^{-1}[\{\mathcal{M}\}]$ *is* $\mathcal{S}^+$-*closed.*

By theorem 1.5.3 we only need to show that $\bar{f}^{-1}[\{M\}]$ is S^+-convex. To show this, take $L_0, L_1 \in \bar{f}^{-1}[\{M\}]$ and choose

$$P \in I_{S^+}(L_0, L_1).$$

Assume that $P \notin \bar{f}^{-1}[\{M\}]$. We will derive a contradiction. As $\bar{f}(P) \neq \bar{f}(L_0)$ there are $T_0, T_1 \in T$ such that

$$T_0 \cap T_1 = \emptyset,$$

and

$$f^{-1}[T_0] \in P \quad \text{and} \quad f^{-1}[T_1] \in L_0.$$

Take $V_0, V_1 \in T$ such that $f^{-1}[T_0] \cap f^{-1}[V_1] = \emptyset = f^{-1}[V_0] \cap f^{-1}[T_1]$ and $f^{-1}[V_0] \cup f^{-1}[V_1] = X$. This is possible since T is normal and f is surjective. Since L_1 is a maximal linked system, either $f^{-1}[V_0] \in L_1$ or $f^{-1}[V_1] \in L_1$. If $f^{-1}[V_0] \in L_1$ then

$$V_0 \in P\underline{L}_1 \subset \bar{f}(L_1) = M,$$

and since $V_0 \cap T_1 = \emptyset$ this is a contradiction. On the other hand, if $f^{-1}[V_1] \in L_1$ then $f^{-1}[V_1]$ is an element both of L_0 and L_1. Consequently

$$I_{S+}(L_0, L_1) \subset f^{-1}[V_1]^+,$$

and since $P \in I_{S+}(L_0, L_1)$ it follows that $f^{-1}[V_1] \in P$. However, this is also a contradiction since $f^{-1}[T_0] \cap f^{-1}[V_1] = \emptyset$.

By corollary 1.5.12 (a) it now follows that $\bar{f}^{-1}[\{M\}]$ is a retract of $\lambda(X,S)$ and as $\lambda(X,S)$ is an AR (theorem 3.2.1) the fiber $f^{-1}[\{M\}]$ is an AR too.

This completes the proof of the theorem. □

3.2.4. <u>COROLLARY</u>. *Let* X *and* Y *be metrizable continua and let* f: X → Y *be a continuous surjection. Then* $\lambda(f)\colon \lambda X \longrightarrow \lambda Y$ *(cf. lemma 3.1.22) is cellular.*

3.2.5. This corollary explicates a fundamental difference between 2^X and λX. For all compact metric spaces X and Y and for each continuous function $f\colon X \to Y$ there is natural extension $2^f\colon 2^X \longrightarrow 2^Y$ of f defined by

$$2^f(A) := f[A].$$

The mappings 2^f are not cellular in general. For example, let $X = [0,1]$

and let Y be the space obtained from X by identifying 0 and 1. Let $f: X \to Y$ be the quotient mapping. Then

$$(2^f)^{-1}[\{f(0)\}] = \{\{0\},\{1\},\{0,1\}\},$$

which is not connected.

3.2.6. A Q-*manifold* is a separable metric space modelled on Q, i.e. a space which admits an open covering by sets homeomorphic to open subsets of the Hilbert cube Q. CHAPMAN [30] has shown that the class of Q-manifolds coincides with the class of spaces of the form $K \times Q$, where K is a locally finite polyhedron. Moreover CHAPMAN showed that each cell-like mapping between Q-manifolds is a near-homeomorphism. This is a consequence of his papers [28] and [29]. This powerful result will be very important for us.

If (X_i,f_i) is an inverse sequence, then the inverse limit $\varprojlim(X_i,f_i)$ is the subspace $\{x \in \Pi_i X_i \mid f_i(x_{i+1}) = x_i \ \ (i \in \mathbb{N})\}$ of $\Pi_i X_i$. BROWN [25] has shown that the inverse limit $\varprojlim(X_i,f_i)$ of compact metric spaces X_i, all homeomorphic to a given space X, such that each bonding map f_i is a near-homeomorphism is homeomorphic to X.

Combining the results of CHAPMAN and BROWN it follows that the *inverse limit of a sequence of Hilbert cubes with cellular bonding maps is again a Hilbert cube.*

This observation yields the following:

3.2.7. <u>THEOREM</u>. *Let* X *be homeomorphic to* $\varprojlim(X_i,f_i)$ *where the bonding maps* f_i *are surjective. If* $\lambda X_i \approx Q$ $(i \in \mathbb{N})$ *then* $\lambda X \approx Q$.

<u>PROOF</u>. Identify X and $\varprojlim(X_i,f_i)$ and let $\pi_i: X \to X_i$ $(i \in \mathbb{N})$ be the projections. Since π_i is a continuous surjection, for each $i \in \mathbb{N}$, there is an extension

$$\lambda(\pi_i)\colon \lambda X \longrightarrow \lambda X_i.$$

It is easily seen that $\lambda(f_i)\circ\lambda(\pi_{i+1}) = \lambda(\pi_i)$ since $f_i \circ \pi_{i+1} = \pi_i$ $(i \in \mathbb{N})$ and consequently the mapping

$$e\colon \lambda X \longrightarrow \varprojlim\big(\lambda X_i,\lambda(f_i)\big)$$

defined by $e(M)_i = \lambda(\pi_i)(M)$ is a continuous surjection. We claim that e is one to one. For this, choose $M,N \in \lambda X$ such that $M \neq N$. Also, choose disjoint $M \in \mathcal{M}$ and $N \in \mathcal{N}$. By the compactness of the spaces X_i ($i \in \mathbb{N}$) (cf. corollary 2.5.4) there is an $i_0 \in \mathbb{N}$ such that $\pi_{i_0}[M] \cap \pi_{i_0}[N] = \emptyset$. Then, clearly

$$\lambda(\pi_{i_0})(\mathcal{M}) \neq \lambda(\pi_{i_0})(\mathcal{N}),$$

since $\pi_{i_0}[M] \in \lambda(\pi_{i_0})(\mathcal{M})$ and $\pi_{i_0}[N] \in \lambda(\pi_{i_0})(\mathcal{N})$. It follows that $e(\mathcal{M})_{i_0} \neq e(\mathcal{N})_{i_0}$ and consequently e is one to one.

We conclude that λX is homeomorphic to $\varprojlim(\lambda X_i, \lambda(f_i))$. Since $\lambda X_i \approx Q$ ($i \in \mathbb{N}$) the spaces X_i are metrizable continua (cf. corollary 2.5.4); corollary 3.2.4 implies that the mappings $\lambda(f_i)$ are cellular. It now follows that $\varprojlim(\lambda X_i, \lambda(f_i)) \approx Q$ (cf. 3.2.6). Therefore $\lambda X \approx Q$. □

3.2.8. In section 3.4 we will show that λI is homeomorphic to the Hilbert cube Q. Therefore, theorem 3.2.7 implies that a space such as

$$Y = \{(0,y) \mid -1 \le y \le 1\} \cup \{(x, \sin\tfrac{1}{x}) \mid 0 < x \le 1\}$$

has the property that its superextension is homeomorphic to the Hilbert cube. This is of interest since 2^Y is not homeomorphic to the Hilbert cube, not being locally connected.

3.3. Some $\lambda(I,\mathcal{S})$ is a Hilbert cube

In this section we construct an easy to describe subbase $\mathcal{S}$ for $I = [0,1]$ with the property that $\lambda(I,\mathcal{S})$ is homeomorphic to the Hilbert cube Q. The space $\lambda(I,\mathcal{S})$ will be the first step in an inverse limit representation of λI, the superextension of the closed unit interval.

3.3.1. We start with a general method in order to construct superextensions of I as subspaces of I^2. For this, let $\mathcal{T}$ denote the canonical binary subbase for I^2, i.e.

$$\mathcal{T} = \{A \subset I^2 \mid A = \pi_i^{-1}[0,x] \vee A = \pi_i^{-1}[x,1] \ (i \in \{0,1\});\ x \in I\}.$$

Assume that I is imbedded in I^2, preserving arc-length, as indicated in the following figure:

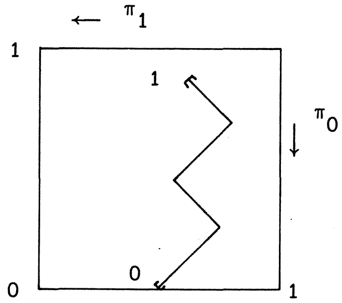


Figure 9.

We are interested in $\lambda(I,\mathcal{T}_0)$, where $\mathcal{T}_0$ is the restriction of $\mathcal{T}$ to I, i.e.

$$\mathcal{T}_0 = \{T \cap I \mid T \in \mathcal{T}\}.$$

(Here I denotes the embedded copy of I in I^2.)
It is easy to see that $\mathcal{T}_0$ is a supernormal T_1-subbase (cf. 2.2.1). We assert that $\lambda(I,\mathcal{T}_0)$ is homeomorphic to the space X indicated in figure 10

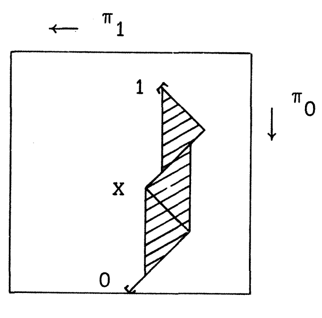


Figure 10.

To prove this, define an interval structure (cf. definition 1.3.2) I_X on X by

$$I_X(x,y) := \bigcap\{T \in \mathcal{T} \mid x,y \in T\} \cap X.$$

The verification that I_X indeed is an interval structure is routine and follows immediately from figure 10, since for all $x,y,z \in X$ we have

$$I_{\mathcal{T}}(x,y) \cap I_{\mathcal{T}}(x,z) \cap I_{\mathcal{T}}(y,z) \subset X.$$

Consequently, each element of $\mathcal{T} \cap X = \{T \cap X \mid T \in \mathcal{T}\}$ is I_X-convex. We conclude that $\mathcal{T} \cap X$ is a binary subbase for X (cf. theorem 1.3.3). It is easily seen that for all $A_0, A_1 \in \mathcal{T} \cap X$ with $A_0 \cap A_1 \neq \emptyset$ also $A_0 \cap A_1 \cap I \neq \emptyset$, due to the special choice of X. Theorem 2.2.5 now implies that $\lambda(I,\mathcal{T}\cap I) \approx X$.

If we consider the proof of theorem 2.2.5 we see that the homeomorphism between $\lambda(I,\mathcal{T}\cap I)$ and X is very "direct". For instance the point p in figure 11 represents the $\mathcal{T} \cap I$ mls $\mathcal{M}$ for which

$$\{[0,e],[e,1],[a,b] \cup [c,d],[0,a] \cup [b,c] \cup [d,1]\}$$

is a pre-mls.

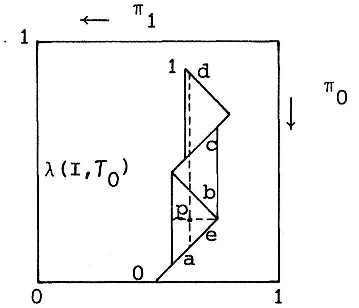


Figure 11.

3.3.2. We will now construct the announced subbase $\mathcal{S}$ for I. Define

$$E := \{-2.3^k \mid k \in \{0,1,2,\dots\}\}.$$

For each $n \in E$ let I be embedded in I^2, preserving arc-length, as indicated in the following figure.

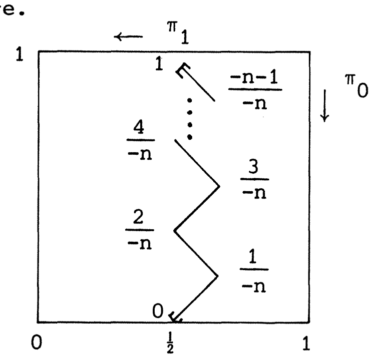


Figure 12.

All angles are $\frac{1}{2}\pi$ except the one at $(\frac{1}{2},0)$ which is $\frac{1}{4}\pi$. Define A_n by

$$A_n := \{T \cap I \mid T \in T\}.$$

Then, using the same technique as in 3.3.1, it follows that $\lambda(I,A_n)$ is the convex-hull of the embedded copy of I in I^2.

Notice that A_n $(n \in E)$ is a supernormal subbase for I and hence that $\lambda(I,\bigcup_{n\in E} A_n)$ can be embedded in $\Pi_{n\in E}\ \lambda(I,A_n)$ in a very canonical way; cf. theorem 2.3.13 and lemma 2.3.14. We will make two identifications. First we consider $\lambda(I,\bigcup_{n\in E} A_n)$ to be a subspace of $\Pi_{n\in E}\ \lambda(I,A_n)$. Second, we identify $\lambda(I,A_n)$ and the subspace of I^2 indicated in figure 12 $(n \in E)$.

3.3.3. PROPOSITION. $\lambda(I,\bigcup_{i\in E} A_i)$ *is a (linearly) convex subspace of* $\Pi_{i\in E}\ \lambda(I,A_i)$.

PROOF. Suppose that $\lambda(I,\bigcup_{i\in E} A_i)$ is not a convex subspace of $\Pi_{i\in E}\ \lambda(I,A_i)$. Then there exist $x,y \in \lambda(I,\bigcup_{i\in E} A_i)$ and $\alpha,\beta \in \mathbb{R}$ with $\alpha+\beta = 1$, $\alpha \geq 0$, $\beta \geq 0$ such that

$$\alpha x + \beta y \notin \lambda(I,\bigcup_{i\in E} A_i).$$

Since for all $i \in E$ the point $\alpha x_i + \beta y_i$ belongs to $\lambda(I,A_i)$ it follows that the system $\bigcup_{i\in E}(\alpha x_i+\beta y_i)$ is not linked (cf. lemma 2.3.14). Notice that we identify $\alpha x_i+\beta y_i$ and the mls which is represented by $\alpha x_i+\beta y_i$ $(i \in E)$. Choose two indices i_0 and i_1 such that

$$(\alpha x_{i_0}+\beta y_{i_0}) \cup (\alpha x_{i_1}+\beta y_{i_1})$$

is not linked. Hence there exists an $M \in (\alpha x_{i_0}+\beta y_{i_0})$ and an $N \in (\alpha x_{i_1}+\beta y_{i_1})$ such that $M \cap N = \emptyset$.

If in the copy of I^2 corresponding to i_0 we draw a horizontal line through x_{i_0} and determine its intersection p_0 with the embedded copy of I, and we do the same in the copy of I^2 corresponding to i_1, thus obtaining p_1, then p_0 and p_1 are derived from the same point of I; for if not, then it is easy to see that $x_{i_0} \cup x_{i_1}$ is not linked. In the same way, straight horizontal lines through y_{i_0} and y_{i_1} also must determine the same point on the embedded copies of I and consequently the same is true for horizontal lines through $\alpha x_{i_0} + \beta y_{i_0}$ and $\alpha x_{i_1} + \beta y_{i_1}$ because of the specially chosen embeddings of I. Hence it follows that the situation drawn in the following figure is the only possibility (except for interchanging i_0 and i_1).

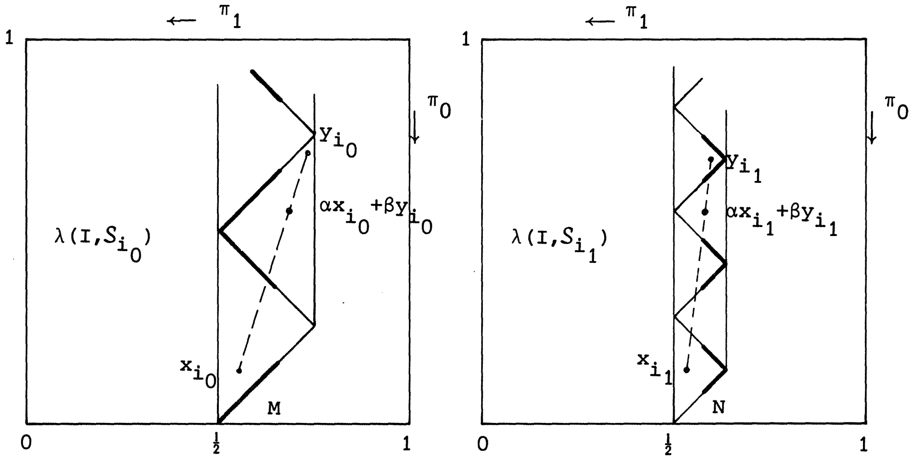


Figure 13.

REMARKS.

(i) M meets any set of the form $\pi_0^{-1}[\frac{1}{2},x] \cap I$ with $x \geq \pi_0(\alpha x_{i_1}+\beta y_{i_1})$ in the point 0 of the embedded copy of I.

(ii) N meets any set of the form $\pi_0^{-1}[x,1] \cap I$ with $x \leq \pi_0(\alpha x_{i_0}+\beta y_{i_0})$ in the point $\frac{1}{2}$ of the embedded copy of I.

(iii) It is possible that an element of $\alpha x_{i_0} + \beta y_{i_0}$ containing M, and an element of $\alpha x_{i_1} + \beta y_{i_1}$, containing N, have a void intersection. In that case of course the sets M and N also have a void intersection.

(iv) In figure 13 we have drawn the points x_{i_0}, y_{i_0}, x_{i_1} and y_{i_1} in such a way that $\pi_0\, x_{i_0} < \pi_0\, y_{i_0}$ and $\pi_0\, x_{i_1} < \pi_0\, y_{i_1}$. This is done because in the cases $\pi_0\, x_{i_0} = \pi_0\, y_{i_0}$ or $\pi_0\, x_{i_1} = \pi_0\, y_{i_1}$ or ($\pi_0\, x_{i_0} < \pi_0\, y_{i_0}$ and $\pi_0\, x_{i_1} > \pi_0\, y_{i_1}$) or ($\pi_0\, x_{i_0} > \pi_0\, y_{i_0}$ and $\pi_0\, x_{i_1} < \pi_0\, y_{i_1}$) it is easy to see that M and N intersect, as the reader can easily verify.

Without loss of generality we may assume that $\pi_0\, y_{i_1} - \pi_0\, x_{i_1} \leq \pi_0\, y_{i_0} - \pi_0\, x_{i_0}$. It then follows that $\pi_0^{-1}[\pi_0\, x_{i_1},1] \cap I \subset \pi_0^{-1}(\pi_0\, x_{i_0},1] \cap I$ since $N \subset I\backslash M$ and since $\pi_0(\alpha x_{i_1}+\beta y_{i_1}) - \pi_0\, x_{i_1} \leq \pi_0(\alpha x_{i_0}+\beta y_{i_0}) - \pi_0\, x_{i_0}$. However, this is a contradiction since $x_{i_0} \cup x_{i_1}$ is linked. □

3.3.4. PROPOSITION. *$\lambda(I,\bigcup_{i\in E} A_i)$ is infinite dimensional.*

PROOF. We will show that $\lambda(I,\bigcup_{i\in E} A_i)$ contains a copy of the Hilbert cube. For each $n \in E$ let I_n be defined by

$$I_n := \left[\frac{1}{2} + \frac{1}{3\sqrt{2}\cdot -n}, \ \frac{1}{2} + \frac{2}{3\sqrt{2}\cdot -n}\right].$$

Define a map $\phi\colon \Pi_{i\in E}\, I_i \longrightarrow \Pi_{i\in E}\, I^2$ by

$$(\phi(x))_i = (x_i, \tfrac{1}{4}\sqrt{2}).$$

Notice that for each $i \in E$ the point $(\phi(x))_i$ is an element of $\lambda(I, A_i)$ for all $x \in \Pi_{i\in E}\, I_i$. Furthermore it is obvious that ϕ is an embedding.

It suffices to show that the image of $\Pi_{i\in E}\, I_i$ is contained in $\lambda(I, \cup_{i\in E}\, A_i)$ and for this it suffices to show that

$$\bigcup_{i\in E} (\phi(x))_i$$

is linked for all $x \in \Pi_{i\in E}\, I_i$ (cf. lemma 2.3.14). Assume to the contrary that for some $x \in \Pi_{i\in E}\, I_i$ the system $\cup_{i\in E}\, (\phi(x))_i$ were not linked. Then there exist indices n_0 and n_1 such that

$$(\phi(x))_{n_0} \cup (\phi(x))_{n_1}$$

is not linked. Choose $M \in (\phi(x))_{n_0}$ and $N \in (\phi(x))_{n_1}$ with a void intersection. Then there are two possibilities drawn in figure 14 and figure 15. Without loss of generality we may assume that $n_1 < n_0$.

[A]

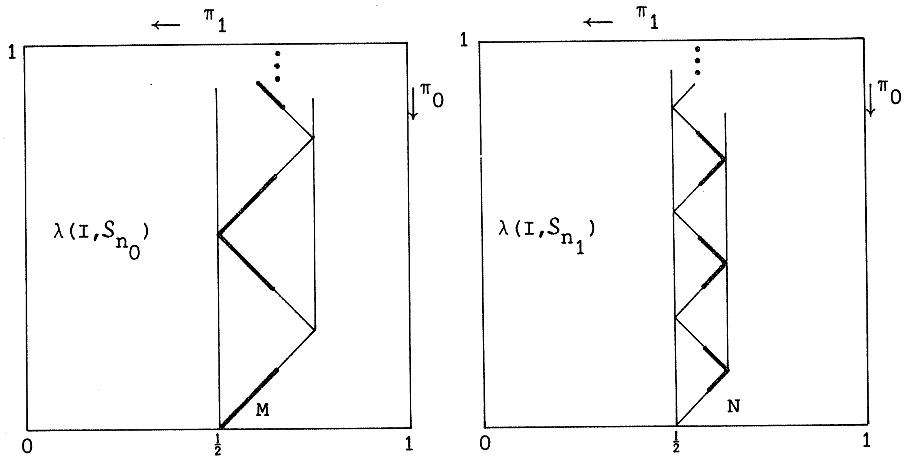


Figure 14.

This shows that $\pi_0^{-1}[\frac{1}{2}, \pi_0(\phi(x))_{n_0}] \cap I \subset \pi_0^{-1}[\frac{1}{2}, \pi_0(\phi(x))_{n_1}) \cap I$. Since

$n_1 < n_0$ it follows that

$$\sqrt{2}(\pi_0(\phi(x))_{n_1} - \tfrac{1}{2}) < \frac{1}{-n_1} \leq \frac{1}{-3n_0} \leq \sqrt{2}(\pi_0(\phi(x))_{n_0} - \tfrac{1}{2})$$

and therefore

$$\sqrt{2}(\pi_0(\phi(x))_{n_1} - \tfrac{1}{2}) < \sqrt{2}(\pi_0(\phi(x))_{n_0} - \tfrac{1}{2})$$

which shows that the component containing 0 of $\pi_0^{-1}[\frac{1}{2},\pi_0(\phi(x))_{n_0}] \cap I$ cannot be contained in the component containing 0 of $\pi_0^{-1}[\frac{1}{2},\pi_0(\phi(x))_{n_1}] \cap I$. This is a contradiction.

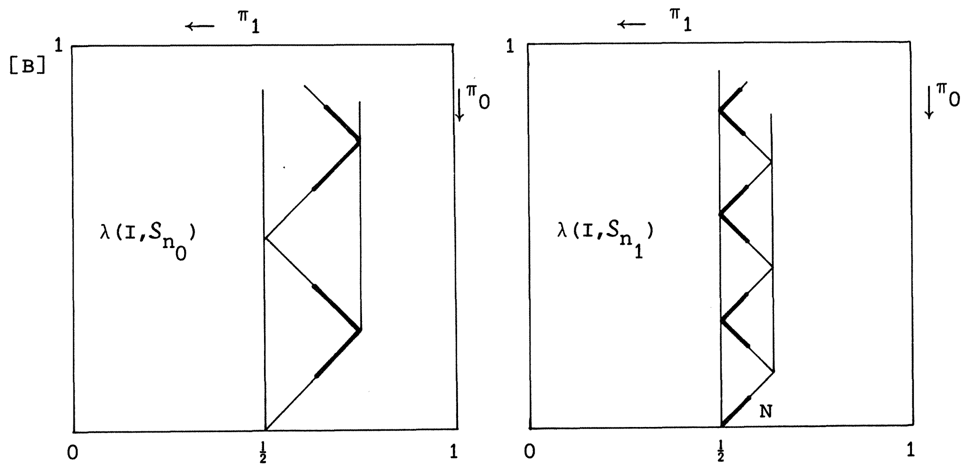


Figure 15.

Now, $\pi_0^{-1}[\pi_0(\phi(x))_{n_0},1] \cap I \subset \pi_0^{-1}(\pi_0(\phi(x))_{n_0},1] \cap I$. Since $-n_0 < -n_1$ it follows that the component containing $\frac{1}{2}$ of $\pi_0^{-1}[\pi_0(\phi(x))_{n_0},1] \cap I$ cannot be contained in the component containing $\frac{1}{2}$ of $\pi_0^{-1}(\pi_0(\phi(x))_{n_1},1] \cap I$. This is a contradiction. □

Proposition 3.3.3 and proposition 3.3.4 now give the desired result.

3.3.5. <u>THEOREM</u>. $\lambda(I,\bigcup_{i\in E} A_i)$ *is homeomorphic to the Hilbert cube.*

<u>PROOF</u>. According to a theorem of KELLER [68] each infinite dimensional (linearly) convex compact subspace of the separable Hilbert space is homeomorphic to the Hilbert cube. □

3.4. The superextension of the closed unit interval is homeomorphic to the Hilbert cube

In this section we show that the superextension of the closed unit interval λI is homeomorphic to the Hilbert cube. We represent λI as the inverse limit of a sequence of Hilbert cubes with cellular bonding maps. It then follows that λI itself is a Hilbert cube.

3.4.1. For the closed unit interval I, define

$$S := \{G \subset I \mid G \text{ is the union of finitely many closed intervals with rational endpoints}\}.$$

It is clear that S separates the closed subsets of I and hence it follows that λI and $\lambda(I,S)$ are homeomorphic (cf. theorem 2.4.2). Define

$$F := \{(S_0,S_1) \mid S_i \in S \ (i \in \{0,1\}) \text{ and } S_0 \cap S_1 = \emptyset\}.$$

Clearly F is countable; we enumerate F using a bijection of F onto $\mathbb{N}\backslash\{1\}$. If $(S_0,S_1) \in F$, then $\varepsilon = d(S_0,S_1) > 0$ and also $\delta = \frac{1}{2}\varepsilon\sqrt{2} > 0$. Consider the following embedding, depending on (S_0,S_1), of I preserving arc-length in I^2.

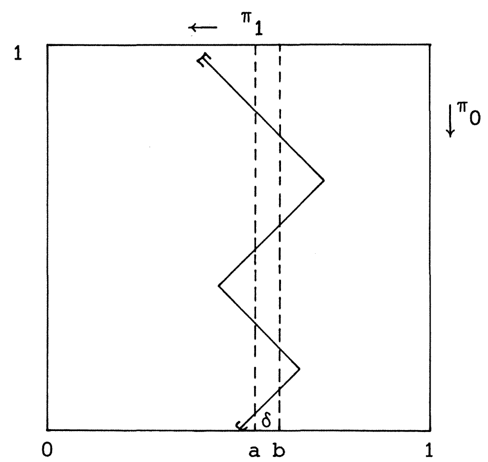


Figure 16.

All angles are $\frac{1}{2}\pi$ except the one at $(\frac{1}{2},0)$ which is $\frac{1}{4}\pi$. Also $b-a = \delta$ and $S_0 \subset \pi_0^{-1}[0,a] \cap I$ and $S_1 \subset \pi_0^{-1}[b,1] \cap I$. Since S_0 and S_1 are *finite* unions of closed intervals, such an embedding is always possible.

As in section 3.3 define

$$T := \{A \subset I^2 \mid A = \pi_i^{-1}[0,x] \vee A = \pi_i^{-1}[x,1] \ (i \in \{0,1\}),\ x \in I\}.$$

Then $\lambda(I, T\cap I)$ is the space designed in figure 17 (cf. 3.3.1).

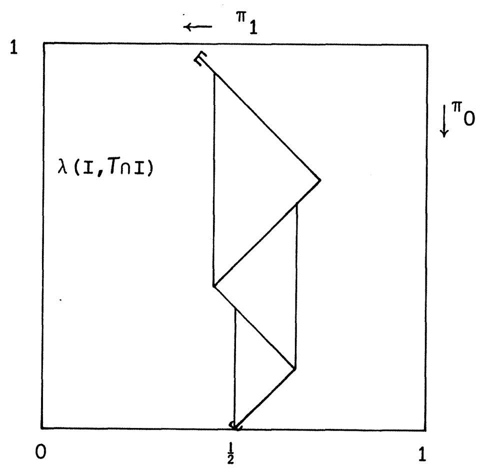


Figure 17.

If (S_0, S_1) is the n^{th} element of F, let

$$\lambda(I, S_n)$$

be the superextension of I as indicated in figure 17. In addition put

$$S_1 := \cup_{i\in E} A_i,$$

where the A_i's are defined as in section 3.3 (cf. 3.3.2).

The hardest part of our program is to show that for each $n \in \mathbb{N}$ the superextension $\lambda(I, \cup_{i=1}^{n} S_i)$ is a Q-manifold, the proof of which will be postponed till the end of this section. Notice that for each $n \in \mathbb{N}$ the subbase $\cup_{i=1}^{n} S_i$ is *supernormal* (cf. 2.2.1 (iv)) and hence that we can apply the results derived in 2.3.10 - 2.3.15.

3.4.2. <u>PROPOSITION</u>. *For each* $n \in \mathbb{N}$ *the superextension* $\lambda(I, \cup_{i=1}^{n} S_i)$ *is a compact Q-manifold.*

Now an interesting result of CHAPMAN [27] is applicable to show that $\lambda(I, \cup_{i=1}^{n} S_i)$ is even a Hilbert cube.

3.4.3. <u>LEMMA</u>. *For each* $n \in \mathbb{N}$ *the superextension* $\lambda(I, \cup_{i=1}^{n} S_i)$ *is a Hilbert cube.*

<u>PROOF</u>. The normality of $\cup_{i=1}^{n} S_i$ (cf. theorem 2.3.13) implies that $\lambda(I, \cup_{i=1}^{n} S_i)$ is an AR (cf. theorem 3.2.1). In particular $\lambda(I, \cup_{i=1}^{n} S_i)$ is contractible. Therefore $\lambda(I, \cup_{i=1}^{n} S_i)$ is a contractible compact Q-manifold by proposition 3.4.2. However, CHAPMAN [27] has shown that a compact contractible Q-manifold is a Hilbert cube, which proofs the lemma. $\square$

3.4.4. Consider the following inverse limit system

$$\lambda(I,S_1) \xleftarrow{g_1} \lambda(I,S_1 \cup S_2) \xleftarrow{g_2} \lambda(I,S_1 \cup S_2 \cup S_3) \xleftarrow{g_3} \cdots\cdots$$

where the bonding maps g_n are defined by

$$g_n(M) := M \cap \bigcup_{i=1}^{n} S_i$$

($n \in \mathbb{N}$). These mappings are well-defined, cf. corollary 2.3.12.

3.4.5. <u>LEMMA</u>. λI *is homeomorphic to* $\varprojlim(\lambda(I, \cup_{i=1}^{n} S_i), g_i)$.

<u>PROOF</u>. For each $n \in \mathbb{N}$ define a mapping $\xi_n \colon \lambda I \to \lambda(I, \cup_{i=1}^{n} S_i)$ by

$$\xi_n(M) := M \cap \bigcup_{i=1}^{n} S_i.$$

This mapping is well-defined, cf. corollary 2.3.12. We claim that for each $n \geq 2$ the diagram

$$\begin{tikzcd} & \lambda(I, \cup_{i=1}^{n} S_i) \arrow[d, "g_n"] \\ \lambda I \arrow[ur, "\xi_n"] \arrow[dr, "\xi_{n-1}"'] & \\ & \lambda(I, \cup_{i=1}^{n-1} S_i) \end{tikzcd}$$

commutes.

Indeed, take $M \in \lambda I$. Then

$$g_n(\xi_n(M)) = g_n(M \cap \bigcup_{i=1}^{n} S_i)$$
$$= (M \cap \bigcup_{i=1}^{n} S_i) \cap \bigcup_{i=1}^{n-1} S_i$$
$$= M \cap \bigcup_{i=1}^{n-1} S_i$$
$$= g_{n-1}(M).$$

Consequently, the mapping e: $\lambda I \longrightarrow \varprojlim(\lambda(I,\cup_{i=1}^{n} S_i),g_n)$ defined by $e(M)_n := \xi_n(M)$ ($n \in \mathbb{N}$) is a continuous closed surjection. It remains to show that e is one to one. Choose distinct $M,N \in \lambda I$ and choose $M \in M$ and $N \in N$ such that $M \cap N = \emptyset$. Since S separates the closed subsets of I there are $S_0,S_1 \in S$ with $M \subset S_0$ and $N \subset S_1$ and $S_0 \cap S_1 = \emptyset$. Now, $(S_0,S_1) \in F$, say the n^{th} element, and therefore S_0 and S_1 are separated by elements of S_n. It now follows that $\xi_n(M) \neq \xi_n(N)$, since $S_n \subset \cup_{i=1}^{n} S_i$. This proves that e is one to one; consequently e is a homeomorphism. □

3.4.6. This lemma completes the proof of the fact $\lambda I \approx Q$, since the theorem 3.2.3 implies that the bonding maps in the inverse sequence are cellular. They are even cellular in a very strong way: in [79] we showed that each point inverse of g_n ($n \in \mathbb{N}$) either is a point or is homeomorphic to an interval. We will not give the argument here, since there is no use fot it. But it is a nice fact.

We did not check whether the bonding maps are *strictly-cellular*, i.e. have the additional property that the point inverses are Z-sets. Probably this is the case.

3.4.7. THEOREM. *The superextension of the closed unit interval is homeomorphic to the Hilbert cube.*

PROOF. As indicated above, the bonding maps g_n ($n \in \mathbb{N}$) are cellular. Hence $\lambda I \approx \varprojlim(\lambda(I,\cup_{i=1}^{n} S_i),g_n) \approx Q$ (cf. lemma 3.4.3, lemma 3.4.5 and 3.2.6). □

3.4.8. PROOF OF PROPOSITION 3.4.2. Choose

$$x \in \lambda(I,\bigcup_{i=1}^{n} S_i) \subset \prod_{i=1}^{n} \lambda(I,S_i) \subset \prod_{i\in E} \lambda(I,A_i) \times \prod_{i=2}^{n} \lambda(I,S_i).$$

Let $\{p_i \mid i \in E\} \cup \{p_i \mid i \in \{2,\ldots,n\}\}$ denote the projection maps of the latter product. For each $i \in \{2,3,\dots,n\}$ the projection of $\lambda(I,S_i)$ onto the first coordinate axis of I^2 is an interval, say $[c_i^0,c_i^1]$. Assume that for each $i \in \{2,3,\dots,q\}$ where $q \le n$, the projection $\pi_0 x_i \in (c_i^0,c_i^1)$ and that for $i \in \{q+1,q+2,\dots,n\}$ we have $\pi_0 x_i \notin (c_i^0,c_i^1)$. Define

$$\varepsilon := \min\{d(\pi_0 x_i,c_i^j) \mid i \in \{2,3,\dots,q\};\ j \in \{0,1\}\}.$$

Let $A := \{2,3,\dots,n\}$. If $i \in A$ and $M \in x_i$ define

$$M^* := \mathrm{cl}_I\ \mathrm{int}_I(M)$$

(here I refers to the copy of $[0,1]$ embedded in $\lambda(I,S_i) \subset I^2$). Also, for $i \in A$, put

$$F(x_i) := \{M^* \mid (M=\pi_0^{-1}[0,\pi_0 x_j] \cap I \text{ or } M=\pi_0^{-1}[\pi_0 x_j,1] \cap I)\ (j \in A\backslash\{i\}) \text{ and } M^* \cap \pi_0^{-1}\pi_0 x_i = \emptyset\}.$$

Notice that $F(x_i)$ is finite. If $i \in \{2,3,\dots,q\}$ then choose a subinterval (a_i,b_i) of (c_i^0,c_i^1) (an interval is non-degenerate in our terminology) such that

(i) $\pi_0 x_i \in (a_i,b_i)$;
(ii) $a_i - c_i^0 > \frac{1}{4}\varepsilon$ and $c_i^1 - b_i > \frac{1}{4}\varepsilon$;
(iii) $\pi_0^{-1}[a_i,b_i] \cap \lambda(I,S_i)$ consists of two closed convex subspaces D_i^0 and D_i^1 such that $\pi_0 D_i^0 = [a_i,\pi_0 x_i]$ and $\pi_0 D_i^1 = [\pi_0 x_i,b_i]$;
(iv) $\pi_0^{-1}[a_i,b_i] \cap \bigcup F(x_i) = \emptyset$;
(v) for each subinterval $[e_1,e_2]$ of $[a_i,\pi_0 x_i)$ and for each subinterval $[d_1,d_2]$ of $(\pi_0 x_i,b_i]$ we have that $\pi_0^{-1}[e_1,e_2] \cap I$ and $\pi_0^{-1}[d_1,d_2] \cap I$ both have no isolated points.

If $i \in A\backslash\{2,3,\dots,q\}$ then choose a subinterval $[a_i,b_i]$ of $[c_i^0,c_i^1]$ such that
(i) $\pi_0^{-1}[a_i,b_i] \cap \lambda(I,S_i)$ is convex in $\lambda(I,S_i)$;
(ii) x_i is an interior point of $\pi_0^{-1}[a_i,b_i] \cap \lambda(I,S_i)$ in $\lambda(I,S_i)$;
(iii) $\pi_0^{-1}[a_i,b_i] \cap \bigcup F(x_i) = \emptyset$;
(iv) for each subinterval $[e_1,e_2]$ of $[a_i,b_i]$ we have that $\pi_0^{-1}[e_1,e_2] \cap I$ has no isolated points;

(one should convince oneself that in all cases suitable a_i,b_i do indeed exist!).

We will show that the closed neighborhood

$$B(x) = \bigcap_{i=2}^{n} p_i^{-1}[\pi_0^{-1}[a_i,b_i] \cap \lambda(I,S_i)] \cap \lambda(I,\bigcup_{i=1}^{n} S_i)$$

of x is a Q-manifold, which will establish the proof of proposition 3.4.2 (there is an open U in $\lambda(I,\bigcup_{i=1}^{n} S_i)$ such that $x \in U \subset B(x)$ and as B(x) is a compact Q-manifold, there is also an open O in $\lambda(I,\bigcup_{i=1}^{n} S_i)$ such that $x \in O \subset U \subset B(x)$ and O is homeomorphic to an open subset of Q).

Let us first anatomize B(x). Consider $F = \{0,1\}^{\{2,3,\ldots,q\}}$ and for each $\sigma = (\sigma_i)_i \in F$ define

$$X(\sigma) := \bigcap_{i=2}^{q} p_i^{-1}[D_i^{\sigma_i}] \cap \bigcap_{i=q+1}^{n} p_i^{-1}[\pi_0^{-1}[a_i,b_i] \cap$$

$$\cap \lambda(I,S_i)] \cap \lambda(I,\bigcup_{i=1} S_i).$$

It then is clear that

$$\bigcup_{\sigma\in F} X(\sigma) = B(x).$$

<u>CLAIM</u> 1. *For each* $\sigma \in F$ *the set* $X(\sigma)$ *is closed and convex in* $\lambda(I,\bigcup_{i=1}^{n} S_i)$.

Indeed, assume to the contrary that for some $\sigma \in F$ the set $X(\sigma)$ were not convex. Then there exist $y,z \in X(\sigma)$ and $\alpha,\beta \in \mathbb{R}$ with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$ such that $\alpha y + \beta z \notin X(\sigma)$. We claim that

$$\bigcup_{i\in E} (\alpha y+\beta z)_i \cup \bigcup_{i=2}^{n} (\alpha y+\beta z)_i$$

is not linked, for else it would follow that $\alpha y + \beta z \in \lambda(I,\bigcup_{i=1}^{n} S_i)$ (cf. lemma 2.3.14), and as $(\alpha y+\beta z)_i = \alpha y_i + \beta z_i$ for each i, it is easily seen that even $\alpha y + \beta z \in X(\sigma)$. Therefore there exist two indices i_0, j_0 such that

$$(\alpha y+\beta z)_{i_0} \cup (\alpha y+\beta z)_{j_0}$$

is not linked and consequently there exists an $M \in (\alpha y+\beta z)_{i_0}$ and an $N \in (\alpha y+\beta z)_{j_0}$ such that $M \cap N = \emptyset$. Now, if i_0 and j_0 are both elements of $E \cup \{q+1,q+2,\ldots,n\}$ then, using the same technique as in proposition 3.3.3, it follows that M and N must intersect, for we have chosen the intervals $[a_i,b_i]$ $(i \in \{q+1,q+2,\ldots,n\})$ is such a way that $\pi_0^{-1}[e_1,e_2] \cap I$ has no isolated points for every subinterval $[e_1,e_2]$ of $[a_i,b_i]$.

Therefore, let us assume that $i_0 \in \{2,3,\dots,q\}$. Since straight horizontal lines through $(\alpha y+\beta z)_{i_0}$ and $(\alpha y+\beta z)_{j_0}$ must intersect the embedded copies of I in the same point (cf. the proof of proposition 3.3.3), the situation sketched in figure 18 is the only possibility (except for an interchange of the indices i_0 and j_0, which induces a similar situation).

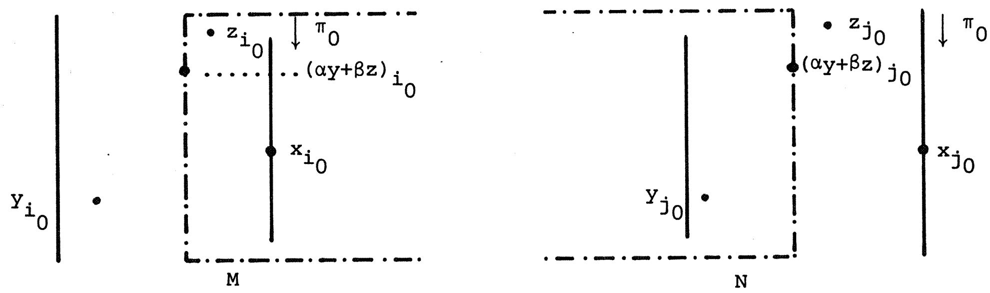


Figure 18.

REMARKS.

(i) It is possible that an element of $(\alpha y+\beta z)_{i_0}$, containing M, and an element of $(\alpha y+\beta z)_{j_0}$, containing N, have a void intersection. In that case the sets M and N of course also have a void intersection.

(ii) In figure 18 we have drawn the points y_{i_0}, z_{i_0}, x_{i_0}, y_{j_0}, z_{j_0} and x_{j_0} in such a way that $\pi_0 y_{i_0} < \pi_0 z_{i_0} < \pi_0 x_{i_0}$ and $\pi_0 y_{j_0} < \pi_0 z_{j_0} < \pi_0 x_{j_0}$. This is not the only possible configuration. More generally, we may assume that either ($\pi_0 y_{i_0} < \pi_0 z_{i_0} \le \pi_0 x_{i_0}$ and $\pi_0 y_{j_0} < \pi_0 z_{j_0} \le \pi_0 x_{j_0}$) or ($\pi_0 x_{i_0} \le \pi_0 y_{i_0} < \pi_0 z_{i_0}$ and $\pi_0 x_{j_0} \le \pi_0 y_{j_0} < \pi_0 z_{j_0}$) (these two cases are similar), for in all other cases it is easy to see that $(\alpha y+\beta z)_{i_0} \cup (\alpha y+\beta z)_{j_0}$ is linked. The lack of generality in our diagram will cause no trouble, as will appear from the proof.

We distinguish two subcases:

(a) $\pi_0 z_{i_0} - \pi_0 y_{i_0} \le \pi_0 z_{j_0} - \pi_0 y_{j_0}$.

Since $M \subset \pi_0^{-1}(\pi_0(\alpha y+\beta z)_{j_0},1] \cap I$, it follows that

$$\pi_0^{-1}[\pi_0 y_{i_0},1] \cap I \subset \pi_0^{-1}(\pi_0 y_{j_0},1],$$

since $\pi_0^{-1}[\pi_0\, y_{i_0},1] \cap I$ has no isolated points and since

$$\pi_0(\alpha y+\beta z)_{i_0} - \pi_0 y_{i_0} \le \pi_0(\alpha y+\beta z)_{j_0} - \pi_0 y_{j_0}.$$

However, this is a contradiction since $y_{i_0} \cup y_{j_0}$ is linked.

(b) $\pi_0 z_{j_0} - \pi_0 y_{j_0} \le \pi_0 z_{i_0} - \pi_0 y_{i_0}$.

Since $N \subset \pi_0^{-1}[0,\pi_0(\alpha y+\beta z)_{i_0}) \cap I$, we conclude that

$$(\pi_0^{-1}[0,\pi_0 z_{j_0}] \cap I)^* \subset \pi_0^{-1}[0,\pi_0 z_{i_0}) \cap I,$$

since $\pi_0(\alpha y+\beta z)_{j_0} - \pi_0 y_{j_0} \le \pi_0(\alpha y+\beta z)_{i_0} - \pi_0 y_{i_0}$. Therefore, if $\pi_0^{-1}\pi_0 z_{j_0} \cap I$ contains no isolated points of $\pi_0^{-1}[0,\pi_0 z_{j_0}] \cap I$, then this is a contradiction by the linkedness of $z_{i_0} \cup z_{j_0}$. If $\pi_0^{-1}\pi_0 z_{j_0} \cap I$ contains an isolated point of $\pi_0^{-1}[0,\pi_0 z_{j_0}] \cap I$, then $\pi_0 z_{j_0} = \pi_0 x_{j_0}$, for if not, then $\pi_0^{-1}[0,\pi_0 z_{j_0}] \cap I$ is not perfect, which is a contradiction.

Now, since

$$(\pi_0^{-1}[0,\pi_0 x_{j_0}] \cap I)^* \cap \pi_0^{-1}[a_{i_0},\pi_0 x_{i_0}] = \emptyset$$

it follows that also $\pi_0 y_{i_0} = \pi_0 x_{j_0}$, for if not, then $y_{i_0} \cup y_{j_0}$ is not linked. However, this implies that also $\pi_0(\alpha y+\beta z)_{j_0} = \pi_0 x_{j_0}$ and consequently $N \in z_{j_0}$. This is a contradiction, since $z_{i_0} \cup z_{j_0}$ is linked.

It now follows that the neighborhood $B(x)$ of x is a finite union of closed (and hence compact) convex subspaces. By a theorem of QUINN & WONG ([94], theorem 3.4) it follows that $B(x)$ is a Q-manifold provided that for all nonvoid subsets F_0 of F the set $\bigcap_{\sigma\in F_0} X(\sigma)$ either is void or is a Hilbert cube.

<u>CLAIM</u> 2. *Let F_0 be a nonvoid subset of F. Then $\bigcap_{\sigma\in F_0} X(\sigma)$ either is void or is a Hilbert cube.*

Assume that $\bigcap_{\sigma\in F_0} X(\sigma)$ were nonvoid. It suffices to show that $\bigcap_{\sigma\in F_0} X(\sigma)$ is infinite dimensional, for an infinite dimensional compact convex subset of the separable Hilbert space is a Hilbert cube (cf. KELLER [68]). We will show that $\bigcap_{\sigma\in F_0} X(\sigma)$ contains a copy of the Hilbert cube. Choose $y \in \bigcap_{\sigma\in F_0} X(\sigma)$. We again distinguish two subcases:

(a) For each $i \in \{2,3,\dots,n\}$ the point $\pi_0 y_i$ is an element of (c_i^0, c_i^1).

Assume that y is such that for every coordinate y_i $(i \in E \cup \{2,3,\dots,n\})$ a straight horizontal line through y_i doesnot intersect I in 0 or 1 (this assumption is justified by the fact that if $y=0$ or $y=1$, then $\bigcap_{\sigma \in F_0} X(\sigma)$ is the intersection of a finite number of sets, each of which intersects I in a neighborhood of y). This intersection, say f, must be the same point for every coordinate. Define

$$\delta_0 := \min\{|y_i - c_i^0| \mid i \in \{2,3,\dots,n\}\},$$

$$\delta_1 := \min\{|y_i - c_i^1| \mid i \in \{2,3,\dots,n\}\}$$

and choose $n_0 \in E$ such that

$$-\frac{1}{n_0} < \frac{1}{4}\min\{\delta_0\sqrt{2}, \delta_1\sqrt{2}, f, 1-f\}.$$

For all $j \in E$, let I_j be defined as in proposition 3.3.4. It is easy to show, using the same technique as in the proof of proposition 3.3.4, that for all $j \in E$ with $j \le n_0$ and for each point $d \in I_j \times \{\frac{f}{\sqrt{2}}\}$ we have that

$$\bigcup_{i=2}^{n} y_i \cup d$$

is linked (notice that indeed $I_j \times \{\frac{f}{\sqrt{2}}\} \subset \lambda(I, A_j)$).

Now, by induction, for each $k \in \{m \in E \mid n_0 \le m\}$ we will construct a point $h_k \in \lambda(I, A_k)$ with the following property:

*) for all $j \in E$ with $j \le n_0$ there exists a (nondegenerate) subinterval I_j^k of I_j such that for every point $d_j^k \in I_j^k \times \{\frac{f}{\sqrt{2}}\}$ the system

$$\bigcup_{i=2}^{n} y_i \cup \bigcup_{\substack{j \in E \\ k \le j}} h_j \cup \bigcup_{\substack{j \in E \\ j \le n_0}} d_j^k$$

is linked.

For each $j \in E$ with $j \le n_0$ let $\bar{a}_j$ be the middle of the interval $I_j \times \{\frac{f}{\sqrt{2}}\}$. Then the linked system

$$\bigcup_{i=2}^{n} y_i \cup \bigcup_{\substack{j \in E \\ j \le n_0}} a_j$$

is contained in at least one maximal linked system $g_0 \in \lambda(I,\bigcup_{i=1}^n S_i)$. Define $h_{-2} := (g_0)_{-2}$. The intervals I_j^{-2} $(j \le n_0)$ now can be found in the following way:

(i) $I_j^{-2} := I_j$ if $\pi_0 h_{-2} \in I_{-2}$;

(ii) $I_j^{-2} := [\frac{1}{2},\pi_0\bar{a}_j] \cap I_j$ if $\pi_0 h_{-2} \in [\frac{1}{2},\pi_0\bar{a}_j]\backslash I_j$;

(iii) $I_j^{-2} := [\pi_0\bar{a}_j,1] \cap I_j$ if $\pi_0 h_{-2} \in [\pi_0\bar{a}_j,1]\backslash I_j$.

It is easy to verify that the intervals I_j^{-2} $(j \le n_0)$, defined in this way, satisfy our requirements.

Let all points h_k be defined for all $k \ge \ell$ $(\ell,k \in \{m\in E \mid n_0 \le m\})$. For each $j \in E$ with $j \le n_0$ let $\bar{a}_j^{\ell}$ be the middle of the interval $I_j^{\ell} \times \{\frac{f}{\sqrt{2}}\}$. Then the linked system

$$\bigcup_{i=r}^{n} y_i \cup \bigcup_{\substack{j\in E\\ \ell\le j}} h_j \cup \bigcup_{\substack{j\in E\\ j\le n_0}} \bar{a}_j^{\ell}$$

is contained in at least one maximal linked system $p_0 \in \lambda(I,\bigcup_{i=1}^n S_i)$. Define $h_{3\ell} := (p_0)_{3\ell}$. The intervals $I_j^{3\ell}$ $(j \le n_0)$ now can be found in the following way:

(i) $I_j^{3\ell} := I_j^{\ell}$ if $\pi_0 h_{3\ell} \in I_{3\ell}$;

(ii) $I_j^{3\ell} := [\frac{1}{2},\pi_0\bar{a}_j^{\ell}] \cap I_j^{\ell}$ if $\pi_0 h_{3\ell} \in [\frac{1}{2},\pi_0\bar{a}_j^{\ell}]\backslash I_j$;

(iii) $I_j^{3\ell} := [\pi_0\bar{a}_j^{\ell},\frac{1}{2}] \cap I_j^{\ell}$ if $\pi_0 h_{3\ell} \in [\pi_0\bar{a}_j^{\ell},1]\backslash I_j$.

Again, it is easy to verify that the intervals $I_j^{3\ell}$ $(j \le n_0)$, defined in this way satisfy our requirements.

Now, it is obvious that $\bigcap_{\sigma\in F_0} X(\sigma)$ contains a copy of $\prod_{\substack{j\in E\\ j\le n_0}} I_j^{n_0/3}$, which shows that $\bigcap_{\sigma\in F_0} X(\sigma)$ is infinite dimensional.

(b) There exists a coordinate $i_0 \in \{2,3,\ldots,n\}$ such that $\pi_0 y_{i_0} \notin (c_{i_0}^0,c_{i_0}^1)$.

We will construct a point $g \in \bigcap_{\sigma\in F_0} X(\sigma)$ such that $\pi_0 g_i \in (c_i^0,c_i^1)$ for all $i \in \{2,3,\ldots,n\}$. Then case (a) is applicable to show that $\bigcap_{\sigma\in F_0} X(\sigma)$ is infinite dimensional.

Without loss of generality we may assume that

$$\bigcap_{\sigma\in F_0} X(\sigma) = \bigcap_{i=2}^{n} p_i^{-1}[S_i] \cap \lambda(I,\bigcup_{i=1}^n S_i),$$

where each S_i $(2 \le i \le n)$ is convex in $\lambda(I,S_i)$, while, moreover, for each $i > q$ we have that $S_i = \pi_0^{-1}[H_i] \cap \lambda(I,S_i)$ for some (nondegenerate!) interval H_i. As in case (a), we may assume that a straight horizontal line through y_i does not intersect I in 0 or 1. Let this intersection be f. Define

$$V := \{i \in \{2,3,\dots,n\} \mid \pi_0 y_i \notin (c_i^0, c_i^1)\}.$$

Clearly $V \subset \{q+1,q+2,\dots,n\}$. Now, for every $i \in V$ there is a subinterval L_i of H_i such that $\pi_0 y_i \in L_i$ and $L_i \times \{\frac{f}{\sqrt{2}}\} \subset \lambda(I,S_i)$. Let δ_i denote the length of this interval ($i \in V$). Let

$$\delta := \min\{\delta_i \mid i \in V\}.$$

Moreover define

$$\rho_0 := \min\{|\pi_0 y_i - c_0^j| \mid i \in \{2,3,\dots,n\}\backslash V;\ j \in \{0,1\}\}$$

and

$$\rho := \frac{1}{4}\min\{\delta,\rho_0\}.$$

Choose for each $i \in V$ a point $g_i \in L_i \times \{\frac{f}{\sqrt{2}}\} \subset \lambda(I,S_i)$ such that

$$|\pi_0 y_i - \pi_0 g_i| = \rho.$$

Recall that $A = \{2,3,\dots,n\}$. We will show that

$$L = \bigcup_{i\in V} g_i \cup \bigcup_{i\in A\backslash V} y_i$$

is linked; consequently each mls $g \in \lambda(I,\bigcup_{i=1}^n S_i)$ which comtains L is a point of $\bigcap_{\sigma\in F_0} X(\sigma)$ such that $\pi_0 g_i \in (c_i^0,c_i^1)$ for all $i \in \{2,3,\dots,n\}$.

Assume that L were not linked. We again distinguish two subcases:

CASE 1. There exist two indices $i_0,j_0 \in V$ such that $g_{i_0} \cup g_{j_0}$ is not linked.

Choose $M \in g_{i_0}$ and $N \in g_{j_0}$ such that $M \cap N = \emptyset$. There are two subcases:

(i) One of the sets M,N contains the corresponding projection of y, say $y_{i_0} \in N$.

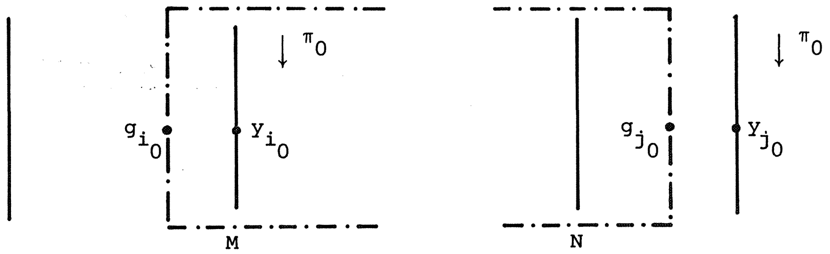


Figure 19.

Since $N \subset \pi_0^{-1}[0,\pi_0 g_{i_0}) \cap I$ and since $|\pi_0 g_{i_0}-\pi_0 y_{i_0}| = |\pi_0 g_{j_0}-\pi_0 y_{j_0}|$ it follows that $\pi_0^{-1}[0,\pi_0 y_{j_0}] \cap I \subset \pi_0^{-1}[0,\pi_0 y_{i_0}) \cap I$. However, this is a contradiction since $\pi_0^{-1}[0,\pi_0 y_{j_0}] \cap I = I$ and $I \cap \pi_0^{-1}\pi_0 y_{i_0} \neq \emptyset$.

(ii) None of the sets M,N contains the corresponding projection of y.

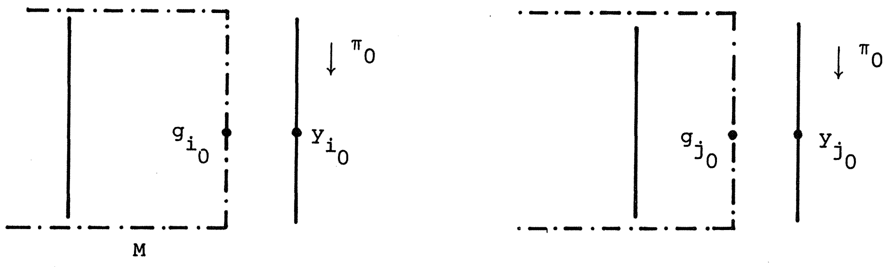


Figure 20.

It now follows that, for example, $M \subset \pi_0^{-1}(\pi_0 g_{j_0},1] \cap I$. However, this is a contradiction since M contains a component of length at least $\frac{3}{4}\rho\sqrt{2}$ while all components of $\pi_0^{-1}(\pi_0 g_{j_0},1] \cap I$ have length less than or equal to $\frac{2}{4}\rho\sqrt{2}$ since $\pi_0^{-1}[H_{j_0}] \cap I$ contains no isolated points and the same is true for each subinterval of H_{j_0}.

CASE 2. There exist indices $i_0 \in V$ and $j_0 \in A\backslash V$ such that $g_{i_0} \cup y_{j_0}$ is not linked.

This can be treated in the same way as case 1 (ii).

This completes the proof of the proposition. $\square$

3.4.9. REMARK. As announced it now follows from theorem 3.4.7, corollary 3.2.4 and the remarks in 3.2.6 that each continuous surjection $f\colon I \longrightarrow I$ extends to a near-homeomorphism $\lambda(f)\colon \lambda I \longrightarrow \lambda I$.

3.5. Pseudo-interiors of superextensions

In this section we concentrate on pseudo-interiors and capsets of superextensions. For any metrizable continuum X we define

$$\lambda_{cap}(X) := \{M \in \lambda X \mid M \text{ is defined on some } M \in 2^X \setminus \{X\}\}.$$

We show that if X has a binary normal subbase then $\lambda_{cap}(X)$ is a $B(Q)$ factor, i.e. $\lambda_{cap}(X) \times Q \approx B(Q)$. From results derived in the previous chapter it follows that $\lambda_{cap}(I) \approx B(Q)$ and also that $\lambda_{comp}(\mathbb{R})$, the subspace of $\lambda\mathbb{R}$ consisting of those mls's $M \in \lambda\mathbb{R}$ which are defined on some compact subset of $\mathbb{R}$, is homeomorphic to $B(Q)$. As a consequence a conjecture of VERBEEK [119] turns out to be false.

3.5.1. A subset M of the Hilbert cube Q is called a *capset* (cf. ANDERSON [5]) if M can be written as $M = \bigcup_{i=1}^{\infty} M_i$, where M_i is a Z-set in Q, with $M_i \subset M_{i+1}$ $(i \in \mathbb{N})$ while in addition the following absorption property holds: for each $\varepsilon > 0$ and $i \in \mathbb{N}$ and for every Z-set $K \subset Q$ there exists a $j > i$ and an embedding $h\colon K \longrightarrow M_j$ such that $h \restriction (K\cap M_i) = \mathrm{id}_{K\cap M_i}$ and $d(h,\mathrm{id}_K) < \varepsilon$. It is known that *every capset of* Q *is equivalent to* $B(Q) = \{x \in Q \mid \exists i \in \mathbb{N}\colon |x_i| = 1\}$ *the pseudo-boundary of* Q, *under an autohomeomorphism of* Q (cf. ANDERSON [5]). The complement of a capset is called a *pseudo-interior* of Q and is homeomorphic to ℓ_2, the separable Hilbert space (cf. ANDERSON [5]).

Recall that an mls $M \in \lambda X$ is said to be *defined* on $A \in 2^X$ if $M \cap A \in M$ for all $M \in M$. For any space X the space $\lambda_{comp}(X)$ is the subspace of λX consisting of those mls's which are defined on some compact subset of X (cf. VERBEEK [119] (cf. also 2.7.10).

3.5.2. LEMMA. *If* X *is locally compact and* σ*-compact then* $\lambda_{comp}(X)$ *is* σ*-compact.*

PROOF. Write $X = \bigcup_{n=1}^{\infty} X_n$, where $X_n \subset X_{n+1}$ $(n \in \mathbb{N})$ and each X_n is compact $(n \in \mathbb{N})$, while in addition each compact $C \subset X$ is contained in some X_n.

CLAIM. $\lambda_{comp}(X) = \bigcup_{n=1}^{\infty} \lambda X_n$.

(Notice that X, being Lindelöf, is normal and hence that for each $A \in 2^X$ the superextension λA can be embedded in a natural way in λX (cf. lemma 3.1.15)).

Indeed, choose $M \in \lambda_{comp}(X)$ and let $C \subset X$ be a compact defining set for M. Choose $n \in \mathbb{N}$ such that $C \subset X_n$. Then lemma 3.1.15 implies that $M \in \lambda X_n$. Therefore $M \in \bigcup_{n=1}^{\infty} \lambda X_n$.

On the other hand choose $M \in \bigcup_{n=1}^{\infty} \lambda X_n$. Let $n \in \mathbb{N}$ be such that $M \in \lambda X_n$.

It now is easily seen that X_n is a (compact) defining set for $\mathcal{M}$, i.e. $\mathcal{M} \in \lambda_{comp}(X)$. □

3.5.3. For any topological space X, define

$$\lambda_{cap}(X) := \{\mathcal{M} \in \lambda X \mid \mathcal{M} \text{ is defined on some } A \in 2^X\backslash\{X\}\}.$$

3.5.4. LEMMA. *If* X *is a compact metric space, then* $\lambda_{cap}(X)$ *is* σ*-compact. If moreover* X *is connected then* $\lambda_{cap}(X)$ *is a countable union of* Z*-sets in* λX.

PROOF. Let $\{B_n \mid n \in \mathbb{N}\}$ be a countable closed basis for X, such that $B_n \neq X$ for all $n \in \mathbb{N}$. With the same technique as in lemma 3.5.2 it now follows that

$$\lambda_{cap}(X) = \bigcup_{n=1}^{\infty} \lambda B_n,$$

showing that $\lambda_{cap}(X)$ is σ-compact.

If moreover X is connected then λB_n is a Z-set in λX for each $n \in \mathbb{N}$ (cf. theorem 3.1.17). Hence $\lambda_{cap}(X)$ is a countable union of Z-sets. □

In [71] KROONENBERG gave an alternative characterization of capsets in Q and we will use this characterization to show that $\lambda_{cap}(X)$ is a B(Q)-factor in case X is a metrizable continuum with a binary normal subbase.

3.5.5. LEMMA ([71]). *Suppose* M *is a* σ*-compact subset of* Q *such that*

(i) *for every* $\varepsilon > 0$, *there exists a map* $h: Q \to Q\backslash M$ *such that* $d(h, id_Q) < \varepsilon$;

(ii) M *contains a family of compact subsets* $M_1 \subset M_2 \subset \dots$ *such that each* M_i *is a copy of* Q *and* M_i *is a* Z*-set in* M_{i+1} $(i \in \mathbb{N})$, *and such that for each* $\varepsilon > 0$ *there exists an integer* $i \in \mathbb{N}$ *and a map* $h: Q \to M_i$ *such that* $d(h, id_Q) < \varepsilon$.

Then M *is a capset for* Q.

We need some simple results.

3.5.6. LEMMA. *Let* (X,d) *be compact metric and let* $f: X \to X$ *be continuous. Then* $d(f, id_X) = \bar{d}(\lambda(f), id_{\lambda X})$.

PROOF. Since $\lambda(f) : \lambda X \longrightarrow \lambda X$ is an extension of f and since $\underline{i}: X \hookrightarrow \lambda X$ is an isometry (cf. VERBEEK [119]) we find that

$$d(f,id_X) \leq \bar{d}(\lambda(f),id_{\lambda X}).$$

Assume that $d(f,id_X) < \bar{d}(\lambda(f),id_{\lambda X})$. Let $\varepsilon := d(f,id_X)$. Then there is an $\mathcal{M} \in \lambda X$ such that

$$\bar{d}(\mathcal{M},\lambda(f)(\mathcal{M})) > \varepsilon.$$

Choose $M \in \mathcal{M}$ such that $B_\varepsilon(M) \notin \lambda(f)(\mathcal{M})$ (cf. lemma 3.1.11). Also take $M_0 \in \mathcal{M}$ with $B_\varepsilon(M) \cap f[M_0] = \emptyset$ (cf. lemma 3.1.22). As $\mathcal{M}$ is a linked system there is an $x \in M \cap M_0$. Then $f(x) \in f[M_0]$ and consequently

$$d(x,f(x)) > \varepsilon,$$

which is a contradiction. □

3.5.7. THEOREM. *Let* (X,d) *be a non-degenerate metrizable continuum which admits a binary normal subbase. Then there is a sequence* $M_1 \subset M_2 \subset \ldots$ *of subcontinua of* X *such that:*

(i) M_i *is a proper subcontinuum of* M_{i+1} $(i \in \mathbb{N})$;

(ii) *for each* $\varepsilon > 0$ *there exists an* $i \in \mathbb{N}$ *and a retraction* $r: X \to M_i$ *such that* $d(r,id_X) < \varepsilon$.

PROOF. Let $\mathcal{S}$ be a binary normal subbase for X. Then $H(X,\mathcal{S})$, the hyperspace of $\mathcal{S}$-closed sets (cf. section 2.10), is a compact densely ordered (by inclusion) subspace of 2^X (cf. theorem 2.10.5 and theorem 1.5.22). Fix a point $p \in X$ and let J be a maximal chain in $H(X,\mathcal{S})$ containing $\{p\}$. Then J is homeomorphic to the closed unit interval [0,1] since 2^X is metrizable (cf. WARD [124]). Let

$$\{M_i\}_{i=1}^{\infty} \subset J\backslash\{X\}$$

be a sequence which converges to X and which is indexed in such a way that M_n is properly contained in M_k if and only if $n < k$. It is clear that this is possible.

We claim that the sequence $\{M_i\}_{i=1}^{\infty}$ defined above satisfies (i) and (ii). The claim that each M_i is a proper subcontinuum of X is trivial since each $\mathcal{S}$-closed subset $A \subset X$ is a retract of X (cf. corollary 1.5.12 (a)).

This proves (i).

To prove (ii) choose $\varepsilon > 0$. Let $F \subset X$ be a finite set, say $F = \{x_1,\ldots,x_n\}$ satisfying

$$X = \bigcup_{i=1}^{n} U_{\frac{1}{2}\varepsilon}(x_i).$$

Choose a finite refinement $\{A_1,\ldots,A_m\}$, consisting of S-closed sets with nonempty interior, of $\{U_{\frac{1}{2}\varepsilon}(x_i) \mid 1 \le i \le n\}$ (that this is possible is an easy consequence of the fact that S is a normal T_1-closed subbase for the compact space X). Let $\delta > 0$ be such that for each $i \le j \le m$ there is an $y_j \in A_j$ with

$$B_\delta(y_j) \subset A_j.$$

Choose $i \in \mathbb{N}$ such that $d_H(M_i,X) < \frac{1}{2}\delta$. Then M_i intersects all members from the covering $\{A_1,\ldots,A_m\}$. Now let $r\colon X \to M_i$ be the retraction of theorem 1.5.2, in formula

$$\{r(x)\} = \bigcap_{y \in M_i} I_S(x,y) \cap M_i.$$

We claim that r moves the points less than ε. Indeed, take $x \in X$. Choose $1 \le k \le n$ such that $x \in A_k$. Since A_k intersects M_i, there is a $z \in A_k \cap M_i$. Then

$$\{r(x)\} = \bigcap_{y \in M_i} I_S(x,y) \cap M_i \subset I_S(x,z) \cap M_i \subset A_k;$$

consequently x and $r(x)$ both belong to A_k. Since A_k is contained in $U_{\frac{1}{2}\varepsilon}(x_\ell)$ for some $1 \le \ell \le n$ we conclude that

$$d\big(x,r(x)\big) \le d(x,x_\ell) + d\big(x_\ell,r(x)\big) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This completes the proof of the theorem. $\square$

We now prove the main result in this section.

3.5.8. <u>THEOREM</u>. *Let* (X,d) *be a metrizable continuum. If there is a sequence* $M_1 \subset M_2 \subset \ldots$ *of subcontinua of* X *satisfying:*

(i) M_i *is a proper subcontinuum of* M_{i+1} $(i \in \mathbb{N})$;

(ii) *for each* $\varepsilon > 0$ *there is an* $i \in \mathbb{N}$ *and a map* $h\colon X \to M_i$ *with* $d(h,\mathrm{id}_X) < \varepsilon$, *then* $\lambda_{cap}(X) \times Q$ *is a capset for* $\lambda X \times Q$. *In particular,*

$\lambda_{cap}(X)$ *is a* $B(Q)$*-factor.*

PROOF. First notice that the spaces λX and λM_i ($i \in \mathbb{N}$) are AR's (cf. theorem 3.2.1) and hence that they are Q-factors (cf. EDWARDS [45]). Therefore

$$\lambda M_1 \times Q \subset \lambda M_2 \times Q \subset \ldots$$

is a sequence of Hilbert cubes. Moreover $\lambda M_i \times Q$ is a Z-set in $\lambda M_{i+1} \times Q$ ($i \in \mathbb{N}$) by theorem 3.1.17 (ii). Let ρ be a metric for X. Then ρ_0, defined by

$$\rho_0\big((x_0,y_0),(x_1,y_1)\big) := \max\{\bar{d}(x_0,x_1),\rho(y_0,y_1)\}$$

is a metric for $\lambda X \times Q$.

We claim that the family $\{\lambda M_i \times Q \mid i \in \mathbb{N}\}$ satisfies the conditions of lemma 3.5.5. To prove 3.5.5 (i) choose $\varepsilon > 0$. Also choose two disjoint finite $F_0, F_1 \subset X$ such that $d_H(F_i,X) < \frac{1}{2}\varepsilon$ ($i \in \{0,1\}$). By induction for each $i \in \mathbb{N}$ choose a point $p_i \in M_{i+1} \setminus M_i$ such that $P = \{p_i \mid i \in \mathbb{N}\}$ has a void intersection with $F_0 \cup F_1$. It is clear that this is possible.

Now define a linked system $\mathcal{M} = \{N_k \mid k > 1\}$ on P by

$$N_k := \{p_2,p_4,\ldots,p_k,p_{k+1}\} \qquad \text{if } k \text{ is even}$$

and

$$N_k := \{p_1,p_3,\ldots,p_k,p_{k+1}\} \qquad \text{if } k \text{ is odd.}$$

It is clear that $\mathcal{M}$ is a linked system and also that

$$N_k \cap N_{k+1} = \{p_{k+1}\}$$

for all $k > 1$. Define for all $k > 1$ sets G_k by

$$G_k := N_k \cup F_0 \qquad \text{if } k \text{ is even}$$

and

$$G_k := N_k \cup F_1 \qquad \text{if } k \text{ is odd.}$$

Then $\{G_k \mid k > 1\}$ is a linked system of finite subsets of X and hence there is a retraction $r\colon \lambda X \to \bigcap_{k>1} G_k^+$ defined by

$$\{r(\mathcal{L})\} := \bigcap\{L^+ \mid L \in \mathcal{L} \text{ and } L \cap G_k \neq \emptyset\ (k > 1)\} \cap \bigcap_{k>1} G_k^+$$

(cf. theorem 1.5.2 and theorem 3.1.13). Then

$$\bar{d}(r,\mathrm{id}_{\lambda X}) \le \sup_{k>1} d_H(G_k,X) \le \tfrac{1}{2}\varepsilon < \varepsilon$$

(cf. theorem 3.1.13) and moreover

$$r[\lambda X] \cap \bigcup_{i=1}^{\infty} \lambda M_i = \emptyset.$$

For choose $L \in r[\lambda X]$ and $k \in \mathbb{N}$. Then G_k and G_{k+1} both belong to L. Also $G_k \cap G_{k+1} \cap M_k = \{p_{k+1}\} \cap M_k = \emptyset$, since $P \cap (F_0 \cup F_1) = \emptyset$. Now, lemma 3.1.15 implies that $L \notin \lambda M_k$. This completes the proof of 3.5.5 (i), since

$$r \times \mathrm{id}_Q \colon \lambda X \times Q \longrightarrow \bigcap_{k>1} G_k^+ \times Q$$

clearly is a retraction which moves the points less than ε and whose image is disjoint from $\bigcup_{k=1}^{\infty}(\lambda M_k \times Q)$.

To prove 3.5.5 (ii) choose $\varepsilon > 0$. Then there is, by assumption, an $i \in \mathbb{N}$ and a map $h\colon X \longrightarrow M_i$ with $d(h,\mathrm{id}_X) < \varepsilon$. Then $\lambda(h)\colon \lambda X \longrightarrow \lambda M_i$ and also $\bar{d}(\lambda(h),\mathrm{id}_{\lambda X}) < \varepsilon$ by lemma 3.5.6. Therefore

$$\lambda(h) \times \mathrm{id}_Q \colon \lambda X \times Q \longrightarrow \lambda M_i \times Q,$$

is the desired mapping. □

3.5.9. <u>COROLLARY</u>. *Let* X *be a metrizable continuum with a binary normal subbase. Then* $\lambda_{\mathrm{cap}}(X) \times Q$ *is a capset for* $\lambda X \times Q$. *In particular,* $\lambda_{\mathrm{cap}}(X)$ *is a* $B(Q)$*-factor.*

<u>PROOF</u>. This follows from theorem 3.5.7 and theorem 3.5.8. □

3.5.10. The metrizable continua with a binary normal subbase are not the only compacta with a sequence of subcontinua as described in theorem 3.5.8, for it is easy to see that a space such as

$$Y = \{(0,y) \mid -1 \le y \le 1\} \cup \{(x,\sin\tfrac{1}{x}) \mid 0 < x \le 1\}$$

also has such a sequence (notice that Y does not possess a binary normal subbase since Y is not locally connected; cf. corollary 1.5.8 (iii)).

The technique used in the proof of theorem 3.5.8 can also be used to obtain the following results.

3.5.11. THEOREM.

(i) $\lambda_{comp}(0,1)$ *is a capset for* λI;

(ii) $\lambda_{cap}(I)$ *is a capset for* λI.

PROOF. Define $M_i = [0+\frac{1}{i}, 1-\frac{1}{i}]$ $(i > 2)$ and then use the same technique as in the proof of theorem 3.5.8 and the fact that $\lambda M_i \approx Q$ $(i > 2)$ (theorem 3.4.7). □

3.5.12. COROLLARY. $\lambda I \backslash \lambda_{cap}(I)$ *is homeomorphic to* ℓ_2. □

As noted in the introduction of this section theorem 3.5.11 (i) disproves a conjecture of VERBEEK [119].

We conjecture the following:

3.5.13. CONJECTURE. $\lambda_{cap}(X)$ *is homeomorphic to* $B(Q)$ *for any metrizable continuum with a binary normal subbase.*

In connection with this conjecture we also have the following question:

3.5.14. QUESTION. *Let* X *be the* 1-*sphere* S_1. *Is* $\lambda_{cap}(X)$ *homeomorphic to* $B(Q)$? *Or is it a capset of* λX *(if* $\lambda X \approx Q$)? *Is* $\lambda_{cap}(X) \times Q$ *a capset of* $\lambda X \times Q$?

3.6. Some subspaces of λX homeomorphic to the Hilbert cube

We show that in case λX is homeomorphic to the Hilbert cube the open basis $\{\bigcap_{i\le n} \mid X\backslash U_i \in 2^X;\ n \in \mathbb{N}\}$ of λX has the property that the closure of a nonvoid element of it is again homeomorphic to the Hilbert cube.

3.6.1. In this section we assume that (X,d) is a compact metric space such that the space λX is homeomorphic to the Hilbert cube. From results of VERBEEK [119] (cf. also corollary 2.5.4) it then follows that X is a non-degenerate metrizable continuum.

For simplicity of notation we will write A^- for the closure of the subset A of the topological space Y.

3.6.2. LEMMA. *Let* $\{U_1,\dots,U_n\}$ *be a finite linked system of open subsets in* X. *Then* $(\bigcap_{i\le n} U_i^+)^-$ *equals* $\bigcap_{i\le n} (U_i^-)^+$.

PROOF. Clearly $\bigcap_{i\le n} U_i^+ \subset \bigcap_{i\le n}(U_i^-)^+$ so that in any case

$$\left(\bigcap_{i\le n} U_i^+\right)^- \subset \bigcap_{i\le n}\left(U_i^-\right)^+.$$

Choose a point $M \in \bigcap_{i\le n}(U_i^-)^+ \setminus (\bigcap_{i\le n} U_i^+)^-$. Choose finitely many open sets O_j $(j \le m)$ in X such that $M \in \bigcap_{j\le m} O_j^+$ and

$$\bigcap_{j\le m} O_j^+ \cap \bigcap_{i\le n} U_i^+ = \emptyset.$$

Then $\{O_j \mid j \le m\} \cup \{U_i \mid i \le n\}$ is not a linked system for otherwise $\bigcap_{j\le m} O_j^+ \cap \bigcap_{i\le n} U_i^+ \ne \emptyset$ (see VERBEEK [119]). Hence, since clearly $\{O_j \mid j \le m\}$ is linked, there are $j_0 \le m$ and $i_0 \le n$ such that

$$O_{j_0} \cap U_{i_0} = \emptyset.$$

Then $O_{j_0} \cap U_{i_0}^- = \emptyset$ and consequently $O_{j_0}^+ \cap (U_{i_0}^-)^+ = \emptyset$. This is a contradiction, since $M \in O_{j_0}^+ \cap (U_{i_0}^-)^+$. □

3.6.3. COROLLARY. *Let* $\{U_1,\ldots,U_n\}$ *be a finite linked system of open sets in* X. *Then* $(\bigcap_{i\le n} U_i^+)^-$ *is a Hilbert cube factor.*

PROOF. By lemma 3.6.2 $(\bigcap_{i\le n} U_i^+)^-$ is a retract of λX (cf. theorem 3.1.13) and consequently it is an AR. Now the EDWARDS [45] theorem gives the desired result. □

As in section 2.7 the subspace $\{\bigcap\{M^+ \mid M \in \mathcal{M}\} \mid \mathcal{M} \subset 2^X$ is a linked system of $2^{\lambda X}\}$ will be denoted by $K(\lambda X)$. An element $S \in K(\lambda X)$ is called *convex* for short (theorem 3.1.13 motivates this terminology). We need a simple lemma.

3.6.4. LEMMA. *Let* $S_1,\ldots,S_n$ *be a finite collection of convex sets in* λX *such that* $\bigcap_{i\le n} S_i \ne \emptyset$. *Then* $\bigcup_{i\le n} S_i$ *is an* AR.

PROOF. We will prove the lemma by induction on n. The lemma is true for $n = 1$ (cf. theorem 3.1.13).

Suppose that the lemma is true for unions of $n-1$ convex sets. Let $S_i \subset \lambda X$ $(i \le n)$ be convex such that $\bigcap_{i\le n} S_i \ne \emptyset$. Write $\bigcup_{i\le n} S_i = (\bigcup_{i\le n-1} S_i) \cup S_n$. Then $\bigcup_{i\le n-1} S_i$ is an AR by induction hypothesis. Also S_n is an AR. As $(\bigcup_{i\le n-1} S_i) \cap S_n = \bigcup_{i\le n-1} (S_i \cap S_n)$ and as the intersection of two convex sets again is convex, the intersection $(\bigcup_{i\le n-1} S_i) \cap S_n$ also is an AR by induction hypothesis. But then $\bigcup_{i\le n} S_i$ is the union of two AR's the intersection of which also is an AR. By a theorem of BORSUK [20] it now follows that $\bigcup_{i\le n} S_i$ is an AR too. □

We need the following compactification result of WEST [127].

3.6.5. THEOREM. *Suppose that* X *is a compactification of a* Q*-manifold* M *such that*

(i) X *is a* Q*-factor;*

(ii) $X\backslash M$ *is a* Q*-factor;*

(iii) $X\backslash M$ *is a* Z*-set in* X.

Then X *is a Hilbert cube.*

This theorem is the basic tool in proving the main result in this section.

3.6.6. THEOREM. *Let* (X,d) *be a compact metric for which* λX *is homeomorphic to the Hilbert cube* Q. *Then for each finite linked system* $\{U_1,\dots,U_n\}$ *of open sets in* X *the closure (in* λX*) of* $\bigcap_{i\le n} U_i^+$ *is homeomorphic to the Hilbert cube.*

PROOF. Let $\{U_1,\dots,U_n\}$ be a finite linked system of open sets in X. Fix a point $p \in X$ and define $V_i := U_i\backslash\{p\}$ $(i \le n)$. Then, since X is connected $\{V_1,\dots,V_n\}$ is again a linked system. Hence

$$\left(\bigcap_{i\le n} V_i^+\right)^- = \left(\bigcap_{i\le n} U_i^+\right)^-,$$

since $(\bigcap_{i\le n} V_i^+)^- = \bigcap_{i\le n} (V_i^-)^+ = \bigcap_{i\le n} (U_i^-)^+ = (\bigcap_{i\le n} U_i^+)^-$ by lemma 3.6.2. We will show that $(\bigcap_{i\le n} V_i^+)^-$ is a Hilbert cube. Without loss of generality we may assume that $V_i^- \not\subset V_j^-$ for all $i,j \le n$. Define

$$A := \left(\bigcap_{i\le n} V_i^+\right)^- \setminus \bigcap_{i\le n} V_i^+$$

CLAIM 1. A *equals* $\bigcup_{j\le n}\left((X\backslash V_j^+) \cap (\bigcap_{i\le n} V_i^+)^-\right)$.

Indeed, assume that $M \in A$. Then $M \notin \bigcap_{i\le n} V_i^+$ and hence there is an $i_0 \le n$ such that $M \notin V_{i_0}^+$. But then $M \in (X\backslash V_{i_0})^+$ (cf. proposition 2.2.3 (v)). Consequently $M \in \bigcup_{j\le n} ((X\backslash V_j)^+ \cap (\bigcap_{i\le n} V_i^+)^-)$.

On the other hand, if $M \in (X\backslash V_{j_0})^+ \cap (\bigcap_{i\le n} V_i^+)^-$ for some $j_0 \le n$

then clearly $M \notin V_{j_0}^+$ and consequently $M \in A$.

CLAIM 2. $\bigcap_{j\le n} \left((X\backslash V_j)^+ \cap (\bigcap_{i\le n} V_i^+)^-\right) \ne \emptyset$.

The linked system $\{X\backslash V_j \mid j \le n\} \cup \{V_i^- \mid i \le n\}$ is contained in at least one maximal linked system

$$M \in \bigcap_{j\le n} \left((X\backslash V_j)^+ \cap \bigcap_{i\le n} (V_i^-)^+\right).$$

Now lemma 3.6.2 establishes claim 2.

Lemma 3.6.2 also implies that A is a finite union of convex sets; hence, by claim 2 and by lemma 3.6.4, A is an AR.

CLAIM 3. *A is a Z-set in* $(\bigcap_{i\le n} V_i^+)^-$.

For each $i,j \le n$ choose a point $p_{ij} = p_{ji} \in V_i \cap V_j$. Define $P_i := \{p_{ij} \mid j \le n\}$. Then $\{P_i \mid i \le n\}$ is a finite linked system of finite subsets of X such that $P_i \subset V_i$ for all $i \le n$.

Fix $\varepsilon > 0$ and for each $i \le n$ choose a finite $F_i \subset V_i$ such that $d_H(F_i, V_i^-) < \frac{1}{2}\varepsilon$. Define $L_i := F_i \cup P_i$ $(i \le n)$. Let

$$r\colon \lambda X \longrightarrow \bigcap_{i\le n} L^+$$

be the retraction onto $\bigcap_{i\le n} L_i^+$ of theorem 3.1.13. Let f_ε be the restriction of r to $(\bigcap_{i\le n} V_i^+)^-$. Notice that $f_\varepsilon[(\bigcap_{i\le n} V_i^+)^-] \subset \bigcap_{i\le n} V_i^+$. We will show that f_ε moves the points less than ε. Indeed, choose $M \in (\bigcap_{i\le n} V_i^+)^-$. Then

$$PM = \{M \in M \mid \forall i \le n\colon M \cap L_i \ne \emptyset\} \cup \{L_i \mid i \le n\}$$

is a pre-mls for $r(M) = f_\varepsilon(M)$ (cf. the proof of theorem 3.1.13; see also theorem 1.5.2). Also

$$\bar{d}(M, f_\varepsilon(M)) = \min\{a \ge 0 \mid \forall S \in PM\colon B_a(S) \in M\}$$

(cf. lemma 3.1.11). Therefore $\bar{d}(M, f_\varepsilon(M)) < \varepsilon$. Indeed, choose $S \in PM$: if $S \in M$ then also $B_{\frac{1}{2}\varepsilon}(S) \in M$ since $S \subset B_{\frac{1}{2}\varepsilon}(S)$; if $S \in \{L_i \mid i \le n\}$, say $S = L_{i_0}$, then $V_{i_0}^- \subset B_{\frac{1}{2}\varepsilon}(S)$ and consequently $B_{\frac{1}{2}\varepsilon}(S) \in M$ since $V_{i_0}^- \in M$ by lemma 3.6.2. This yields in any case

$$\bar{d}(M, f_\varepsilon(M)) \le \tfrac{1}{2}\varepsilon < \varepsilon.$$

By corollary 3.6.3 $(\bigcap_{i\leq n} V_i^+)^-$ is a Hilbert cube factor which is a compactification of the Q-manifold $\bigcap_{i\leq n} V_i^+$ such that the remainder A is an AR (and hence a Q-factor) which is a Z-set in $(\bigcap_{i\leq n} V_i^+)^-$ (claim 2 and claim 3). Therefore $(\bigcap_{i\leq n} V_i^+)^- \approx Q$ by theorem 3.6.5. □

3.7. Notes

The techniques derived in the previous chapter to show that the superextension of the closed unit interval is homeomorphic to the Hilbert cube are not applicable to show that the superextension of any non-degenerate metrizable continuum is homeomorphic to the Hilbert cube. We can show that the superextension of any finite tree is the Hilbert cube and, more generally, using the approximation result in section 3.2, that the superextension of any dendron is homeomorphic to the Hilbert cube (it is easily seen that any dendron is the inverse limit of a sequence of finite trees with elementary collapses as bonding maps). Also, if X is the topological sum of finitely many dendra, then λX is a Q-manifold; in fact it is a topological sum of finitely many Hilbert cubes.

Recently we have shown that the superextension of any finite connected graph is the Hilbert cube. Unfortunately this result could not be included in the previous chapter.

Theorem 3.1.19 is taken from VAN MILL & SCHRIJVER [80].

CHAPTER IV

COMPACTIFICATION THEORY

In this chapter we deal with the following two questions:

a) *Is every Hausdorff compactification of a Tychonoff space a Wallman compactification?*

b) *Is every Hausdorff compactification of a Tychonoff space a* GA *compactification?*

Question a) was posed by FRINK [51], who used Wallman-type compactifications (cf. also SHANIN [106a]) to obtain an internal characterization of complete regularity. It is unsolved until now, although many partial results suggest that the question can be answered affirmatively (cf. AARTS [1], STEINER & STEINER [109],[111],[112],[113], HAMBURGER [62], MISRA [85], NJÅSTAD [89], VAN MILL [77]). *)

DE GROOT & AARTS [57] generalized FRINK's technique and considerably strengthened his characterization of complete regularity. They also used a compactification method, which is related to the Wallman compactification technique and which is now known as the "GA *compactification method*" (cf. HURSCH [65], DE GROOT, HURSCH & JENSEN [58]). A.B. PAALMAN-DE MIRANDA posed question b) (cf. VERBEEK [119] question V.3.9). It remains as yet unsolved (however, see 4.7).

In 4.1 we will derive some preliminary results on Wallman compactifications, results which are all known but which are included for completeness sake. The next section contains the main result of this chapter; we show that *every Hausdorff compactification of a Tychonoff space in which the collection of multiple points is Lindelöf semi-stratifiable is a z-compactification* (a compactification obtainable as the ultrafilter space of a normal base consisting of zero-sets). Sections 4.3, 4.4 and the last part of section 4.2 deal with regular Wallman spaces. Among other things we show that *every Hausdorff compactification of a locally compact metrizable*

*) There is a rumour going that Uljanov and Shapiro have constructed a counterexample.

space with zero-dimensional remainder is regular Wallman (cf. also BAAYEN & VAN MILL [11]). Closely related to regular Wallman spaces are regular supercompact superextensions; they are considered in section 4.5.

The sections 4.6 and 4.7 deal with GA compactifications. We will characterize the class of GA compactifications of a given topological space and from an analogous characterization of Wallman compactifications (cf. STEINER [114]) it follows that any Wallman compactification is a GA compactification. This implies that the questions a) and b) are related. Finally we show, using the characterization announced above, that *any compact Hausdorff space of weight at most* $\mathfrak{c}$ *is a* GA *compactification of each dense subspace*.

4.1. Wallman compactifications; some preliminaries

This section contains some preliminary results concerning Wallman compactifications. Most of the results are taken from STEINER [114].

4.1.1. Let $\mathcal{S}$ be a T_1-subbase (cf. definition 2.2.1) for the topological space X. Define

$$\omega(X,\mathcal{S}) := \{\mathcal{A} \subset \mathcal{S} \mid \mathcal{A} \text{ is maximally centered}\}.$$

For each $S \in \mathcal{S}$ define

$$S^* := \{\mathcal{A} \in \omega(X,\mathcal{S}) \mid S \in \mathcal{A}\}$$

and define a topology on $\omega(X,\mathcal{S})$ by taking

$$\mathcal{S}^* := \{S^* \mid S \in \mathcal{S}\}$$

as a closed subbase. With this topology $\omega(X,\mathcal{S})$ is called the *Wallman compactification of* X *relative* $\mathcal{S}$. If $\mathcal{S}$ is the collection of all closed sets in X then $\omega(X,\mathcal{S})$ is denoted by ωX and is called *the* Wallman compactification of X (cf. WALLMAN [121]). That $\omega(X,\mathcal{S})$ is a compactification of X is shown in STEINER [114]. We mention the following result (recall that $\wedge.\vee.\mathcal{S}$ is the ring generated by $\mathcal{S}$, cf. 0.A):

4.1.2. <u>THEOREM</u>. *Let* $\mathcal{S}$ *be a* T_1*-subbase for the topological space* X. *Then* $\omega(X,\mathcal{S})$ *is compact and is homeomorphic to* $\omega(X,\wedge.\vee.\mathcal{S})$. *Moreover the mapping*

$\underline{i}$: $X \longrightarrow \omega(X,\mathcal{S})$ *defined by* $\underline{i}(x) := \{S \in \mathcal{S} \mid x \in S\}$ *is an embedding.* □

4.1.3. In case the subbase $\mathcal{S}$ is a separating ring (cf. 0.A) and is normal (cf. 2.2.1) it is called a *normal base*. Notice that a base may very well be a normal subbase without being a normal base. The best known example of a normal base is the ring of zero-sets $Z(X)$ of a Tychonoff space X.

The following result is also taken from STEINER [114].

4.1.4. THEOREM. *Let* $\mathcal{S}$ *be a* T_1*-subbase for* X. *Then* $\omega(X,\mathcal{S})$ *is Hausdorff if* $\mathcal{S}$ *is normal. Moreover* $\omega(X,\mathcal{S})$ *is Hausdorff if and only if* $\wedge.\vee.\mathcal{S}$ *is a normal base.* □

4.1.5. A compactification αX of a topological space X is called a *Wallman compactification* if it is equivalent to a compactification $\omega(X,\mathcal{S})$ for some T_1-subbase $\mathcal{S}$ for X.

Let X be a space and let Y be a subspace of X. A family $\mathcal{T}$ of closed subsets of X has the *trace property with respect to* Y (cf. STEINER [114]) provided that for any finite $\mathcal{F} \subset \mathcal{T}$ with $\cap\mathcal{F} \neq \emptyset$ also $\cap\mathcal{F} \cap Y \neq \emptyset$. STEINER [114] gives the following useful characterization of Wallman compactifications.

4.1.6. THEOREM. *A compactification* αX *of* X *is a Wallman compactification if and only if* αX *possesses a separating family of closed sets with the trace property with respect to* X. □

Many compactifications are Wallman compactifications, for example, this is true for all metric compactifications (cf. AARTS [1] and STEINER & STEINER [109]).

4.1.7. In the above characterization of Wallman compactifications the separating family $\mathcal{F}$ of closed sets in αX with the trace property with respect to X can be chosen in such a way that $\{F \cap X \mid F \in \mathcal{F}\} \subset Z(X)$ then we say that αX is a *z-compactification*. Many compactifications are z-compactifications, cf. STEINER & STEINER [112] and HAMBURGER [62].

4.1.8. Let αX be a compactification of X and let ξ denote the unique projection mapping of βX, the Čech-Stone compactification of X, onto αX which on X is the identity. We say that a point $p \in \alpha X \backslash X$ is a *multiple point* of αX (cf. NJÅSTAD [89]) if $\xi^{-1}(p)$ consists of more than one point.

Every compactification in which the set of multiple points is countable (this is usually called a *countable multiple point compactification*) is a z-compactification (cf. STEINER & STEINER [112]). This result is strengthened in section 4.2.

4.1.9. A compact topological space X is called *regular Wallman* if it possesses a separating ring consisting of regular closed sets (cf. STEINER [114]). From theorem 4.1.6 it follows that a regular Wallman space is Wallman compactification of each dense subspace. Many compact Hausdorff spaces are regular Wallman, for example all compact metric spaces (cf. STEINER & STEINER [109]). The first example of a compact Hausdorff space which is not regular Wallman was obtained by SOLOMON [107].

4.1.10. Let $\kappa > \omega$ be any uncountable cardinal. A topological space X is called *strongly κ compact* if for each subset A of X with $|A| \geq \kappa$ and for each total order < on A there exists a $y \in A$ such that for each open neighborhood U of y both $U \cap \{x \in A \mid x < y\}$ and $U \cap \{x \in A \mid x > y\}$ are nonvoid. It is very easy to show that a space of weight κ is strongly κ^+ compact. Hence each separable metric space is strongly ω_1 compact.

The following theorem is due to BERNEY [15]. For completeness sake we will include its proof.

4.1.11. THEOREM. *A strongly ω_1 compact space is hereditarily strongly ω_1 compact. Moreover it is hereditarily separable and hereditarily Lindelöf.*

PROOF. Let X be a strongly ω_1 compact space. That X is hereditarily strongly ω_1 compact is trivial. Hence we only need to show that X is both separable and Lindelöf.

If X is not separable then there is a sequence $P = \{x_\alpha \mid \alpha \in \omega_1\}$ of elements of X such that for each $\alpha \in \omega_1$ the point x_α is not in the closure of $\{x_\beta \mid \beta < \alpha\}$. Choose $\alpha_0 \in \omega_1$ such that x_{α_0} is limit point from below of P. But x_{α_0} is not in the closure of $\{x_\beta \mid \beta < \alpha_0\}$, which is a contradiction.

If X is not Lindelöf then there is a sequence $\mathcal{U} = \{U_\alpha \mid \alpha \in \omega_1\}$ of open subsets of X such that for all $\alpha \in \omega_1$

$$U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$$

is nonvoid. For each $\alpha \in \omega_1$ choose $x_\alpha \in U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$ and define

$$P := \{x_\alpha \mid \alpha \in \omega_1\}.$$

Choose $\alpha_0 \in \omega_1$ such that $U_{\alpha_0} \cap \{x_\beta \mid \alpha_0 < \beta\} \neq \emptyset$. Then there is a $\beta_0 \in \omega_1$ such that $\alpha_0 < \beta_0$ and $x_{\beta_0} \in U_{\alpha_0}$. This is a contradiction. □

4.1.12. A topological space X is called *semi-stratifiable* if to each open subset U of X, one can assign a sequence $\{U_n\}_{n=1}^\infty$ of closed subsets of X such that

(a) $\bigcup_{n=1}^\infty U_n = U$;

(b) if $U \subset V$ and $\{V_n\}_{n=1}^\infty$ is the sequence assigned to V, then $U_n \subset V_n$ for all $n \in \mathbb{N}$.

It is easily seen that each metric space (X,d) is semi-stratifiable; for each open subset $V \subset X$ and each $n \in \mathbb{N}$ let V_n be defined by

$$V_n := V \backslash U_{\frac{1}{n}}(X \backslash V).$$

In [33] CREEDE showed that each Lindelöf semi-stratifiable space is strongly ω_1 compact. Hence we have the following implications:

separable metric ⇒ Lindelöf semi-stratifiable ⇒ strongly ω_1 compact ⇒

⇒ hereditarily separable and hereditarily Lindelöf.

Since CREEDE's theorem is very important for us, cf. section 4.2, we will include a proof of it. The proof presented here was suggested to me by J.M. VAN WOUWE.

4.1.13. THEOREM. *A Lindelöf semi-stratifiable space is strongly* ω_1 *compact.*

PROOF. Let X be a Lindelöf semi-stratifiable space and assume there exist a totally ordered subset A of X such that $|A| \geq \omega_1$ and such that for each $x \in A$ there exists an open neighborhood U_x such that either $U_x \cap \{a \in A \mid a < x\} = \emptyset$ or $U_x \cap \{a \in A \mid a > x\} = \emptyset$. Since $|A| \geq \omega_1$, we may assume, without loss of generality, that for each $x \in A$ we have

$$U_x \cap \{a \in A \mid a > x\} = \emptyset.$$

For each $x \in A$ and for each $n \in \mathbb{N}$ define

$$O_x^n := U_x \setminus (\bigcup_{a<x} U_a)_n.$$

It is clear that O_x^n is an open neighborhood of x such that $O_x^n \cap \{a \in A \mid a > x\} = \emptyset$ for all $n \in \mathbb{N}$. Since a Lindelöf semi-stratifiable space is hereditarily Lindelöf (cf. CREEDE [34]), for each $n \in \mathbb{N}$ the covering

$$\mathcal{O}_n = \{O_x^n \mid x \in A\}$$

of A has a countable subcover $\{O^n_{x(i,n)} \mid i \in \mathbb{N}\}$. Define

$$F := \{x(i,n) \mid n \in \mathbb{N},\ i \in \mathbb{N}\}.$$

As F is countable there is an $a^* \in A\backslash F$. Since $a^* \in \bigcup_{b \leq a^*} U_b$, there is an $n_0 \in \mathbb{N}$ such that

$$a^* \in \left(\bigcup_{b \leq a^*} U_b\right)_{n_0}.$$

Consider the sequence $\{x(i,n_0) \mid i \in \mathbb{N}\}$. Since $a^* \in A\backslash F$ it follows that $a^* \neq x(i,n_0)$ for all $i \in \mathbb{N}$. Now fix $i \in \mathbb{N}$. If $x(i,n_0) < a^*$ then

$$a^* \notin O^{n_0}_{x(i,n_0)}.$$

If $x(i,n_0) > a^*$, then

$$\begin{aligned} O^{n_0}_{x(i,n_0)} &= U^{n_0}_{x(i,n_0)} \Big\backslash \left(\bigcup_{b<x(i,n_0)} U_b\right)_{n_0} \\ &\subset U^{n_0}_{x(i,n_0)} \Big\backslash \left(\bigcup_{b \leq a^*} U_b\right)_{n_0}. \end{aligned}$$

Hence it again follows that $a^* \notin O^{n_0}_{x(i,n_0)}$.

It now follows that $\mathcal{O}_{n_0}$ is not a covering of A, which is a contradiction. □

4.1.14. E.S. BERNEY [16] has introduced the concept of strongly κ compactness in the theory of Wallman compactifications. His techniques turn out to be very powerful and will be used in section 4.2 and section 4.7.

4.2. Compactifications in which the collection of multiple points is Lindelöf semi-stratifiable

In this section we show that any compactification αX of a Tychonoff space X in which the collection of multiple points is Lindelöf semi-stratifiable is a z-compactification. If in addition X is also Lindelöf semi-stratifiable then αX is regular Wallman. In particular, βX is regular Wallman if X is Lindelöf semi-stratifiable (cf. BERNEY [16]).

4.2.1. In this section we assume that αX is a compactification of the Tychonoff space X. The set of multiple points of αX is denoted by M.

We start with some simple results.

4.2.2. LEMMA. *Let Y be a subspace of βX such that $X \subset Y \subset \beta X$. If $Z_0, Z_1 \in Z(X)$ then $\mathrm{cl}_Y(Z_0) \cap \mathrm{cl}_Y(Z_1) = \mathrm{cl}_Y(Z_0 \cap Z_1)$.*

PROOF. $\mathrm{cl}_{\beta X}(Z_0) \cap \mathrm{cl}_{\beta X}(Z_1) = \mathrm{cl}_{\beta X}(Z_0 \cap Z_1)$. $\square$

Let $\xi\colon \beta X \longrightarrow \alpha X$ be the unique projection which extends id_X.

4.2.3. LEMMA. *Let $Z \in Z(X)$. If $\partial\mathrm{cl}_{\alpha X}(Z) \cap M = \emptyset$, then $\xi^{-1}[\mathrm{cl}_{\alpha X}(Z)] = \mathrm{cl}_{\beta X}(Z)$.*

PROOF. Assume there exists an

$$x_0 \in \xi^{-1}[\mathrm{cl}_{\alpha X}(Z)] \backslash \mathrm{cl}_{\beta X}(Z).$$

Then $\xi(x_0) \in \mathrm{cl}_{\alpha X}(Z) \cap M$ and consequently $\xi(x_0) \in \mathrm{int}_{\alpha X}\, \mathrm{cl}_{\alpha X}(Z)$ since $\partial\mathrm{cl}_{\alpha X}(Z) \cap M = \emptyset$. Therefore

$$x_0 \in \xi^{-1}[\mathrm{int}_{\alpha X}\, \mathrm{cl}_{\alpha Z}(Z)] \subset \mathrm{int}_{\beta X}\, \xi^{-1}[\mathrm{cl}_{\alpha X}(Z)].$$

Let O be any open neighborhood of x_0 in βX. Then clearly

$$O \cap \mathrm{int}_{\beta X}\, \xi^{-1}[\mathrm{cl}_{\alpha X}(Z)] \cap X \neq \emptyset.$$

As ξ is the identity on X it follows that $O \cap Z \neq \emptyset$. We conclude that $x_0 \in \mathrm{cl}_{\beta X}(Z)$. This is a contradiction. $\square$

If $f \in C(\alpha X, I)$ then we will write $U(\delta, f)$ in stead of $f^{-1}[0,\delta)$ $(\delta \in (0,1])$.

4.2.4. LEMMA. *Let* $f \in C(\alpha X, I)$ *and assume that* M *is strongly* ω_1 *compact. Then* $\{\delta \in (0,1) \mid cl_{\alpha X}(f^{-1}[0,\delta] \cap X) \cap M \neq cl_{\alpha X}(U(\delta,f)) \cap M\}$ *is countable.*

PROOF. Assume to the contrary that it were uncountable. If for some $\delta \in (0,1)$ we have that $cl_{\alpha X}(f^{-1}[0,\delta] \cap X) \cap M \neq cl_{\alpha X}(U(\delta,f)) \cap M$ then there is an

$$a(\delta) \in \big(cl_{\alpha X}(f^{-1}[0,\delta] \cap X) \setminus cl_{\alpha X}(U(\delta,f))\big) \cap M.$$

Let B be the set of $a(\delta)$ chosen in this way. Since $f(a(\delta)) = \delta$ for all $a(\delta) \in B$ it follows that $\delta_1 \neq \delta_2$ implies that $a(\delta_1) \neq a(\delta_2)$ and therefore B is uncountable. Also, a total order $<$ is defined on B by putting

$$a(\delta_0) < a(\delta_1) \iff \delta_0 < \delta_1.$$

Since $B \subset M$ and since M is strongly ω_1 compact it follows that B has a limit point $a(\delta_0)$ from below.

Let U be any open neighborhood of $a(\delta_0)$. Since $a(\delta_0)$ is limit point from below there is an $a(\delta_1) \in U \cap B$ such that $a(\delta_1) < a(\delta_0)$. This shows that $a(\delta_1) \in U(\delta_0,f) \cap U$ and in particular $U(\delta_0,f) \cap U \neq \emptyset$. Hence $a(\delta_0) \in cl_{\alpha X}(U(\delta_0,f)) \cap M$, which is a contradiction. □

The following lemma is due to BERNEY [16]; for completeness sake we will include its proof.

4.2.5. LEMMA. *Let* $f \in C(\alpha X, I)$ *and let* U *be open in* αX. *If* $A \subset \alpha X$ *is strongly* ω_1 *compact, then*

$$\{\delta \in (0,1) \mid cl_{\alpha X}(U) \cap cl_{\alpha X}(U(\delta,f)) \cap A \neq cl_{\alpha X}(U \cap U(\delta,f)) \cap A\}$$

is countable.

PROOF. Assume that it were uncountable. If for some $\delta \in (0,1)$ we have that

$$cl_{\alpha X}(U) \cap cl_{\alpha X}(U(\delta,f)) \cap A \neq cl_{\alpha X}(U \cap U(\delta,f)) \cap A$$

then there is an $a(\delta) \in \big((cl_{\alpha X}(U) \cap cl_{\alpha X}(U(\delta,f))) \setminus cl_{\alpha X}(U \cap U(\delta,f))\big) \cap A$. Let B be the set of $a(\delta)$ chosen in this way. Clearly $f(a(\delta)) = \delta$ for all $a(\delta) \in B$ which implies that B is uncountable and also that the order $<$ on B defined by

$$a(\delta_0) < a(\delta_1) \iff \delta_0 < \delta_1$$

is a total ordering. Since $B \subset A$ and since A is strongly ω_1 compact, there is an $a(\delta_0)$ in B which is a limit point from below. Let O be any open neighborhood of $a(\delta_0)$. Then there is an $a(\delta_1) \in O$ with $\delta_1 < \delta_0$. Then $a(\delta_1) \in U(\delta_0,f) \cap O$ and consequently $\emptyset \neq U(\delta_0,f) \cap O \cap U = O \cap (U(\delta_0,f) \cap U)$, since $a(\delta_1) \in cl_{\alpha X}(U)$. It now follows that $a(\delta_0) \in cl_{\alpha X}(U \cap U(\delta_0,f)) \cap A$, which is a contradiction. □

We now can prove the main result in this chapter.

4.2.6. THEOREM. *Any compactification of a topological space X in which the collection of multiple points is strongly ω_1 compact is a z-compactification.*

4.2.7. COROLLARY. *Any compactification of a topological space X in which the collection of multiple points is Lindelöf semi-stratifiable is a z-compactification.*

PROOF. Let M^* denote the closure of M in αX. Then M^* is a compactification of M and since M is separable we have that the weight of M^* is at most c. Let $\mathcal{B}$ be an open base for the topology of M^* which is closed under finite intersections and finite unions and which contains at most c members. Define

$$\mathcal{C} := \{cl_{\alpha X}(B_0),\ cl_{\alpha X}(B_1)) \mid B_0,B_1 \in \mathcal{B} \text{ and } cl_{\alpha X}(B_0) \cap cl_{\alpha X}(B_1) = \emptyset\}.$$

For each $(cl_{\alpha X}(B_0), cl_{\alpha X}(B_1)) \in \mathcal{C}$, choose an $f \in C(\alpha X,I)$ such that $f[cl_{\alpha X}(B_0)] = 0$ and $f[cl_{\alpha X}(B_1)] = 1$. Let $\mathcal{F}$ denote the set of mappings obtained in this way. Write $\mathcal{F} = \{f_\kappa \mid \kappa \in c\}$.

By transfinite induction we will construct for each $\kappa \in c$ a $\delta_\kappa \in (0,1)$ such that

(i) $cl_{\alpha X}(f_\kappa^{-1}[0,\delta_\kappa) \cap X) \cap M = cl_{\alpha X}(U(\delta_\kappa,f_\kappa)) \cap M$;

(ii) $cl_{\alpha X}(U(\delta_\kappa,f_\kappa)) \cap cl_{\alpha X}(V) \cap M = cl_{\alpha X}(U(\delta_\kappa,f_\kappa) \cap V) \cap M$,

for all $V \in \wedge.\vee.\{U(\delta_\beta,f_\beta) \mid \beta < \kappa\}$.

Let $\kappa \in c$ and assume that δ_β is defined for all $\beta < \kappa$. If $\kappa = 0$ then choose $\delta \in (0,1)$ such that

$$cl_{\alpha X}(f_\kappa^{-1}[0,\delta] \cap X) \cap M = cl_{\alpha X}(U(\delta,f_\kappa)) \cap M.$$

Such a choice for δ is possible (cf. lemma 4.2.4). Define $\delta_\kappa := \delta$. If $\kappa \neq 0$, let $\mathcal{V} := \wedge.\vee.\{U(\delta_\beta,f_\beta) \mid \beta < \kappa\}$. Then if $V \in \mathcal{V}$ we have that

$$\left|\{\delta \in (0,1) \mid cl_{\alpha X}(U(\delta,f_\kappa)) \cap cl_{\alpha X}(V) \cap M \neq cl_{\alpha X}(U(\delta,f_\kappa) \cap V) \cap M\}\right| \leq \omega,$$

by lemma 4.2.5 and consequently

$$\left|\bigcup_{V \in \mathcal{V}}\{\delta \in (0,1) \mid cl_{\alpha X}(U(\delta,f_\kappa)) \cap cl_{\alpha X}(V) \cap M \neq cl_{\alpha X}(U(\delta,f_\kappa) \cap V) \cap M\}\right| < \mathfrak{c}.$$

From lemma 4.2.4 it now follows that there exists a $\delta \in (0,1)$ such that for all $V \in \mathcal{V}$ we have that $cl_{\alpha X}(U(\delta,f_\kappa)) \cap cl_{\alpha X}(V) \cap M = cl_{\alpha X}(U(\delta,f_\kappa) \cap V) \cap M$ and also that $cl_{\alpha X}(f_\kappa^{-1}[0,\delta] \cap X) \cap M = cl_{\alpha X}(U(\delta,f_\kappa)) \cap M$. Now define $\delta_\kappa := \delta$. This completes the inductive construction.

Now, for each $\alpha \in \mathfrak{c}$ define $H_\alpha := f_\alpha^{-1}[0,\delta_\alpha] \cap X$. Notice that $H_\alpha \in Z(X)$ for all $\alpha \in \mathfrak{c}$. Finally define $\mathcal{H} := \{H_\alpha \mid \alpha \in \mathfrak{c}\}$ and

$$\mathcal{L} := \{Z \in Z(X) \mid cl_{\alpha X}(Z) \cap M^* = \emptyset \text{ or } M^* \subset int_{\alpha X}\, cl_{\alpha X}(Z)\} \cup \mathcal{H}.$$

Using the compactness of αX it is easy to show that

$$\wedge.\vee.\{cl_{\alpha X}(L) \mid L \in \mathcal{L}\}$$

is a separating ring. We will show that for each finite number of elements $L_0,L_1,\ldots,L_n \in \mathcal{L}$ the equality

$$(*) \qquad cl_{\alpha X}\left(\bigcap_{i\leq n} L_i\right) = \bigcap_{i\leq n} cl_{\alpha X}(L_i)$$

holds, which then proves the theorem (cf. theorem 4.1.6).

If $L_i \notin \mathcal{H}$ $(i \leq n)$ then apply lemma 4.2.3 and use the analogous equality

$$(**) \qquad cl_{\beta X}\left(\bigcap_{i\leq n} L_i\right) = \bigcap_{i\leq n} cl_{\beta X}(L_i)$$

in βX. Notice that equality (**) holds because $L_i \in Z(X)$ $(i \leq n)$. So it suffices to prove equality (*) in case $L_1,L_2,\ldots,L_n \in \mathcal{H}$ and $L_0 \notin \mathcal{H}$ (if all $L_i \in \mathcal{H}$ then enlarge $\{L_0,L_1,\ldots,L_n\}$ with $L_{n+1} = X$ and renumber them). Suppose that equality (*) does not hold; then there exists an

$$x_0 \in \bigcap_{i\leq n} cl_{\alpha X}(L_i) \setminus cl_{\alpha X}\left(\bigcap_{i\leq n} L_i\right).$$

We have to consider two cases:

<u>CASE</u> 1. $cl_{\alpha X}(L_0) \cap M^* = \emptyset$.

Since $x_0 \in \bigcap_{i\leq n} cl_{\alpha X}(L_i) \subset cl_{\alpha X}(L_0)$ it follows that $x_0 \notin M^*$. Let $Y := \alpha X \setminus M$. Notice that Y is homeomorphic to $\xi^{-1}[Y]$. Then

$$x_0 \in \bigcap_{i\le n} cl_{\alpha X}(L_i) \cap Y = \bigcap_{i\le n} cl_Y(L_i) \qquad \text{(lemma 4.2.2)}$$

$$= cl_{\alpha X}\left(\bigcap_{i\le n} L_i\right) \cap Y \subset cl_{\alpha X}\left(\bigcap_{i\le n} L_i\right);$$

this is a contradiction.

CASE 2. $M^* \subset int_{\alpha X}\, cl_{\alpha X}(L_0)$.

Let $L_i = f_{\kappa_i}^{-1}[0,\delta_{\kappa_i}] \cap X$ ($i \in \{1,2,\ldots,n\}$). If $x_0 \notin M$ then use the same technique as in case 1 in order to derive a contradiction. Next, suppose $x_0 \in M$; then $x_0 \in \bigcap_{i=1}^n cl_{\alpha X}(f_{\kappa_i}^{-1}[0,\delta_{\kappa_i}] \cap X) \cap cl_{\alpha X}(L_0) \cap M$ and consequently (i)

$$x_0 \in \bigcap_{i=1}^n cl_{\alpha X}(U(\delta_{\kappa_i}, f_{\kappa_i})) \cap cl_{\alpha X}(L_0) \cap M \qquad \text{((ii))}$$

$$= cl_{\alpha X}\left(\bigcap_{i=1}^n U(\delta_{\kappa_i}, f_{\kappa_i})\right) \cap cl_{\alpha X}(L_0) \cap M$$

$$= cl_{\alpha X}\left(\bigcap_{i=1}^n U(\delta_{\kappa_i}, f_{\kappa_i})\right) \cap int_{\alpha X}\, cl_{\alpha X}(L_0) \cap M$$

$$\subset cl_{\alpha X}\left(\bigcap_{i=1}^n U(\delta_{\kappa_i}, f_{\kappa_i}) \cap int_{\alpha X}\, cl_{\alpha X}(L_0)\right) \cap M$$

$$\subset cl_{\alpha X}\left(\bigcap_{i\le n} L_i\right) \cap M$$

$$\subset cl_{\alpha X}\left(\bigcap_{i\le n} L_i\right),$$

which is a contradiction. This completes the proof of the theorem. □

Since separable metric spaces and countable spaces are Lindelõf semi-stratifiable we have the following corollaries:

4.2.8. COROLLARY (cf. [1],[109]). *Every metric compactification is a Wallman compactification.*

4.2.9. COROLLARY (cf. [112]). *Every countable multiple point compactification is a z-compactification.*

4.2.10. We will now prove that certain compactifications of strongly ω_1 compact spaces are regular Wallman. For this, we assume for the remainder of this section that X is a strongly ω_1 compact space and that αX is a compactification of X. As before M denotes the set of multiple points of αX. If $B \subset X$ then B^- denotes the closure of B in X.

We need a simple lemma.

4.2.11. LEMMA. *Let* U *and* V *be open subsets of* αX *such that*

(i) $(U \cap X)^- \cap (V \cap X)^- = (U \cap V \cap X)^-$;

(ii) $\mathrm{cl}_{\alpha X}(U) \cap \mathrm{cl}_{\alpha X}(V) \cap M = \mathrm{cl}_{\alpha X}(U \cap V) \cap M$; *then*

$$\mathrm{cl}_{\alpha X}(U) \cap \mathrm{cl}_{\alpha X}(V) = \mathrm{cl}_{\alpha X}(U \cap V).$$

PROOF. Suppose to the contrary that there exists an

$$x_0 \in (\mathrm{cl}_{\alpha X}(U) \cap \mathrm{cl}_{\alpha X}(V)) \setminus \mathrm{cl}_{\alpha X}(U \cap V).$$

Let $Y := \alpha X \setminus M$. Since X is Lindelöf (cf. theorem 4.1.10) X is normal and consequently

$$\begin{aligned} \mathrm{cl}_{\beta X}((U \cap X)^-) \cap \mathrm{cl}_{\beta X}((V \cap X)^-) &= \mathrm{cl}_{\beta X}((U \cap X)^- \cap (V \cap X)^-) \\ &= \mathrm{cl}_{\beta X}((U \cap V \cap X)^-) \\ &= \mathrm{cl}_{\beta X}(U \cap V \cap X). \end{aligned}$$

Hence it follows that $\mathrm{cl}_Y(U \cap X) \cap \mathrm{cl}_Y(V \cap X) = \mathrm{cl}_Y(U \cap V \cap X)$ and therefore $x_0 \notin Y$. It is also clear that $x_0 \notin M$. Contradiction. $\square$

This lemma implies the following theorem.

4.2.12. THEOREM. *Any compactification of a strongly* ω_1 *compact space in which the collection of multiple points is also strongly* ω_1 *compact, is regular Wallman.*

4.2.13. COROLLARY. *Any compactification of a Lindelöf semi-stratifiable space in which the collection of multiple points is also Lindelöf semi-stratifiable, is regular Wallman.*

PROOF. Since X is separable it follows that the weight of αX is at most $\mathfrak{c}$. Let $\mathcal{B}$ an open basis for αX, closed under finite intersections and finite unions, which has at most $\mathfrak{c}$ members. Define

$$\mathcal{C} := \{\mathrm{cl}_{\alpha X}(B_0), \mathrm{cl}_{\alpha X}(B_1)) \mid B_0, B_1 \in \mathcal{B} \text{ and } \mathrm{cl}_{\alpha X}(B_0) \cap \mathrm{cl}_{\alpha X}(B_1) = \emptyset\}.$$

For each $(cl_{\alpha X}(B_0), cl_{\alpha X}(B_1)) \in \mathcal{C}$ choose an $f \in C(\alpha X, I)$ such that

$$f[cl_{\alpha X}(B_0)] = 0 \quad \text{and} \quad f[cl_{\alpha X}(B_1)] = 1.$$

Let F denote the set of mappings obtained in this way; write $F = \{f_\kappa \mid \kappa \in \mathcal{C}\}$. By transfinite induction we can construct, in a similar manner as in the proof of theorem 4.2.6, for each $\kappa \in \mathcal{C}$ a $\delta_\kappa \in (0,1)$ such that

(i) $cl_{\alpha X}(U(\delta_\kappa, f_\kappa)) \cap cl_{\alpha X}(V) \cap M = cl_{\alpha X}(U(\delta_\kappa, f_\kappa) \cap V) \cap M$ for all $V \in \wedge.\vee.\{U(\delta_\beta, f_\beta) \mid \beta < \kappa\}$;

(ii) $(U(\delta_\kappa, f_\kappa) \cap V \cap X)^- = (U(\delta_\kappa, f_\kappa) \cap X)^- \cap (V \cap X)^-$ for all $V \in \wedge.\vee.\{U(\delta_\beta, f_\beta) \mid \beta < \kappa\}$.

Here we use lemma 4.2.5 in case $A = X$. From lemma 4.2.11 we deduce that $\wedge.\vee.\{cl_{\alpha X}(U(\delta_\kappa, f_\kappa)) \mid \kappa \in \mathcal{C}\}$ is a ring of regular closed sets in αX. □

4.2.14. COROLLARY (cf. [16]). *βX is regular Wallman if X is regular Lindelöf semi-stratifiable.*

4.2.15. COROLLARY to COROLLARY (cf. [85]). *βX is regular Wallman if X is separable metric.*

4.3. Compactifications of locally compact spaces with zero-dimensional remainder

For a locally compact space X we give a necessary and sufficient condition for every compactification αX of X with zero-dimensional remainder to be regular Wallman. As an application it follows that the Freudenthal compactification of a locally compact metrizable space is regular Wallman. The results in this section are taken from BAAYEN & VAN MILL [11].

4.3.1. For shortness, from now on a *separating* ring of regular closed sets of a topological space X will be called an *s-ring*.

4.3.2. PROPOSITION. *Any open subspace of a regular Wallman space possesses an s-ring.*

PROOF. Let U be an open subspace of the regular Wallman space X and let F be an s-ring for X. Then it is easy to see that $S := \{F \cap U \mid F \in F\}$ is an s-ring in U. □

4.3.2. Notice that a closed subspace of a regular Wallman space need not have an s-ring, for SOLOMON's [107] example can be embedded in a product of closed unit segments and each product of closed unit segments is regular Wallman (cf. STEINER & STEINER [109]).

4.3.3. When A and B are open subsets of the topological space X and $A \cap B = \emptyset$, we will write $A + B$ instead of $A \cup B$. If X is a locally compact space and $\mathcal{F}$ is an s-ring in X then we will write

$$\mathcal{F}^* := \{F \in \mathcal{F} \mid F \text{ is compact or } (X\backslash F) \text{ is relatively compact}\}.$$

Clearly $\mathcal{F}^*$ is an s-ring. In addition, if αX is any compactification of X, we define a collection $\alpha\mathcal{F}$ of subsets of X in the following manner:

$S \in \alpha\mathcal{F} :\Longleftrightarrow$ there are $F \in \mathcal{F}^*$, compact $K \subset X$ and open subsets V_1, V_2 of αX such that:
(i) $F \cap K = \emptyset$,
(ii) $\alpha X \backslash K = V_1 + V_2$ and $S = F \cap V_1$.

4.3.4. <u>LEMMA</u>. *Let* X *be a locally compact space,* αX *a compactification of* X*, and* $\mathcal{F}$ *an* s*-ring in* X*. Then* $\alpha\mathcal{F}$ *is closed under finite intersections, and* $\vee.\alpha\mathcal{F}$ *is again an* s*-ring.*

<u>PROOF</u>. First notice that $\alpha\mathcal{F}$ consists of regular closed sets. Secondly we show that $\alpha\mathcal{F}$ is closed under finite intersections. Take $S_0, S_1 \in \alpha\mathcal{F}$. Then for $i \in \{0,1\}$ there exist $F_i \in \mathcal{F}^*$, compact $K_i \subset X$ and open $U_i, V_i \subset \alpha X$ such that $\alpha X\backslash K_i = U_i + V_i$ and $F_i \cap K_i = \emptyset$ and $S_i = F_i \cap U_i$. Then $S_0 \cap S_1 = (F_0 \cap F_1) \cap (U_0 \cap U_1)$. Since $K_0 \cup K_1$ is compact, $(F_0 \cap F_1) \cap (K_0 \cup K_1) = \emptyset$, and

$$\begin{aligned}\alpha X\backslash (K_0 \cup K_1) &= (\alpha X\backslash K_0) \cap (\alpha X\backslash K_1) \\ &= (U_0 + V_0) \cap (U_1 + V_1) \\ &= (U_0 \cap U_1) + \{(U_0 \cap V_1) \cup (V_0 \cap U_1) \cup (V_0 \cap V_1)\}\end{aligned}$$

it follows that $S_0 \cap S_1 \in \alpha\mathcal{F}$.

Trivially $\mathcal{F}^* \subset \alpha\mathcal{F}$ and hence $\alpha\mathcal{F}$ is separating if $\mathcal{F}^*$ is. To prove the latter, let $x \in X$ and let G be a closed set in X such that $x \notin G$. Take an open $U \subset X$ such that $x \in U \subset cl_X(U)$ and $cl_X(U) \cap G = \emptyset$, while moreover $cl_X(U)$ is compact. This is possible since X is locally compact. Now, $\mathcal{F}$ is separating and therefore there exist $F_0, F_1 \in \mathcal{F}$ such that $x \in F_0$, $X \setminus U \subset F_1$ and $F_0 \cap F_1 = \emptyset$. Evidently $F_0, F_1 \in \mathcal{F}^*$ and hence $\mathcal{F}^*$ is separating.

Since the union of finitely many regular closed sets is again regular closed it now follows that $\vee.\alpha\mathcal{F} = \wedge.\vee.\alpha\mathcal{F}$ is an s-ring. $\square$

4.3.5. THEOREM. *Let* X *be a locally compact space. Then the following assertions are equivalent:*

(i) X *possesses an s-ring;*

(ii) *any compactification* αX *of* X *with zero-dimensional remainder* $\rho X = \alpha X \setminus X$ *is regular Wallman.*

PROOF. (ii) $\Rightarrow$ (i). This follows from proposition 4.3.2.

(i) $\Rightarrow$ (ii). Let $\mathcal{F}$ be an s-ring in X and let $\mathcal{S} := \{cl_{\alpha X}(S) \mid S \in \alpha\mathcal{F}\}$. We will show that $\vee.\mathcal{S}$ is an s-ring in αX, which implies that αX is regular Wallman.

Let $F \in \mathcal{F}^*$ and let K be a compact subset of X such that $\alpha X \setminus K = V_0 + V_1$ and $F \cap K = \emptyset$; we put $S_i = F \cap V_i$ $(i \in \{0,1\})$.

CLAIM 1. *Either* $cl_{\alpha X}(S_i) = S_i$ *or* $cl_{\alpha X}(S_i) = S_i \cup (V_i \cap \rho X)$ $(i \in \{0,1\})$.

Indeed, if F is compact, then also S_i is compact; consequently $cl_{\alpha X}(S_i) = S_i$. If $X \setminus F$ is relatively compact, then $cl_{\alpha X}(F) = F \cup \rho X$ and consequently

$$\begin{aligned} cl_{\alpha X}(S_i) &= cl_{\alpha X}(F \cap V_i) \subset (F \cup \rho X) \cap cl_{\alpha X}(V_i) \\ &\subset (F \cup \rho X) \cap (V_i \cup K) = (F \cap V_i) \cup (\rho X \cap V_i) \\ &= S_i \cup (\rho X \cap V_i). \end{aligned}$$

Since $cl_{\alpha X}(S_0 \cup S_1) \cap \rho X = \rho X$ and $cl_{\alpha X}(S_0) \cap cl_{\alpha X}(S_1) = \emptyset$ it follows that $cl_{\alpha X}(S_i) = S_i \cup (\rho X \cap V_i)$ $(i \in \{0,1\})$.

CLAIM 2. *For all* $S_0, S_1 \in \alpha\mathcal{F}$ *we have* $cl_{\alpha X}(S_0) \cap cl_{\alpha X}(S_1) = Cl_{\alpha X}(S_0 \cap S_1)$.

If S_0 or S_1 is compact, then this is a triviality. Therefore suppose neither is compact. For $i \in \{0,1\}$ let K_i be a compact subset of X, $F_i \in \mathcal{F}^*$ and U_i, V_i open subsets of αX such that $S_i = F_i \cap V_i$, while $\alpha X \setminus K_i = V_i + U_i$ and $F_i \cap K_i = \emptyset$. Then

$$\begin{aligned} cl_{\alpha X}(S_0) \cap cl_{\alpha X}(S_1) &= (S_0 \cup (V_0 \cap \rho X)) \cap (S_1 \cup (V_1 \cap \rho X)) \\ &= (S_0 \cap S_1) \cup (\rho X \cap V_0 \cap V_1). \end{aligned}$$

Suppose that there exists an $x \in (cl_{\alpha X}(S_0) \cap cl_{\alpha X}(S_1)) \backslash cl_{\alpha X}(S_0 \cap S_1)$. Then $x \in V_0 \cap V_1$. Now, as $cl_{\alpha X}(F_0 \cap F_1) \cap \rho X = \rho X$, it follows (cf. the proof of lemma 4.3.4) that

$$x \in V_0 \cap V_1 \cap cl_{\alpha X}(F_0 \cap F_1) \subset cl_{\alpha X}((V_0 \cap V_1) \cap (F_0 \cap F_1))$$
$$= cl_{\alpha X}(S_0 \cap S_1),$$

which is a contradiction.

It now follows that $\mathcal{T} := \vee.\mathcal{S}$ is a ring consisting of regular closed sets.

CLAIM 3. *$\mathcal{T}$ is separating.*

Let $x_0 \in \alpha X$ and let G be a closed set of αX such that $x_0 \notin G$. If $x_0 \in X$, then the existence of $T_0, T_1 \in \mathcal{T}$ such that $x_0 \in T_0$ and $G \subset T_1$ and $T_0 \cap T_1 = \emptyset$ is evident. So, we may assume that $x_0 \in \rho X$. Since ρX is zero-dimensional it possesses a base of open and closed sets. Let C be an open and closed set of ρX such that $x_0 \in C$ and $C \cap G = \emptyset$. Define $C_0 = \rho X \backslash C$. Then C and C_0 are disjoint closed subsets in αX such that $C_0 \cup C = \rho X$. As αX is normal, there exist open $U_0, U_1 \subset \alpha X$ such that $C_0 \cup G \subset U_0$, $C \subset U_1$ and $U_0 \cap U_1 = \emptyset$. Then $K = \alpha X \backslash (U_0 \cup U_1)$ is a compact subset of X such that $K \cap G = \emptyset$. Choose a relatively compact open O in X such that $K \subset O \subset cl_X(O)$ and $cl_X(O) \cap (G \cap X) = \emptyset$. As $\mathcal{F}^*$ is separating we conclude that

$$X \backslash O = \bigcap\{F \in \mathcal{F}^* \mid X \backslash O \subset F\}$$

and consequently, by the compactness of K, there exists an $F \in \mathcal{F}^*$ such that $X \backslash O \subset F$ and $F \cap K = \emptyset$. Define $S_0 := F \cap U_0$ and $S_1 := F \cap U_1$. From claim 1 it now follows that $x_0 \in cl_{\alpha X}(S_1)$ and $G \subset cl_{\alpha X}(S_0)$ and $cl_{\alpha X}(S_0) \cap cl_{\alpha X}(S_1) = \emptyset$.

This completes the proof of the theorem. □

4.3.6. COROLLARY. *Let X be a topological space and let αX be a compactification of X such that the set M of multiple points is compact and zero-dimensional. If βX is regular Wallman, then also αX is regular Wallman.*

PROOF. By proposition 4.3.2 $\alpha X \backslash M$ possesses an s-ring and hence, as αX is a compactification of $\alpha X \backslash M$, the space αX is regular Wallman (cf. theorem 4.3.5). □

4.3.7. In [85] MISRA showed that $\beta(\sum_{i\in I} X_i)$ is regular Wallman if βX_i is regular Wallman for all $i \in I$. It is well known that any locally compact metrizable space is a topological sum of locally compact separable metric spaces. As βX is regular Wallman if X is separable metric (cf. MISRA [85], also corollary 4.2.15) this implies that βX is regular Wallman if X is locally compact and metrizable. This yields the following:

4.3.8. COROLLARY. *Let X be a locally compact metrizable space. Then each bouding system compactification of Gould, all finite and countable compactifications, all finite multiple point compactifications and the Freudenthal compactification are regular Wallman.*

PROOF. Bounding system compactifications of Gould have only one multiple point (cf. NJÅSTAD [88]) and the Freudenthal compactification has zero-dimensional remainder. □

4.3.9. In [85] MISRA also showed that βX is regular Wallman in case X is normal and homeomorphic to a finite product of locally compact ordered spaces. Thus the above corollaries also hold for these spaces.

4.4. Tree-like spaces and Wallman compactifications

We show that the Čech-Stone compactification βX of a peripherally compact tree-like space X, which has at most c closed subsets, is regular Wallman.

4.4.1. Let X be a peripherally compact tree-like space (cf. 1.3.16). For all distinct $a,b \in X$ define

$$S(a,b) := \{x \in X \mid x \text{ separates } a \text{ and } b\} \cup \{a,b\}.$$

It is well known that S(a,b) is an orderable connected subspace of X with two endpoints (cf. PROIZVOLOV [92]; also KOK [70]) and therefore S(a,b) is compact (cf. KELLEY [69]).

In [93] PROIZVOLOV proved that any two disjoint closed sets A and B are separated by a closed discrete set $C = \{x_\alpha \mid \alpha \in \kappa\}$; that is $X\backslash C$ is the union of two disjoint open sets U_0 and U_1 such that $A \subset U_0$ and $B \subset U_1$. The set C is not uniquely determined. In fact, each x_α is a point arbitrarily chosen from $S(a_\alpha,b_\alpha)\backslash\{a_\alpha,b_\alpha\}$ for certain $a_\alpha,b_\alpha \in X$ ($\alpha \in \kappa$). Hence it follows that for each x_α there are at least $\mathfrak{c}$ different choices.

This observation will be used in the proof of the following theorem.

4.4.2. THEOREM. *Let* X *be a peripherally compact tree-like space. Suppose* X *has at most* $\mathfrak{c}$ *closed subsets. Then* βX *is regular Wallman.*

PROOF. Let $\mathcal{B}$ the collection of closed subsets of X. Define

$$\mathcal{A} := \{(A,B) \mid A,B \in \mathcal{B} \text{ and } A \cap B = \emptyset\}.$$

Write $\mathcal{A} = \{(A_\alpha,B_\alpha) \mid \alpha \in \mathfrak{c}\}$. For each $\alpha \in \mathfrak{c}$ we will construct an open subset U_α of X such that:

(i) $A_\alpha \subset U_\alpha \subset \mathrm{cl}_X(U_\alpha) \subset X\backslash B_\alpha$;

(ii) ∂U_α is discrete;

(iii) $\beta < \alpha$ implies that $\partial U_\beta \cap \partial U_\alpha = \emptyset$.

Suppose that all U_α are defined for $\beta < \alpha$. If $\alpha = 0$, choose an open O in X with discrete boundary such that $A_0 \subset O \subset \mathrm{cl}_X(O) \subset X\backslash B_0$ and define $U_0 := O$. If $\alpha \neq 0$, then define

$$\mathcal{H} := \wedge.\vee.\{U_\beta \mid \beta < \alpha\}.$$

It is clear that $\mathcal{H}$ is a family of less than $\mathfrak{c}$ open sets with discrete boundary. Let $C = \{x_i \mid i \in I\}$ be a discrete set separating A_α and B_α, and, for each $i \in I$, let $S(a_i,b_i)$ be selected in such a way that $x_i \in S(a_i,b_i)\backslash\{a_i,b_i\}$ while, moreover, for any choice of $y_i \in S(a_i,b_i)\backslash\{a_i,b_i\}$ ($i \in I$) the set $D = \{y_i \mid i \in I\}$ is again a closed discrete set separating A_α and B_α (cf. 4.4.1). Since $S(a_i,b_i)$ is compact we have that

$$|\partial H \cap S(a_i,b_i)| < \omega$$

for all $H \in \mathcal{H}$ and consequently

$$\left|\bigcup_{H\in\mathcal{H}}(\partial H \cap S(a_i,b_i))\right| < \mathfrak{c}.$$

For each $i \in I$ choose $x_i' \in S(a_i,b_i)\backslash\{a_i,b_i\}$ such that $x_i' \notin \bigcup_{H\in\mathcal{H}}(\partial H \cap S(a_i,b_i))$. It is clear that such a choice is possible. Define $C' = \{x_i' \mid i \in I\}$. Let O be an open subset of X such that $A_\alpha \subset O \subset \mathrm{cl}_X(O) \subset O \cup C' \subset X\backslash B_\alpha$ and define $U_\alpha := O$. This completes the transfinite construction.

Finally define $\mathcal{V} := \wedge.\vee.\{U_\alpha \mid \alpha \in \mathfrak{c}\}$. As the intersection of two regular closed sets with disjoint boundaries, is again regular closed it immediately follows that $\{cl_X(V) \mid V \in \mathcal{V}\}$ is a ring consisting of regular closed sets of X, while moreover it separates (in the sense of 2.3.1) the closed subsets of X. Since X is normal, βX is regular Wallman (cf. MISRA [85], theorem 3.4). $\square$

4.4.3. The proof of the previous theorem is a modification of the proof of theorem 1.4.8. There we showed that a *compact* tree-like space of weight at most $\mathfrak{c}$ is regular supercompact, hence, in particular, is regular Wallman. This suggests the following question.

4.4.4. QUESTION. *Are all compact tree-like spaces regular Wallman?*

4.5. Regular supercompact superextensions

In section 1.4 we defined a space X to be regular supercompact provided that X possesses a binary subbase $\mathcal{T}$ such that $\wedge.\vee.\mathcal{T}$ is a ring consisting of regular closed sets. Since superextensions are supercompact in a canonical way, it is natural to ask in what cases spaces λX are regular supercompact. We will prove that in case βX is regular Wallman, $\lambda(X,Z(X))$ is regular supercompact. Hence for a normal space X it follows that λX is regular supercompact if βX is regular Wallman.

4.5.1. LEMMA. *Let* X *be a topological space and let* $\mathcal{F}$ *be a separating ring of regular closed subsets of* X. *If* $\mathcal{M} = \{F_1,\ldots,F_n\} \subset \mathcal{F}$ *is a finite linked system then there is a finite linked system* $\mathcal{M}' = \{F_1',\ldots,F_n'\} \subset \mathcal{F}$ *such that* $F_i' \subset int_X(F_i)$ *for all* $i \le n$.

PROOF. For $i,j \le n$ choose $F_{ij} = F_{ji} \in \mathcal{F}$ such that $F_{ij} \subset int_X(F_i \cap F_j)$ and $F_{ij} \ne \emptyset$. This is possible since $\mathcal{F}$ is separating and is a ring consisting of regular closed sets and $\mathcal{M}$ is linked. Define

$$F_i' = \bigcup_{j=1}^{n} F_{ij}$$

for all $i \le n$. It is clear that $\mathcal{M}' = \{F_1',\ldots,F_n'\}$ has the desired properties. $\square$

2.5.2. THEOREM. *Let X be regular Wallman and Hausdorff. Then λX is regular supercompact (and hence regular Wallman).*

PROOF. Let $\mathcal{F}$ be a separating ring of regular closed subsets of X. Then $\mathcal{F}$ separates the closed subsets of X (cf. 0.A) and consequently λX and $\lambda(X,\mathcal{F})$ are equivalent (cf. theorem 2.4.2). Hence it suffices to show that $\lambda(X,\mathcal{F})$ is regular supercompact.

CLAIM 1. *Choose $F \in \mathcal{F}$. Then $(\mathrm{int}_X(F))^+$ is open in $\lambda(X,\mathcal{F})$.*

Indeed, let $\mathcal{M} \in (\mathrm{int}_X(F))^+$. Choose $M_0 \in \mathcal{M}$ such that $M_0 \subset \mathrm{int}_X(F)$. As $\mathcal{F}$ separates the closed subsets of X there is an $F_0 \in \mathcal{F}$ such that $X\setminus(\mathrm{int}_X(F)) \subset F_0$ and $F_0 \cap M_0 = \emptyset$. Therefore $\mathcal{M} \in (X\setminus F_0)^+ \subset (\mathrm{int}_X(F))^+$.

CLAIM 2. *$\{(\mathrm{int}_X(F))^+ \mid F \in \mathcal{F}\}$ is an open subbase for $\lambda(X,\mathcal{F})$.*

Choose $\mathcal{M} \in \bigcap_{i\le n} U_i^+$ with $X\setminus U_i \in \mathcal{F}$ $(i \le n)$. Fix $i \le n$ and choose $M \in \mathcal{M}$ such that $M \subset U_i$. By normality of X there is an open subset $O \subset X$ such that

$$M \subset O \subset \mathrm{cl}_X(O) \subset U_i.$$

Choose $F_i \in \mathcal{F}$ such that $\mathrm{cl}_X(O) \subset F_i \subset U_i$. Then

$$\mathcal{M} \in \bigcap_{i\le n} (\mathrm{int}_X(F_i))^+ \subset \bigcap_{i\le n} F_i^+ \subset \bigcap_{i\le n} U_i^+ .$$

CLAIM 3. *$\wedge.\vee.\{F^+ \mid F \in \mathcal{F}\}$ is a regular ring.*

It suffices to prove that $\bigcap_{i\le n} F_i^+$ $(n \in \omega)$ is regular closed in $\lambda(X,\mathcal{F})$ for arbitrary $F_i \in \mathcal{F}$ $(i \le n)$. Let $\mathcal{M} \in \bigcap_{i\le n} F_i^+$ and let U be any open neighborhood of $\mathcal{M}$. Without loss of generality, by claim 2,

$$U = \bigcap_{j\le m} (\mathrm{int}_X(T_j))^+$$

where $T_j \in \mathcal{F}$ $(j \le m)$. Clearly

$$\mathcal{M} \in \bigcap_{j\le m} T_j^+ \cap \bigcap_{i\le n} F_i^+$$

and consequently $\{T_j \mid j \le m\} \cup \{F_i \mid i \le n\}$ is linked. By lemma 2.11.1 there are $T_j' \in \mathcal{F}$ $(j \le m)$ and $F_i' \in \mathcal{F}$ $(i \le n)$ such that

$$T'_j \subset \mathrm{int}_X(T_j) \quad \text{and} \quad F'_i \subset \mathrm{int}_X(F_i) \qquad (j \le m,\ i \le n);$$

$$\{T'_j \mid j \le m\} \cup \{F'_i \mid i \le n\} \text{ is linked.}$$

Choose $L \in \lambda(X,F)$ such that $L \in \bigcap_{j\le m} T'^{+}_j \cap \bigcap_{i\le n} F'^{+}_i$. Then $L \in \bigcap_{j\le m} (\mathrm{int}_X(T_j))^+ \cap \bigcap_{i\le n} (\mathrm{int}_X(F_i))^+$. In particular

$$U \cap \bigcap_{i\le n} (\mathrm{int}_X(F_i))^+ \ne \emptyset.$$

It follows that $\bigcap_{i\le n} F_i^+$ is the closure (in $\lambda(X,F)$) of $\bigcap_{i\le n}(\mathrm{int}_X(F_i))^+$; consequently $\bigcap_{i\le n} F_i^+$ is regular closed. □

4.5.3. COROLLARY.

(i) *If* βX *is regular Wallman then* $\lambda(X,Z(X))$ *is regular supercompact;*

(ii) λX *is regular supercompact if* X *is a regular Lindelöf semi-stratifiable space;*

(iii) λX *is regular supercompact if* X *is normal and homeomorphic to a finite product of locally compact ordered spaces.*

PROOF. (i) This follows from corollary 2.2.6 and theorem 2.5.2. (ii) This follows from corollary 4.2.14. (iii) MISRA [85] showed that βX is regular Wallman if X is normal and homeomorphic to a finite product of locally compact ordered spaces. □

Finally we prove that a regular supercompact space is a superextension of each of its dense subspaces.

4.5.4. THEOREM. *A regular supercompact space is a superextension of each dense subspace.*

PROOF. This immediately follows from the definition of regular supercompactness and from theorem 2.2.5. □

4.6. GA compactifications; some preliminaries

This section contains some preliminary results concerning GA compactifications. These results will be used in section 4.7 to show that each compact Hausdorff space of weight at most c is GA compactification of each dense subspace.

4.6.1. As noted in section 2.2, the GA compactification $\beta(X,\mathcal{S})$ of the topological space X relative the closed T_1-subbase $\mathcal{S}$ is the closure of X in the superextension $\lambda(X,\mathcal{S})$. One of the basic properties of the GA compactification $\beta(X,\mathcal{S})$ is that it is Hausdorff in case $\mathcal{S}$ is weakly normal (cf. 2.2.1 (ii)) (cf. DE GROOT & AARTS [57]). As mentioned earlier DE GROOT & AARTS [57] used this fact to obtain a new intrinsic characterization of complete regularity: *a topological space is completely regular if and only if it possesses a weakly normal closed* T_1*-subbase.* This result considerably strengthened FRINK's [51] result and it motivates the interest in GA compactifications. It is unknown whether there exists a direct proof of the above characterization, i.e. a proof without using compactifications. For FRINK's [51] result there are several direct proofs (cf. STEINER [115], VAN MILL & WATTEL [84]).

4.6.2. LEMMA. *Let* $\mathcal{S}$ *be a closed* T_1*-subbase for the topological space* X. *Then the following assertions are equivalent:*

(i) $\beta(X,\mathcal{S})$ *is Hausdorff;*

(ii) $\mathcal{S}$ *is weakly normal;*

(iii) $\{S^+ \cap \lambda(X,\mathcal{S}) \mid S \in \mathcal{S}\}$ *is weakly normal.*

PROOF. (i) $\Rightarrow$ (ii). Assume that $\beta(X,\mathcal{S})$ is Hausdorff and take $S_0, S_1 \in \mathcal{S}$ such that $S_0 \cap S_1 = \emptyset$. Then $(S_0^+ \cap \beta(X,\mathcal{S})) \cap (S_1^+ \cap \beta(X,\mathcal{S})) = \emptyset$ and hence there exist open disjoint $U_i \subset \beta(X,\mathcal{S})$ such that

$$S_i^+ \cap \beta(X,\mathcal{S}) \subset U_i \qquad (i \in \{0,1\}).$$

Then $\beta(X,\mathcal{S})\setminus U_i$ is closed in $\beta(X,\mathcal{S})$ and as $\beta(X,\mathcal{S})$ is closed in $\lambda(X,\mathcal{S})$ it is closed in $\lambda(X,\mathcal{S})$ too $(i \in \{0,1\})$. Since $\mathcal{S}^+$ is a closed subbase for the compact space $\lambda(X,\mathcal{S})$ there exist $T_{ij} \in \mathcal{S}$ and $T'_{ij} \in \mathcal{S}$ $(i,j \le n,\ n \in \omega)$ such that

(i) $\beta(X,\mathcal{S})\setminus U_0 \subset \bigcup_{i\le n} \bigcap_{j\le n} T_{ij}^+$; $\beta(X,\mathcal{S})\setminus U_1 \subset \bigcup_{i\le n} \bigcap_{j\le n} T'^+_{ij}$;

(ii) $\bigcup_{i\le n} \bigcap_{j\le n} T_{ij}^+ \cap S_0^+ = \emptyset = \bigcup_{i\le n} \bigcap_{j\le n} T'^+_{ij} \cap S_1^+$.

(Notice that a finite intersection of finite unions of subbase elements also can be represented as a finite union of finite intersections of subbase elements.) As $\mathcal{S}^+$ is binary, for each $i \le n$ there is a $j_0(i) \le n$ such that $T^+_{ij_0(i)} \cap S^+_0 = \emptyset$ and a $j_1(i) \le n$ such that $T'^+_{ij_1(i)} \cap S^+_1 = \emptyset$; writing T_i for $T_{ij_0(i)}$ and T'_i for $T'_{ij_1(i)}$ we find that

(i) $\beta(X,S)\backslash U_0 \subset \bigcup_{i\le n} T^+_i$; $\beta(X,S)\backslash U_1 \subset \bigcup_{i\le n} T'^+_i$;

(ii) $\bigcup_{i\le n} T^+_i \cap S^+_0 = \emptyset = \bigcup_{i\le n} T'^+_i \cap S^+_1$.

Then

$$X \subset \beta(X,S) \subset \bigcup_{i\le n} T^+_i \cup \bigcup_{i\le n} T'^+_i.$$

and consequently

$$X = \bigcup_{i\le n} (T^+_i \cap X) \cup \bigcup_{i\le n} (T'^+_i \cap X) = \bigcup_{i\le n} T_i \cup \bigcup_{i\le n} T'_i.$$

Moreover it is obvious that $\bigcup_{i\le n} T_i \cap S_0 = \emptyset = \bigcup_{i\le n} T'_i \cap S_1$. This implies that S is weakly normal.

(ii) ⇒ (i). See DE GROOT & AARTS [57, lemma 9] or VERBEEK [119, Theorem 11.2.3].

(ii) ⇒ (iii). Choose $S^+_0, S^+_1 \in S^+$ such that $S^+_0 \cap S^+_1 = \emptyset$. As $S_0 \cap S_1 = \emptyset$, there exist $T_i \in S$ and $T'_i \in S$ $(i \le n)$ such that

(i) $S_0 \cap \bigcup_{i\le n} T'_i = \emptyset = S_1 \cap \bigcup_{i\le n} T_i$;

(ii) $\bigcup_{i\le n} T'_i \cup \bigcup_{i\le n} T_i = X$.

Then it follows that $S^+_0 \cap \bigcup_{i\le n} T'^+_i = \emptyset = S^+_1 \cap \bigcup_{i\le n} T^+_i$ and that

$$X \subset \beta(X,S) \subset \bigcup_{i\le n} T'^+_i \cup \bigcup_{i\le n} T^+_i,$$

and consequently $\beta(X,S) = \bigcup_{i\le n}(T'^+_i \cap \beta(X,S)) \cup \bigcup_{i\le n}(T^+_i \cap \beta(X,S))$.

(iii) ⇒ (ii). This can be proved in a similar way. □

4.6.3. <u>THEOREM</u>. *A Hausdorff compactification αX of X is a GA compactification if and only if αX possesses a weakly normal closed T_1-subbase T such that for all $T_0, T_1 \in T$ with $T_0 \cap T_1 \ne \emptyset$ we have $T_0 \cap T_1 \cap X \ne \emptyset$.*

<u>PROOF</u>. (⇒). This follows from lemma 4.5.2 and from the trivial observation that if $\alpha X = \beta(X,S)$, then $\{S^+ \cap \beta(X,S) \mid S \in S\}$ is a closed T_1-subbase for $\beta(X,S)$.

(⇐). Suppose that αX possesses a weakly normal closed T_1-subbase T such that for all $T_0, T_1 \in T$ with $T_0 \cap T_1 \ne \emptyset$ we have that $T_0 \cap T_1 \cap X \ne \emptyset$.

Define

$$\mathcal{T} \restriction X = \{T \cap X \mid T \in \mathcal{T}\}.$$

We will show that αX is equivalent to $\beta(X, \mathcal{T} \restriction X)$

For all $x \in \alpha X$ define $M(x) := \{T \cap X \mid T \in \mathcal{T}$ and $x \in T\}$.

CLAIM 1. *$M(x)$ is a maximal linked system (in $\mathcal{T} \restriction X$).*

That $M(x)$ is a linked system is evident. Assume that there is a $T \in \mathcal{T}$ such that $M(x) \cup \{T \cap X\}$ is linked and $x \notin T$. Then there is a $T_0 \in \mathcal{T}$ such that $x \in T_0$ and $T_0 \cap T = \emptyset$, since $\mathcal{T}$ is a T_1-subbase. Now $T_0 \cap X \in M(x)$ and $(T_0 \cap X) \cap (T \cap X) = \emptyset$, which is a contradiction.

Define a mapping $f\colon \alpha X \longrightarrow \lambda(X, \mathcal{T} \restriction X)$ by $f(x) := M(x)$.

CLAIM 2. *f is one to one and continuous and is the identity on X.*

Choose distinct $x, y \in X$. Choose disjoint $T_0, T_1 \in \mathcal{T}$ such that $x \in T_0$ and $y \in T_1$. Then $T_0 \cap X \in M(X)$, $T_1 \cap X \in M(y)$ and $(T_0 \cap X) \cap (T_1 \cap X) = \emptyset$; consequently $M(x) \neq M(y)$.

The continuity of f follows from the following observation: $x \in f^{-1}[(T \cap X)^+] \iff f(x) \in (T \cap X)^+ \iff (T \cap X) \in M(x) \iff x \in T$.

Finally, choose $x \in X$. Then $f(x) = M(x) = \{T \cap X \mid T \in \mathcal{T}$ and $x \in T\} = x$, which shows that f is the identity on X.

CLAIM 3. *f is a closed mapping.*

As f is one to one, we need only show that $f[T]$ is closed in $\lambda(X, \mathcal{T} \restriction X)$ for all $T \in \mathcal{T}$. This however is a triviality, since it is easy to show that $f[T] = (T \cap X)^+ \cap \beta(X, \mathcal{T} \restriction X)$ for all $T \in \mathcal{T}$.

Since f is the identity on X we conclude that $f\colon \alpha X \longrightarrow \beta(X, \mathcal{T} \restriction X)$ is a homeomorphism. $\square$

We conclude this section with a sufficient condition for extending continuous functions over GA compactifications. (We refer to 2.3.1 for the definition of the relation $\sqsubset$ between closed subbases.)

4.6.4. THEOREM. *Let $\mathcal{S}$ be a T_1-subbase for X and let $\mathcal{T}$ be a weakly normal T_1-subbase for Y and let $f\colon X \to Y$ be a continuous map such that*

$\{f^{-1}[T] \mid T \in \mathcal{T}\} \sqsubset \mathcal{S}$.

Then f *can be extended to a continuous map* $\bar{f}$: $\beta(X,\mathcal{S}) \longrightarrow \beta(Y,\mathcal{T})$. *Moreover, if* f *is onto then* $\bar{f}$ *is onto.*
If f *is* 1 - 1 *and* $\{f[S] \mid S \in \mathcal{S}\} \sqsubset \mathcal{T}$ *then* $\bar{f}$ *is an embedding.*

PROOF. The proof is almost the same as the proof of theorem 2.3.4, except for some replacements of two elements covers by finite covers. □

In a similar manner one obtains an analogue of corollary 2.3.5.

4.6.5. COROLLARY. *Let* $\mathcal{S}$ *be a separating ring of closed subsets of* X, *and let* $\mathcal{T}$ *be a weakly normal* T_1*-subbase for* Y *and let* f: X → Y *be a continuous surjection. Then the following assertions are equivalent:*

(i) *there is a continuous surjection* $\bar{f}$: $\beta(X,\mathcal{S}) \longrightarrow \beta(Y,\mathcal{T})$ *such that* $\bar{f} \restriction X = f$;
(ii) $\{f^{-1}[T] \mid T \in \mathcal{T}\} \sqsubset \mathcal{S}$. □

4.7. Every compactification of a separable space is a GA compactification

In this section we show that any compact space of weight at most $\mathfrak{c}$ is a GA compactification of each dense subspace. First we show that any compact space of weight at most $\mathfrak{c}$ is a GA compactification of each dense open subspace. Then using a technique of ŠAPIRO [101a] (cf. also ÜNLÜ [117a], STEINER & STEINER [113]) we derive the above result.

4.7.1. For technical reasons we need to define a new class of compactifications.

DEFINITION. Let X be a topological space and let αX be a compactification of X. Then αX is called a GA^* *compactification* of X provided that αX possesses a family $\mathcal{T}$ of closed sets satisfying:

(i) for every pair of disjoint closed sets $A_0, A_1 \subset \alpha X$ there are disjoint $T_0, T_1 \in \mathcal{T}$ with $A_i \subset T_1$ $(i \in \{0,1\})$ (i.e. $2^{\alpha X} \sqsubset \mathcal{T}$, cf. 2.3.1);
(ii) for all $T_0, T_1 \in \mathcal{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$.

4.7.2. LEMMA. *Each Wallman compactification is a* GA^* *compactification and each* GA^* *compactification is a GA compactification.*

PROOF. That every Wallman compactification is a GA^* compactification follows from theorem 4.1.4 and theorem 4.1.6.

Let αX be a compactification of X and let $\mathcal{T}$ be a family of closed sets of αX satisfying (i) and (ii) of definition 4.7.1. Clearly $\mathcal{T}$ is a closed base which is T_1. We will show that $\mathcal{T}$ is normal, which suffices to prove the lemma (cf. 4.6.3). Choose disjoint $T_0, T_1 \in \mathcal{T}$ and let U_0 and U_1 be disjoint neighborhoods of T_0 and T_1 respectively. Then, by 4.7.1 (i) there are $T_0', T_1' \in \mathcal{T}$ such that $\alpha X \backslash U_i \subset T_i'$ and $T_i \cap T_i' = \emptyset$ $(i \in \{0,1\})$. Consequently $\mathcal{T}$ is normal. □

The following proposition was the main result in VAN MILL [78].

4.7.3. PROPOSITION. *Let αX be a compactification of a locally compact space X such that weight $(\alpha X) \le \mathfrak{c}$. Then αX is a GA^* compactification of X.*

PROOF. Let $\mathcal{B}$ be an open basis for αX such that $|\mathcal{B}| \le \mathfrak{c}$. Without loss of generality we may assume that $\mathcal{B}$ is closed under finite intersections and finite unions. Define

$$\mathcal{C} := \{(\mathrm{cl}_{\alpha X}(B_0), \mathrm{cl}_{\alpha X}(B)) \mid B_0, B_1 \in \mathcal{B} \text{ and } \mathrm{cl}_{\alpha X}(B_0) \cap \mathrm{cl}_{\alpha X}(B_1) = \emptyset\}.$$

For each pair $(\mathrm{cl}_{\alpha X}(B_0), \mathrm{cl}_{\alpha X}(B_1)) \in \mathcal{C}$ choose an $f \in C(\alpha X, I)$ such that $f[\mathrm{cl}_{\alpha X}(B_0)] = 0$ and $f[\mathrm{cl}_{\alpha X}(B_1)] = 1$. Let $\mathcal{F}$ denote the set of mappings obtained in this way; write $\mathcal{F} = \{f_\gamma \mid \gamma \in \mathfrak{c}\}$. For each $\gamma \in \mathfrak{c}$ we will construct a $\delta_\gamma \in (0,1)$ such that

$$(*) \qquad \mathrm{cl}_{\alpha X}(f_\gamma^{-1}[0,\delta_\gamma)) \cap \mathrm{cl}_{\alpha X}(f_\rho^{-1}[0,\delta_\rho)) \ne \emptyset \Rightarrow$$

$$\mathrm{cl}_{\alpha X}(f_\gamma^{-1}[0,\delta_\gamma)) \cap \mathrm{cl}_{\alpha X}(f_\rho^{-1}[0,\delta_\rho)) \cap X \ne \emptyset$$

for all $\rho < \gamma$.

Let $f_\gamma \in \mathcal{F}$ and define $\mathcal{M} := \{f_\gamma^{-1}(p) \backslash X \mid p \in f_\gamma[\alpha X \backslash X]\} \cup \{\{x\} \mid x \in X\}$ and let $\alpha_\gamma(X)$ be the decomposition space of $\mathcal{M}$. It is easily seen that $\alpha_\gamma(X)$ is Hausdorff; consequently $\alpha_\gamma(X)$ is a compactification of X with $f_\gamma[\alpha X \backslash X]$ is a remainder. Let P_γ denote the projection map. Then P_γ is the identity on X. Finally define $h_\gamma \colon \alpha_\gamma(X) \to I$ by $h_\gamma = f \circ P_\gamma^{-1}$. Then h_γ is continuous and the diagram

$$\begin{tikzcd} \alpha X \arrow[r, "P_\gamma"] \arrow[d, "f_\gamma"'] & \alpha_\gamma(X) \arrow[dl, "h_\gamma"] \\ I & \end{tikzcd}$$

commutes.

Define $\delta_0 := \frac{1}{2}$ and assume that all δ_ρ have been defined for $\rho < \gamma$ ($\gamma \in C$) such that (*) is satisfied. If $B \subset \alpha_\gamma(X)$, then $\bar{B}$ denotes the closure of B in $\alpha_\gamma(X)$. As in the proof of theorem 4.2.6 there is a $\delta \in (0,1)$ such that

$$\overline{f_\rho^{-1}[0,\delta_\rho) \cap X} \cap \overline{h_\gamma^{-1}[0,\delta)} \cap (\alpha_\gamma(X)\setminus X) =$$

$$\overline{f_\rho^{-1}[0,\delta_\rho) \cap h_\gamma^{-1}[0,\delta) \cap X} \cap (\alpha_\gamma(X)\setminus X)$$

for all $\rho < \gamma$ (notice that $\alpha_\gamma(X)\setminus X$ is homeomorphic to a closed subset of the real line and hence is strongly ω_1 compact). Define $\delta_\gamma := \delta$. We claim that (*) is satisfied. Take $\rho < \gamma$ and assume that $cl_{\alpha X}(f_\gamma^{-1}[0,\delta_\gamma)) \cap cl_{\alpha X}(f_\rho^{-1}[0,\delta_\rho)) \neq \emptyset$. Then

$$\overline{P_\gamma f_\gamma^{-1}[0,\delta_\gamma)} \cap \overline{f_\rho^{-1}[0,\delta_\rho) \cap X} \neq \emptyset ,$$

since it is easily seen that $P_\gamma(cl_{\alpha X}(U)) = \overline{U \cap X}$ for each open $U \subset \alpha X$. Therefore

$$\overline{h_\gamma^{-1}[0,\delta_\gamma)} \cap \overline{f_\rho^{-1}[0,\delta_\rho) \cap X} \neq \emptyset.$$

Now assume that $\overline{h_\gamma^{-1}[0,\delta_\gamma)} \cap \overline{f_\rho^{-1}[0,\delta_\rho) \cap X} \cap X = \emptyset$. It then follows that

$$\overline{h_\gamma^{-1}[0,\delta_\gamma)} \cap \overline{f_\rho^{-1}[0,\delta_\rho) \cap X} \cap (\alpha_\gamma(X)\setminus X) =$$

$$\overline{h_\gamma^{-1}[0,\delta_\gamma) \cap f_\rho^{-1}[0,\delta_\rho) \cap X} \cap (\alpha_\gamma(X)\setminus X) \neq \emptyset$$

and consequently $\overline{h_\gamma^{-1}[0,\delta_\gamma)} \cap \overline{f_\rho^{-1}[0,\delta_\rho)} \cap X \neq \emptyset$, which is a contradiction. Therefore $\overline{h_\gamma^{-1}[0,\delta_\gamma)} \cap \overline{f_\rho^{-1}[0,\delta_\rho) \cap X} \cap X \neq \emptyset$. Let

$$x \in \overline{h_\gamma^{-1}[0,\delta_\gamma)} \cap \overline{f_\rho^{-1}[0,\delta_\rho) \cap X} \cap X;$$

then $x \in \mathrm{cl}_{\alpha X}(f_\gamma^{-1}[0,\delta_\gamma)) \cap \mathrm{cl}_{\alpha X}(f_\rho^{-1}[0,\delta_\rho)) \cap X$. Thus (*) holds indeed for δ_γ; this completes the construction of the δ_γ $(\gamma \in \mathcal{C})$.

Define $\mathcal{A} := \{\mathrm{cl}_{\alpha X}(f_\gamma^{-1}[0,\delta_\gamma)) \mid \gamma \in \mathcal{C}\}$. It is easy to see that $\mathcal{A}$ separates the closed subsets of αX; consequently αX is a GA^* compactification of X. □

The following lemma is straightforward generalization of a lemma due to ÜNLÜ ([117a]; cf. also STEINER & STEINER [113]).

4.7.4. <u>LEMMA</u>. *Let $\alpha_0 X_0$ and $\alpha_1 X_1$ be compactifications of X_0 and X_1, respectively. Let $f\colon \alpha_0 X_0 \to \alpha_1 X_1$ be a continuous surjection such that $f[X_0] = X_1$, and $f \restriction (\alpha_0 X_0 \setminus X_0)$ is one to one. If $\alpha_0 X_0$ is a GA^* compactification of X_0 then $\alpha_1 X_1$ is a GA^* compactification of X_1.*

<u>PROOF</u>. Let $\mathcal{T}$ be a family of closed sets in $\alpha_0 X_0$ satisfying (i) and (ii) of definition 4.7.1. Define $\mathcal{S} := \{f[T] \mid T \in \mathcal{T}\}$. We will show that $\mathcal{S}$ satisfies the conditions of definition 4.7.1. Indeed, take disjoint closed sets $A_0, A_1 \subset \alpha_1 X_1$ and take disjoint open neighborhoods U_0, U_1 of them. By 4.7.1 (i) there are $T_0, T_1 \in \mathcal{T}$ such that $f^{-1}[A_i] \subset T_i \subset f^{-1}[U_i]$ $(i \in \{0,1\})$. Then $A_i \subset f[T_i] \subset U_i$ $(i \in \{0,1\})$. Clearly $\mathcal{S}$ consists of closed subsets of $\alpha_1 X_1$.

Take $T_0, T_1 \in \mathcal{T}$ such that $f[T_0] \cap f[T_1] \neq \emptyset$. Suppose that $f[T_0] \cap f[T_1] \cap X_1 = \emptyset$. Then there is a $y \in f[T_0] \cap f[T_1] \cap (\alpha_1 X_1 \setminus X_1)$. Choose $x_i \in T_i$ such that $f(x_i) = y$ $(i \in \{0,1\})$. Clearly $x_i \notin X_0$ $(i \in \{0,1\})$ since $f[X_0] = X_1$ so that $x_0 = x_1$, since $f \restriction (\alpha_0 X_0 \setminus X_0)$ is one to one. We conclude that $T_0 \cap T_1 \neq \emptyset$ and consequently $T_0 \cap T_1 \cap X \neq \emptyset$. Therefore $f[T_0] \cap f[T_1] \cap f[X_0] = f[T_0] \cap f[T_1] \cap X_1 \neq \emptyset$, which is a contradiction. □

The next lemma is a straightforward generalization of a lemma due to ŠAPIRO [101a].

4.7.5. <u>LEMMA</u>. *Suppose that $X = Y \cup Z$ and that αX is a compactification of X. If $\mathrm{cl}_{\alpha X}(Y)$ and $\mathrm{cl}_{\alpha X}(Z)$ both are GA^* compactifications of Y and Z,* *then* αX *is a* GA^* *compactification of* X.

PROOF. Let S and T be families of closed sets of $cl_{\alpha X}(Y)$ and of $cl_{\alpha X}(Z)$, satisfying 4.7.1 (i)(ii). Let $W := cl_{\alpha X}(Y) \cap cl_{\alpha X}(Z)$. Define

$$F := \{S \cup T \mid S \in S,\ T \in T \text{ and } S \cap W \subset T\}.$$

We will show that F satisfies 4.7.1 (i)(ii).

Indeed, choose disjoint closed sets $A_0, A_1 \subset \alpha X$. Choose disjoint $S_0, S_1 \in S$ such that $A_i \cap cl_{\alpha X}(Y) \subset S_i$ $(i \in \{0,1\})$. In addition, choose disjoint $T_0, T_1 \in T$ such that $(A_i \cap cl_{\alpha X}(Z)) \cup (S_i \cap W) \subset T_i$ $(i \in \{0,1\})$. Then $S_i \cup T_i \in F$ while moreover $A_i \subset S_i \cup T_i$ $(i \in \{0,1\})$ and $(S_0 \cup T_0) \cap (S_1 \cup T_1) = \emptyset$.

Let $F_i = S_i \cup T_i \in F$ $(i \in \{0,1\})$ such that $F_0 \cap F_1 \neq \emptyset$. If $S_0 \cap T_0 \neq \emptyset$ or $T_0 \cap T_1 \neq \emptyset$ then clearly $F_0 \cap F_1 \cap X \neq \emptyset$. Therefore assume that $S_0 \cap T_1 \neq \emptyset$. Then $(S_0 \cap W) \cap T_1 \neq \emptyset$ and consequently, by definition, also $T_0 \cap T_1 \neq \emptyset$. The case $S_1 \cap T_0 \neq \emptyset$ can be treated analogously. □

We now can prove the main result in this section. The technique of proof is again due to ŠAPIRO [101a].

4.7.6. THEOREM. *Every compact Hausdorff space of weight at most* $\mathfrak{c}$ *is a* GA^* *compactification of each dense subspace.*

PROOF. Let X be a compact Hausdorff space of weight at most $\mathfrak{c}$ and let Y be a dense subspace of X. Let D be the set of isolated points of Y. Define $E := Y \setminus cl_Y(D)$. Then E is an open subspace of Y without isolated points.

CLAIM. $cl_X(E)$ *is a* GA^* *compactification of* E.

Indeed, let $Z := cl_X(E)$ and let A be a dense subspace of E of cardinality at most $\mathfrak{c}$. Topologize $B := (Z \times \{0\}) \cup (A \times \{1\})$ by taking as an open base the collection

$$V := \{(a,1) \mid a \in A\} \cup \{(U \times \{0\}) \cup ((U \cap A) \setminus (a,1)) \mid U \text{ open in } Z \text{ and } a \in U \cap A\}$$

(cf. ENGELKING [49]). Clearly B is a compact Hausdorff space of weight at most $\mathfrak{c}$. Also $A \times \{1\}$ is dense in B, since E has no isolated points. Now, by proposition 4.7.3 B is a GA^* compactification of $A \times \{1\}$. Define a mapping $f: B \to Z$ by

$$\begin{cases} f((x,0)) = x & (x \in Z) \\ f((e,1)) = e & (e \in E). \end{cases}$$

Then f clearly is continuous. By lemma 4.7.4 it now follows that Z is a GA^* compactification of A. By an obvious argument it now follows that Z is a GA^* compactification of E too.

By proposition 4.7.3 it also follows that $cl_X(D)$ is a GA^* compactification of D. Thus, lemma 4.7.5 implies that X is a GA^* compactification of $D \cup E$. By an obvious argument it now follows that X is a GA^* compactification of Y. □

4.7.7. COROLLARY. *Let* X *be a separable space. Then all compactifications of* X *are* GA^* *compactifications.* □

4.7.8. QUESTION. *Is there a* GA *compactification which is not a* GA^* *compactification?*

4.7.9. REMARK. Using the same technique as above it can be shown that every compactification is a GA^* compactification if and only if every compactification of a discrete space is a GA^* compactification.

4.8. Notes

In the present chapter we have given partial answers to questions posed by FRINK and PAALMAN-DE MIRANDA. Interesting is the connection between Wallman compactifications and GA compactifications. Our technical but natural proof of proposition 4.7.3 unfortunately only "works" for GA compactifications.

As noted before, some of the techniques used in the present chapter are inspired on ideas of BERNEY [16].

The results in section 4.3 were taken from BAAYEN & VAN MILL [11].

CHAPTER V

A SURVEY OF RECENT RESULTS

In this final chapter we give a survey of recent results; moreover we mention some important results on superextensions which were proved by VERBEEK [119]. References are to be found at the end of this chapter; they are not included in the list of references for the first 4 chapters.

5.1. Cardinal functions on superextensions (cf. VERBEEK [*10*], VAN MILL [*4*]).

Let X be a topological space. The definitions of the following cardinal functions on X can be found in JUHÁSZ [67]; let

$d(X)$ denote the density of X;
$t(X)$ denote the tightness of X;
$c(X)$ denote the cellularity of X;
$w(X)$ denote the weight of X;
$\chi(X)$ denote the character of X.

5.1.1. THEOREM (a) (cf. VERBEEK [*10*]). *Let* X *be a topological space. Then*

(i) $d(\lambda X) \le d(X)$;

(ii) *if* X *is compact and Hausdorff then* $w(X) = w(\lambda X)$;

(iii) *if* X *is an infinite Hausdorff space then*
$c(X) \le c(\lambda X) = \sup\{c(X^n) \mid n \in \mathbb{N}\} = c(X^\omega)$.

(b) (cf. VAN MILL [*4*]). *Let* X *be a normal topological space. Then*

(i) $t(\lambda X) = \chi(\lambda X)$;

(ii) *if* X *has a binary normal subbase then*
$\chi(X) \le d(X) \cdot t(X)$.

5.2. Metrizability in superextensions (cf. VAN DOUWEN [*3*])

The following theorem answers some questions posed in 2.11.

5.2.1. <u>THEOREM</u>. *Let* X *be a normal topological space. Then the following assertions are equivalent:*

(i) X *is compact and metrizable;*

(ii) λX *is metrizable;*

(iii) λX *is perfectly normal;*

(iv) βX *is a* G_δ *in* λX*;*

(v) λX *is hereditarily normal.*

5.2.2. <u>THEOREM</u>. *Let* X *be a normal topological space for which* λX *is first countable. Then* X *is compact, hereditarily separable and perfectly normal.*

5.3. <u>The compactness number of a compact topological space</u> (cf. BELL & VAN MILL [*2*])

BELL & VAN MILL [*2*] define the *compactness number* cmpn(X) of a compact Hausdorff space X in the following manner:

cmpn(X) $\le$ k (k $\in$ $\mathbb{N}$) provided that X admits an open subbase $\mathcal{U}$ such that each covering of X with elements of $\mathcal{U}$ contains a subcovering of at most k elements of $\mathcal{U}$;

cmpn(X) = k if cmpn(X) $\le$ k and cmpn(X) $\not\le$ k;

cmpn(X) = ∞ if cmpn(X) $\ne$ n for all n $\in$ $\mathbb{N}$.

5.3.1. <u>THEOREM</u>.(a) *Let* X *be a non-pseudocompact space. If* Y *is a compact Hausdorff space which can be mapped continuously onto* βX*, then* cmpn(Y) = ∞.

(b) *For each* k $\in$ $\mathbb{N}$ *there is a compact Hausdorff space* X_k *for which* cmpn(X_k) = k.

5.3.2. <u>THEOREM</u>. *There is a non-compact, locally compact and* σ*-compact topological space* X *all compactifications of which have infinite compactness number.*

5.4. <u>A cellular constraint in supercompact Hausdorff spaces</u> (cf. BELL [*1*])

The following result is quite unexpected.

5.4.1. <u>THEOREM</u>. *Let* X *be a compact Hausdorff space which is a neighborhood retract of a supercompact Hausdorff space. If* D *is any dense subspace of* X *then* $c(X\backslash D) \le w(D)$.

Notice that the above theorem implies that if $\gamma\mathbb{N}$ is a supercompact compactification of $\mathbb{N}$ then $\gamma\mathbb{N}\setminus\mathbb{N}$ satisfies the countable chain condition.

5.4.2. THEOREM. *$2^{\beta\mathbb{N}}$ and $2^{\beta\mathbb{N}\setminus\mathbb{N}}$ are not supercompact.*

5.5. An external characterization of spaces which admit binary normal subbases (cf. VAN MILL & WATTEL [7])

5.5.1. THEOREM. *Let S be a normal T_1-subbase for the topological space X. Let p,q be distinct elements of X. Then there is a function f: $X \to [0,1]$ such that $f(p) = 0$ and $f(q) = 1$ while for every $t \in [0,1]$ the sets $f^{-1}[0,t]$ and $f^{-1}[t,1]$ are countable intersections of members from S.*

This theorem is used to give an unexpected characterization of spaces which admit binary normal subbases. First we give a definition. If $x,y,z \in I = [0,1]$ then let $m(x,y,z)$ be the unique point in $[x,y] \cap [y,z] \cap [x,z]$. We call a subset X in a product of unit segments I^A *triple convex* provided that for all $x,y,z \in X$ the point p of I^A defined by

$$p_\alpha := m(x_\alpha, y_\alpha, z_\alpha) \qquad (\alpha \in A)$$

also belongs to X. We now get the following characterization of spaces which admit a binary normal subbase.

5.5.2. THEOREM. *A compact space X admits a binary normal subbase if and only if it can be embedded as a triple-convex set in a product of closed unit segments.*

5.6. Some elementary proofs in fixed point theory (cf. VAN DE VEL [9])

Let X be a space with a binary normal subbase S. A mapping $f: X \to X$ is called *convexity preserving* (cp map) (cf. VAN MILL & WATTEL [7]) provided that $f^{-1}(S) \in H(X,S)$ for all $S \in S$.

As noted in chapter 1, each connected space with a binary normal subbase has the fixed point property for continuous functions. This was proved by VAN DE VEL [118] using methods from algebraic topology. Recently VAN DE VEL has found an elementary proof of a special case of the above theorem.

5.6.1. THEOREM. *Let X be a normally supercompact connected space. Then* *each* cp *map* $f: X \to X$ *has a fixed point.*

5.7. Reductions of the generalized De Groot conjecture (cf. VAN MILL & VAN DE VEL [*6*])

The generalized De Groot conjecture states that $\lambda X \approx Q$ iff X is a nondegenerate metrizable continuum. We have two reductions.

5.7.1. THEOREM. *The following assertions are equivalent:*

(i) *the generalized De Groot conjecture;*

(ii) $\lambda P \approx Q$ *for all nondegenerate compact connected polyhedra.*

5.7.2. THEOREM. *The following assertions are equivalent:*

(i) *the generalized De Groot conjecture;*

(ii) *for each compact connected polyhedron* P *and for each continuous surjection* $f: P \to P$ *the Jensen extension* $\lambda(f): \lambda P \longrightarrow \lambda P$ *is a near-homeomorphism.*

5.8. More about convexity (cf. VAN DE VEL [*8*])

VAN DE VEL has proved the following remarkable result.

5.8.1. THEOREM. *Let* X *be a space with a binary normal subbase* $\mathcal{S}$. *Let* O *be an open subset of* X. *Then the following properties are equivalent:*

(i) *for each pair* $x,y \in O$: $I_{\mathcal{S}}(x,y) \subset O$.

(ii) *for each closed set* $D \subset O$: $I_{\mathcal{S}}(D) \subset O$.

By an example it is demonstrated that the restriction to open subsets of X is essential.

5.9. Convexity preserving mappings in subbase convexity theory (cf. VAN MILL & VAN DE VEL [*5*])

Convexity preserving mappings are very important in the theory of normally supercompact spaces. Examples of cp maps are the nearest point mappings.

5.9.1. THEOREM. *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be normal* T_1*-subbases for the spaces* X *and* Y, *respectively, and let* $f: X \to Y$ *be a mapping such that* $f^{-1}(T) \in \mathcal{S}$ *for each* $T \in \mathcal{T}$. *Then the induced Jensen mapping*

$$\lambda(f) = \lambda(f;\mathcal{S},\mathcal{T}) : \lambda(X,\mathcal{S}) \longrightarrow \lambda(Y,\mathcal{T})$$

is a cp mapping extending f. *Moreover, if* f *is surjective, then* $\lambda(f)$ *is the unique surjective cp mapping which extends* f.

Due to the fact that a space X is usually not dense in $\lambda(X,\mathcal{S})$ (e.g. if X is compact and if $\mathcal{S}$ is not binary), there may as well exist more than one *continuous* extension of the map f. Within the category of surjective cp mappings, the extension is unique. Hence, superextension theory can be regarded as an extension of "ordinary compactification theory" to the appropriate category.

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