European Journal of Combinatorics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On the linear extension complexity of stable set polytopes for perfect graphs[☆]

Hao Hu [a], Monique Laurent [a,b]

[a] Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
[b] Centrum Wiskunde & Informatica (CWI), Science Park 123, Postbus 94079, 1090 GB Amsterdam, The Netherlands

ARTICLE INFO

Article history:
Available online xxxx

ABSTRACT

We study the linear extension complexity of stable set polytopes of perfect graphs. We make use of known structural results permitting to decompose perfect graphs into basic perfect graphs by means of two graph operations: 2-joins and skew partitions. Exploiting the link between extension complexity and the nonnegative rank of an associated slack matrix, we investigate the behavior of the extension complexity under these graph operations. We show bounds for the extension complexity of the stable set polytope of a perfect graph G depending linearly on the size of G and involving the depth of a decomposition tree of G in terms of basic perfect graphs.

1. Introduction

The polyhedral approach is a classical, fundamental approach to solve combinatorial optimization problems, which aims to represent the convex hull of feasible solutions by linear inequalities and then to use linear programming to solve the optimization problem. One of the major difficulties is that the explicit linear description of the corresponding polytope may need an exponentially large number of linear inequalities (facets) in its natural space. This is the case, e.g., for spanning tree polytopes or for matching polytopes while the corresponding combinatorial optimization problems are in fact polynomially solvable. A widely investigated approach consists in searching for a compact extension (aka extended formulation) of a given polytope P, i.e., searching for another polytope Q lying in a

[☆] This work is dedicated to the memory of Michel Deza, with gratitude for his stimulating role during the early career of the second author.

E-mail addresses: h.hu@tilburguniversity.edu (H. Hu), monique@cwi.nl (M. Laurent).

https://doi.org/10.1016/j.ejc.2018.02.014

higher dimensional space, which projects onto P and has less facets than P. The smallest number of facets of such an extension is known as the *extension complexity* of P, investigated in the seminal work of Yannakakis [32]. The interest in this parameter lies in the fact that linear optimization over P amounts to linear optimization over Q.

Understanding which classes of polytopes have small extension complexity has received considerable attention recently (see, e.g., the surveys [11,21]). Well known classes admitting polynomial extension complexity include ℓ_1-balls, spanning tree polytopes [23,31], permutahedra [17]. On the negative side Rothvoß [28] showed the existence of 0/1 polytopes whose extension complexity grows exponentially with the dimension and Fiorini et al. [15] show that this is the case for classes of combinatorial polytopes including cut, traveling salesman and stable set polytopes of graphs. In particular, a class of graphs on n vertices is constructed in [15] whose stable set polytope has extension complexity at least $2^{\Omega(\sqrt{n})}$. While the latter polytopes correspond to hard combinatorial optimization problems, Rothvoß [29] shows that the matching polytope of the complete graph K_n has exponentially large extension complexity $2^{\Omega(n)}$, answering a long-standing open question of Yannankakis [32].

Yannakakis [32] investigated the extension complexity of stable set polytopes for perfect graphs. While their linear inequality description is explicitly known (given by nonnegativity and clique constraints [10]) it may involve exponentially many inequalities since perfect graphs may have exponentially many maximal cliques. This is the case, for instance, for double-split graphs (see Section 3.1). Using a reformulation of the extension complexity in terms of the nonnegative rank of the so-called slack matrix and a link to communication complexity Yannakakis [32] proved that the extension complexity for a perfect graph on n vertices is in the order $n^{O(\log n)}$. It is an open problem whether this is the right regime or whether the extension complexity can be polynomially bounded in terms of n. This question is even more puzzling in view of the fact that compact *semidefinite* extensions (instead of linear ones) do exist. Indeed the stable set polytope of a perfect graph on n vertices can be realized as projection of an affine section of the cone of $(n+1)\times(n+1)$ positive semidefinite matrices (using the so-called theta body, see [19]). In fact the only known polynomial-time algorithms for the maximum stable set problem in perfect graphs are based on semidefinite programming, and it is open whether efficient algorithms exist that are based on linear programming. We note that it has also been shown in [24] that compact semidefinite extensions do not exist for cut, traveling salesman and stable set polytopes for general graphs.

In this paper we revisit the problem of finding upper bounds for the extension complexity of stable set polytopes for perfect graphs. We make use of the recent decomposition results for perfect graphs by Chudnovsky et al. [7], who proved that any perfect graph can be decomposed into *basic perfect graphs* by means of two graph operations (special 2-joins and skew partitions). There are five classes of basic perfect graphs: bipartite graphs and their complements, line graphs of bipartite graphs and their complements, and double-split graphs. As a second crucial ingredient we use the fundamental link established by Yannakakis [32] between the extension complexity of a polytope and the nonnegative rank of its slack matrix. We investigate how the nonnegative rank of the slack matrix behaves under the graph operation used for decomposing perfect graphs. This allows to upper bound the extension complexity of the stable set polytope of a perfect graph in terms of its number n of vertices and the depth of a decomposition tree. As an application the extension complexity is polynomial for the class of perfect graphs admitting a decomposition tree whose depth is logarithmic in n.

The paper is organized as follows. In Section 2 we recall definitions and preliminary results that we need in the paper. First we consider extended formulations and slack matrices and we recall the result of Yannakakis [32] expressing the extension complexity in terms of the nonnegative rank of the slack matrix. After that we consider perfect graphs and their stable set polytopes and recall the structural decomposition result of Chudnovsky et al. [7] for perfect graphs. In Section 3 we consider the basic perfect graphs and show that their extension complexity is (at most) linear in the number of vertices and edges.

In Section 4 we consider the behavior of the extension complexity of the stable set polytope under several graph operations: graph substitution, 2-joins and skew partitions. Finally in Section 5 we use these results to upper bound the extension complexity for arbitrary perfect graphs in terms of the number of vertices and edges and of the depth of a decomposition tree into basic perfect graphs.

Please cite this article in press as: H. Hu, M. Laurent, On the linear extension complexity of stable set polytopes for perfect graphs, European Journal of Combinatorics (2018), https://doi.org/10.1016/j.ejc.2018.02.014.

2. Preliminaries

Here we group some definitions and preliminary results that we will need in the rest of the paper. In Section 2.1 we consider extended formulations of polytopes and recall the fundamental result of Yannakakis [32] which characterizes the extension complexity of a polytope in terms of the nonnegative rank of its slack matrix. Then in Section 2.2 we recall results about perfect graphs and their stable set polytopes.

Throughout we use the following notation. We let $\text{conv}(V)$ denote the convex hull of a set $V \subseteq \mathbb{R}^d$. For an integer $n \in \mathbb{N}$, we set $[n] = \{1, \ldots, n\}$. Given a subset $S \subseteq [n]$, $\chi^S \in \{0, 1\}^n$ denotes its characteristic vector. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, $G[S]$ denotes the subgraph of G induced by S, with vertex set S and edges all pairs $\{u, v\} \in E$ with $u, v \in S$.

2.1. Extended formulation, extension complexity and slack matrix

An extended formulation of a polytope is a linear system describing this polytope possibly using additional variables. The interest of extended formulations is due to the fact that one can sometimes reduce the number of inequalities needed to define the polytope when additional variables are allowed.

Definition 2.1 (*Extended Formulation*)**.** Let $P \subseteq \mathbb{R}^d$ be a polytope. The linear system

$$Ex + Ft = g, \ \hat{E}x + \hat{F}t \leq \hat{g}, \tag{1}$$

in the variables $(x, t) \in \mathbb{R}^d \times \mathbb{R}^q$, is called an *extended formulation* of P if the following equality holds:

$$P = \{x \in \mathbb{R}^d : \exists t \in \mathbb{R}^q \text{ s.t. } Ex + Ft = g, \ \hat{E}x + \hat{F}t \leq \hat{g}\}.$$

Here, the matrices $E, \hat{E}$ have d columns, the matrices $F, \hat{F}$ have q columns, the additional variable t is called the *lifting variable*, and the *size* of the extended formulation is defined as the number of inequalities in the system (1) (i.e., the number of rows of the matrices $\hat{E}, \hat{F}$). The extended formulation is said to be in *slack form* if the only inequalities are nonnegativity conditions on the lifting variable t, i.e., if it is of the form:

$$Ex + Ft = g, \ t \geq 0 \tag{2}$$

and then its size is the dimension of the variable t.

Remark 1. Note that the inclusion $P \subseteq \{x \in \mathbb{R}^d : \exists t \in \mathbb{R}^q \text{ s.t. } Ex + Ft = g, \ \hat{E}x + \hat{F}t \leq \hat{g}\}$ holds if and only if for every vertex v of P there exists a lifting variable t_v such that the vector (v, t_v) satisfies (1).

Definition 2.2 (*Extension Complexity*)**.** Let $P \subseteq \mathbb{R}^d$ be a polytope. A polytope $Q \subseteq \mathbb{R}^k$ is called an *extension* of P if there exists a linear mapping $\pi : \mathbb{R}^k \to \mathbb{R}^d$ such that $P = \pi(Q)$. The *size* of the extension Q, denoted by $\text{size}(Q)$, is defined as the number of facets of Q. Then the *extension complexity* of P is the parameter $\text{xc}(P)$ defined as

$$\text{xc}(P) = \min\{\textit{size}(Q) : Q \textit{ is an extension of } P\}.$$

As we recall in Theorem 2.6, extended formulations and extensions are in fact equivalent notions and the extension complexity of P can be computed via the nonnegative rank of its slack matrix.

Definition 2.3 (*Slack Matrix*)**.** Let $P \subseteq \mathbb{R}^d$ be a polytope. Consider a linear system $Ax \leq b$ describing P, i.e., $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ with $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$, and a set $V = \{v_1, \ldots, v_n\} \subseteq P$ containing all the vertices of P, i.e., $P = \text{conv}(V)$. Then the $m \times n$ matrix $S = (S_{i,j})$ with entries

$$S_{i,j} = b_i - A_i^T v_j \quad \text{for } i \in [m], \ j \in [n]$$

is called a *slack matrix* of P, said to be induced by V and the linear system $Ax \leq b$.

Please cite this article in press as: H. Hu, M. Laurent, On the linear extension complexity of stable set polytopes for perfect graphs, European Journal of Combinatorics (2018), https://doi.org/10.1016/j.ejc.2018.02.014.

Note that choosing different point sets and linear systems in Definition 2.3 will induce different slack matrices. However, it will follow from Theorem 2.6 that they all have the same nonnegative rank. So we may speak of *the* slack matrix of P without referring explicitly to the selected point set and linear system if there is no ambiguity.

Definition 2.4 (*Nonnegative Rank*)**.** The *nonnegative rank* of a nonnegative matrix $S \in \mathbb{R}_+^{m\times n}$ is defined as

$$\mathrm{rank}_+(S) = \min\{r : \exists\, T \in \mathbb{R}_+^{m\times r} \;\; \exists\, U \in \mathbb{R}_+^{r\times n} \textit{ such that } S = TU\}.$$

In what follows, a decomposition of the form $S = TU$ as above is called a *nonnegative decomposition with intermediate dimension* r.

We refer, e.g., to [16] for an overview of applications of the nonnegative rank and for further references. The following are easy well-known properties of the nonnegative rank, which we will extensively use later. In what follows we use the notation $(S_1\ S_2)$ to denote the matrix obtained by concatenating two matrices S_1 and S_2 (with appropriate sizes).

Lemma 2.5.

(i) *For* $S \in \mathbb{R}_+^{m\times n}$, *we have* $\mathrm{rank}_+(S) = \mathrm{rank}_+(S^T) \le \min\{m, n\}$.
(ii) *For* $S_1 \in \mathbb{R}_+^{m\times n_1}$, $S_2 \in \mathbb{R}_+^{m\times n_2}$, *we have* $\mathrm{rank}_+(S_1\ S_2) \le \mathrm{rank}_+(S_1) + \mathrm{rank}_+(S_2)$.
(iii) *For* $S \in \mathbb{R}_+^{m\times n}$ *and* $b \in \mathbb{R}_+^{n}$, *we have* $\mathrm{rank}_+(S\ b\ b) = \mathrm{rank}_+(S\ b)$ *and, when* b *is the zero vector,* $\mathrm{rank}_+(S\ b) = \mathrm{rank}_+(S)$.

We can now formulate the following result of Yannakakis [32], which establishes a fundamental link between extended formulations, the extension complexity of a polytope and the nonnegative rank of its slack matrix. We also refer, e.g., to [18] for a detailed exposition in the more general setting of conic factorizations.

Theorem 2.6 ([32])**.** *Let* $P = \{x \in \mathbb{R}^d : Ax \le b\}$ *be a polytope whose dimension is at least one, where* $A \in \mathbb{R}^{m\times d}$ *and* $b \in \mathbb{R}^m$, *let* $V = \{v_1, \ldots, v_n\}$ *be a subset of* P *containing the set of vertices of* P, *and let* $S \in \mathbb{R}^{m\times n}$ *be the induced slack matrix. Let* r *be a positive integer. The following assertions are equivalent:*

(i) $\mathrm{rank}_+(S) \le r$;
(ii) *P has an extension of size at most r;*
(iii) *P has an extended formulation in slack form of size at most r;*
(iv) *P has an extended formulation of size at most r.*

Hence the extension complexity of P can be defined by any of the following formulas:

$$\begin{aligned}\mathrm{xc}(P) &= \min\{r : P \text{ has an extension of size } r\}\\ &= \min\{r : P \text{ has an extended formulation (in slack form) of size } r\}\\ &= \mathrm{rank}_+(S) \;\text{ for any slack matrix of } P.\end{aligned}$$

In particular the extension complexity of a d-dimensional polytope is at least $d + 1$.

2.2. Stable set polytopes and perfect graphs

Given a graph $G = (V, E)$, a *stable set* of G is a subset $I \subseteq V$ where no two elements of I form an edge of G. The maximum cardinality of a stable set in G is the stability number of G, denoted by $\alpha(G)$. The *stable set polytope* STAB(G) of G is defined as the convex hull of the characteristic vectors of the stable sets of G:

$$\mathrm{STAB}(G) := \mathrm{conv}\{\chi^I : I \text{ is stable in } G\} \subseteq \mathbb{R}^{|V|}.$$

Computing the stability number $\alpha(G)$ is an NP-hard problem and accordingly the full linear inequality description of the stable set polytope is not known in general. However, for some classes of graphs, there exist efficient algorithms for computing $\alpha(G)$ and an explicit linear inequality description of STAB(G) is known. This is the case in particular for the class of perfect graphs, as we now recall.

The *chromatic number* $\chi(G)$ is the minimum number of colors that are needed to properly color the vertices of G, in such a way that two adjacent nodes receive distinct colors. The clique number of G is the largest cardinality of a clique in G, denoted by $\omega(G)$. Clearly, $\chi(G) \geq \omega(G)$. Following Berge [2] a graph G is said to be *perfect* if $\chi(G') = \omega(G')$ for each induced subgraph G' of G. A classical result of Lovász [25,26] shows that G is perfect if and only if its complement $\overline{G}$ is perfect.

Going back to the stable set polytope of G, it is clear that for any clique C of G, the following linear inequality $\sum_{v \in C} x_v \leq 1$ (called a *clique inequality*) is valid for the stable set polytope. An early result of Chvátal [9] shows that perfect graphs can be characterized as those graphs for which the clique inequalities together with nonnegativity fully describe the stable set polytope.

Theorem 2.7 ([9]). *A graph $G = (V, E)$ is perfect if and only if* STAB(G) *is characterized by the following linear system, in the variables $x \in \mathbb{R}^{|V|}$:*

$$\sum_{v \in C} x_v \leq 1 \qquad \forall\, C \text{ maximal clique of } G, \tag{3}$$

$$x_v \geq 0 \qquad \forall\, v \in V. \tag{4}$$

Hence, when G is a perfect graph, its stable set polytope STAB(G) can be characterized by the nonnegativity constraints and the maximal clique constraints, as stated in Theorem 2.7. However, the number of maximal cliques of G might be exponentially large, and thus this result does not lead directly to an efficient algorithm for solving the maximum stable set problem in perfect graphs. As a matter of fact, as of today, the only known efficient algorithm for this problem is based on using semidefinite programming, as shown by Grötschel, Lovász and Schrijver [19]. It is not known whether an efficient linear programming based algorithm exists for solving this problem. This motivates our work in this paper to investigate the extension complexity of the stable set polytope of perfect graphs.

Throughout we use the following notation: for a graph G, $\mathcal{I}_G$ denotes the set of stable sets of G and $\mathcal{C}_G$ denotes the set of *maximal* cliques of G. We will use the following slack matrix for the stable set polytope of perfect graphs.

Definition 2.8. Given a graph $G = (V, E)$, S_G denotes the slack matrix of STAB(G), whose rows are indexed by $V \cup \mathcal{C}_G$ (corresponding to the nonnegativity constraints (4) and the maximal clique constraints (3)), and whose columns are indexed by $\mathcal{I}_G$, with entries

$$S_G(v, I) = |\{v\} \cap I|, \quad S_G(C, I) = 1 - |I \cap C| \text{ for } v \in V,\ C \in \mathcal{C}_G,\ I \in \mathcal{I}_G.$$

From Theorem 2.6, we know that $\text{rank}_+(S_G) = \text{xc}(\text{STAB}(G))$ when G is perfect. Hence to study the extension complexity of STAB(G) we need to gain insight on the nonnegative rank of the slack matrix and for this we will use the structural decomposition result for perfect graphs from [4,5,7], that we recall below.

Berge [2] observed that if a graph G is perfect then neither G nor $\overline{G}$ contains an induced cycle of odd length at least 5, and he asked whether the converse is true. This was answered in the affirmative recently by Chudnovsky et al. [7], a result known as the *strong perfect graph theorem*. The proof of this result in [7] relies on a structural decomposition result for perfect graphs. We need some definitions to be able to state this decomposition result.

First we introduce double-split graphs, which form an additional class of basic graphs considered in [7], next to bipartite graphs, line graphs of bipartite graphs and their complements.

Definition 2.9 ([7]). Consider integers $p, q \geq 2$ and sets $L_1, \ldots, L_p \subseteq [q]$. A graph $G = (V, E)$ is a *double-split graph*, with parameters $(p, q, L_1, \ldots, L_p)$, if V can be partitioned as $V = V_1 \cup V_2$, where $V_1 = \{a_1, b_1, \ldots, a_p, b_p\}$, $V_2 = \{x_1, y_1, \ldots, x_q, y_q\}$ and

- $G[V_1] = (V_1, E_1)$ is a disjoint union of edges, $G[V_2] = (V_2, E_2)$ is the complement of a disjoint union of edges, say

$$E_1 = \{\{a_i, b_i\} : i \in [p]\}, \quad E_2 = \{\{x_i, y_j\}, \{x_i, x_j\}, \{y_i, y_j\} : i \neq j \in [q]\};$$

- The only edges between V_1 and V_2 are the pairs $\{a_i, x_j\}$, $\{b_i, y_j\}$ for $i \in [p], j \in L_i$, and the pairs $\{a_i, y_j\}$, $\{b_i, x_j\}$ for $i \in [p], j \in [q] \setminus L_i$.

Note that double-split graphs may have at the same time exponentially many maximal cliques and exponentially many maximal independent sets (when choosing, e.g., $p = q$).

The decomposition result for perfect graphs needs two graph operations: 2-joins and skew partitions.

Definition 2.10 ([13]). A 2-join of $G = (V, E)$ is a partition of V into (V_1, V_2) together with disjoint nonempty subsets $A_k, B_k \subseteq V_k$ (for $k = 1, 2$) such that every vertex of A_1 (resp., B_1) is adjacent to every vertex of A_2 (resp., B_2) and there are no other edges between V_1 and V_2.

Definition 2.11 ([7,10]). A *skew partition* of $G = (V, E)$ is a partition of V into four nonempty sets (A_1, B_1, A_2, B_2) such that every vertex in A_1 is adjacent to every vertex in A_2, and there are no edges between vertices in B_1 and vertices in B_2.

The following decomposition result for perfect graphs involves 2-joins and skew partitions with refined properties, namely *proper* 2-joins and *balanced* skew partitions. As these additional properties will play no role in our treatment, we do not include the exact definitions.

Theorem 2.12 ([7, Statement 1.4]). *Let G be a perfect graph. Then, either G belongs to one of the following five basic classes: bipartite graphs and their complements, line graphs of bipartite graphs and their complements, double-split graphs; or one of G or $\bar{G}$ admits a proper 2-join; or G admits a balanced skew partition.*

In this paper we investigate how the extension complexity of the stable set polytope of a perfect graph G can be upper bounded depending on the two decomposition operations (2-joins and skew partitions) that are needed to build G from the basic graph classes.

3. Extension complexity for basic perfect graphs

In this section we show bounds for the extension complexity of the stable set polytope for the basic classes of perfect graphs. Recall the definition of the slack matrix S_G introduced in Definition 2.8. From Theorem 2.6, we know that when G is perfect, the extension complexity of its stable set polytope is given by the nonnegative rank of the matrix S_G:

$$\mathrm{xc}(\mathrm{STAB}(G)) = \mathrm{rank}_+(S_G).$$

So in order to upper bound $\mathrm{xc}(\mathrm{STAB}(G))$ it suffices to upper bound $\mathrm{rank}_+(S_G)$. The following upper bound follows directly from Lemma 2.5(i) (since S_G has $|V(G)| + |\mathcal{C}|$ rows).

Lemma 3.1 ([32]). *Let $G = (V, E)$ be a perfect graph and let $\mathcal{C}$ denote its set of maximal cliques. Then we have:* $\mathrm{xc}(\mathrm{STAB}(G)) \le |V| + |\mathcal{C}|$.

As an example of application of Lemma 3.1, $\mathrm{xc}(\mathrm{STAB}(G)) \le 2|V|$ when G is a chordal graph (i.e., has no induced cycle of length at least 4). Indeed, if G is chordal then G has at most $|V|$ maximal cliques; this is well-known and can easily be seen using the fact that G has a perfect elimination ordering. Moreover, for the complete graph K_p, $\mathrm{xc}(\mathrm{STAB}(K_p)) \le p + 1$, since K_p has a unique maximal clique. As $\mathrm{STAB}(K_p)$ has dimension p, the reverse inequality holds and thus $\mathrm{xc}(\mathrm{STAB}(K_p)) = p + 1$. For the complement of K_p, $\mathrm{xc}(\mathrm{STAB}(\overline{K_p})) \le 2p$, since $\overline{K_p}$ has p maximal cliques. In fact as $\mathrm{STAB}(\overline{K_p}) = [0, 1]^p$ we have $\mathrm{xc}(\mathrm{STAB}(\overline{K_p})) = 2p$ [14].

Furthermore, using Lemma 2.5(ii), one can verify that when G is perfect the extension complexity of the stable set polytope of G and its complement $\overline{G}$ are linearly related.

Lemma 3.2 ([32]). *Let $G = (V, E)$ be a perfect graph and let $\overline{G}$ be its complement. Then*

$$\mathrm{xc}(\mathrm{STAB}(\overline{G})) \le \mathrm{xc}(\mathrm{STAB}(G)) + |V|.$$

In what follows we first present a simple bounding technique for the extension complexity which we then apply to double-split graphs. After that we consider the extension complexity for the other four basic classes of perfect graphs.

3.1. A simple bounding technique and double-split graphs

We begin with a simple bounding technique based on considering a partition $V = V_1 \cup V_2$ of the vertex set of $G = (V, E)$.

Below and later in the paper we will use the following notation. For $k = 1, 2$ we let $G_k = G[V_k]$ denote the subgraph of G induced by V_k, $\mathcal{C}_k$ denotes the set of maximal cliques of G_k and $\mathcal{I}_k$ denotes the set of independent sets of G_k. In addition $\mathcal{C}_{12}$ (resp., $\mathcal{I}_{12}$) denotes the set of 'mixed' maximal cliques (resp., 'mixed' independent sets) of G, i.e., those that meet both V_1 and V_2. Finally, we set $\mathcal{R}_k = V_k \cup \mathcal{C}_k$, so that the rows of the slack matrix S_G of the stable polytope of G are indexed by the set $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{C}_{12}$, while the columns of S_G are indexed by the set $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_{12}$. With respect to these partitions of its row and column index sets, the matrix S_G has the following block form:

$$S_G = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{C}_{12} \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_{12} \end{array} \\ \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} \\ S_{2,1} & S_{2,2} & S_{2,3} \\ S_{3,1} & S_{3,2} & S_{3,3} \end{pmatrix} \end{array}. \tag{5}$$

Note that, for convenience, we let the column indexed by the empty set appear twice, once as column indexed by $\emptyset \in \mathcal{I}_1$ and once as column indexed by $\emptyset \in \mathcal{I}_2$.

Lemma 3.3. *Let $G = (V, E)$ be a perfect graph and let $V = V_1 \cup V_2$ be a partition of its vertex set. Then we have*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(G[V_1])) + \mathrm{xc}(\mathrm{STAB}(G[V_2])) + |\mathcal{C}_{12}|,$$

where $\mathcal{C}_{12}$ denotes the set of maximal cliques of G that meet both V_1 and V_2.

Proof. We use the form of the slack matrix S_G in (5). By construction, for $k = 1, 2$, we have $S_{k,k} = S_{G_k}$, each column of $S_{k,3}$ is the copy of a column of $S_{k,k}$, and each column of $S_{1,2}$ (resp., $S_{2,1}$) coincides with the column of $S_{1,1}$ (resp., $S_{2,2}$) indexed by the empty set. Hence $\mathrm{rank}_+(S_{k,1}\ S_{k,2}\ S_{k,3}) = \mathrm{rank}_+(S_{G_k})$ holds for $k = 1, 2$ (using Lemma 2.5(iii)). Finally, we have $\mathrm{rank}_+(S_{3,1}\ S_{3,2}\ S_{3,3}) \leq |\mathcal{C}_{12}|$ since this matrix has $|\mathcal{C}_{12}|$ rows. Combining these and applying Lemma 2.5 to S_G, we obtain the desired inequality. □

As an example of application, if G is the disjoint union of two graphs G_1 and G_2 then we obtain the well-known bound $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2))$. As another application, we can upper bound the extension complexity for double-split graphs.

Lemma 3.4. *If $G = (V, E)$ is a double-split graph with parameters $(p, q, L_1, \ldots, L_p)$ then*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 5p + 5q \leq 5|V|/2 \text{ and } \mathrm{xc}(\mathrm{STAB}(\overline{G})) \leq 5p + 5q \leq 5|V|/2.$$

Proof. The inequality $5p + 5q \leq 5|V|/2$ is clear since $|V| = 2p + 2q$. As G is perfect and its complement is again a double-split graph (exchanging p and q), it suffices to show the inequality $\mathrm{rank}_+(S_G) \leq 5p + 5q$. For this we use Lemma 3.3, with the partition $V = V_1 \cup V_2$ in the definition of a double-split graph from Definition 2.9. As $G_1 = G[V_1]$ is a disjoint union of p edges, G_1 has p maximal cliques and $2p$ vertices, which implies $\mathrm{rank}_+(S_{G_1}) \leq 3p$ (by Lemma 3.1). As $G_2 = G[V_2]$ is the complement of the disjoint union of q edges, we obtain $\mathrm{rank}_+(S_{G_2}) \leq 3q + 2q = 5q$ (using Lemma 3.2). Finally there are $2p$ maximal cliques in $\mathcal{C}_{12}$, given by the sets $\{a_i\} \cup x_{L_i} \cup y_{\overline{L_i}}$, and $\{b_i\} \cup x_{\overline{L_i}} \cup y_{L_i}$ for $i \in [p]$. Hence, applying Lemma 3.3 we obtain that $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2)) + |\mathcal{C}_{12}|$ is upper bounded by $\mathrm{rank}_+(S_{G_1}) + \mathrm{rank}_+(S_{G_2}) + |\mathcal{C}_{12}| \leq 3p + 5q + 2p = 5p + 5q$. □

3.2. Bipartite graphs and their line graphs and complements

We just saw in Lemma 3.4 that the extension complexity of the stable set polytope of double-split graphs is linear in $|V|$. We now consider the other classes of basic perfect graphs.

The next bound for bipartite graphs and their complements is well known and follows directly from Lemma 3.1 combined with Lemma 3.2.

Lemma 3.5. *Let $G = (V, E)$ be a bipartite graph. Then*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq |V| + |E| \text{ and } \mathrm{xc}(\mathrm{STAB}(\overline{G})) \leq 2|V| + |E|.$$

Recently Aprile et al. [1] showed the following alternative upper bound for bipartite graphs: $\mathrm{xc}(\mathrm{STAB}(G)) = O(|V|^2/\log|V|)$, which is thus sharper than the bound $|V| + |E|$ when the number of edges is quadratic in $|V|$. Moreover a class of bipartite graphs G is constructed in [1] for which $\mathrm{xc}(\mathrm{STAB}(G)) = \Omega(|V|\log|V|)$. Finding the exact regime of the extension complexity for bipartite graphs is still open.

Next we see that for line graphs of bipartite graphs and their complements, the extension complexity is linear in $|V|$.

Lemma 3.6. *Let $G = (V, E)$ be the line graph of a bipartite graph. Then*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 2|V| \text{ and } \mathrm{xc}(\mathrm{STAB}(\overline{G})) \leq 3|V|.$$

Proof. Assume G is the line graph of a bipartite graph $G' = (V', E')$. Then $V(G) = E'$ and $\mathrm{STAB}(G)$ is the matching polytope $M(G')$ of G'. For $v \in V'$, let $\delta(v)$ denote the set of edges in G' incident to v, called the star of v, and let W be the set of vertices $v \in V'$ for which $\delta(v)$ is maximal (i.e., not strictly contained in the star of another vertex of G'). Then, as G' is bipartite, $\mathrm{STAB}(G) = M(G')$ is defined by the nonnegativity constraints $x_e \geq 0$ ($e \in E'$) and the star constraints $\sum_{e\in\delta(v)} x_e \leq 1$ for $v \in W$. We show that in the description of $M(G')$ we need to consider at most $|E'|$ star constraints. Clearly we may assume that G' is connected (else consider each connected component). If some node $v \in W$ is adjacent to a unique other node $u \in V'$ then G' consists only of the edge $\{u, v\}$ and it is clear that one star constraint suffices. Otherwise we may assume that each node $v \in W$ has degree at least 2, which implies $|E'| \geq |W|$ and thus the number of star constraints is at most $|E'|$. Summarizing, the matching polytope of G' is defined by at most $2|E'|$ linear constraints, which shows that $\mathrm{STAB}(G)$ is defined by at most $2|E'| = 2|V|$ linear constraints. The inequality $\mathrm{xc}(\mathrm{STAB}(\overline{G})) \leq 3|V|$ follows using Lemma 3.2. □

Finally we show an upper bound which is uniform for all basic perfect graphs, which we will use in Section 5 to deal with general perfect graphs.

Corollary 3.7. *For every basic perfect graph $G = (V, E)$, $\mathrm{xc}(\mathrm{STAB}(G)) \leq 2(|V| + |E|)$ holds.*

Proof. The claim is obvious if G is bipartite or the line graph of a bipartite graph. Assume now $G = (V, E)$ is bipartite and $\overline{G} = (V, \overline{E})$ is not bipartite (thus $n \geq 3$). It suffices to show $|E| \leq 2|\overline{E}|$, which then implies $\mathrm{xc}(\mathrm{STAB}(\overline{G})) \leq 2|V| + |E| \leq 2(|V| + |\overline{E}|)$. As $|\overline{E}| = \binom{|V|}{2} - |E|$, $|E| \leq 2|\overline{E}|$ is equivalent to $|E| \leq |V|(|V| - 1)/3$, which follows from $|E| \leq |V|^2/4$.

Consider now the case when G is the line graph of a bipartite graph G'. By Lemma 3.6, $\mathrm{xc}(\mathrm{STAB}(\overline{G})) \leq 3|V|$. We show that $\mathrm{xc}(\mathrm{STAB}(\overline{G})) \leq 2(|V| + |\overline{E}|)$, which follows if we can show $|V| \leq 2|\overline{E}|$. If $\overline{G}$ has no isolated vertex then $|V| \leq 2|\overline{E}|$ indeed holds. Assume now $\overline{G}$ has an isolated vertex. Then G has a vertex adjacent to all other vertices, which means G' has an edge incident to all other edges of G'. This implies that G is the union of two cliques intersecting at a single vertex and thus $\overline{G}$ is a bipartite graph, so we are done as this case was treated above.

Finally if G is a double-split graph, then it has no isolated vertex and thus, by Lemma 3.4, $\mathrm{xc}(\mathrm{STAB}(G)) \leq 5|V|/2 \leq 2(|V| + |E|)$. □

4. Graph operations

We now consider some graph operations that play an important role when dealing with perfect graphs. The operation of "graph substitution" was first considered by Lovász [26] as crucial tool for his *perfect graph theorem*, stating that the class of perfect graphs is closed under taking graph complements. After that we consider the two graph operations: 2-joins and skew partitions, that are used in the structural characterization of perfect graphs by Chudnovsky et al. [7].

Please cite this article in press as: H. Hu, M. Laurent, On the linear extension complexity of stable set polytopes for perfect graphs, European Journal of Combinatorics (2018), https://doi.org/10.1016/j.ejc.2018.02.014.

4.1. Graph substitution

In this section, we consider the behavior of the extension complexity of STAB(G) when G is obtained from two other graphs G_1 and G_2 via the "graph substitution" operation. This operation preserves perfect graphs: if G_1 and G_2 are perfect, then G is perfect [9].

Definition 4.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex-disjoint graphs and let u be a vertex of G_1. *Substituting G_2 in G_1 at u* produces the graph $G = \mathcal{S}(G_1, u, G_2)$, where $G = (V, E)$ with

$$V = (V_1 \setminus \{u\}) \cup V_2,$$

$$E = E(G_1[V_1 \setminus \{u\}]) \cup E_2 \cup \bigcup_{v \in V_2} \big\{\{v, w\} : \{u, w\} \in E_1\big\}.$$

We show that the extension complexity of STAB(G) is bounded by the sum of the extension complexities of STAB(G_1) and STAB(G_2). We will use the following lemma.

Lemma 4.2. *Let P be a nonempty polytope. Consider an extended formulation of P:*

$$Ex + Fs = g, \;\; s \geq 0. \tag{6}$$

If the pair (x_0, s_0) satisfies $Ex_0 + Fs_0 = 0$ and $s_0 \geq 0$, then $x_0 = 0$.

Proof. As $P \neq \emptyset$ there exists a feasible solution (x, s) of (6). For any $\lambda \geq 0$, $(x, s) + \lambda(x_0, s_0)$ also satisfies (6), which implies $x + \lambda x_0 \in P$ and thus $x_0 = 0$ since P is bounded. □

Theorem 4.3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex-disjoint graphs and $u \in V_1$. If $G = \mathcal{S}(G_1, u, G_2)$ is the graph obtained by substituting G_2 in G_1 at u, then we have*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2)).$$

Proof. We use the following notation: for $x \in \mathbb{R}^{|V|}$ and $S \subseteq V$, $x(S) = (x_v)_{v \in S}$ denotes the restriction of x to its entries indexed by S. For $i = 1, 2$, set $r_i = \mathrm{xc}(\mathrm{STAB}(G_i))$ and assume the linear system

$$E_i x_i + F_i s_i = g_i, \;\; s_i \geq 0 \tag{7}$$

is an extended formulation in slack form of STAB(G_i) of size r_i (whose existence follows from Theorem 2.6), with variables $x_i \in \mathbb{R}^{|V_i|}$ and lifting variables $s_i \in \mathbb{R}^{r_i}$.

Consider now variables $y_i \in \mathbb{R}^{|V_i|}$ and $t_i \in \mathbb{R}^{r_i}$ for $i = 1, 2$. For convenience set $y_1(\bar{u}) = y_1(V_1 \setminus \{u\})$, so that $y_1 = (y_1(\bar{u}), y_1(u))$ and the vector $(y_1(\bar{u}), y_2)$ is indexed by the vertex set V of G. We claim that the linear system

$$\begin{cases} E_1 y_1 + F_1 t_1 = g_1, & t_1 \geq 0, \\ E_2 y_2 + F_2 t_2 - g_2 \cdot y_1(u) = 0, & t_2 \geq 0 \end{cases} \tag{8}$$

provides an extended formulation of STAB(G), with lifting variables $(t_1, t_2, y_1(u))$. As its size is equal to $r_1 + r_2$ this implies the desired inequality $\mathrm{xc}(\mathrm{STAB}(G)) \leq r_1 + r_2$.

To prove that (8) is an extended formulation of STAB(G), we have to show that a vector $(y_1(\bar{u}), y_2)$ belongs to STAB(G) if and only if there exists a vector $(t_1, t_2, y_1(u))$ in $\mathbb{R}^{r_1+r_2+1}$ for which the vector (y_1, y_2, t_1, t_2) satisfies the linear system (8), where we set $y_1 = (y_1(\bar{u}), y_1(u))$.

We first show the "only if" part. In view of Remark 1 we may assume that $(y_1(\bar{u}), y_2)$ is a vertex of STAB(G). Then $(y_1(\bar{u}), y_2)$ is the characteristic vector χ^I of a stable set I in G. Then the set $I_1 = I \cap V_1$ is a stable set in G_1, contained in $V_1 \setminus \{u\}$, and the set $I_2 = I \cap V_2$ is stable in G_2. We consider the following two cases depending on whether the set $I_1 \cup \{u\}$ is stable in G_1.

(i) If $I_1 \cup \{u\}$ is stable in G_1, then there exists a nonnegative vector $t_1 \in \mathbb{R}^{r_1}$ for which the vector $(y_1, t_1) = (\chi^{I_1 \cup \{u\}}, t_1)$ satisfies the system $E_1 y_1 + F_1 t_1 = g_1$. Similarly, since I_2 is stable in G_2, there exists a nonnegative vector $t_2 \in \mathbb{R}^{r_2}$ for which the vector $(y_2, t_2) = (\chi^{I_2}, t_2)$ satisfies the linear system $E_2 y_2 + F_2 t_2 = g_2$. As $y_1(u) = 1$, the vector (y_1, y_2, t_1, t_2) satisfies the linear system (8).

Please cite this article in press as: H. Hu, M. Laurent, On the linear extension complexity of stable set polytopes for perfect graphs, European Journal of Combinatorics (2018), https://doi.org/10.1016/j.ejc.2018.02.014.

(ii) If $I_1 \cup \{u\}$ is not stable in G_1, then u is adjacent to one vertex in I_1 and thus $I_2 = \emptyset$. As I_1 is stable in G_1, there exists a nonnegative vector $t_1 \in \mathbb{R}^{r_1}$ such that $(y_1, t_1) = (\chi^{I_1}, t_1)$ satisfies the system $E_1 y_1 + F_1 t_1 = g_1$. Note that $y_1(u) = 0$ as $u \notin I_1$. Taking $y_2 = 0$ and $t_2 = 0$, we have that (y_2, t_2) satisfies $E_2 y_2 + F_2 t_2 - g_2 \cdot y_1(u) = 0$. Thus (y_1, y_2, t_1, t_2) satisfies the linear system (8).

In both cases, we have constructed lifting variables $(t_1, t_2, y_1(u)) \in \mathbb{R}^{r_1+r_2+1}$ such that the vector $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_1|+|V_2|+r_1+r_2}$ satisfies the linear system (8).

We now show the "if part". Assume $(y_1, y_2, t_1, t_2) \in \mathbb{R}^{|V_1|+|V_2|+r_1+r_2}$ satisfies the linear system (8). Assume first $y_1(u) = 0$. Then the conditions $E_2 y_2 + F_2 t_2 = 0$, $t_2 \geq 0$ imply $y_2 = 0$ (by Lemma 4.2 applied to $\mathrm{STAB}(G_2)$). Moreover the conditions $E_1 y_1 + F_1 t_1 = g_1$, $t_1 \geq 0$ imply that $y_1 \in \mathrm{STAB}(G_1)$. Hence y_1 is a convex combination of characteristic vectors of stable sets $I_1 \subseteq V_1 \setminus \{u\}$, which also gives a decomposition of the vector $(y_1(\bar{u}), y_2)$ as a convex combination of characteristic vectors of stable sets in G.

We may now assume $y_1(u) \neq 0$. As (y_1, y_2, t_1, t_2) satisfies the system (8) we deduce that $y_1 \in \mathrm{STAB}(G_1)$ and $\frac{1}{y_1(u)} y_2 \in \mathrm{STAB}(G_2)$. Say

$$y_1 = \sum_{I \in \mathcal{I}_1} \lambda_I \chi^I, \quad y_2 / y_1(u) = \sum_{J \in \mathcal{I}_2} \mu_J \chi^J,$$

where all sets in $\mathcal{I}_1$ (resp., $\mathcal{I}_2$) are stable sets in G_1 (resp., G_2), $\sum_I \lambda_I = \sum_J \mu_J = 1$ and $\lambda_I, \mu_J > 0$. Then $y_1(u) = \sum_{I \in \mathcal{I}_1 : u \in I} \lambda_I$ and we have the identity

$$\sum_{I \in \mathcal{I}_1 : u \notin I} \lambda_I \begin{pmatrix} \chi^I \\ 0 \end{pmatrix} + \sum_{I \in \mathcal{I}_1 : u \in I} \sum_{J \in \mathcal{I}_2} \lambda_I \mu_J \begin{pmatrix} \chi^{I \setminus \{u\}} \\ \chi^J \end{pmatrix} = \begin{pmatrix} y_1(\bar{u}) \\ y_2 \end{pmatrix}.$$

All coefficients are nonnegative and their sum is $\sum_{I \in \mathcal{I}_1 : u \notin I} \lambda_I + \sum_{I \in \mathcal{I}_1 : u \in I} \sum_{J \in \mathcal{I}_2} \lambda_I \mu_J = 1 - y_1(u) + y_1(u) = 1$. Moreover, if I is a stable set of G_1 with $u \in I$ and J is a stable set of G_2, then the set $(I \setminus \{u\}) \cup J$ is stable in G. So we have shown that the vector $(y_1(\bar{u}), y_2)$ belongs to the stable set polytope of G. □

Remark 2. One can show a slightly tighter upper bound for $\mathrm{xc}(\mathrm{STAB}(G))$ when G is obtained by substituting at a vertex of G_1 the graph $G_2 = K_p$ or $\overline{K_2}$. Indeed, one can show that $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + p$ when $G_2 = K_p$, and $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + 3$ when $G_2 = \overline{K_2}$, see [20] for details. This is a slight improvement over the result from Theorem 4.3 which would, respectively, give the bounds $\mathrm{xc}(\mathrm{STAB}(G_1)) + p + 1$ and $\mathrm{xc}(\mathrm{STAB}(G_1)) + 4$, using the fact that $\mathrm{xc}(\mathrm{STAB}(K_p)) = p + 1$ and $\mathrm{xc}(\mathrm{STAB}(\overline{K_2})) = 4$.

We conclude with some applications of this bounding technique for graph substitution.

Lemma 4.4. *(i) If G is the complete bipartite graph $K_{p,q}$ then* $\mathrm{xc}(\mathrm{STAB}(G)) \leq 2p + 2q + 3$. *(ii) If G is the complement of the disjoint union of p edges then* $\mathrm{xc}(\mathrm{STAB}(G)) \leq 4p + 1$.

Proof. (i) The complete bipartite graph $G = K_{p,q}$ can be obtained by considering an edge $\{u, v\}$ for G_1 and successively substituting $\overline{K_p}$ at u and $\overline{K_q}$ at v. Applying Theorem 4.3 we obtain $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(\overline{K_p})) + \mathrm{xc}(\mathrm{STAB}(\overline{K_q})) + \mathrm{xc}(\mathrm{STAB}(K_2)) = 2p + 2q + 3$.
(ii) If G is the complement of the union of p edges then G can be obtained by successively substituting $\overline{K_2}$ at each vertex of the complete graph K_p. By Remark 2 we obtain that $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(K_p)) + 3p = p + 1 + 3p = 4p + 1$. □

Using Lemma 4.4(ii) one can sharpen the bound of Lemma 3.4 when G is a double-split graph with parameters $(p, q, L_1, \ldots, L_p)$ and show $\mathrm{xc}(\mathrm{STAB}(G)) \leq 5p + 4q + 3$ $(\leq 5p + 5q + 2)$.

As $K_{p,q}$ has pq edges, the bound from Lemma 3.5 is quadratic in the number of vertices while by Lemma 4.4(i) the extension complexity of $\mathrm{STAB}(K_{p,q})$ is linear in the number of vertices.

Please cite this article in press as: H. Hu, M. Laurent, On the linear extension complexity of stable set polytopes for perfect graphs, European Journal of Combinatorics (2018), https://doi.org/10.1016/j.ejc.2018.02.014.

4.2. 2-join decompositions

Here we consider how the extension complexity of the stable set polytope behaves under 2-join decompositions.

Theorem 4.5. *Let G be a perfect graph and let (V_1, V_2) be a partition of V providing a 2-join decomposition of G as in Definition 2.10. Then we have*

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 3 \cdot \mathrm{xc}(\mathrm{STAB}(G[V_1])) + 3 \cdot \mathrm{xc}(\mathrm{STAB}(G[V_2])).$$

Proof. As G is perfect we need to show $\mathrm{rank}_+(S_G) \leq 3 \cdot \mathrm{rank}_+(S_{G_1}) + 3 \cdot \mathrm{rank}_+(S_{G_2})$. For this we examine the block structure of the slack matrix S_G from (5). As we have no control on the size of the set $\mathcal{C}_{12}$ of maximal mixed cliques, we examine in more detail how the mixed cliques and independent sets arise. For $k = 1, 2$, let A_k, B_k be the subsets of V_k as in Definition 2.10 and set $D_k = V_k \setminus (A_k \cup B_k)$.

Any mixed maximal clique is of the form $C = C_1 \cup C_2$ where, either $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ (call $\mathcal{C}_A$ the set of such maximal cliques), or $C_1 \subseteq B_1$ and $C_2 \subseteq B_2$ (call $\mathcal{C}_B$ the set of such maximal cliques), so that $\mathcal{C}_{12} = \mathcal{C}_A \cup \mathcal{C}_B$. One can verify that $\mathcal{I}_{12} = \mathcal{I}_3 \cup \mathcal{I}_4 \cup \mathcal{I}_5 \cup \mathcal{I}_6$, where $\mathcal{I}_3$ (resp., $\mathcal{I}_4, \mathcal{I}_5, \mathcal{I}_6$) contains the independent sets of the form $I \cup J$ with $I \subseteq D_1$ and $J \subseteq V_2$ (resp., with $I \subseteq D_1 \cup A_1$ and $J \subseteq D_2 \cup B_2$, $I \subseteq D_1 \cup B_1$ and $J \subseteq D_2 \cup A_2$, $I \subseteq V_1$ and $J \subseteq D_2$). Recall that $\mathcal{R}_k = V_k \cup \mathcal{C}_k$ for $k = 1, 2$. With respect to these partitions of its row and column index sets the matrix S_G has the block form:

$$S_G = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{C}_A \\ \mathcal{C}_B \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_3 & \mathcal{I}_4 & \mathcal{I}_5 & \mathcal{I}_6 \end{array} \\ \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & S_{1,5} & S_{1,6} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} & S_{2,5} & S_{2,6} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} & S_{3,5} & S_{3,6} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} & S_{4,5} & S_{4,6} \end{pmatrix} \end{array}.$$

To conclude the proof it suffices to make the following observations. For $k = 1, 2$, we have $S_{k,k} = S_{G_k}$, each column of $S_{k,3}, S_{k,4}, S_{k,5}, S_{k,6}$ is copy of a column of $S_{k,k}$, and each column of $S_{1,2}$ (resp., $S_{2,1}$) coincides with the column of $S_{1,1}$ (resp., $S_{2,2}$) indexed by the empty set. Moreover, for $k = 3, 4$, $S_{k,1}$ is a submatrix of S_{G_1}, $S_{k,2}$ is a submatrix of S_{G_2}, and each column of $S_{k,3}, S_{k,4}, S_{k,5}, S_{k,6}$ is copy of a column of $S_{k,1}$ or $S_{k,2}$. Combining these observations with Lemma 2.5 gives the desired inequality. □

Note that the result of Theorem 4.5 still holds when some set A_k or B_k $(k = 1, 2)$ is empty. Moreover we can then show a sharper bound. For instance, if A_1 or A_2 is empty then we have $\mathcal{C}_A = \emptyset$ and thus one can show the upper bound $\mathrm{xc}(\mathrm{STAB}(G)) \leq 2 \cdot \mathrm{xc}(\mathrm{STAB}(G[V_1])) + 2 \cdot \mathrm{xc}(\mathrm{STAB}(G[V_2]))$.

4.3. Skew partitions

We examine now the behavior of the extension complexity under skew partitions.

Theorem 4.6. *Let $G = (V, E)$ be a perfect graph and let (A_1, B_1, A_2, B_2) be a partition of V providing a skew partition decomposition of G as in Definition 2.11. Then we have*

$$\begin{aligned} \mathrm{xc}(\mathrm{STAB}(G)) \leq\ & 2 \cdot \mathrm{xc}(\mathrm{STAB}(G[A_1 \cup B_1])) + 2 \cdot \mathrm{xc}(\mathrm{STAB}(G[A_2 \cup B_2])) \\ & + \mathrm{xc}(\mathrm{STAB}(G[A_1 \cup B_2])) + \mathrm{xc}(\mathrm{STAB}(G[A_2 \cup B_1])). \end{aligned}$$

Proof. It suffices to show that $\mathrm{rank}_+(S_G)$ is at most

$$2\,\mathrm{rank}_+(S_{G[A_1 \cup B_1]}) + 2\,\mathrm{rank}_+(S_{G[A_2 \cup B_2]}) + \mathrm{rank}_+(S_{G[A_1 \cup B_2]}) + \mathrm{rank}_+(S_{G[A_2 \cup B_1]}).$$

For this we exploit the block structure of S_G in (5), using the partition $V = V_1 \cup V_2$ with $V_k = A_k \cup B_k$ for $k = 1, 2$. The mixed maximal cliques of G are of the form $C_1 \cup C_2$, either with $C_1 \subseteq A_1 \cup B_1$ and $C_2 \subseteq A_2$ (call their set $\mathcal{C}_3$), or with $C_1 \subseteq A_1$ and $C_2 \subseteq A_2 \cup B_2$ (call their set $\mathcal{C}_4$). The mixed independent sets of G are of the form $I_1 \cup I_2$, either with $I_1 \subseteq A_1 \cup B_1$ and $I_2 \subseteq B_2$ (call their set $\mathcal{I}_3$), or with $I_1 \subseteq B_1$

and $I_2 \subseteq A_2 \cup B_2$ (call their set $\mathcal{I}_4$). With respect to these partitions of its row and column index sets, the slack matrix S_G has the block form:

$$S_G = \begin{array}{c c} & \begin{array}{cccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_3 & \mathcal{I}_4 \end{array} \\ \begin{array}{c} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \end{array} & \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} \end{pmatrix} \end{array}.$$

As in earlier proofs, we have $\text{rank}_+(S_{k,1}\ S_{k,2}\ S_{k,3}\ S_{k,4}) \leq \text{rank}_+(S_{G[A_k \cup B_k]})$ for $k = 1, 2$. Moreover, by looking at the shape of the mixed cliques and independent sets one can make the following observations: $\text{rank}_+(S_{3,1}\ S_{3,3}) \leq \text{rank}_+(S_{G[A_1 \cup B_1]})$ since $S_{3,1} = S_{3,3}$ is a submatrix of $S_{G[A_1 \cup B_1]}$, $\text{rank}_+(S_{3,2}\ S_{3,4}) \leq \text{rank}_+(S_{G[A_2 \cup B_1]})$ since each column of $S_{3,2}$ is copy of a column of $S_{3,4}$ which in turn is a submatrix of $S_{G[A_2 \cup B_1]}$, $\text{rank}_+(S_{4,2}\ S_{4,4}) \leq \text{rank}_+(S_{G[A_2 \cup B_2]})$ since $S_{4,2} = S_{4,4}$ is a submatrix of $S_{G[A_2 \cup B_2]}$, and $\text{rank}_+(S_{4,1}\ S_{4,3}) \leq \text{rank}_+(S_{G[A_1 \cup B_2]})$ since each column of $S_{4,1}$ is a copy of a column of $S_{4,3}$ which in turn is a submatrix of $S_{G[A_1 \cup B_2]}$. □

Here too observe that the result still holds if some set A_k or B_k is empty, in which case one may show a sharper bound. For instance, if $A_1 = \emptyset$ then $\mathcal{C}_4 = \emptyset$ and we have $\text{xc}(\text{STAB}(G)) \leq 2 \cdot \text{xc}(\text{STAB}(G[A_1 \cup B_1])) + \text{xc}(\text{STAB}(G[A_2 \cup B_2])) + \text{xc}(\text{STAB}(G[A_2 \cup B_1]))$.

5. Application to perfect graphs

We now use the above results to upper bound the extension complexity of the stable set polytope of a perfect graph G. For this we use the decomposition result of [7] (Theorem 2.12), which claims that if G is not basic then, either G has a skew partition, or G or $\bar{G}$ has a 2-join. Hence G can be decomposed into basic perfect graphs by means of skew partitions and 2-joins and one can represent this decomposition process using a tree (such a decomposition tree may not be unique). Recall that basic perfect graphs are bipartite graphs or their complements, line graphs of bipartite graphs or their complements, and double-split graphs for which we know that $\text{xc}(\text{STAB}(G)) \leq 2(|V| + |E|)$ (by Corollary 3.7).

Theorem 5.1. *Let $G = (V, E)$ be a perfect graph. Let d be the depth of a decomposition tree representing a decomposition of G into basic perfect graphs by means of 2-join and skew partition decompositions. Then we have*

$$\text{xc}(\text{STAB}(G)) \leq 4^{d+1}(|V| + |E|).$$

Proof. We use induction on the depth $d \geq 0$ of the decomposition tree. If $d = 0$ then G is a basic perfect graph and the result holds by Corollary 3.7. Assume now $d \geq 1$. Then G admits a skew partition decomposition, or G or $\bar{G}$ admits a 2-join decomposition.

We first consider the case when G admits a skew partition (A_1, B_1, A_2, B_2). We use Theorem 4.6, which implies that

$$\text{xc}(\text{STAB}(G)) \leq 2 \cdot (\text{xc}(\hat{G}_1) + \text{xc}(\hat{G}_2) + \text{xc}(\hat{G}_3) + \text{xc}(\hat{G}_3)),$$

where $\hat{G}_1, \hat{G}_2, \hat{G}_3, \hat{G}_4$ are induced subgraphs of G such that $\sum_{k=1}^4 |V(\hat{G}_k)| = 2|V|$ and $\sum_{k=1}^4 |E(\hat{G}_k)| \leq 2|E|$. By the induction assumption, for each $k = 1, 2, 3, 4$ we have: $\text{xc}(\text{STAB}(\hat{G}_k)) \leq 4^d(|V(\hat{G}_k)| + |E(\hat{G}_k)|)$. Combining with the above relations we obtain the desired inequality:

$$\text{xc}(\text{STAB}(G)) \leq 2 \cdot 4^d \sum_{k=1}^4 (|V(\hat{G}_k)| + |E(\hat{G}_k)|) \leq 4^{d+1}(|V| + |E|).$$

Next we consider the case when G admits a 2-join decomposition (V_1, V_2). Then, by Theorem 4.5, we have

$$\text{xc}(\text{STAB}(G)) \leq 3 \cdot \text{xc}(\text{STAB}(G[V_1])) + 3 \cdot \text{xc}(\text{STAB}(G[V_2])).$$

By the induction assumption, we have $\mathrm{xc}(\mathrm{STAB}(G[V_k])) \leq 4^d(|V_k| + |E_k|)$ for each $k = 1, 2$. As $|V| = |V_1| + |V_2|$ and $|E_1| + |E_2| \leq |E|$, we obtain the desired bound: $\mathrm{xc}(\mathrm{STAB}(G)) \leq 3 \cdot 4^d(|V_1| + |E_1| + |V_2| + |E_2|) \leq 4^{d+1}(|V| + |E|)$.

Finally we consider the case when $\bar{G}$ admits a 2-join decomposition (V_1, V_2).

Then, using Lemma 3.2 and Theorem 4.5 applied to the 2-join decomposition (V_1, V_2) of $\bar{G}$, we obtain

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq |V| + \mathrm{xc}(\mathrm{STAB}(\bar{G})) \leq |V| + 3(\mathrm{xc}(\mathrm{STAB}(\bar{G}[V_1])) + \mathrm{xc}(\mathrm{STAB}(\bar{G}[V_2]))),$$

which, using again Lemma 3.2, implies

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 4|V| + 3(\mathrm{xc}(\mathrm{STAB}(G[V_1])) + \mathrm{xc}(\mathrm{STAB}(G[V_2]))).$$

We now use the induction assumption applied to $G[V_1]$ and $G[V_2]$ combined with $|V| = |V_1| + |V_2|$ and $|E_1| + |E_2| \leq |E|$ to derive the desired inequality

$$\begin{aligned}\mathrm{xc}(\mathrm{STAB}(G)) &\leq 4|V| + 3 \cdot 4^d(|V_1| + |E_1| + |V_2| + |E_2|) \leq 4|V| + 3 \cdot 4^d(|V| + |E|)\\ &\leq 4^{d+1}(|V| + |E|). \quad \square\end{aligned}$$

As $|V| + |E| \leq |V|^2$, we derive the bound $\mathrm{xc}(\mathrm{STAB}(G)) \leq 4^{d+1}|V|^2$ when G has a decomposition tree of depth d. In particular, for the class of perfect graphs G admitting a decomposition tree whose depth d is logarithmic in $|V|$, say $d \leq c \log|V|$ for some constant $c > 0$, the extension complexity of the stable set polytope is polynomial in V:

$$\mathrm{xc}(\mathrm{STAB}(G)) \leq 4|V|^{c+2}.$$

To the best of our knowledge it is not known whether upper bounds exist on the depth of a decomposition tree for perfect graphs in terms of 2-joins and skew partitions, which are polynomial in terms of the number of nodes. We also do not know whether there exists a class of perfect graphs which does not admit a decomposition tree of logarithmic depth.

In fact, it is still an open problem whether the decomposition result of [7] can be used to derive a polynomial time algorithm for optimally coloring perfect graphs, which is purely combinatorial (in contrast with the polynomial time algorithm of [19], which relies on the ellipsoid method). Recent results in this area can be found, e.g., in [6,8,30]. Kennedy and Reed [22] give a polynomial time algorithm for finding a skew partition in any graph (if some exists). In [6] the authors present a refined polynomial time algorithm for finding a balanced skew partition in perfect graphs, which they use to design an efficient combinatorial coloring algorithm for perfect graphs with bounded clique number.

To conclude let us remark that other graph operations are known that preserve perfect graphs and can be used to give structural characterizations for subclasses of perfect graphs. This is the case in particular for the "graph amalgam" operation considered in [3]. The behavior of the amalgam operation is studied in [12] (see also [20]): if G is the amalgam of two perfect graphs G_1 and G_2 then $\mathrm{xc}(\mathrm{STAB}(G)) \leq \mathrm{xc}(\mathrm{STAB}(G_1)) + \mathrm{xc}(\mathrm{STAB}(G_2))$. Meyniel [27] introduced a class of perfect graphs, known as *Meyniel graphs*. Burlet and Fonlupt [3] introduce a notion of *basic* Meyniel graph and show that any Meyniel graph can be decomposed into basic Meyniel graphs using graph amalgams. It follows from results in Conforti et al. [12] that the extension complexity of the stable set polytope is polynomial in the number of vertices for Meyniel graphs.

In summary, the question of deciding whether the extension complexity of the stable set polytope is polynomial for all perfect graphs remains wide open.

Acknowledgments

We are grateful to Ronald de Wolf for useful discussions and comments about the topic of this paper. The second author thanks Maria Chudnovsky for bringing the works [6,8] to our attention.

References

[1] M. Aprile, Y. Faenza, S. Fiorini, T. Huynh, M. Macchia, Extension complexity of stable set polytopes of bipartite graphs, 2017. arXiv:1702.08741v1.
[2] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe 114 (1961).
[3] M. Burlet, J. Fonlupt, Polynomial Algorithm to Recognize a Meyniel Graph, in: North-Holland mathematics studies, vol. 88, 1984, pp. 225–252.
[4] M. Chudnovsky, Berge Trigraphs and their Applications, (Ph.D. thesis), Princeton University, Princeton, NJ, 2003.
[5] M. Chudnovsky, Berge trigraphs, J. Graph Theory 53 (1) (2006) 1–55.
[6] M. Chudnovsky, A. Lagoutte, P. Seymour, S. Spirkl, Colouring perfect graphs with bounded clique number, J. Combin. Theory Ser. B 122 (2017) 757–775.
[7] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas, The strong perfect graph theorem, Ann. Math. 164 (1) (2006) 51–229.
[8] M. Chudnovsky, N. Trotignon, T. Trunck, K. Vušković, Coloring perfect graphs with no balanced skew-partitions, J. Combin. Theory Ser. B 115 (2015) 26–65.
[9] V. Chvátal, On certain polytopes associated with graphs, J. Combin. Theory Ser. B 18 (2) (1975) 138–154.
[10] V. Chvátal, Star-cutsets and perfect graphs, J. Combin. Theory Ser. B 39 (3) (1985) 189–199.
[11] M. Conforti, G. Cornuéjols, G. Zambelli, Extended formulations in combinatorial optimization, 4OR 8 (2010) 1–48.
[12] M. Conforti, B. Gerards, K. Pashkovich, Stable sets and graphs with no even holes, Math. Program. 153 (1) (2015) 13–39.
[13] G. Cornuéjols, W.H. Cunningham, Compositions for perfect graphs, Discrete Math. 55 (1985) 245–254.
[14] S. Fiorini, V. Kaibel, K. Pashkovich, D.O. Theis, Combinatorial bounds on negative rank and extended formulations, Discrete Math. 313 (1) (2013) 67–83.
[15] S. Fiorini, S. Massar, S. Pokutta, H.R. Tiwary, R. de Wolf, Exponential lower bounds for polytopes in combinatorial optimization, J. ACM 62 (2) (2015) Article No. 17.
[16] N. Gillis, Introduction to nonnegative matrix factorization, Optima 25 (2017) 7–16.
[17] M. Goemans, Smallest compact formulation for the permutahedron, Math. Program. 153 (1) (2015) 5–11.
[18] J. Gouveia, P. Parrilo, R. Thomas, Lifts of convex sets and cone factorizations, Math. Oper. Res. 38 (2) (2013) 248–264.
[19] M. Grötschel, L. Lovász, A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (2) (1981) 169–197.
[20] H. Hu, On the Extension Complexity of Stable Set Polytopes for Perfect Graphs (Master thesis), Utrecht University, 2015.
[21] V. Kaibel, Extended formulations in combinatorial optimization, Optima 85 (2011) 2–7.
[22] W. Kennedy, B. Reed, Fast skew partition recognition, in: Computational Geometry and Graph Theory, in: Lecture Notes in Computer Science, vol. 4535, 2008, pp. 101–107.
[23] R. Kipp Martin, Using separation algorithms to generate mixed integer model reformulations, Oper. Res. Lett. 10 (3) (1991) 119–128.
[24] J.R. Lee, P. Raghavendra, D. Steurer, Lower bounds on the size of semidefinite programming relaxations, in: Proceedings of the 47th Annual ACM Symposium on Theory of Computing, 2015, pp. 567–576.
[25] L. Lovász, A characterization of perfect graphs, J. Combin. Theory Ser. B 13 (2) (1972) 95–98.
[26] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (3) (1972) 253–267.
[27] H. Meyniel, On the perfect graph conjecture, Discrete Math. 16 (4) (1976) 339–342.
[28] T. Rothvoß, Some 0/1 polytopes need exponential size extended formulations, Math. Program. Ser. A 142 (2013) 255–268.
[29] T. Rothvoß, The matching polytope has exponential extension complexity, in: Proceedings of the 46th Annual ACM Symposium on Theory of Computing, 2014, pp. 263–272.
[30] N. Trotignon, K. Vušković, Combinatorial optimization with 2-joins, J. Comb. Optim. B. 1029 (1) (2012) 153–185.
[31] R.T. Wong, Integer programming formulations of the traveling salesman problem, in: Proceedings of 1980 IEEE International Conference on Circuits and Computers, 1980, pp. 149–152.
[32] M. Yannakakis, Expressing combinatorial optimization problems by linear programs, in: Proceedings of the 20th Annual ACM Symposium on Theory of Computing, 1988, pp. 223–228.

Please cite this article in press as: H. Hu, M. Laurent, On the linear extension complexity of stable set polytopes for perfect graphs, European Journal of Combinatorics (2018), https://doi.org/10.1016/j.ejc.2018.02.014.