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RICHARD ASKEY

EIGHT LECTURES ON ORTHOGONAL POLYNOMIALS

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Contents

Preface

These notes are a partial summary of lectures I gave at the Mathematisch Centrum in 1969-70. They are not intended to be a complete survey of recent work on the classical orthogonal polynomials, but they should serve as an introduction to some of the current work.
Many references to further work are given in my survey paper, Orthogonal polynomials and positivity, SIAM symposium on Special Functions, to appear in 1970. Due to recent work this paper is out of date before it has appeared and hopefully this field will settle down in a couple of years, so that a complete treatment of these problems can be given.

A preliminary version of these lectures was written and elaborated by Mr. Bakker for providing me with a good record of what I said in these lectures.

Finally I would like to thank Mr. Bavinck for helping to read the final version of these lecture notes and the Mathematisch Centrum for giving me the opportunity to present these lectures.

Amsterdam, April 1970

Lecture 1

Introduction

In studying special functions you should go back to the simple special functions and examine their properties in details. We will give here some elementary properties of $\sin\theta$ and $\cos\theta$ and see what problems they lead to for orthogonal polynomials.

Starting with $\cos\theta$ the addition formula gives

$$\cos(n+m)\theta = \cos n\theta \cos m\theta - \sin n\theta \sin m\theta, \tag{1}$$

$$\cos(n-m)\theta = \cos n\theta \cos m\theta + \sin n\theta \sin m\theta. \tag{2}$$

Adding (1) and (2) gives

$$\cos n\theta \cos m\theta = \frac{1}{2}\left[\cos(n+m)\theta + \cos(n-m)\theta\right]. \tag{3}$$

For $m = 1$ we have

$$\cos\theta \cos n\theta = \frac{1}{2}\left[\cos(n+1)\theta + \cos(n-1)\theta\right]. \tag{4}$$

Using (4) we can show by induction that $\cos n\theta = T_n(\cos\theta)$, where $T_n(x)$ is a polynomial of degree n in x. It is usually called the Tchebycheff polynomial. Notice that

$$T_0(x) = 1, \quad T_1(x) = x. \tag{5}$$

(4) then becomes

$$xT_n(x) = \frac{1}{2}T_{n+1}(x) + \frac{1}{2}T_{n-1}(x). \tag{6}$$

Recall that

$$\int_0^\pi \cos n\theta \cos m\theta \, d\theta = 0, \text{ if } m \neq n. \tag{7}$$

Letting $x = \cos\theta$ in (7) we see that

$$(8) \qquad \int_{-1}^{1} T_n(x)\, T_m(x)\, (1-x^2)^{-\frac{1}{2}}\, dx = 0, \quad \text{if } m \neq n.$$

(8) can be generalized easily. We assume that we have a nonnegative measure $d\alpha(x)$ and define an inner product (f,g) by

$$(9) \qquad (f,g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, d\alpha(x).$$

We assume that the measure $d\alpha(x)$ has absolute moments of all order, i.e. that $\int_{-\infty}^{\infty} |x|^n\, d\alpha(x)$ exists for $n = 0,1,2,\ldots$. Then we can find a sequence of polynomials $P_n(x)$, $P_n(x)$ of degree n for which

$$(10) \qquad (P_n,P_m) = \int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, d\alpha(x) = \delta_{nm}.$$

We call such polynomials orthonormal. If we do not require that $(P_n,P_n) = 1$, then we call them orthogonal. These polynomials are unique up to a factor of ± 1 and we will standardize them by requiring that

$$P_n(x) = k_n\, x^n + \ldots, \quad k_n > 0.$$

For general orthogonal polynomials we can generalize (6). $xP_n(x)$ is a polynomial of degree n+1 and so we can write it as

$$xP_n(x) = \sum_{k=0}^{n+1} \alpha_{k,n}\, P_k(x).$$

Multiplying by $P_j(x)$ and using (10) we see that

$$\int_{-\infty}^{\infty} xP_n(x)\, P_j(x)\, d\alpha(x) = \alpha_{j,n} \int_{-\infty}^{\infty} P_j^2(x)\, d\alpha(x) = \alpha_{j,n}.$$

If $j < n-1$ then $xP_j(x)$ is a polynomial of degree less than n and so from (10) and the fact that any polynomial of degree (j+1) can be written as a sum of $P_k(x)$, $(k = 0,1,\ldots,j+1)$ with constant coefficients we have $\alpha_{j,n} = 0$ for $j = 0,1,\ldots,n-2$. Thus (6) generalizes to

(11) $xP_n(x) = \alpha_{n+1,n} P_{n+1}(x) + \alpha_{n,n} P_n(x) + \alpha_{n-1,n} P_{n-1}(x).$

Since our polynomials were normalized to have positive highest coefficients we have $\alpha_{n+1,n} > 0$ and $\alpha_{n-1,n} > 0$, since $\alpha_{n-1,n} = \alpha_{n,n-1}$.

For many problems we want to normalize these polynomials in a different way. In particular it is often convenient to have $p_n(x) = x^n + \ldots$. Then (11) takes the form

(12) $xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x).$

In (12) we have $\beta_n > 0$ and α_n real. A famous theorem of Favard [1] says that if we are given a sequence of polynomials $p_n(x) = x^n + \ldots$ which satisfies (12) with $\beta_n > 0$, α_n real, then there is a non-negative measure $d\alpha(x)$ with finite absolute moments of all order for which

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, d\alpha(x) = 0, \quad \text{if } m \neq n.$$

Unfortunately there is no constructive method to obtain $d\alpha(x)$ when we are given α_n and β_n. In fact $d\alpha(x)$ may not even be unique. There is a refinement of Favard's theorem due to Shohat [2]. Shohat proved that if $|\alpha_n| \leq A$, $\beta_n \leq B$, A and B finite, then $d\alpha(x)$ was supported on a compact set. In this case the measure is unique but a construction of $d\alpha(x)$ is still lacking. When either α_n or β_n is unbounded then the measure has mass on an unbounded set and it may or may not uniquely be determined. Many of these results and others are given by Freud [3].

Thus we have satisfactorily generalized (6) to all orthogonal polynomials. Next we ask if we can generalize (3), or

(13) $T_n(x)\, T_m(x) = \frac{1}{2} T_{n+m}(x) + \frac{1}{2} T_{|n-m|}(x).$

There is a trivial generalization to

$$p_n(x)\, p_m(x) = \sum_{k=0}^{n+m} \alpha(k,m,n)\, p_k(x)$$

which holds for any sequence of polynomials. If the polynomials are orthogonal we have

$$p_n(x)\, p_m(x) = \sum_{k=|n-m|}^{n+m} \alpha(k,m,n)\, p_k(x).$$

This is enough for some problems but for other problems we want to know more about $\alpha(k,m,n)$. In particular we would like to have a formula for $\alpha(k,m,n)$ in terms of α_n and β_n. It seemingly is possible to obtain such a formula, which is not surprising. However there are some problems where it is not necessary to have $\alpha(k,m,n)$ exactly but only to know something about it. In the next lecture we will show how it is sometimes possible to prove that $\alpha(k,m,n) \geq 0$ for all k, m and n. There we will also give some applications.

There is one other simple set of orthogonal polynomials for which can find $\alpha(k,m,n)$. The addition formula for $\sin\theta$ is

$$\sin(n+m)\theta = \sin m\theta \cos n\theta + \cos m\theta \sin n\theta.$$

Letting $m = \pm 1$ and adding we get

$$\sin n\theta \cos\theta = \frac{1}{2}\left[\sin(n+1)\theta + \sin(n-1)\theta\right].$$

Dividing by $\sin\theta$ we get

$$\cos\theta \frac{\sin n\theta}{\sin\theta} = \frac{1}{2}\frac{\sin(n+1)\theta}{\sin\theta} + \frac{1}{2}\frac{\sin(n-1)\theta}{\sin\theta}. \tag{14}$$

An easy induction using (14) shows that

$$\frac{\sin n\theta}{\sin\theta} = U_{n-1}(\cos\theta),$$

where $U_n(x)$ is a polynomial of degree n in x. It satisfies the recurrence formula

$$xU_n(x) = \frac{1}{2} U_{n+1}(x) + \frac{1}{2} U_{n-1}(x). \tag{15}$$

Observe that this is the same recurrence formula satisfied by $T_n(x)$. The difference is in initial conditions. We have

(16) $$U_0(x) = 1 \quad \text{and} \quad U_1(x) = 2x,$$

while $T_1(x) = x$. By Favard's theorem $U_n(x)$ are orthogonal. In this case we can find the weight function. We have

$$\int_0^\pi \sin(n+1)\theta \sin(m+1)\theta \, d\theta =$$

$$= \int_0^\pi \frac{\sin(n+1)\theta}{\sin\theta} \cdot \frac{\sin(m+1)\theta}{\sin\theta} \sin^2\theta \, d\theta = 0, \quad \text{if } m \neq n.$$

Letting $x = \cos\theta$ we see that

(17) $$\int_{-1}^{1} U_n(x)\, U_m(x)\, (1-x^2)^{\frac{1}{2}}\, dx = 0, \quad \text{if } n \neq m.$$

To find $\alpha(k,m,n)$ set $x = \cos\theta$ to get

$$\frac{\sin(n+1)\theta}{\sin\theta} \cdot \frac{\sin(m+1)\theta}{\sin\theta} = \sum_{k=0}^{n+m} \alpha(k,m,n) \frac{\sin(k+1)\theta}{\sin\theta}.$$

Multiply by $\sin^2\theta$ and use

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

to get

(18) $$\cos(n+m+2)\theta - \cos(n-m)\theta = \sum_{k=0}^{n+m} \alpha(k,m,n) \left[\cos(k+2)\theta - \cos k\theta\right].$$

From (18) it follows immediately that

$$\alpha(n+m,m,n) = 1, \quad \alpha(n+m-1,m,n) = 0 \quad \text{and}$$

$$\alpha(k-2,m,n) = \alpha(k,m,n),$$

$k = n-m+2,\ldots,n+m$ and $\alpha(k,m,n) = 0$ for $k < n-m$. Thus we have

$$(19)\qquad U_n(x)\,U_m(x) = \sum_{k=0}^{m} U_{n+m-2k}(x), \quad n \geq m.$$

Observe that in this case we also have $\alpha(k,m,n) \geq 0$.

The recurrence formula (12) is a second order difference equation. The polynomials $T_n(x)$ and $U_n(x)$ also satisfy second order differential equations. Thus we should try to see if there are results analogous with n and x interchanged. For $T_n(x)$ the result is trivial. We have

$$(20)\qquad \cos n\theta \cos n\phi = \frac{1}{2}\left[\cos n(\theta+\phi) + \cos n(\theta-\phi)\right].$$

The following positivity result is the essential positivity result. Let $f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$, $|x| \leq 1$ and $\sum_{n=0}^{\infty} |a_n| < \infty$. Then $f(x) \geq 0$ iff

$$f(x;y) = \sum_{n=0}^{\infty} a_n T_n(x)\, T_n(y) \geq 0, \quad -1 \leq x,y \leq 1.$$

This follows immediately from (20). The corresponding result for $U_n(x)$ is more interesting. We can still form the series

$$f(x) = \sum_{n=0}^{\infty} a_n U_n(x).$$

However to form the corresponding function of the two variables we now form

$$f(x;y) = \sum_{n=0}^{\infty} a_n U_n(x)\, U_n(y)/U_n(1).$$

In either of these cases $f(x;1) = f(x)$ and so $f(x) \geq 0$ is a necessary condition for $f(x;y) \geq 0$, $-1 \leq x,y \leq 1$. The surprising fact is that it is also sufficient. For $T_n(x)$ it is obvious but it is far from obvious for $U_n(x)$. In fact it was first stated in 1933 by L. Féjér [4]. It was also used implicitly by Kogbetliantz [5]. Féjér's statement is the following:

Let $\sum_{n=1}^{\infty} n|a_n| < \infty$, $f(\theta) = \sum_{n=1}^{\infty} n\, a_n \sin n\theta$, $0 \leq \theta \leq \pi$.

Then $f(\theta) \geq 0$ iff

$$f(\theta;\phi) = \sum_{n=1}^{\infty} a_n \sin n\phi \sin n\theta \geq 0, \quad 0 \leq \theta, \phi \leq \pi.$$

Since $U_n(\cos\theta) = \sin(n+1)\theta / \sin\theta$ and $U_n(1) = n+1$, Féjér's statement is equivalent to the result we stated above.

In one direction it is easy to proof since $\lim_{\phi\to 0} \frac{f(\theta;\phi)}{\phi} = f(\theta)$. To obtain the other implication we consider $f(\theta+\phi) + f(\theta-\phi)$ and integrate. Then we have

$$\int_0^{\phi} [f(\theta+\psi) + f(\theta-\psi)]\, d\psi = 2 \sum_{n=1}^{\infty} n\, a_n \sin n\theta \int_0^{\phi} \cos n\psi\, d\psi =$$

$$= 2 \sum_{n=1}^{\infty} a_n \sin n\theta \sin n\phi = 2f(\theta;\phi).$$

We can integrate term by term because of uniform convergence of the series. Thus if $0 \leq \phi \leq \theta$ and $\phi + \theta \leq \pi$ we have $f(\theta;\phi) \geq 0$. $f(\theta;\phi) = f(\phi;\theta)$ so we may remove the restriction $\phi < \theta$. Also $f(\pi-\theta;\pi-\phi) = f(\theta;\phi)$ so we may remove the restriction $\phi + \theta \leq \pi$ and obtain $f(\theta;\phi) \geq 0$, $0 \leq \theta, \phi < \pi$.

We have made the assumption that $\sum_{n=1}^{\infty} n|a_n| < \infty$ only for convenience. Actually no assumption is needed but then we must be careful about what we mean when we state $\sum_{n=1}^{\infty} n\, a_n \sin n\theta \geq 0$. The easiest way to define this is a positive distribution. We will say more about this point in lecture 5. Let us assume for the moment that we have removed the assumption and give an application of Féjér's theorem. In the April 1967 issue of SIAM Review the following problem was posed. For all real x show that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left\{\frac{\sin n\pi x}{n}\right\}^{2r} \geq 0, \quad r = 1,2,\ldots .$$

This is an even function of period 2 so it is sufficient to prove the nonnegativity for $0 \leq x \leq 1$. This is a nonlinear problem and it is often easier to solve a linear problem in several variables.

We will show that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin \pi n x_1}{n} * \ldots * \frac{\sin \pi n x_k}{n} \geq 0, \; 0 \leq x_j \leq 1.$$

Using Féjér's result (k-1) times we see that it is sufficient to prove

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\pi x_1}{n} \geq 0, \quad 0 \leq x_1 \leq \pi.$$

But

$$\frac{\pi x}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\pi x}{n}, \quad 0 \leq x \leq 1,$$

and this is obviously nonnegative.

We will be interested in one other positivity result for orthogonal polynomials. We will try to find expansions of one orthogonal polynomial in terms of a second orthogonal polynomial with a nonnegative kernel. We will give two illustrations using $T_n(x)$ and $U_n(x)$.
The first example is very old

$$\frac{\sin(n+1)\theta}{\sin \theta} = \sum_{k=0}^{n} \cos(n-2k)\theta.$$

So $U_n(x) = \sum_{k=0}^{n} T_{n-2k}(x)$ which we proved before. Another result is

$$\frac{\sin(n+1)\theta}{(n+1)\sin \theta} = \int_0^{\pi} \cos n\phi \, d\mu_\theta(\phi),$$

where $d\mu_\theta(\phi) \geq 0$.
In fact $d\mu_\theta(\phi) = K(\theta,\phi)\, d\phi$, where

$$K(\theta,\phi) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin(n+1)\theta \cos n\phi}{(n+1) \sin \theta}.$$

A new proof uses the relation

$$\frac{d}{d\theta} \frac{\sin(\alpha+1)\theta}{(\alpha+1)(\cos\theta)^{\alpha+2}} = \frac{\cos\alpha\theta}{(\cos\theta)^{\alpha+2}} .$$

In lecture 3 we will treat this subject more extensively.

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Lecture 2

Linearization of the product of two orthogonal polynomials.

From the first lecture we know that for a set of orthogonal polynomials $\{p_n(x)\}$ the following recurrence formulas hold:

(1) $$x\, p_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x),$$

where $\beta_n > 0$, α_n real and $p_n(x) = x^n + \ldots.$ and

(2) $$p_n(x)\, p_m(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)\, p_k(x) h_k,$$

where $a(k,m,n) = \int_E p_n(x) p_m(x) p_k(x)\, d\alpha(x)$

and $h_k^{-1} = \int_E p_k^2(x)\, d\alpha(x).$

For Legendre polynomials the coefficients $a(k,m,n)$ are known explicitly. They are the product of a large number of gamma functions. See for instance Hobson, p. 86. *)

There are a number of methods which can be used to calculate these linearization coefficients for the classical polynomials. The most powerful method seems to involve the differential equation. For any second order Sturm-Liouville equation

(3) $$a(x)y'' + b(x)y' + \lambda_n y = 0,$$

with $a(x)$ and $b(x)$ sufficiently differentiable, there exists a fourth order differential equation with as solutions the product of solutions of (3) for two different values of λ_n.
That is, if $p(x,\lambda_n)$ and $q(x,\lambda_n)$ are two linearly independent solutions of (3), then $p(x,\lambda_n) * p(x,\lambda_m)$, $p(x,\lambda_n) * q(x,\lambda_m)$, $p(x,\lambda_m) * q(x,\lambda_n)$ and $q(x,\lambda_n) * q(x,\lambda_m)$ are solutions of this fourth order differential equation.

*) E.W. Hobson, The theory of spherical and ellipsodal harmonics, Cambridge University Press.

A related result is given in Watson, Bessel Functions, 5.4. The details will not be given here, since the calculation is lengthy and not very enlightening. Using a differential equation of this type (actually a fifth order equation found by Hylleraas) Gasper has been able to say something about the coefficients a(k,m,n) in

$$P_n^{(\alpha,\beta)}(x)\ P_m^{(\alpha,\beta)}(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)\ P_k^{(\alpha,\beta)}(x).$$

These coefficients had been obtained for $\alpha = \beta$ and Hylleraas found them for $\alpha = \beta+1$ using his differential equation with a series of the type $\sum a(k)\ P_k^{(\alpha,\beta)}(x)$. This method is well known when a power series is used instead of a Jacobi series and the method is the same in the more general case, the details are just more complicated and so will not be given here. For $\beta = -\frac{1}{2}$ these coefficients may also be obtained as the product of gamma functions when one uses

$$\frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)} = \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)}.$$

For other values of (α,β) it seems impossible to obtain a(k,m,n) as a product of simple functions. They have been computed as Appell hypergeometric functions of two variables but this expression seems to be useless. What Gasper did was to completely solve the question of finding the value (α,β) for which $a(k,m,n) \geq 0$ for all k,m,n. The region is slightly larger then $\alpha \geq \beta$, $\alpha+\beta+1 \geq 0$, $\beta > -1$, with a similar region for $\beta > \alpha$ when the polynomials are normalized to be positive at $x = -1$ instead of $x = 1$. For many problems the nonnegativity of the linearization coefficients is all that is needed. This very important work will appear in two papers in the Canadian Journal of Mathematics.

For Jacobi polynomials this method is the most powerful and it is amazing that with it we can completely solve the nonnegativity problem.

Unfortunately the only orthogonal polynomials for which a simple differential equation exists are the Jacobi polynomials and their limit, Laguerre and Hermite polynomials. More will be said about Laguerre and Hermite polynomials later, but now we would like to give another method of attacking problems of this type.

As was mentioned at the beginning of this lecture, orthogonal polynomials are characterized by the recurrence formula

$$x\, p_n(x) = p_{n+1}(x) + \alpha_n'\, p_n(x) + \beta_n\, p_{n-1}(x).$$

Adding a constant times $p_n(x)$ to both sides and recalling that $p_1(x) = x-\alpha_0'$ we see that

$$(4) \qquad p_1(x)\, p_n(x) = p_{n+1}(x) + \alpha_n\, p_n(x) + \beta_n\, p_{n-1}(x).$$

This is a special case of

$$p_n(x)\, p_m(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)\, p_k(x).$$

We are interested in proving $a(k,m,n) \geq 0$ and that $\alpha(n) \geq 0$, $\beta(n) > 0$ is the necessary and sufficient condition that $p_n(x)$ defined by (1) are orthogonal polynomials. These conditions are not sufficient, but the following theorem can be proved.

Theorem 1. Let $p_n(x)$ be defined by (4), $p_0(x) = 1$, $p_1(x) = x + c$ and assume

$$\alpha(n) \geq 0,\ \beta(n) > 0,\ \alpha(n+1) \geq \alpha(n),\ \beta(n+1) \geq \beta(n).$$

Then

$$p_m(x)\, p_n(x) = \sum_{k=|m-n|}^{m+n} a(k,m,n)\, p_k(x),$$

with $a(k,m,n) \geq 0$.

The proof is by induction on m, assuming $m < n$.

Then

$$p_{m+1}\, p_n = [p_1 p_m - \alpha(m) p_m - \beta(m) p_{m-1}]\, p_n =$$

$$= p_m [p_{n+1} + \alpha(n) p_n + \beta(n) p_{n-1}] -$$

$$- \alpha(m) p_m\, p_n - \beta(m) p_{m-1}\, p_n =$$

$$p_m\, p_{n+1} + [\alpha(n) - \alpha(m)]\, p_m\, p_n + [\beta(n) - \beta(m)] \quad p_m\, p_{n-1}$$

$$+ \beta(n)\ [p_m\, p_{n-1} - p_{m-1}\, p_n].$$

By induction and montonicity of $\alpha(n)$ and $\beta(n)$ the first three terms on the right have nonnegative coefficients when written as a sum of $p_k(x)$. We also have

$$p_{m+1}\, p_n - p_m\, p_{n+1} = [\alpha(n) - \alpha(m)]\, p_m\, p_n +$$

$$+ [\beta(n) - \beta(m)]\, p_m\, p_{n-1} + \beta(n) [p_m\, p_{n-1} - p_{m-1}\, p_n]$$

and continuing in this fashion we have nonnegative terms on the right except for the last term which is $\beta(n)\ \beta(n-1) \ldots\ \beta(n-m+1)\, [p_1\, p_{n-m} - p_{n-m+1}]$ and this is $\beta(n)\ \beta(n-1) \ldots\ \beta(n-m+1)\, [\alpha(n-m) p_{n-m} + \beta(n-m) p_{n-m-1}]$ and these coefficients are also nonnegative.

The same proof gives a slightly more general result for difference equations but rather than repeat it here we will give the partial difference equation approach of the problem.

There are a number of other orthogonal polynomials which should be called classical polynomials and considered at the same time. They are orthogonal on a discrete point set and the measures are important measures in probability theory, the binomial, Poisson, negative binomial and hypergeometric distributions, the last including the uniform distribution on N equally spaced points as a special case.

The Charlier polynomials, the polynomials orthogonal with respect to the Poisson distribution which assigns mass $e^{-a}\, x^a/x!$ to the point x ($x = 0,1,2, \ldots ; a > 0$), are covered by Theorem 1. In this case $\alpha(n) = n$, $\beta(n) = an$, $C_0(x;a) = 1$ and $C_1(x;a) = x-a$. As we will see later (lecture 5) this result is not as interesting as the same result for polynomials orthogonal on a bounded set and the more interesting result given there for Laguerre polynomials is still unknown for Charlier polynomials. However, for the Krawtchouk polynomials, the polynomials orthogonal with respect to the binomial distribution, the linearization theorem with nonnegative coefficients is true. It was first proved by Eagleson using generating functions, but there is a proof using a variant of Theorem 1 which we will now give.

The binomial distribution puts mass $\binom{N}{x}\, p^x\, (1-p)^{N-x}$ $(0 < p < 1)$ on the points $x = 0,1,2,\ldots,N$. The corresponding orthogonal polynomials $K_n(x;p,N)$ satisfy

$$K_1(x)\, K_n(x) = K_{n+1}(x) + n(q-p)\, K_n(x) + \beta(n)K_{n-1}(x),$$

where $q = 1-p$, $K_0(x) = 1$, $K_1(x) = x-pN$, and

$$\beta(n) = pq\, n(N+1-n).$$

$\beta(n) > 0$ for $n = 1,2,\ldots, N$ but $\beta(N+1) = 0$. This forces the polynomials $K_n(x;p,N)$ to be orthogonal on a finite set of points and also means that the assumption of Theorem 1 that $\beta(n) \leq \beta(n+1)$ cannot be satisfied. However, $\beta(n)$ satisfies

$$\beta(n) = \beta(N+1-n)$$

and this suggests that there should be a theorem with some assumption of this type. It is

Theorem 2. Let $p_n(x)$ be defined by (4) with $\alpha(n)$ and $\beta(n)$ satisfying

$1^\circ \quad 0 \le \alpha(n) \le \alpha(n+1),\ \alpha(n) \le \alpha(N+1-n),\ n=1,2,\ldots,\ \left[\frac{N+1}{2}\right],$

$2^\circ \quad 0 \le \beta(n) \le \beta(n+1),\ \beta(n) \le \beta(N+1-n),\ n=1,2,\ldots,\ \left[\frac{N+1}{2}\right],$

then

$$p_n(x)\, p_m(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)\, p_k(x), \quad n+m \le N,$$

with $a(k,m,n) \ge 0$.

We will prove the following theorem which will easily imply Theorem 2:

Theorem 3. Let $a(n,m)$ satisfy the difference equation

$$(5) \qquad \Delta_n\, a(n,m) = \Delta_m\, a(n,m),$$

where $\Delta_n\, k(n) = k(n+1) + \alpha(n)\, k(n) + \beta(n)\, k(n-1)$.

Then if $\beta_0 = \beta_{N+1} = 0$,

$$(\alpha) \qquad 0 \le \alpha_n \le \alpha_{n+1}, \quad \alpha_{N+1-n} \ge \alpha_n, \quad n = 1,2,\ldots, \left[\frac{N+1}{2}\right],$$

$$(\beta) \qquad 0 < \beta_n \le \beta_{n+1}, \quad \beta_n \le \beta_{N+1-n}, \quad n = 1,2,\ldots, \left[\frac{N+1}{2}\right]$$

and if $a(n,0) = a(0,n) \ge 0$, $a(-1,n) = a(n,-1) = 0$, $n = 0,1,\ldots,N$, then

$$(6) \qquad a(n,m) \ge 0, \quad n,m = 1,2,\ldots, \quad n + m \le N.$$

The proof is by induction on m. Assume we have proven (6) for $0,1\ldots,m$ and consider $a(n,m+1)$. From (5) we have

$$a(n,m+1) + \alpha_m a(n,m) + \beta_m a(n,m-1) = a(n+1,m) + \alpha_n a(n,m) + \beta_n a(n-1,m),$$

so

$$a(n,m+1) = a(n+1,m) + (\alpha_n-\alpha_m)\, a(n,m) + (\beta_n-\beta_m)\, a(n-1,m)$$
$$+ \beta_m \left[a(n-1,m) - a(n,m-1)\right].$$

Since $a(n,m) = a(m,n)$, we may assume that $m + 1 \leq n$ or $m < n$. Also we have $m + n \leq N$ so $m < N + 1 - n$. Thus from (α) we have

$$\alpha_n - \alpha_m \geq 0 \quad \text{if} \quad m < n \leq \left[\frac{N}{2}\right]$$

and

$$\alpha_n - \alpha_m \geq \alpha_{N+1-n} - \alpha_m \geq 0 \text{ if } \left[\frac{N+1}{2}\right] \leq n \leq N, \text{ since } m < N+1-n.$$

Similarly $\beta_n - \beta_m \geq 0$. Also we can estimate $a(n-1,m) - a(n,m-1)$ by recurrence; for

$$\begin{aligned} a(n,m+1) - a(n+1,m) &\geq \beta_m\left[a(n-1,m) - a(n,m-1)\right] \\ &\geq \beta_m\beta_{m-1}\left[a(n-2,m-1) - a(n-1,m-2)\right] \\ &\geq \beta_m\beta_{m-1}\beta_1\left[a(n-m-1,1) - a(n-m,0)\right] \\ &\geq \beta_m\beta_{m-1} \cdots \beta_1\beta_{n-m-1}a(n-m-2,0) \geq 0. \end{aligned}$$

Thus $a(n,m) \geq 0$ for $n,m = 1,2,\ldots,$ $n + m \leq N$.

To obtain theorem 2 we observe that

$$p_n(x)\ p_m(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)p_k(x),$$

then

$$a(k,m,n) = \frac{\int p_n(x)\ p_m(x)\ p_k(x)\ d\alpha(x)}{\int p_k^2(x)\ d\alpha(x)},$$

for a nonnegative measure $d\alpha(x)$. In our case the measure is a finite number of point masses but is not necessary for this result.

(7) $\qquad \Delta_n a(k,m,n) = \Delta_m a(k,m,n),$

and that

$$a(k,0,n) = a(k,n,0) \geq 0,$$

$$a(k,-1,n) = a(k,n,-1) = 0.$$

(7) follows from the recurrence formula for

$$\Delta_n a(k,m,n) = \frac{\int p_n(x)\ p_m(x)\ p_k(x)\ p_1(x)\ d\alpha(x)}{\int p_k^2(x)\ d\alpha(x)} = \Delta_m a(k,m.n).$$

Also

$$a(k,n,0) = a(k,0,n) = \frac{\int p_n(x)\ p_k(x)\ d\alpha(x)}{\int p_k^2(x)\ d\alpha(x)} = \begin{cases} 0 & \text{if } n \neq k \\ 1 & \text{if } n = k. \end{cases}$$

If $p_{-1}(x)$ is defined to be zero then the recurrence formula holds, so we have $a(k,-1,n) = a(k,n,-1) = 0$.

Lecture 3

Connexions between orthogonal polynomials of different classes .

We are now interested in the possibility of representing an orthogonal polynomial as the sum of orthogonal polynomials of a different class with nonnegative coefficients i.e.

$$p_n(x) = \sum_{k=0}^{n} \alpha_{kn}\, q_k(x), \text{ with } \alpha_{kn} \geq 0.$$

At first we will give some simple examples of polynomials with that property.

a. We will show, that

$$P_n(x) = \sum_{k=0}^{n} \alpha_{kn}\, T_k(x),\ \alpha_{kn} \geq 0.$$

Proof: We will use the generating function of $P_n(x)$:

$$(1-2xr+r^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) r^n, \quad |x| \leq 1,\ r < 1.$$

Put $x = \cos\theta$. Since $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, we have

$$(1-2xr+r^2)^{-\frac{1}{2}} = (1-re^{i\theta})^{-\frac{1}{2}}\,(1-re^{-i\theta})^{-\frac{1}{2}} =$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} e^{-ni\theta} r^n * \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_k}{n!} e^{ni\theta} r^n =$$

$$= \sum_{n=0}^{\infty} \Big[\sum_{k=0}^{n} \frac{(\frac{1}{2})_k\,(\frac{1}{2})_{n-k}}{k!\,(n-k)}\, e^{-ki\theta}.e^{(n-k)i\theta}\Big]\, r^n,$$

which implies

$$P_n(\cos\theta) = \sum_{k=0}^{n} \frac{(\frac{1}{2})_k\,(\frac{1}{2})_{n-k}}{k!\,(n-k)!}\, e^{(n-2k)i\theta} =$$

$$= \sum_{k=0}^{n} \frac{(\frac{1}{2})_k\,(\frac{1}{2})_{n-k}}{k!\,(n-k)!}\, \cos\,(n-2k)\theta.$$

Since $x = \cos\theta$ and $\cos(n-2k)\theta = T_{n-2k}(x)$, we have

$$P_n(x) = 2\frac{(\frac{1}{2})_n}{n!}\ T_n(x) + 2\frac{(\frac{1}{2})_1\ (\frac{1}{2})_{n-1}}{1!\ (n-1)!}\ T_{n-2}(x) +$$

$+ \ldots + (\quad)\,T_{n-2[\frac{n}{2}]}(x)$. The term of the lowest degree equals $2\left(\frac{(\frac{1}{2})_{\frac{n}{2}}}{(n/2)!}\right)^2$, if n is even and equals $\frac{(\frac{1}{2})_{\frac{n-1}{2}}\ (\frac{1}{2})_{\frac{n+1}{2}}}{(\frac{n-1}{2})!\ (\frac{n+1}{2})!}\ x$, if n is odd.

All the coefficients are nonnegative so the proof is complete.

b. Next we will show that

$$L_n^{(\alpha+\beta+1)}(x) = \sum_{k=0}^{n} a(k,n;\beta)\ L_k^{(\alpha)}(x)\ .$$

Again we use the generating function.
We know that

$$(1-r)^{-\alpha-1} \exp\frac{-xr}{1-r} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\ r^n.$$

So we have

$$\sum_{n=0}^{\infty} L_n^{(\alpha+\beta+1)}(x+y)r^n = (1-r)^{-\alpha-\beta-2} \exp - \frac{(x+y)r}{1-r} =$$

$$= (1-r)^{-\alpha-1} \exp - \frac{xr}{1-r} * (1-r)^{-\beta-1} \exp - \frac{yr}{1-r} =$$

$$= \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)r^n * \sum_{n=0}^{\infty} L_n^{(\beta)}(y)r^n =$$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} L_k^{(\alpha)}(x)\ L_{n-k}^{(\beta)}(y)\right] r^n,$$

which implies

$$\sum_{n=0}^{n} L_k^{(\alpha)}(x)\ L_{n-k}^{(\beta)}(0) = L_n^{(\alpha+\beta+1)}(x).$$

Since $L_{n-k}^{(\beta)}(0) = \binom{n-k+\beta}{n-k}$, we have our theorem.

These two examples would follow from the following more general conjecture provided it is true. Unfortunately it has not yet been proven.

Conjecture: Let $w(x)$ be a weight function on $[a,b]$, a finite, $\{p_n(x)\}_{n=0}^{\infty}$ a set of polynomials orthogonal on $[a,b]$ with respect to $w(x)$ and standardized by $p_n(a) > 0$. Let $\{p_n^{\mu}(x)\}_{n=0}^{\infty}$ be a set polynomials orthogonal on $[a,b]$ with respect to $(x-a)^{\mu} w(x)$ and standardized by $p_n^{\mu}(a) > 0$. Then the following positivity relation should hold :

$$p_n^{\mu}(x) = \sum_{k=0}^{n} \alpha_{kn}\, p_k(x), \text{ with } \alpha_{kn} \geq 0 \text{ for } \mu \geq 0.$$

For $\mu = 1,2,\ldots$ this follows from two well known results of Christoffel. For noninteger μ the conjecture is still open.

Now we would like to find a similar connection between two different classes of orthogonal polynomials and their respective weight functions.

Assume $\{p_n(x)\}_{n=0}^{\infty}$ and $\{q_n(x)\}_{n=0}^{\infty}$ are two sets of polynomials orthonormal on E with respect to $w(x)$ and $v(x)$.

If

$$q_n(x) = \sum_{k=0}^{n} \alpha_{kn}\, p_k(x), \text{ with } \alpha_{kn} \geq 0$$

then

$$p_k(x)\, w(x) = \sum_{n=k}^{\infty} \alpha_{kn}\, q_n(x)\, v(x) \quad .$$

Proof:

$$q_n(x) = \sum_{k=0}^{n} \alpha_{kn}\, p_k(x) \quad \text{so}$$

$$\int_E p_k(x)\, q_n(x)\, w(x)dx = \alpha_{kn}.$$

Now we want

$$p_k(x)\ w(x) = \sum_{n=k}^{\infty} \beta_{nk}\ q_n(x)\ v(x),$$

so

$$\int_E p_k(x)\ q_m(x)\ w(x)dx = \beta_{mk} = \alpha_{km},$$

which completes our proof. The fact that the series expansion of p(x) w(x) starts only at n = k follows from the fact that α_{kn} vanishes for k > n. For any specific set of weight functions the series (1) may not converge, so this is just a formal result. The convergence (say in L^2) must be proven in any specific case.

Next we will give an illustration:

$$h^{-\frac{1}{2}}_{n,\alpha+\beta+1} * L_n^{(\alpha+\beta+1)}(x) = \sum_{k=0}^{n} \gamma(k,n;\alpha,\beta) * L_k^{(\alpha)}(x) * h^{-\frac{1}{2}}_{k,\alpha},$$

where

$$h_{n,\alpha} = \int_0^{\infty} x^{\alpha}e^{-x}\ [L_n^{(\alpha)}(x)]^2\ dx \quad .$$

So

$$x^{\alpha}e^{-x}\ L_k^{(\alpha)}(x)\quad h^{-\frac{1}{2}}_{k,\alpha} =$$

$$= \sum_{n=k}^{\infty} \gamma(k,n;\ )h^{-\frac{1}{2}}_{n,\alpha+\beta+1}\ L_n^{(\alpha+\beta+1)}(x)\ e^{-x}\ x^{\alpha+\beta+1} \quad .$$

Applications.

We will start this section with the formulation of a problem involving orthogonal polynomials and its dual problem. Then we will give an example in mathematical physics and its dual, which has no connection with any practical problem. However, this dual problem can be solved easily and the dual of this method was first found by B. Noble. The details in Noble's proof are much more complicated and our proof is a good introduction to Noble's method.

Now we procede to the formulation of the problem:

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a set of polynomials orthogonal on a measurable set E with respect to the weight function $w(x)$. Then compute $(a_n)_{n=0}^{\infty}$ from the following data

$$\sum_{n=0}^{\infty} a_n\, p_n(x) = f(x), \qquad x \in E_1$$

and

$$\sum_{n=0}^{\infty} t_n\, a_n\, p_n(x) = g(x), \qquad x \in E_2,$$

where the functions $f(x)$ and $g(x)$ and the sequence $(t_n)_{n=0}^{\infty}$ are given and E_1 and E_2 are two measurable subsets of E whose union equals E.

The dual problem is:

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a set of polynomials orthogonal on E with respect to $w(x)$. Then determine $f(x)$ from the following data

$$\int_E f(x)\, p_n(x)\, w(x)dx = a_n, \qquad n \in N_1,$$

and

$$\int_E f(x)\, g(x)\, p_n(x)\, w(x)dx = b_n, \qquad n \in N_2,$$

where $(a_n)_{n \in N_1}$, $(b_n)_{n \in N_2}$ and $g(x)$ are given and N_1 and N_2 are two disjoint sets of integers whose union equals the whole set of non-negative integers.

Specific examples now follow.

i. The function u is harmonic in the interior of the unit circle. Solve u from the following boundary conditions:

$$u(1,\theta) = f(\theta), \qquad 0 \leq |\theta| < \alpha,$$

and

$$\frac{\partial u}{\partial n}(1,\theta) = g(\theta), \qquad \alpha < |\theta| \leq \pi.$$

Translated into terms of Fourier series:

Compute $(a_n)_{n=0}^{\infty}$ from the following data:

$$\sum_{n=0}^{\infty} a_n \cos n\theta = f(\theta), \qquad 0 \le \theta < \alpha,$$

$$\sum_{n=1}^{\infty} n\, a_n \cos n\theta = g(\theta), \qquad \alpha \le \theta \le \pi.$$

ii. can be dualized to the following:

If

$$\int_0^{\pi} f(\theta) \cos n\theta \, d\theta = a_n, \qquad n = 0,1,\dots,N,$$

and

$$\int_0^{\pi} \sin\theta \, f(\theta) \cos n\theta \, d\theta = b_n, \qquad n = N+1, \dots ,$$

then compute $f(\theta)$.

The next example is even easier to solve than ii and the method that we use to solve it can be used on many problems of this type and in particular it can be used to solve i. and generalizations of it to Jacobi polynomials. This is the method due to B. Noble.

iii. Solve $f(x)$ from the following data:

a) $$\int_0^{\infty} x^c f(x) L_n^{(\alpha)}(x)\, x^{\alpha} e^{-x} dx = a_n, \qquad n = 0,\dots,N, \ c > 0,$$

b) $$\int_0^{\infty} f(x) L_n^{(\alpha)}(x)\, x^{\alpha} e^{-x}\, dx = b_n, \qquad n = N+1,\dots .$$

We know that $L_n^{(\alpha+c)}(x) = \sum_{k=0}^{n} \alpha_{kn} L_k^{(\alpha)}(x), \qquad n = 0,\dots,N.$

So we have

$$A_n = \int_0^\infty f(x)\, L_n^{(\alpha+c)}(x)\, x^{\alpha+c} e^{-x} dx = \sum_{k=0}^{n} \alpha_{kn}\, a_k.$$

Furthermore we know from the former section that

$$x^c\, L_n^{(\alpha+c)}(x) = \sum_{k=n}^{\infty} \beta_{kn}\, L_k^{(\alpha)}(x) \qquad (n=N+1,\ldots)$$

and this series converges if $c > 0$. So

$$B_n = \int_0^\infty f(x)\, L_n^{(\alpha+c)}(x)\, x^{\alpha+c} e^{-x}\, dx = \sum_{k=n}^{\infty} \beta_{kn}\, b_k.$$

Since we know the coefficients a_n, b_n, α_{kn}, and β_{kn}, we can compute A_n and B_n and hence we can expand $f(x)$ into an infinite series of Laguerre polynomials.

The problem we just solved is an example where the coefficients α_{kn} and β_{kn} are known. Unfortunately the coefficients are usually unknown. Nonetheless, we would like to be able to say something about the coefficients.

We recall that

$$L_n^{(\beta)}(x) = \sum_{k=0}^{n} \alpha_{kn}\, L_k^{(\alpha)}(x).$$

Now if $\beta > \alpha$ then $\alpha_{kn} \geq 0$. We would like to find a general theorem which would imply this. At present we do not have such a theorem.

We will close this part of lecture 3 by giving two more examples without comment.

1. For Jacobi polynomials Szegö has proven

$$P_n^{(\gamma+\mu,\delta)}(x) = \sum_{k=0}^{n} \alpha_{kn}\, P_k^{(\gamma,\delta)}(x),$$

with $\alpha_{kn} \geq 0$ for $\mu > 0$.

2. Gegenbauer has proven

$$P_n^{(\gamma+\mu,\gamma+\mu)}(x) = \sum_{k=0}^{n} \alpha_{kn} P_k^{(\gamma,\gamma)}(x),$$

with $\alpha_{kn} \geq 0$ for $\mu > 0$.

In both cases α_{kn} were computed explicitly.

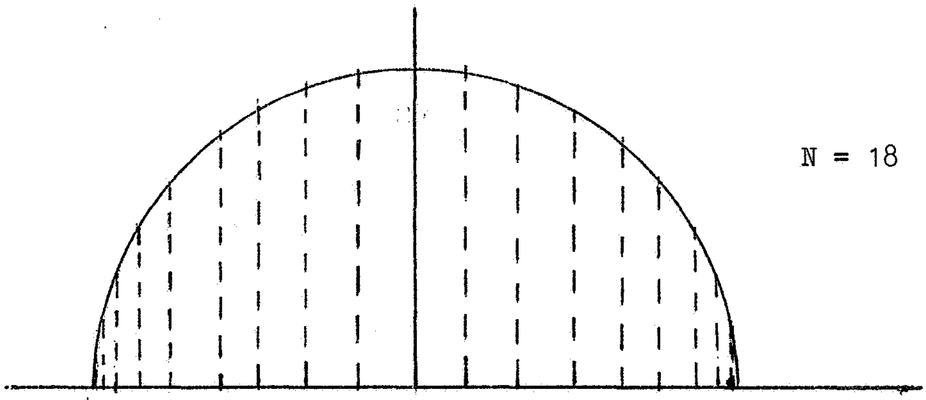
Next we will consider a recent theorem of M. Wayne Wilson. Wilson has studied some discrete orthogonal polynomials that approximate Legendre polynomials. The standard classical example is the set of discrete Tchebycheff polynomials. They are defined as follows:
Divide the unit segment into N equal parts. Then give each of the points $\frac{x}{N}(x = 0,..,N)$ the weight $\frac{1}{N+1}$. If we let $x_i = {}^i/N$ the discrete Tchebycheff polynomials are defined by the orthogonality relation

$$\frac{1}{N+1} \sum_{i=0}^{N} t_n(x_i;N)\, t_m(x_i;N) = 0, \quad (m \neq n)$$

and the standardization $t_n(0;N) = 1$.
It is not hard to show that as the division of the unit segment is finer, $t_n(Nx;N)$ converges to $P_n(1-2x)$. Unfortunately the convergence is not very good, for in another paper Wilson has given the formula

$$t_n(Nx;N) = P_n(1-2x) + \frac{x(1-x)}{2N} P_n'(1-2x) + O(\frac{1}{N^2}) \; .$$

But $P_n'(1-2x)$ grows in n like $n^{\frac{3}{2}}$ for fixed x, $0 < x < 1$ and an infinite number of n, while $P_n(1-2x)$ decreases like $n^{-\frac{1}{2}}$.
Thus, unless $N \geq cn^2$ for some $c > 0$, it is not clear, that $t_n(Nx;N) - P_n(1-2x)$ is small and in general it is not. To obtain a nicer set of polynomials, Wilson has constructed a new discrete polynomial as follows. He divided the upper part of the unit circle into N equal parts and prejected the division points into the x-axis.



Now x_i has the mass density $\mu(x_i) = x_{i+1} - x_i$ for $i = 0,\ldots,N$. Since $x_i = -\cos\frac{\pi i}{N}$ the precise value of $\mu(x_i)$ is $\cos\frac{\pi i}{N} - \cos\frac{\pi(i+1)}{N}$

Wilson defined his polynomials by the orthogonality relation:

$$\sum_{i=0}^{N} w_n(x_i;N)\, w_m(x_i;N)\, \mu(x_i) = 0, \quad n \neq m.$$

Now the segment $[-1,1]$ is not uniformly distributed,but the convergence of $w_n(Nx;N)$ to $P_n(1-2x)$ is much more rapid. Also, $w_n(Nx;N)$ behaves qualitatively like a Legendre polynomial for $N \geq n$ and not just $N \geq cn^2$ as in the case of discrete Tchebycheff polynomials, in the sense that these polynomials take on their largest value on the interval of orthogonality at the end points of that interval. The discrete Tchebycheff polynomials do not in general.

For the investigation of his polynomials Wilson used the following theorem.

Theorem: Let $\{p_n(x)\}$ be a set of polynomials orthonormal on E with respect to $w(x)$ and let $\{q_n(x)\}$ be a set polynomials orthonormal on E with respect to $v(x)$.

If

$$\int_E p_n(x)\, p_m(x)\, v(x)dx < 0, \quad n \neq m,$$

then

$$q_n(x) = \sum_{k=0}^{n} \alpha_{kn}\, p_k(x), \quad \alpha_{kn} > 0.$$

For the proof Wilson used Stieltjes' theorem:

If A is a symmetrical matrix with positive elements in the main diagonal and negative elements elsewhere, then its inverse has only positive elements.

These new polynomials are very interesting and much work remains to be done. We are still lacking most of the standard properties of orthogonal polynomials. For example we have no explicit expression and we do not know the coefficients in the recurrence formula.

We will close this lecture with a theorem about positivity, which comes from the recurrence formulas.

Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be two sets of polynomials orthogonal on E with respect to the respective weight functions $w(x)$ and $v(x)$. Let $p_n(x)$ and $q_n(x)$ satisfy:

$$x\, p_n(x) = p_{n+1}(x) + \alpha_n\, p_n(x) + \beta_n\, p_{n-1}(x),\quad \beta_0 = 0, \beta_n > 0 \quad \text{for } n \geq 1,$$

$$x\, q_n(x) = q_{n+1}(x) + \gamma_n\, q_n(x) + \delta_n\, q_{n-1}(x),\quad \delta_0 = 0, \delta_n > 0 \quad \text{for } n \geq 1 \ .$$

Then $q_{n+1}(x) = \sum_{k=0}^{n+1} = a(k,n+1)\, p_k(x) =$

$$= (x-\gamma_n) \sum_{k=0}^{n} a(k,n)\, p_k(x) - \delta_n \sum_{k=0}^{n-1} a(k,n-1)\, p_k(x) =$$

$$= \sum_{k=0}^{n} a(k,n) \left[p_{k+1}(x) + (\alpha_k-\gamma_n)\, p_k(x) + \beta_k\, p_{k-1}(x)\right] -$$

$$- \sum_{k=0}^{n-1} \delta_n\, a(k,n-1)\, p_k(x).$$

So $a(k,n+1) = a(k-1,n) + (\alpha_k-\gamma_n)\, a(k,n) + \beta_{k+1}\, a(k+1,n) - \delta_n\, a(k,n-1).$

A more surveyable result is

$$a(k,n+1) - a(k-1,n) = (\alpha_k-\gamma_n)\ a(k,n) +$$

$$+ \ (\beta_{k+1}-\delta_n)\ a(k+1,n) + \delta_n\big[a(k+1,n) - a(k,n-1)\big].$$

Now, if $\alpha_k > \gamma_n$ and $\beta_{k+1} > \delta_n$ for $k = 0,\ldots,n$, a simple induction shows that the coefficients $a(k,n)$ are nonnegative.

One application is the following:
For Legendre polynomials we have

$$x\, P_n(x) = P_{n+1} + \frac{n^2}{4n^2-1}\ P_{n-1}(x).$$

Here $\frac{1}{4} < \beta_n < \beta_{n-1}$.
Define the associated polynomials $P_n(x;\nu)$ by

$$x\, P_n(x;\nu) = P_{n+1}(x;\nu) + \frac{(n+\nu)^2}{4(n+\nu)^2-1}\, P_{n-1}(x;\nu).$$

Then we have

$$P_n(x;\nu) = \sum_{k=0}^{n} \alpha_{kn}\, P_k(x), \qquad \alpha_{kn} \geq 0.$$

The coefficients are positive here. This was already known from an explicit expression of α_{kn} found by Barrucand and Dickinson. Other examples are given in TW 114 note II.

Literature.

R. Askey. Orthogonal polynomials and positivity, to appear in proceedings of symposium on special functions, SIAM.

R. Askey. Three notes on orthogonal polynomials. Edited by the Mathematical Centre at Amsterdam as TW 114.
For Stieltjes' theorem, see Szegö's book.

Lecture 4

Hypergeometric functions and their applications.

This lecture will deal with hypergeometric functions and their applications.

At first we define $(a)_n$:

$$(a)_0 = 1 \text{ and } (a)_n = a(a+1)\dots(a+n-1), \text{ for } n \geq 1.$$

If a is not equal 0, -1, -2,..., $(a)_n$ is also given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} .$$

Now we define the hypergeometric function ${}_2F_1\,(a,b;c;x)$ as follows:

$$(1) \qquad {}_2F_1\,(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\; n!}\; x^n,$$

for $|x| < 1$.
Using Stirling's formula for $\Gamma(a+n)$ we can investigate the behavior of the series on the unit circle for all possible values of a,b and c. It appears then that the series converges absolutely for $\mathrm{Re}(a+b-c) < 0$, conditionally for $0 \leq \mathrm{Re}(a+b-c) < 1$ with a pole at $x = 1$ and diverges for $\mathrm{Re}(a+b-c) \geq 1$. Gauss has computed ${}_2F_1(a,b;c;1)$ and found that

$$(2) \qquad {}_2F_1\,(a,b;c;1) = \frac{\Gamma(c)\;\Gamma(c-a-b)}{\Gamma(c-a)\;\Gamma(c-b)} .$$

Another special case is the important formula

$$(3) \qquad {}_2F_1\,(a,b;b;x) = (1-x)^{-a} .$$

Now we consider ${}_2F_1\,(a,b;c;xy)$.

The function

$$(4) \qquad \int_0^1 {}_2F_1\,(a,b;c;xy)\; y^{\alpha}(1-y)^{\beta}\; dy$$

is analytic in a,b,c and x for the values mentioned above. Before evaluating this integral we will give a generalization of the hypergeometric function.

For p and q positive integers and $b_i \neq 0,-1,-2,\dots$ $(i=1,\dots,q)$ we define

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n\; n!}\; x^n.$$

Evaluation of (4) gives

$$\int_0^1 {}_2F_1(a,b;c;xy)\; y^{\alpha}(1-y)^{\beta}\; dy =$$

$$\sum_{n=0}^{\infty} \frac{(a)_n\;(b)_n}{(c)_n\; n!}\; x^n \int_0^1 y^{n+\alpha+1-1}(1-y)^{\beta+1-1}\; dy =$$

$$\sum_{n=0}^{\infty} \frac{(a)_n\;(b)_n}{(c)_n\; n!}\; \frac{\Gamma(n+\alpha+1)\;\Gamma(\beta+1)}{\Gamma(n+\alpha+\beta+2)}\; x^n =$$

$$\frac{\Gamma(\alpha+1)\;\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \sum_{n=0}^{\infty} \frac{(a)_n\;(b)_n\;(\alpha+1)_n}{(c)_n\;(\alpha+\beta+2)_n}\; \frac{x^n}{n!} =$$

$$\frac{\Gamma(\alpha+1)\;\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\; {}_3F_2(a,b,\alpha+1;c,\alpha+\beta+2;x).$$

So we have

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\;\Gamma(\beta)} \int_0^1 {}_2F_1(a,b;c;xy)\; y^{\alpha-1}\;(1-y)^{\beta-1}\; dy =$$

$$(5) \qquad = {}_3F_2(a,b,\alpha;\alpha+\beta,c;x).$$

If we put b = c we get Euler's formula

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \int_0^1 (1-xy)^{-a}\ y^{\alpha-1}\ (1-y)^{\beta-1}\ dy =$$

$$(6) \qquad = {}_2F_1(a,\alpha;\alpha+\beta;x),$$

which gives us a nice integral representation of the hypergeometric function. Letting x = 1 and applying (6) we have

$${}_2F_1(a,\alpha;\alpha+\beta;1) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)} \int_0^1 y^{\alpha-1}\ (1-y)^{\beta-a-1}\ dy =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\ \Gamma(\beta)}\ \frac{\Gamma(\alpha)\ \Gamma(\beta-a)}{\Gamma(\alpha+\beta-a)} = \frac{\Gamma(\alpha+\beta)\ \Gamma(\beta-a)}{\Gamma(\beta)\ \Gamma(\alpha+\beta-a)},$$

which is Gauss' result (2).

Note. There exists a generalization of the hypergeometric series due to Heine. He defined the operator Δ_q by

$$\Delta_q\ f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

For x^n we have

$$\Delta_q\ x^n = \frac{q^n - 1}{q - 1}\ x^{n-1}.$$

The basic hypergeometric series of Heine is defined by

$$\sum_{n=0}^{\infty} \frac{[a]_{n,q}\ [b]_{n,q}}{[c]_{n,q}\ [1]_{n,q}}\ x^n,$$

where

$$a_{n,q} = \frac{q^a - 1}{q - 1}\ \frac{q^{a+1} - 1}{q - 1} \cdots \frac{q^{a+n-1} - 1}{q - 1}.$$

If $q \to 1$, then $[a]_{n,q} \to (a)_n$ and $\Delta_q \to \frac{d}{dx}$. It would be useful to obtain fractional integration theorems for these Heine series using the inverse operator to Δ_q to define an integral.

After this intermediar note we will apply the theory of hypergeometric series to the Jacobi polynomials using the important relation

$$(7) \qquad \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}) .$$

We know that in some cases a Jacobi polynomial of a certain class can be expressed as the sum of Jacobi polynomials of a different class with nonnegative coefficients.

Example

$$P_n^{(\alpha+\mu,\beta)}(x) = \sum_{k=0}^{n} \alpha_{kn} P_k^{(\alpha,\beta)}(x) ,$$

with $\alpha_{kn} \geq 0$ if $\mu \geq 0$.

Now we want to derive some continuous analogues of this and other formulas by means of the hypergeometric series, i.e. we want nonnegative kernels $K(x,y)$, for which e.g. the relation

$$(8) \qquad \frac{P_n^{(\alpha+\mu,\beta)}(x)}{P_n^{(\alpha+\mu,1)}(1)} = \int_{-1}^{1} K(x,y) \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} dy$$

holds.

Before proving (8), we will derive some other relations.

We know already, that

$$\frac{\Gamma(c+\mu)}{\Gamma(c)\,\Gamma(\mu)} \int_0^1 y^{c-1} (1-y)^{\mu-1} {}_2F_1(a,b;c;xy)dy =$$

$$= {}_2F_1(a,b;c+\mu;x).$$

Letting $a = -n$, $b = n+\alpha+\beta+1$, $c = \alpha+1$ and $xy = \frac{1}{2}(1-xy)$ and recalling (7), we get after some substitutions

$$\frac{P_n^{(\alpha+\mu,\beta-\mu)}(x)}{P_n^{(\alpha+\mu,\beta-\mu)}(1)} (1-x)^{\alpha+\mu} =$$

$$(9) \qquad = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\ \Gamma(\mu)} \int_x^1 (1-y) \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} (y-x)^{\mu-1}\, dy .$$

So

$$K(x,y) = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\ \Gamma(\mu)} (1-y)^{\alpha} (y-x)^{\mu-1}, \qquad y \geq x$$

$$= 0 \quad \text{elsewhere.}$$

Since $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ we have

$$(1+x)^{\beta+\mu} \frac{P_n^{(\alpha-\mu,\beta+\mu)}(x)}{P_n^{(\alpha-\mu,\beta+\mu)}(1)} =$$

$$(10) \qquad = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\ \Gamma(\mu)} \int_{-1}^{x} (1+y)^{\beta} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(-1)} (x-y)^{\mu-1} dy.$$

In all these cases μ should be positive.
In (10) μ should also be smaller than $\alpha + 1$.

Another formula can be derived as follows:

Since $_2F_1(a,b-\mu;c;x) =$

$$= \frac{\Gamma(b)}{\Gamma(\mu)\ \Gamma(b-\mu)} \int_0^1 y^{b-\mu-1}(1-y)^{\mu-1}\ {}_2F_1(a,b;c;xy)dy,$$

we have after some suitable substitutions

$$(1+x)^{n+\alpha+\beta}\ P_n^{(\alpha-\mu,\beta)}(x) =$$

(11) $$= \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1-\mu)\ \Gamma(\mu)} \int_{-1}^{x} (1+y)^{n+\alpha+\beta-\mu} P_n^{(\alpha,\beta)}(y)\ (x-y)^{\mu-1} dy.$$

Again using the formula for $P_n^{(\alpha,\beta)}(-x)$, we have

$$(1-x)^{n+\alpha+\beta}\ P_n^{(\alpha,\beta-\mu)}(x) =$$

(12) $$= \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1-\mu)\ \Gamma(\mu)} \int_{x}^{1} (x-y)^{n+\alpha+\beta-\mu}\ P_n^{(\alpha,\beta)}(y)\ (y-x)^{\mu-1}\ dy.$$

Now we will derive the formula we initially wanted. We have therefore to derive some auxiliary formulas, involving hypergeometric functions. We recall that

$${}_2F_1(a,b;c;x) =$$

$$= \frac{\Gamma(b)}{\Gamma(b)\ \Gamma(c-b)} \int_0^1 (1-xy)^{-a}\ y^{b-1}\ (1-y)^{c-b-1}\ dy.$$

Substituting 1 - s for y we have

(13) $${}_2F_1(a,b;c;x) = (1-x)^{-a}\ {}_2F_1(a,c-b;c;\tfrac{x}{x-1}).$$

Because of the symmetry in a and b we have

(14) $${}_2F_1(a,b;c;x) = (1-x)^{-b}\ {}_2F_1(c-a,b;c;\ \tfrac{x}{x-1}) \quad .$$

Using (13) on (14) gives

$$(15) \qquad {}_2F_1(a,b;c;x) = (1-x)^{c-a-b} \; {}_2F_1(c-a,c-b;c;x).$$

Now it is very easy to see that

$$(1-x)^{-a} \; {}_2F_1(a,c+\mu-b;c+\mu;\frac{x}{x-1}) =$$

$$(16) \qquad = \frac{\Gamma(c+\mu)}{\Gamma(c)\;\Gamma(\mu)} \int_0^1 y^{c-1}\,(1-y)^{\mu-1}\,(1-xy)^{-a} \; {}_2F_1(a,c-b;c;\frac{xy}{xy-1})\,dy \;.$$

Letting t = x/(x-1) and s = xy/(xy-1) and replacing c-b by b we get

$$\frac{\Gamma(c+\mu)}{\Gamma(c)\;\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\,(1-s)^{a-c-\mu}\,s^{c-1} \; {}_2F_1(a,b;c;s)\,ds =$$

$$= t^{c+\mu-1}(1-t)^{\alpha-c} \; {}_2F_1(a,b+\mu;c+\mu;t).$$

By the substitutions $s = \frac{1}{2}(1-y)$, $t = \frac{1}{2}(1-x)$, $a = -n, b = n+\alpha+\beta+1$, $c = \alpha+1$, we finally have

$$\frac{(1-x)^{\alpha+\mu}}{(1+x)^{n+\alpha+1}} \; \frac{P_n^{(\alpha+\mu,\beta)}}{P_n^{(\alpha+\mu,\beta)}(1)} =$$

$$(17) \qquad = \frac{2^{\mu}\;\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\;\Gamma(\mu)} \int_x^1 \frac{(1-y)^{\alpha}}{(1+y)^{n+\alpha+\mu+1}} \; \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} \, (y-x)^{\mu-1} dy.$$

So the kernel wanted in (8) equals

$$2^{\mu}\,\frac{(1-y)^{\alpha}}{(1+y)^{n+\alpha+\mu+1}} \; \frac{(1+x)^{n+\alpha+1}}{(1-x)^{\alpha+\mu}} \; \frac{(y-x)^{\mu-1}}{B(\alpha+1,\mu)} \;, \qquad y \geq x,$$

0 elsewhere.

Using the expression for $P_n^{(\alpha,\beta)}(-x)$ we get

$$\frac{(1+x)^{\beta+\mu}}{(1-x)^{n+\beta+1}} \quad \frac{P_n^{(\alpha,\beta+\mu)}(x)}{P_n^{(\alpha,\beta+\mu)}(-1)} =$$

(18) $$= \frac{2^{\mu}\,\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\,\Gamma(\mu)} \int_{-1}^{x} \frac{(1+y)^{\beta}}{(1-y)^{n+\beta+\mu+1}} \quad \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(-1)}\,(x-y)^{\mu-1}\,dy.$$

An important application of these two integral transforms can be made on the ultraspherical polynomials $C_n^{\lambda}(x)$. We define them as follows

(19) $$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n}\;P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x)\;, \qquad \lambda > 0 \quad .$$

We use the auxiliary relations

(20) $$\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)} = \frac{C_{2n}^{\alpha+\frac{1}{2}}(x)}{C_{2n}^{\alpha+\frac{1}{2}}(1)}\;,$$

(21) $$\frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)} = x\;\frac{P_n^{(\alpha,\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,\frac{1}{2})}(1)} = \frac{C_{2n+1}^{\alpha+\frac{1}{2}}(x)}{C_{2n+1}^{\alpha+\frac{1}{2}}(1)} \quad .$$

If we choose $\beta = \pm\frac{1}{2}$ in (11) and (17) and recall (20) and (21) we see after some tedious calculations the following results:

$$C_n^{\lambda}(x) = \frac{2\,\Gamma(\nu)}{\Gamma(\lambda)\,\Gamma(\nu-\lambda)} \;*$$

(22) $$*\int_{0}^{1} t^{n+2\lambda-1}\;C_n^{\lambda}(xt)(1-t^2)^{\nu-\lambda-1}\,dt,$$

$$(\nu > \lambda > 0)$$

and

$$\frac{C_n^{\nu}(\cos\theta)\,\sin^{2\nu-1}\theta}{C_n^{\nu}(1)\,\cos^{n+2\nu+1}\theta} = 2\,\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})\,\Gamma(\nu-\lambda)} \;*$$

$$(23)\qquad \int_0^{\infty} \frac{\sin^{2\lambda}\psi(\cos^2\psi-\cos^2\theta)^{\nu-\lambda-1}}{\cos^{n+2\nu}\psi}\;\frac{C_n^{\lambda}(\cos\psi)}{C_n^{\lambda}(1)}\,d\psi\;,$$

$$0 < \theta < \frac{\pi}{2} \quad \nu > \lambda.$$

The latter formula is due to Feldheim and Vilenkin and can be used to obtain a number of results. For instance

$$\sum_{n=0}^{N} \frac{C_n^{\lambda}(x)}{C_n^{\lambda}(1)} \geq 0, \quad N = 0,1,\ldots, -1 \leq x \leq 1, \ \lambda \geq \tfrac{1}{2},$$

follows from Féjérs result $\sum_{n=0}^{N} P_n(x) \geq 0$.

It would be very interesting to find a general theorem,which says something about the problem of writing a solution of one Sturm-Liouville equation as an integral of a solution of a different S.-L.-equation. In the dual case - that of second order difference equations - some theorems have been given in lecture 3. These theorems are not very satisfactory but at least they exist.

Literature: An extensive bibliography is given in R. Askey and J. Fitch, Integral representations for Jacobi polynomials and some applications, J. Math. Anal. Appl. 26(1969), 411-437.

Lecture 5

Some more positivity results.

A new result of the type considered in lecture 2 has come up and we will start with it. In Math. Zeit. 37(1933) G. Szegö proved the following conjecture of K. Friedrichs and H. Lewy:

$$\frac{1}{(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)} = \sum_{n,m,k=0}^{\infty} A_{n,m,k}\, r^n s^m t^k,$$

with $A_{n,m,k} \geq 0$. His proof used Bessel functions but he concluded the main part of his paper with the following observation.

Define the Laguerre polynomial $L_n(x)$ by

$$\frac{e^{-\frac{xr}{1-r}}}{1-r} = \sum_{n=0}^{\infty} L_n(x) r^n.$$

Then

$$\frac{e^{-\frac{x}{1-r}}}{1-r} = \sum_{n=0}^{\infty} e^{-x} L_n(x) r^n,$$

and so

$$\frac{e^{-x\left(\frac{1}{1-r}+\frac{1}{1-s}+\frac{1}{1-t}\right)}}{(1-r)(1-s)(1-t)} = \sum_{n,m,k=0}^{\infty} L_n(x)\, L_m(x)\, L_k(x)\, e^{-3x}\, r^n s^m t^k.$$

Integration from 0 to ∞ gives

$$\frac{1}{(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)} = \sum_{n,m,k=0}^{\infty} A_{n,m,k}\, r^n s^m t^k,$$

with

$$A_{n,m,k} = \int_0^{\infty} L_n(x)\, L_m(x)\, L_k(x)\, e^{-3x} dx.$$

This is equivalent to

$$e^{-2x} L_n(x)\ L_m(x) = \sum_{k=0}^{\infty} A_{n,m,k}\ e^{-2x} L_k(x)\ ,$$

or

$$e^{-2x} L_n(x)\ e^{-2x} L_m(x) = \sum_{k=0}^{\infty} A_{n,m,k}\ e^{-2x} L_k(x)$$

and in this form it resembles the problem we considered in lecture 2. There we considered problems like

$$L_n(x)\ L_m(x) = \sum_{k=|n-m|}^{n+m} \alpha(k,m,n)\ L_k(x)$$

and a simple calculation from the recurrence formula for $L_n(x)$ shows

$$(-1)^{k+m+n}\ \ \alpha(k,m,n) \geq 0\ \ .$$

Szegö's result is more interesting, since we have $e^{-2x} L_n(x) = 1$ for $x = 0$. Therefore

$$\sum_{k=0}^{\infty} A_{n,m,k} = 1 \text{ and so } \sum_{k=0}^{\infty} |A_{n,m,k}| = 1,$$

since $A_{n,m,k} \geq 0$, which we will show after some lines. We have

$$\sum_{k=|n-m|}^{n+m} \alpha(k,m,n) = 1, \quad \text{but} \quad \sum_{k=|n-m|}^{n+m} |\alpha(k,m,n)| \text{ is unbounded in m and n}$$

and for many applications, in particular in the construction of Banach algebras from these results, it is exactly the boundedness of $\sum |\alpha(k,m,n)|$ or $\sum |A_{n,m,k}|$ that we need.

We will show that $A_{n,m,k} \geq 0$.

It is possible to use similar methods to show its strict positivity but we do not need the positivity for any applications. The positivity proof, some related monotonicity results and some stronger results will be given in a joint paper with George Gasper.

First recall that for Legendre polynomials

$$P_n(x)\ P_m(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n)(k+\tfrac{1}{2})\ P_k(x),$$

with $a(k,m,n) \geq 0$. But

$$a(k,m,n) = \int_{-1}^{1} P_n(x)\ P_m(x)\ P_k(x)dx.$$

We have

$$\left(\frac{1+x}{2}\right)P_n^{(\alpha,\beta+1)}(x) = A_n\ P_{n+1}^{(\alpha,\beta)}(x) + B_n\ P_n^{(\alpha,\beta)}(x),$$

with A_n, $B_n > 0$. The positivity of A_n follows from $P_n^{(\alpha,\beta)}(1) > 0$ and the fact that all the zeros of $P_n^{(\alpha,\beta)}(x)$ lie in $-1 < x < 1$, so $P_n^{(\alpha,\beta)}(x) = k_n\ x^n + \ldots$, $k_n > 0$. Also $P_n^{(\alpha,\beta)}(-x) = (-1)^n\ P_n^{(\beta,\alpha)}(x)$ and letting $x = -1$ gives the positivity of B_n. Combined with the nonnegativity of $a(k,m,n)$ this gives

$$\int_{-1}^{1} P_n^{(0,j)}(x)\ P_m^{(0,j)}(x)\ P_k^{(0,j)}(x)\ \left(\frac{1+x}{2}\right)^{3j}\ dx \geq 0\ .$$

Now set $x = 1 - 2y/j$ and let $j \to \infty$ using

$$\lim_{\beta\to\infty} P_n^{(0,\beta)}\left(1 - \frac{2x}{\beta}\right) = L_n(x),$$

to get

$$\int_0^{\infty} L_n(x)\ L_m(x)\ L_k(x)e^{-3x}\ dx \geq 0.$$

Szegö generalized this to

$$\int_0^\infty L_{n_1}^\alpha(x) \dots L_{n_k}^a(x)e^{-kx}\,dx \geq 0,\ \alpha \geq -\tfrac{1}{2},$$

and this result also follows from our method. This result of Szegö is equivalent to

$$\frac{1}{[f'(1)]^{\alpha+1}} = \sum A_{n_1,\dots,n_k} x_1^{n_1}\dots x_k^{n_k};\ A_{n_1,\dots,n_k} \geq 0,$$

$$\alpha \geq -\tfrac{1}{2},$$

where $f(x) = (x-x_1)\dots(x-x_k)$.

This problem is a beautiful example of the usefulness of special functions. A direct proof has been given of the original problem of Friedrichs and Lewy, but it is complicated and there seems to be no hope at all of obtaining Szegö's general result on j variables with an arbitrary power by any other than using some properties of special functions. As Gasper and I will show in the promised paper it is possible to use other special functions to obtain stronger results. We will show

$$\int_0^\infty L_n(x)\,L_m(x)\,L_k(x)\,e^{-3x}\,dx > 0.$$

In lecture 2 we proved a result that gives

$$(-1)^{n+m-k}\int_0^\infty L_n(x)\,L_m(x)\,L_k(x)\,e^{-x}\,dx \geq 0 .$$

There are no known results of this type for Charlier or Meixner polynomials. The Meixner polynomials are orthogonal on $x = 0,1,\dots,$ with respect to the mass distributions $\frac{(\beta)_x\,C^x}{x!}$. It is not clear what the theorem should be or even if there is a theorem of this type. They do not always exist.

Now we will consider the dual problem. We want to find a(n) so that

$$\sum_{n=0}^{\infty} a(n)\, L_n(x)\, L_n(y)\, L_n(z) \geq 0, \quad x,y,z \geq 0.$$

The only such a(n) is $a(0) \geq 0$, $a(n) = 0$, $n = 1,2,\ldots$. This follows from the following result of Sarmanov.

Theorem 1. If $\sum_{n=0}^{\infty} c(n)\, L_n(x)\, L_n(y) \geq 0$, $x,y \geq 0$, and $\sum_{n=0}^{\infty} |c(n)|^2 < \infty$, then

$$c(n) = \int_0^1 t^n \, d\mu(t) \quad , \quad d\mu(t) \geq 0 \qquad .$$

The positivity of the series

$$\sum_{n=0}^{\infty} t^n\, L_n(x)\, L_n(y)$$

is well known and theorem 1 says that in some sense these are the only positive bilinear series of Laguerre polynomials. If

$$\sum_{n=0}^{\infty} a(n)\, L_n(x)\, L_n(y)\, L_n(z) \geq 0, \text{ then by theorem 1}$$

we have

$$a(n)\, L_n(z) = \int_0^1 t^n \, d\mu_z(t) \quad , \quad d\mu_z(t) \geq 0 \qquad .$$

But for $n = 1,2,\ldots,$ the left hand side changes sign with z, unless $a(n) = 0$. There are slight technical problems about showing that $\sum_{n=0}^{\infty} |a(n)|^2\, |L_n(z)|^2 < \infty$ but they are easily bypassed using the positivity of

$$\sum_{n=0}^{\infty} t^n\, L_n(x)\, L_n(y), \; 0 < t < 1, \text{ for if}$$

$$\sum_{n=0}^{\infty} a(n)\, L_n(x)\, L_n(y)\, L_n(z) \geq 0,$$

so is

$$\sum_{n=0}^{\infty} t_n\, a(n)\, L_n(x)\, L_n(y)\, L_n(z)\, e^{-\frac{z}{2}}$$

and

$$|L_n(z)\, e^{-\frac{z}{2}}| \leq 1.$$

Furthermore $\sum_{n=0}^{\infty} a(n)$ converges so, $|a(n)| \leq C$.

The Meixner polynomials are self-dual:

$$M_n(x) = M_x(n) = {}_2F_1(-n,-x;\beta;1-\tfrac{1}{2}),$$

and they have Laguerre polynomials as limits. Thus, any result that will be obtained for them will have to have both Szegö's and Sarmanov's results as limiting theorems. Strange as it seems, it is possible to have positive theorems for trilinear expansions and have results of the Sarmanov type as a limit.

Consider the case of ultraspherical polynomials $C_n^{\lambda}(x)$. These polynomials are orthogonal with respect to the measure $(1-x^2)^{\lambda-\frac{1}{2}}dx$ and can be defined by the generating function

$$\frac{1-r^2}{(1-2xr+r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{n+\lambda}{\lambda} C_n^{\lambda}(x)\, r^n, \quad \lambda > 0.$$

If $\lambda \to 0$, we obtain

$$\frac{1-r^2}{1-2xr+r^2} = \frac{1}{2} + \sum_{n=0}^{\infty} r^n \cos n\theta, \quad x = r\cos\theta.$$

$$\text{So } \lim_{\lambda\to 0} \frac{n+\lambda}{\lambda} C_n^{\lambda}(\cos\theta) = \begin{cases} \frac{1}{2}, & \text{if } n = 0, \\ \cos\theta, & \text{if } n = 1,2,\ldots . \end{cases}$$

If we let $x = y\lambda^{-\frac{1}{2}}$ in the weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$, we see that $w(y^{\lambda-\frac{1}{2}}) \to e^{-y^2}$ as $\lambda \to \infty$.

It is easy to show that

$$\lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} C_n^\lambda(x\,\lambda^{-\frac{1}{2}}) = H_n(x)/n!$$

So $\cos n\theta$ and $H_n(x)$ are both contained as limits of $C_n^\lambda(x)$. Also, so is x^n, for

$$\lim_{\lambda\to\infty} \frac{C_n^\lambda(x)}{C_n^\lambda(1)} = x^n.$$

From the addition formula for ultraspherical polynomials Bochner made the following observation. If

(1) $$f(x) \sim \sum_{n=0}^{\infty} a_n \left(\frac{n+\lambda}{\lambda}\right) C_n^\lambda(x),$$

and $f(x) \in L^1$, then the formal series

$$f(x;y) \sim \sum_{n=0}^{\infty} a_n \left(\frac{n+\lambda}{\lambda}\right) C_n^\lambda(x)\, C_n^\lambda(y)/C_n^\lambda(1)$$

is for almost all y, $-1 \leq y \leq 1$, the expansion of an L^1 function $f(x;y)$ and what is decisive for us, if $f(x) \geq 0$ then $f(x;y) \geq 0$. If $f(x)$ has the expansion (1) then a_n is given by

$$a_n = \frac{1}{C_\lambda} \int_{-1}^{1} f(x)\, \frac{C_n^\lambda(x)}{C_n^\lambda(1)}\, (1-x^2)^{\lambda-\frac{1}{2}} dx,$$

where

$$C_\lambda = \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} dx.$$

From this it is an easy observation to the following characterization of nonnegative bilinear sums

$$f(x;y) \sim \sum_{n=0}^{\infty} a_n \left(\frac{n+\lambda}{\lambda}\right) C_n^\lambda(x)\, \frac{C_n^\lambda(y)}{C_n^\lambda(1)} \geq 0,$$

iff $$a_n = \frac{1}{C_\lambda} \int_{-1}^{1} f(x) \frac{C_n^\lambda(x)}{C_n^\lambda(1)} (1-x^2)^{\lambda-\frac{1}{2}} dx,$$

where $f(x) \geq 0$ of course.

Some remarks should be made about the interpretation of the positivity everywhere or almost everywhere if $f(x,y)$ is integrable, or in the sense of distributions. In the last case it is only possible to prove that

$$a_n = \int_{-1}^{1} \frac{C_n^\lambda(x)}{C_n^\lambda(1)} d\mu(x), \quad d\mu(x) \geq 0.$$

If we let $\lambda \to 0$ in this theorem and use

$$\lim_{\lambda\to 0} \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} = \cos n\theta \text{ and } \lim_{\lambda\to 0} \frac{n+\lambda}{\lambda} C_n(\cos\theta) = \begin{cases} \frac{1}{2} & , \text{ if } n=0 \\ \cos n\theta, & \text{if } n\geq 1, \end{cases}$$

we obtain formally the trivial and well known result that

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \geq 0, \quad 0 \leq \theta \leq \pi,$$

iff

$$f(\theta;\phi) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \cos n\phi \geq 0, \ 0 \leq \theta, \ \phi \leq \pi \quad .$$

We proved this at the first lecture already.

It is more interesting to let $\lambda \to \infty$. If we do and we use

$$\lambda^{-\frac{n}{2}} C_n^\lambda(x\lambda^{-\frac{1}{2}}) \quad \to \quad H_n(x)/n! \quad \text{and}$$

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} \quad \to \quad x^n$$

we formally obtain the following theorem of Sarmanov:

$$f(x;y) \sim \sum_{n=0}^{\infty} a_n \frac{H_n(x)\, H_n(y)}{2^n\, n!} \geq 0,$$

iff
$$a_n = \int_{-1}^{1} t^n \, d\mu(t) \quad , \quad d\mu(t) \geq 0.$$

This theorem can be proven if $\sum_{n=0}^{\infty} a_n^2 < \infty$. Thus it is possible to obtain these strong bilinear expansion theorems,which say that the Poisson kernel is essentially the only nonnegative bilinear expansion and a formal limit of results, where there are many of nonnegative bilinear expansions. This makes the case of Meixner polynomials that much more interesting, for in that case I do not know which way I expect the result to be. And until this problem is solved we will not know which of the above results, Szegö's or Sarmanov's, is typical of expansions on an infinite interval.

We should mention that the ultraspherical result of $\lambda = 1$ is especially interesting. It is

$$f(\theta;\phi) \sim \sum_{n=1}^{\infty} a_n \sin n\theta \sin n\phi \geq 0, \ 0 \leq \theta, \ \phi \leq \pi,$$

iff
$$f(\theta) \sim \sum_{n=1}^{\infty} n\, a_n \sin n\theta \geq 0, \ 0 \leq \theta \leq \pi,$$

which we already know from lecture 1.

There exist other methods which can be used to prove Bochner's result. Weinberger has shown that it follows from a maximum theorem for hyperbolic differential equations and Gasper has shown that results of this type follow from transformation and reduction formulas for hypergeometric functions and the classical theorem of Sonine on integrals of three Bessel functions. Gasper's work solves the problem for Jacobi polynomials.

However, we can not use Bochner's proof, since the addition theorem has not yet been found for Jacobi polynomials. Gasper's work suggests, that this addition formula will be essentially more complicated than Gegenbauer's addition formula for ultraspherical polynomials. This involves $C_n^{\lambda}(\cos\theta \cos\phi + \sin\theta \sin\phi \cos\chi)$, while the corresponding result for $P_n^{(\alpha,\beta)}$ probably has elliptic functions instead of trigonometric functions as variables. However, it would be very interesting to obtain this addition theorem. The most promising methods are probably algebraic methods over high dimensional Lie algebras, where by high we mean at least six. The calculations will probably be very complicated, but the result is important enough to justify the extensive calculations, which will be necessary.

Lecture 6

Mean convergence of orthogonal series.

This lecture will deal with mean convergence of orthogonal series and continuity of linear operators. We will state some problems in the simplest case, i.e. the trigonometric series, and hence we will investigate, how far they can be extended to other orthogonal series.

Let $$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx ,$$

and let $$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx} .$$

Let $f(x) \in L^p(-\pi,\pi)$ i.e.

$$\int_{-\pi}^{\pi} |f(x)|^p dx < \infty \quad \text{for } 1 < p < \infty.$$

The question is now:

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |S_N(x) - f(x)|^p dx = 0?$$

M. Riesz proved that the answer is yes.

The problem above is a special case of the multiplier problem: Let $f \in L^p$, $1 < p < \infty$, $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$; let $(t_n)_{n=-\infty}^{\infty}$ be a bounded sequence of complex numbers i.e. $|t_n| \leq t \ \forall n$. Now the linear transform T is defined as follows:

If $$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \text{ then } Tf(x) \sim \sum_{n=-\infty}^{\infty} t_n c_n e^{inx}.$$

For $f \in L^2$ T is bounded and $||T|| \leq t$ by the Riesz-Fischer theorem, since this gives $||T f||_2 \leq t||f||_2$.

The corresponding result for L^p, $p \neq 2$, is easily shown to be false and the problem of finding necessary and sufficient conditions on (t_n) for T to be a bounded operator from L^p to L^p is still open for $1 < p < \infty$, $p \neq 2$.

The M. Riesz conjugate function theorem can be formulated as follows:

If $|t_n| \leq C$ and $\sum_{n=-\infty}^{\infty} |t_n - t_{n+1}| \leq C$ then there exists an A_p with $||T f||_p \leq A_p C||f||_p$, $1 < p < \infty$.

A generalization due to Marcinkievicz is:

If $|t_n| \leq C$ and $\sum_{|n|=2^N}^{2^{N+1}} |t_n - t_{n+1}| < C$, $N = 0,1,\ldots$, then

$||T f||_p \leq A_p C||f||_p$.

After this introduction we will talk about the analogous problem for expansions in some orthogonal polynomials.

Let $p_n(x) = k_n x^n + \ldots$ $(n = 0,1,\ldots)$ be polynomials orthonormal on $[a,b]$ with respect to $d\alpha(x)$. Let $f(x)$ be integrable on $[a,b]$ with respect to $d\alpha(x)$. For $f(x)$ we define $S_n^f(x)$:

$$S_n^f(x) = \sum_{k=0}^{n} a_k p_k(x) \quad ,$$

where

$$a_k = \int_a^b f(x) p_k(x) d\alpha(x).$$

Now we want to show that $||S_n^f||_p \leq A_p||f||_p$, where

$$||f||_p^p = \int_a^b |f(x)|^p d\alpha(x).$$

Using the Christoffel-Darboux formula, we have

$$S_n^f(x) = \sum_{k=0}^{n} a_k\, p_k(x) = \int_a^b f(y) \sum_{k=0}^{n} p_k(x)\, p_k(y)\, d\alpha(y) =$$

$$= \frac{k_n}{k_{n+1}} \int_a^b f(y) \;\frac{p_{n+1}(x)\, p_n(y) - p_{n+1}(y)\, p_n(x)}{x - y}\; d\alpha(y).$$

Now if a and b are finite then $\left|\frac{k_n}{k_{n+1}}\right| \le C$. The proof is simple but very technical and not enlightening and will not be given here.

Since the polynomials are not uniformly bounded and we are not assured that the measure does not grow too fast at any point, we must use some sort of cancellation. We consider a = +1 and b = -1. We may assume $0 \le x \le 1$ since the same type of argument will handle $-1 \le x < 0$. Then if $-1 \le y \le -\varepsilon < 0$ the factor $(x-y)^{-1}$ is bounded and we no longer have a singular integral except at possible singularities in $d\alpha(y)$. We now assume that $d\alpha(y) = w(y)dy = (1-y)^{\alpha}\,(1+y)^{\beta}\, t(y)$ where $0 \le A \le t(y) \le B < \infty$, $\alpha, \beta > -1$ and $|t(x+h) - t(x)| \le A_p h$. We will only consider $\alpha, \beta \ge -\frac{1}{2}$ but the case $\alpha, \beta > -1$ can also be handled. We have now

$$(1-x^2)^{\frac{1}{4}}\, [w(x)]^{\frac{1}{2}}\, [p_n(x)] \le A.$$

Now we can consider each of the terms $p_{n+1}(x)\, p_n(y)$ and $p_n(x)\, p_{n+1}(y)$ separately and then we need to estimate

$$g(x) = \int_{-1}^{-\varepsilon} |f(y)|\, (1-x)^{-\frac{\alpha}{2} - \frac{1}{4}} (1+y)^{-\frac{\beta}{2} - \frac{1}{4} + \beta} dy.$$

We then get

$$\int_0^1 |g(x)|^p\, (1-x)^{\alpha}\, (1+x)^{\beta} dx \le A_p \int_0^1 (1-x)^{\alpha - p(\frac{\alpha}{2}+\frac{1}{4})} dx *$$

$$* \left[\int_{-1}^{-\varepsilon} |f(y)| \ (1+y)^{\frac{\beta}{2}-\frac{1}{4}} dy\right]^p,$$

and applying Hölder's inequality, we have

$$\int_0^1 |g(x)|^p \ (1-x)^\alpha \ (1+x)^\beta dx \le A_p \int_{-1}^1 |f(y)|^p \ (1-y)^\alpha (1+y)^\beta dy,$$

if $\quad p < 4(\alpha+1)\ (2\alpha+1)$ and $p > 4(\beta+1)/(2\beta+3)$.

Next we consider

$$p_{n+1}(x)\ p_n(y) - p_n(x)\ p_{n+1}(y) =$$

$$= \left[\frac{p_{n+1}(x)}{p_{n+1}(1)} - \frac{p_n(x)}{p_n(1)}\right] p_n(y)\ p_{n+1}(1) + \left[\frac{p_n(y)}{p_n(1)} - \frac{p_{n+1}(y)}{p_{n+1}(1)}\right] p_n(x) p_{n+1}(1).$$

We see that

$$\frac{p_n(x)}{p_n(1)} - \frac{p_{n+1}(x)}{p_{n+1}(1)} = c_n(1-x)\ \frac{q_n(x)}{q_n(1)}$$

for some $c_n > 0$ where $q_n(x)$ are the polynomials orthonormal on $[a,b]$ with respect to $(1-x)^{\alpha+1}\ (1+x)^\beta\ t(x) = (1-x)\ w(x)$. Then we also have

$$(1-x^2)^{\frac{1}{4}}\ (1-x)^{\frac{1}{2}}\ [w(x)]^{\frac{1}{2}}\ \ |q_n(x)| \le C.$$

For the continuation of the proof we need an estimate of $p_{n+1}(1)\ c_n/q_n(1)$. Equation of the coefficients of y^{n+1} gives

$$c_n\ \frac{p_{n+1}(1)}{q_n(1)} = \frac{k_{n+1}}{l_n}.$$

where l_n is the highest coefficient of $q_n(x)$.

Szegö has shown,that if

$$\int_{-1}^{1} \frac{|\log w(x)|}{\sqrt{(1-x^2)}}\, dx < \infty,$$

then

$$\frac{k_n}{2^n} \to \frac{1}{\sqrt{\pi}} \exp - \frac{1}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{\sqrt{(1-x^2)}}\, dx,$$

as n goes to infinity. In our case both w(x) and (1-x) w(x) satisfy the above condition,so we have $|k_{n+1}/l_n| \leq A_p$. Thus the integrals are bounded by

$$A_n(x) \int_{-\varepsilon}^{1} \frac{f(x)(1-y)^{\frac{\alpha}{2}-\frac{1}{4}}(1-x)^{\frac{3}{4}-\frac{\alpha}{2}}}{x-y} B_n(y)\, dy$$

and

$$A_n(x) \int_{-\varepsilon}^{1} \frac{f(y)(1-y)^{\frac{\alpha}{2}+\frac{3}{4}}(1-x)^{-\frac{1}{4}-\frac{\alpha}{2}}}{x-y} B_n(y)\, dy,$$

where $A_n(x)$ and $B_n(x)$ are functions bounded both in x and n. Since we are interested in L^p norms,we may ignore them since

$$||f\, B_n||_p \leq A_p||f||_p.$$

Now we have reduced our problem to estimating

$$(1-x)^{-\frac{\alpha}{2}+\frac{1}{4}+\frac{1}{2}} \int_{-\varepsilon}^{1} \frac{f(y)(1-y)^{\frac{\alpha}{2}+\frac{1}{4}-\frac{1}{2}}}{x-y}\, dy,$$

and such integrals are classical. So $||S_n^f||_p \leq A||f||_p$ for same p depending on α and β. The exact range is the same as we encountered before,

$$\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1} \quad \text{and the same inequalities with } \beta$$

instead of α.

Now let us consider $\alpha = \beta = 0$, $t(x) \equiv 1$ as a special case. We have then the Legendre polynomials $P_n(x)$.

Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x),$$

then

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{1} f(x) P_n(x) dx,$$

$$S_n^f(x) = \frac{n+1}{2} \int_{-1}^{1} f(t) \frac{P_{n+1}(x) P_n(t) - P_n(x) P_{n+1}(t)}{x - t} dt$$

and

$$||S_n^f||_p \leq A||f||_p, \text{ for } \frac{4}{3} < p < 4.$$

It is well known that

$$P_n(\cos\theta) \sim \frac{\cos\left[(n+\tfrac{1}{2})\theta - \frac{\pi}{2}\right]}{\sqrt{n}\,(\sin\theta)^{\frac{1}{2}}}, \ n \to \infty.$$

Next we want to compare

$$f(\theta) \sim \sum_{n=0}^{\infty} a_n \cos n\theta$$

and

$$g(\theta) \sim \sum_{n=0}^{\infty} a_n P_n(\cos\theta)\sqrt{n}\,(\sin\theta)^{\frac{1}{2}}.$$

These two functions should have much in common.

We now define $||f||_{p,\alpha}$ as follows:

$$||f||_{p,\alpha} = \left[\int_0^{\pi} |f(\theta)|^p (\sin\theta)^{\alpha} d\theta\right]^{\frac{1}{p}}.$$

We would like to show that

$$||g||_{p,\alpha} \leq A||f||_{p,\alpha} \leq A'||g||_{p,\alpha},$$

$$1 < p < \infty, \ -1 < \alpha < p-1.$$

If we could prove this, we could argue as follows:

Let

$$T\, f \sim \sum_{n=0}^{\infty} t_n\, a_n \cos n\theta,$$

$$T\, g \sim \sum_{n=0}^{\infty} t_n\, a_n\, P_n(\cos\theta)\sqrt{n}\,(\sin\theta)^{\frac{1}{2}}.$$

If it is also true that

$$||T\, f||_{p,\alpha} \le A_1 ||f||_{p,\alpha},$$

then we would have

$$||Tg||_{p,\alpha} \le A_1 ||Tf||_{p,\alpha} \le A_2 ||f||_{p,\alpha} \le A_3 ||g||_{p,\alpha}.$$

Consider the case

$$t_n = 1 \text{ for } n \le N,$$

$$t_n = 0 \text{ for } n > N.$$

Hardy and Littlewood have shown that

$$||S_N^f||_{p,\alpha} = ||T\, f||_{p,\alpha} \le A_p ||f||_{p,\alpha}, \quad 1 < p < \infty, \ -1 < \alpha < p-1,$$

where A_p is independent of N.

So, if the conjecture is true, then

$$\int_0^{\pi} \Big| \sum_{n=0}^{N} a_n \sqrt{n}\, P_n(\cos\theta)\Big|^p (\sin\theta)^{\frac{p}{2}+\alpha}\, d\theta \le A ||f||_p^p .$$

If we choose $\frac{p}{2} + \alpha = 1$ and use $-1 < \alpha < p-1$ we have $\frac{4}{3} < p < 4$. Thus we would have a new proof of Pollard's mean convergence theorem.

This conjecture is true, but rather then give a proof of it, we will sketch a proof of the dual result.

Dual theorem:

Let

$$a_n = \int_0^\pi f(\cos\theta) \cos n\theta \; d\theta ,$$

$$b_n = \int_0^\pi f(\cos\theta) P_n(\cos\theta) \sqrt{n \sin\theta} \; d\theta.$$

We want $||a_n||_{p,\alpha} \approx ||b_n||_{p,\alpha}$, for $1 < p < \infty$, $-1 < \alpha < p-1$,

where $||a_n||_{p,\alpha} = [\sum_{n=0}^{\infty} |a_n|^p (n+1)^\alpha]^{\frac{1}{p}}$.

We will give a sketch of the proof. Formally

$$b_n = \int_0^\pi f(\theta) P_n(\cos\theta) \sqrt{n} (\sin\theta)^{\frac{1}{2}} d\theta =$$

$$= \sum_{k=0}^{\infty} a_k \sqrt{n} \int_0^\pi \cos k\theta \, P_n(\cos\theta) (\sin\theta)^{\frac{1}{2}} d\theta =$$

$$= \sum_{k=0}^{\frac{1}{2}n} \dots + \sum_{k=\frac{1}{2}n+1}^{2n} \dots + \sum_{k=2n+1}^{\infty} \dots .$$

We would like to show that these terms are

$$O(\frac{1}{n} \sum_{k=0}^{n/2} a_k) + \sum_{n/2+1}^{2n} \frac{a_k}{k-n} + O(\sum_{2n+1}^{\infty} \frac{a_k}{k}) .$$

The middle term can be estimated using asymptotic properties of $P_n(\cos\theta)$ and it is the desired term plus smaller terms with bounded $L^{p,\alpha}$ norms.

We consider

$$\sqrt{n} \int_0^{\pi} P_n(\cos\theta) \cos k\theta \, (\sin\theta)^{\frac{1}{2}} \, d\theta.$$

Now $P_n(\cos\theta) = \sum_{j=0}^{n} \alpha_j \cos j\theta$, $\alpha_j \geq 0$, as we proved in the third lecture.

So the above integral equals

$$\tfrac{1}{2}\sqrt{n} \sum_{j=0}^{n} \alpha_j \int_0^{\pi} [\cos(k-j)\theta + \cos(k+j)\theta] (\sin\theta)^{\frac{1}{2}} \, d\theta =$$

$$= O\left(\frac{\sqrt{n}}{k^{3/2}}\right) = O\left(\frac{1}{n}\right) \quad \text{for } k > n.$$

Observe that all we needed about α_j was $\alpha_j \geq 0$ and $\sum_{j=0}^{n} \alpha_j = 1$.

This estimate takes care of the first term. To handle the third term we must expand $P_n(\cos\theta)$ in terms of some functions $\phi_j(x)$, for $j \geq n$, so that the subscripts j and k will stay apart. We use

$$P_n(\cos\theta) = \sum_{j=n+1}^{\infty} a(j,n) \frac{\sin j\theta}{\sin\theta},$$

which was given in lecture 3. In this case we need $a(j,n)$ explicitly but we have them. We also must use

$$\sin j\theta \cos k\theta = \tfrac{1}{2}[\sin(j+k)\theta + \sin(j-k)\theta]$$

and here we again have nonnegative coefficients, this time because $j > k$.

This proof can be extended to ultraspherical polynomials and now we need

$$C_n^{\lambda}(x) \, C_m^{\lambda}(x) = \sum_{k=|n-m|}^{n+m} a(k,m,n) C_k^{\lambda}(x),$$

$$C_n^{\lambda}(x) = \sum_{k=0}^{n} \beta(k,n) \, C_k^{\mu}(x),$$

$$C_k^{\lambda}(x)(1-x^2)^{\lambda-\frac{1}{2}} = \sum_{n=k}^{\infty} \gamma(k,n)\ C_n^{\mu}(x)(1-x^2)^{\mu-\frac{1}{2}},$$

all of which we have considered and a new result

$$C_n^{\lambda+1}(x)\ C_m^{\lambda}(x) = \sum_{k=|n-m|}^{n+m} \delta(k,m,n)\ C_k^{\lambda+1}(x).$$

We have said nothing about this result before, since we still have no general theorems which contain the positivity result for $\delta(k,m,n)$ If $n \geq m-1$ then $\delta(k,m,n) \geq 0$ and this generalizes

$$\sin n\theta \cos m\theta = \tfrac{1}{2}\left[\sin(n+m)\theta + \sin(n-m)\theta\right] .$$

Actually, it was this proof which finally convinced the author, that there should be general theorems of the type given in lectures 2 and 3. There is now an improved proof of this result and of its dual, which not only works for Jacobi polynomials, but for Fourier-Bessel and Dini series and many Sturm-Liouville expansions. However, the above proof has more than historical interest, since it shows how some of the fundamental properties of orthogonal expansions we have been considering can be used.

Lecture 7

Gaussian quadrature.

In this lecture we will show the importance of the zeros of orthogonal polynomials in approximation theory. From the Weierstrass theorem we know that a continuous function $f(x)$ on $[-1,1]$ can be approximated uniformly by polynomials. One natural way to attempt to prove this theorem is by interpolation. Divide $[-1,1]$ into k+1 parts $[x_i, x_{i+1}]$, $i = 0,1,\ldots,k$, $x_0 = -1$, $x_k = 1$. Let $L_k^f(x)$ be the polynomial of degree k-1 with the property $L_k^f(x_j) = f(x_j)$, $j = 1,\ldots,k$.

If $w_j(x) = \dfrac{(x-x_1)\ \ldots\ (x-x_k)}{(x_j-x_1)\ \ldots\ (x_j-x_n)}$, where the terms $(x-x_j)/(x_j-x_j)$ are omitted, then $L_k^f(x) = \sum_{j=1}^{k} f(x_j)\, w_j(x)$. A natural choice for x_j is the set of equispaced points, but this is a very bad choice. Actually, it can be proven, that no choice works for all continuous functions. However, if we ask for less, then we can still obtain interesting theorems for an appropriate choice of x_j.

Suppose we wish to compute $\int_{-1}^{1} f(x)\, d\alpha x$. If $f(x)$ is a polynomial of degree k-1, then $L_k^f(x) = f(x)$ and so $\int_{-1}^{1} L_k^f(x)\, d\alpha(x) = \int_{-1}^{1} f(x)\, d\alpha(x)$.

A surprising result of Gauss and Jacobi is that this identity holds for polynomials of degree 2k-1, if the x_j are suitably chosen. Let $d\alpha(x)$ be a nonnegative measure on $[-1,1]$ and $p_n(x)$ the polynomials orthogonal with respect to $d\alpha(x)$. Choose $x_{j,k}$ $(j=1,\ldots,k)$ as the k zeros of $p_k(x)$. Then, if $f(x)$ is a polynomial of 2k-1, $f(x) - L_k^f(x) = p_k(x)\, q_{k-1}(x)$ (q_{k-1} polynomial of degree k-1), since $f(x_{j,k}) = L_k^f(x_{j,k})$. Then

$$\int_{-1}^{1} [f(x) - L_k^f(x)]\, d\alpha(x) = \int_{-1}^{1} p_k(x)\, q_{k-1}(x)\, d\alpha(x) = 0,$$

because of the orthogonality.

Since we have $\int_{-1}^{1} L_k^f(x)\,d\alpha(x) = \int_{-1}^{1} f(x)\,d\alpha(x)$ for a larger class of polynomials than usual, it is natural to consider $\int_{-1}^{1} L_k^f(x)\,d\alpha(x)$ for an arbitrary continuous function. Stieltjes (1884) proved that

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n^f(x) - f(x)]\,d\alpha(x) = 0.$$

The essential step in the proof is to observe that

$$\int_{-1}^{1} w_j(x)\,d\alpha(x) = \int_{-1}^{1} w_j^2(x)\,d\alpha(x) = \lambda_j \geq 0,$$

since each of these integrals is equal to the same sum and all of the terms in this sum vanish except one term, which we will call λ_j. Erdős and Turan (1935) extended Stieltjes' result to

(1) $$\lim_{n\to\infty} \int_{-1}^{1} [L_n^f(x) - f(x)]^2\,d\alpha(x) = 0,$$

for all continuous functions. This is an extension of Stieltjes' theorem, since (1) implies

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n^f(x) - f(x)]\,d\alpha(x) = 0,$$

which is clearly stronger than Stieltjes' theorem. For $d\alpha(x) = (1-x^2)^{-\frac{1}{2}}dx$ Erdős and Feldheim and independently Marcinkiewicz proved that

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n^f(x) - f(x)]^p \frac{dx}{\sqrt{(1-x^2)}} = 0, \quad p < \infty.$$

For $d\alpha(x) = (1-x^2)^{\frac{1}{2}}dx$ Feldheim showed the existence of a continuous function f, for which

$$\int_{-1}^{1} |L_n^f(x) - f(x)|^4 (1-x^2)^{\frac{1}{2}}dx$$

goes to infinity.

It is also possible to consider interpolation at the zeros of one set of orthogonal polynomials and ask for convergence with respect to a different measure.

Szegö proved that

$$\lim_{n\to\infty} \int_{-1}^{1} [f(x) - L_n^f(x)]\,dx = 0,$$

if the interpolation is taken at the zeros of the polynomials orthogonal with respect to $(1-x)^{\alpha} (1+x)^{\beta} dx$ for $\alpha,\beta \leq 3/2$ and he also proved that this result fails for $\alpha > 3/2$ or $\beta > 3/2$.

The following conjecture would connect these results.

Conjecture. Let $L_n^f(x)$ be defined at the zeros of $P_n^{(\alpha,\beta)}(x)$. Then

$$\lim_{n\to\infty} \left[\int_{-1}^{1} L_n^f(x) - f(x)|^p (1-x)^a (1+x)^b \,dx\right]^{1/p} = 0,$$

for all continuous functions if $\alpha,\beta \geq -\frac{1}{2}$, $a,b > -1$ and $p < \min [4(a+1)/(2\alpha+1),\ 4(b+1)/(2\beta+1]$ and this inequality is best possible.

For certain values we can prove this conjecture. In particular for $a = b = \alpha = \beta \geq -\frac{1}{2}$ it is true. Since the argument holds much more generally (except for one step), we will start with a more general measure. However, we will ask only for L^p convergence with respect to the measure which also determines the interpolation.

It is sufficient to prove

$$\left[\int_{-1}^{1} |L_n^f(x)|^p \,d\alpha(x)\right]^{1/p} \leq A \left[\int_{-1}^{1} |f(x)|^p \,d\alpha(x)\right]^{1/p},$$

for all continuous functions.

We use the converse of Hölder's inequality

$$\left[\int_{-1}^{1} |L_n^f(x)|^p \, d\alpha(x)\right]^{1/p} = ||L_n^f||_p = \sup_{g\in L^{p'}, ||g||_{p'=1}} \int_{-1}^{1} L_n^f(x) \, g(x) \, d\alpha(x),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Now let

$$g(x) \sim \sum_{k=0}^{\infty} b_k \, p_k(x) \quad , \quad \int_{-1}^{1} p_k(x) \, p_j(x) \, d\alpha(x) = \delta_{kj},$$

$S_n^g(x) = \sum_{k=0}^{n-1} b_k \, p_k(x)$. Since $L_n^f(x)$ is polynomial of degree n-1,

$$\int_{-1}^{1} L_n^f(x) \, g(x) \, d\alpha(x) = \int_{-1}^{1} L_n^f(x) \, S_n^g(x) \, d\alpha(x).$$

But $L_n^f(x) \, S_n^g(x)$ is a polynomial of degree 2n-2 and so by the fundamental property of Gaussian quadrature

$$\int_{-1}^{1} L_n^b(x) \, S_n^g(x) \, d\alpha(x) = \sum_{k=1}^{n} L_n^f(x_{k,n}) \, S_n^g(x_{k,n})\lambda_k,$$

where λ_k are the Cotes numbers which are positive.

Recall that $L_n^f(x_{k,n}) = f(x_{k,n})$. Using Hölder's inequality we have

$$\int_{-1}^{1} L_n^f(x) \, g(x) \, d\alpha(x) \leq \left[\sum_{k=1}^{n} |f(x_{k,n})|^p \, \lambda_k\right]^{\frac{1}{p}} *$$

$$\left[\sum_{k=1}^{n} | \, S_n^g(x_{k,n})| ^{p'} \, \lambda_k\right]^{1/p'} .$$

From Stieltjes' result we have

$$\left[\sum_{k=1}^{n} |f(x_{k,n})|^p \, \lambda_k\right]^{1/p} \leq A||f||_p,$$

and if we could bound the other factor by $||S_n||_{p'}$, then the problem would reduce to the partial sum problem, which was considered in the last lecture. For $p' = 2$ and $p' = \infty$ such estimates are easy, but they seem to be hard for other values of p'. It is of course equivalent to showing that

$$\left[\sum_{k=1}^{n} |Q_{n-1}(x_{k,n})|^p \lambda_k\right]^{1/p} \leq A\left[\int_{-1}^{1} |Q_{n-1}(x)|^p \, d\alpha(x)\right]^{1/p},$$

for an arbitrary polynomial of degree $n-1$. One method of attacking this is the following

$$Q_{n-1}(x) = \sum_{k=0}^{n-1} \alpha_k \, p_k(x) = \int_{-1}^{1} Q_{n-1}(y) \, D_{n-1}(x,y) \, d\alpha(y),$$

where

$$D_n(x,y) = \sum_{k=0}^{n} p_k(x) \, p_k(y).$$

If we can add to $D_{n-1}(x,y)$ terms $a_k \, p_k(x) \, p_k(y)$, $n \leq k \leq 2n-1$, so that the resulting kernel, $K_{2n-1}(x,y) = D_{n-1}(x,y) + \sum_{k=n}^{2n-1} a_k \, p_k(x) \, p_k(y)$, is nonnegative for $-1 \leq x,y \leq 1$, then from

$$Q_{n-1}(x) = \int_{-1}^{1} Q_{n-1}(y) \, D_{n-1}(x,y) \, d\alpha(x) =$$

$$= \int_{-1}^{1} Q_{n-1}(y) \, K_{2n-1}(x,y) \, d\alpha(y),$$

we have from Jensen's inequality

$$|Q_{n-1}(x)|^p \leq \int_{-1}^{1} |Q_{n-1}(y)|^p \, K_{2n-1}(x,y) \, d\alpha(y),$$

and so

$$\sum_{k=1}^{n} \lambda_k |Q_{n-1}(x_{k,n})|^p \leq \int_{-1}^{1} |Q_{n-1}(y)|^p \sum_{k=1}^{n} K_{2n-1}(x_{k,n},y)\lambda_k \, d\alpha(y) =$$

$$= \int_{-1}^{1} |Q_{n-1}(y)|^p \int_{-1}^{1} K_{2n-1}(x,y)\, d\alpha(x)\, d\alpha(y) = \int_{-1}^{1} |Q_{2n-1}(y)|^p \, d\alpha(y).$$

One way to construct $K_{2n-1}(x,y) \geq 0$ is to use the nonnegativity of the Cesaro means of some order and the generalized delayed means of de la Vallée-Poussin, Zygmund and Stein.

The following is a reasonable conjecture. If $\alpha + \beta + 1 \geq 0$ and

$$0 \leq f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x),$$

then the (C, $\alpha+\beta+2$) means of the series $\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x)$ are nonnegative. For $\alpha = \beta \geq -\frac{1}{2}$ this is known and best possible. It is also known for $\alpha = -\beta = \frac{1}{2}$. It would follow for $\alpha \geq \beta \geq -\frac{1}{2}$. It would follow for $\alpha \geq \beta \geq -\frac{1}{2}$, if it were known for $\beta = -\frac{1}{2}$, from Bateman's integral which was given in lecture 4. The details of this will not be given here. We will conclude with a method which can be used to form counter examples. For interpolation at the zeros of $P_n^{(\alpha,\beta)}(x)$, Szegö, in his book, has shown the existence of a continuous function f(x) with $L_n^f(1) \geq A\, n^{\alpha+\frac{1}{2}}$. It is also not too hard to show that

$$|Q_n(x)| \leq A * n^{(2\alpha+2)/p} \left| \int_{-1}^{1} |Q_n(x)|^p (1-x)^\alpha (1+x)^\beta dx \right|^{1/p},$$

if $\alpha \geq \beta$, $\alpha \geq -\frac{1}{2}$. Similar inequalities are given in Timan's book on approximation theory. If $Q_n(x)$ is $L_n^f(x)$ then

$$||L_n^f||_p \geq A\, n^{-(2\alpha+2)/p} L_n^f(1) \geq A\, n^{\alpha+\frac{1}{2}-(2\alpha+2)/p}$$

and this exponent is positive if $p > 4(\alpha+1)/(2\alpha+1)$. To show that mean convergence fails for $p = 4(1+\alpha)/(1+2\alpha)$, one must go back to Szegö's construction and examine it in more detail. We spare the reader these tedious calculations.

The best reference at present is Szegö's book, Orthogonal polynomials.

Lecture 8

Some open problems.

This will not be a record of the last lecture. This lecture dealt with a few qualitative results on the classical polynomials and their zeros. The results described were all in the literature and my only contribution was to mention a few simple extensions and some open problems. It is this last that will be given here. Some of the problems mentioned here were not given in the lecture.

We have already mentioned the important problem of finding an addition formula for $P_n^{(\alpha,\beta)}(x)$. This would generalize

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

An even easier result for $\cos\theta$ is

$$\cos^2\theta + \sin^2\theta = 1.$$

In generalizations of this addition formula the functions which replace $\sin\theta$ will be other Jacobi polynomials. In effect $\cos n\theta$ is $P_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta) * a_n$ where $a_n^{-1} = 2^{-2n} * \binom{2n}{n}$ and $\sin n\theta$ is $b_n * P_{n-1}^{(\frac{1}{2},\frac{1}{2})}(\cos\theta) * \sin\theta$ where $b_n^{-1} = (2n+1) * 2^{-2n}\binom{2n}{n}$.

However in generalizations of $\cos^2\theta + \sin^2\theta = 1$, $\cos n\theta$ is $P_n^{(-\frac{1}{2},-\frac{1}{2})}(\cos\theta)$ and $\sin n\theta$ is the second solution to

$$\frac{d^2u}{d\theta^2} + n^2u = 0.$$

This is suggested by Nicholson's formula for Bessel functions:

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t\,dt,\ \operatorname{Re} z > 0.$$

See Watson, Bessel functions 13.73.

Of course many formulas can be given for $J_\nu^2(z) + Y_\nu^2(z)$ or for the corresponding classical polynomials and second solutions to their differential equations, but what makes Nicholson's formula so useful is what can be proven from it. In 13.74 Watson shows that

$$x[J_\nu^2(x) + Y_\nu^2(x)]$$

is a decreasing function of x when $\nu > \frac{1}{2}$ and an increasing function when $0 < \nu < \frac{1}{2}$. Of course for $\nu = \frac{1}{2}$ this function is a constant, as it must be, since it reduces to $\sin^2 x + \cos^2 x = 1$.

Actually, much more is true about $x[J_\nu^2(x) + Y_\nu^2(x)]$ and many interesting consequences for monotonicity properties of Bessel functions have been obtained by L. Lorch and P. Szegö in Acta Math. v. 109 (1963). It would be very interesting to have similar results for the classical polynomials. Probably the easiest to handle will be $L_n^\alpha(x^2)$ and after that

$$P_n(\cos\theta),\ 0 \leq \theta \leq \frac{\pi}{2}\ .$$

The above results deal with properties of $P_n(x)$ at various points x_k. It is also possible to compare $P_n(x)$ and $P_{n+1}(x)$ at various points For example, Szegö has shown that the k^{th} relative maximum of $|P_n(x)|$ is a decreasing function of n for all $n \geq k+1$. If $\mu_{k,n} - \mu_{k,n+1} > \mu_{k,n+1} - \mu_{k,n+2}$, i.e. this sequence is convex as well as monotone, but this has not been proven as yet.

If $0 = \theta_0 < \theta_1 < - \ldots < \theta_{[\frac{n}{2}+1]}$ denote the first $[\frac{n}{2}] + 2$ zeros of $\sin\theta\, P_n(\cos\theta)$ then this is a convex sequence as Szegö has shown, i.e. $\theta_k - \theta_{k-1}$ is an increasing sequence. A slightly harder result is due to Szegö and Turan. They have shown that the sequence $\theta_{\nu,n-1} - \theta_{\nu,n}$ increases, as ν goes from 1 to $[\frac{n-1}{2}]$. Here $\theta_{\nu,n}$ is the ν-th zero of $P_n(\cos\theta)$ in increasing order. Similar inequalities for $\theta_{\nu,n} - \theta_{\nu,n+1}$ as a function of n with ν fixed are even harder to find. The ultimate monotonicity of this of this sequence can be proven from asymptotic formulas. The inequalities involve the various second difference in $\theta_{\nu,n}$, first in ν, next the mixed difference, one in n and one in ν, and finally the second difference in n.

The first is $O(n^{-2})$, the second is $O(n^{-3})$ and the last is $O(n^{-4})$ as is easily shown from asymptotic formulas. Thus this last should be substantially harder to prove and it seems to be.

These results and problems have just been mentioned for Legendre polynomials, but they can of course be asked for Jacobi and Laguerre polynomials as well. Some results are known and in some cases the proposed theorems are false, but we are far from having a good grasp on those questions. An even harder question is to consider functions of the same degree and different parameters. Markoff and Stieltjes have monotonicity theorems for the zeros of the classical polynomials (Markoff's theorem is more general), but there are a large number of questions we cannot answer even for this type of question.

Consider the Charlier polynomials, $C_n(x;a)$, the polynomials orthogonal with respect to $e^{-a} * a^x/x!$ for $x = 0, 1, \ldots, a > 0$. Let $0 < x_{1,n}^{(a)} < \ldots < x_{n,n}^{(a)}$ be the zeros in increasing order. Then it is not too hard to show that $\lim_{n\to\infty} x_{k,n} = k-1$ for fixed k. Also from general theorems $x_{k,n+1}^{(a)} < x_{k,n}^{(a)}$ so $x_{k,n}^{(a)} > k-1$. An upper bound can be shown to be $x_{n,n}^{(a)} < (1+a)n$. It is likely that $x_{k,n}^{(a)}$ is an increasing function of a, but I have only shown this for $k = 1$ and $k = n$. What is needed is a generalization of Markoff's theorem to measures which are not absolutely continuous.

If $\mu_{k,n}(\alpha)$ denotes the k-th relative maximum of $\dfrac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)}$, then it is likely that $\mu_{k,n}(\alpha)$ is a decreasing function of α. The list of problems of this type can be extended edgeless and others will suggest themselves to the reader.

Another problem concerns positivity results. Féjér proved that $\sum_{k=0}^{n} P_k(x) \geq 0$ and Feldheim generalized this to $\sum_{k=0}^{n} \dfrac{P_k^{(\alpha,\alpha)}(x)}{P_k^{(\alpha,\alpha)}(1)} \geq 0$, $\alpha \geq 0$. He also mentioned the problem

$$(1) \qquad \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} \geq 0;$$

this has been proven for $\alpha \geq \beta$, $\alpha \geq 0$, $\beta \geq -\frac{1}{2}$ and for some $\beta \in (-1,-\frac{1}{2})$ by Askey and Fitch. However this is not the right inequality to prove for $\alpha > \beta$. A stronger inequality would be

$$(2) \qquad \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} \geq 0, \; \alpha > \beta, \; -1 \leq x \leq 1.$$

This is easy to prove for $\alpha = \beta+1$, $\beta \geq -\frac{1}{2}$, and from this and the integrals given in lecture 4 it is possible to prove this for $|\beta| \leq \alpha \leq \beta+1$, $\beta \geq -\frac{1}{2}$.

One consequence of this inequality would be the following conjecture.

If $f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \geq 0$, $|x| \leq 1$,

then

$$\sum_{n=0}^{N} a_n r^n P_n^{(\alpha,\beta)}(x) \geq 0, \; |x| \leq 1, \; N = 0, 1, \ldots,$$

$$r \leq \frac{1}{\alpha+\beta+3}.$$

This conjecture is true for $\beta \leq \alpha \leq \beta+1$, $\beta \geq -\frac{1}{2}$, and it may not hold for $\alpha+\beta+1 < 0$. However it probably does hold for $\alpha+\beta+1 \geq 0$ and it is best possible.

By itself this conjecture is not very important, but it was this problem which showed me that the right inequality to prove was (2) rather then (1). The latter inequality only implies the nonnegativity of $\sum_{n=0}^{N} a_n r^n P_n^{(\alpha,\beta)}(x)$ for $0 \leq r \leq \frac{\beta+1}{\alpha+1} \frac{1}{\alpha+\beta+1}$, $\alpha > \beta$ and this is almost surely not best possible for any (α,β). If there is anything I would like the reader to learn from these pages, it is just this, that without some type of application the wrong problems will usually be asked and the wrong formulas be proved. I can now tell why $\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)}$ is a better sum to consider then $\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)}$, but the fact remains that this knowledge was hindsight and so I will not give it here.

I would much rather want the reader to learn the above moral: do not study special functions for their own sakes. Without motivation and problems from some other field this area becomes sterile very fast. Of course this warning is not unique for special functions, but holds for any other specialized field of mathematics. And with this remark I close my series of lectures.

References: The best references to results on zeros and on inequalities for the classical polynomials is Szegö's book. The Szegö-Turan result is in Publicationes Mathematicae, Debrecen, Tom 8 (1961), 326-335. Szegö's monotonicity result for $\mu_{k,n}$ is in Boll. Union. Matem. Ital. ser. III, Anno V(1950), 120-121. The Askey-Fitch result is in J.M.A.A. 26(1969), 411-437 and many references are given in this paper.