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Linear Groups and Distance-transitive Graphs

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ABSTRACT

A detailed treatment is given of the graphs on which a group with simple socle $PSL(n,q)$ acts distance-transitively.

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1. Introduction

This paper may be viewed as a continuation of [5], in which all graphs are determined on which a group with socle $L(n,q)$ for some $n \geq 8$ acts distance-transitively. Here we treat the case where the simple socle is isomorphic to $PSL(n,q)$ for some $n \in \mathbb{N}$ with $2 \leq n \leq 7$. This completes the determination of all graphs on which a group with simple socle isomorphic to some $L(n,q)$ acts distance-transitively. We recall that a group G acting on a graph $\Gamma = (V\Gamma, E\Gamma)$ is said to be *distance-transitive* on Γ if its induced action on each of the sets

$$\{(x,y) \mid x,y \in V\Gamma,\ d(x,y) = i\}$$

is transitive, and that a graph is called *distance-transitive* if its autmorphism group acts distance-transitively on it. Here, d denotes the usual distance in Γ, and i runs through $\{0, \cdots, diam(\Gamma)\}$. For notation, standard terminology and facts concerning distance-transitive graphs, the reader is referred to Bannai & Ito [3], and Brouwer, Cohen & Neumaier [6].

1.1. Theorem. *Let G be a group with $PSL(n,q) \lhd G \leq \mathrm{aut}\,PSL(n,q)$, $n \geq 2$, and $(n,q) \neq (2,2),(2,3)$. If Γ is a connected graph of diameter at least 2 on which G acts primitively and distance-transitively, then either Γ is a Grassmann graph or $K := N_{\mathrm{aut}\,\Gamma}(G^{\infty})$, Γ, and the stabilizer H in K of a vertex are as listed in Table 1, with the understanding that, if $diam\Gamma = 2$, only one of Γ and its complement is listed.*

Table 1.

(n,q)	K	H	index	array	name
(2,4)	Sym_5	$\mathrm{Sym}_3 \times 2$	10	$\{3,2;1,1\}$	Petersen
(2,7)	$PGL(2,7)$	Sym_4	28	$\{3,2,2,1;1,1,1,2\}$	Coxeter
(2,8)	$P\Gamma L(2,8)$	$Frob_{7\cdot 6}$	36	$\{14,6;1,4\}$	$J(9,2)$
(2,9)	$P\Sigma L(2,9)$	$L(2,3)\times 2$	15	$\{6,4;1,3\}$	complement of $J(6,2)$
(2,9)	$P\Gamma L(2,9)$	$AGL(1,5)\times 2$	36	$\{5,4,2;1,1,4\}$	Inv (aut $\mathrm{Sym}_6 \setminus \mathrm{Sym}_6$)
(2,9)	$P\Gamma L(2,9)$	[32]	45	$\{4,2,2,2;1,1,1,2\}$	gen. 8-gon(2,1)
(2,16)	$P\Gamma L(2,16)$	$(2\times L(2,4))\cdot 2$	68	$\{12,10,3;1,3,8\}$	Doro
(2,17)	$PSL(2,17)$	Sym_4	102	$\{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$	Biggs-Smith
(2,19)	$PSL(2,19)$	Alt_5	57	$\{6,5,2;1,1,3\}$	Perkel
(2,25)	$P\Sigma L(2,25)$	$L(2,5).2\times 2$	65	$\{10,6,4;1,2,5\}$	locally Petersen
$(3,q)$	aut $P\Gamma L(3,q)$	$Borel.2$	$(q^2+q+1)(q+1)$	$\{2q,q,q;1,1,2\}$	gen. 6-gon$(q,1)$
(3,4)	aut $PSL(3,4)$	$PSU(3,2).Dih_{12}$	280	$\{9,8,6,3;1,1,3,8\}$	Γ_3 (Herm. forms (3,4))

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Table 1.

(n,q)	K	H	index	array	name
(3,4)	$P\Sigma L(3,4).\langle diag\rangle$	$\mathrm{Alt}_6 \cdot 2^2$	56	{10,9; 1,2}	Gewirtz
(4,2)	Sym_8	$\mathrm{Sym}_6 \times 2$	28	{15,8; 1,6}	complement of $J(8,2)$
(4,2)	Sym_8	$\mathrm{Sym}_5 \times \mathrm{Sym}_3$	56	{15,8,3; 1,4,9}	$J(8,3)$
(4,3)	$PGO^+(6,3)$	$PSp(4,3){:}2\times 2$	117	{36,20; 1,9}	Nonisotropics

For the precise definitions of the graphs listed, the reader is referred to [6]. In most cases, the group in the second column is the full automorphism group of Γ. But, for instance, $J(9,2)$ has automorphism group Sym_9, whereas our group is $P\Gamma L(2,8)$.

The results in HEMMETER [12], BROUWER, COHEN & NEUMAIER [6], VAN BON & BROUWER [4] imply that all imprimitive distance-transitive graphs whose primitive quotients are among those listed in Table 1 are known.

Proof. The proof is given in several steps. In view of Theorem 3.2 in VAN BON & COHEN [5] and known results on small valency, cf. A.A. IVANOV & A.V. IVANOV [15], we may (and shall) assume (without loss of generality) that $n \le 7$ and $k \ge 14$. Throughout the proof, we let $\gamma \in V\Gamma$, $X = \mathrm{soc}\, G = PSL(n,q)$, $H = G_\gamma$ and $Y = H \cap X$. Then $H = N_G(Y)$. Finally, we set $q = p^a$, where p is a prime.

2. The case $n = 2$.

Since the graphs corresponding to Alt_5 are known (cf. IVANOV [14] and LIEBECK, PRAEGER & SAXL [21]) and accord with the statement of the theorem, we may (and shall) take $q \ge 7$. Since G acts doubly transitively on the projective line $\Omega = \{\mathbf{F}v \mid v \in \mathbf{F}_q^2\}$ and the permutation character of G on (the cosets of) H is multiplicity-free, the group H has at most two orbits on Ω, and so is listed in an appendix ('Hering's Theorem') or the conclusion of the main theorem of LIEBECK [20]. It is well known (cf. SUZUKI [23]) that $\mathrm{aut} X = P\Gamma L(2,q)$ has order $q(q^2-1)a$ and that the subgroups of $X = L(2,q)$ come in 7 types, which we have labeled (ia), (ib), (ii), ... , (vi) below.

(i). $H_0 := H \cap PSL(2,q)$ is a dihedral group, of order $|H_0| = 2(q-\varepsilon)/(2,q-1)$, where $\varepsilon \in \{\pm 1\}$. We show that Γ is the Johnson graph $J(9,2)$ and $G = P\Gamma L(2,8)$.

(ia). First, suppose $\varepsilon = 1$. Then as a G-set, $V\Gamma$ may be viewed as the set $\binom{\Omega}{2}$ of pairs of projective points. Furthermore, by Lemma 2.6 of VAN BON & COHEN [5], we may suppose that $G = P\Gamma L(2,q)$ or $diam\Gamma \le 4$. We establish that the latter must hold. To this end, assume that $G = P\Gamma L(2,q)$.

Take $\gamma = \{0,\infty\}$ so that $H_1 = G_\gamma \cap PGL(2,q)$ is generated by the elements h, w with matrices

$$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ζ is a generator of $\mathbf{F}_q^*$. Consider the H_1-orbits on $V\Gamma \setminus \{\gamma\}$. The element h acts on $\{\lambda,\mu\} \in \binom{\Omega}{2}$ by multiplication of its members by ζ and the element w by inversion and multiplication by -1. Clearly, the set X_γ of all vertices meeting γ in a singleton is a single orbit of size $2(q-1)$. Each of the remaining $\binom{q}{2}$ vertices in $V\Gamma \setminus \{\gamma\}$ is H_1-conjugate to a vertex of shape $\{1,\zeta^j\}$ for some j $(1 \le j \le (q-1)/2)$.

Now h^i fixes $\{1,\mu\}$ iff $\mu = -1 \ne 1$, and $h^i\{1,\mu\}$ coincides with $w\{1,\mu\}$ iff either $-1 = \zeta^i$ and $-\mu^{-1} = \zeta^i\mu$, or $-1 = \zeta^i\mu$ and $-\mu^{-1} = \zeta^i$. In the first case we have again $\mu = -1 \ne 1$, in the second case there is an i for each μ. This information determines the order of vertex stabilizers in H_1, and yields that on $V\Gamma \setminus (X_\gamma \cup \{\gamma\})$ we have $(q-3)/2$ orbits of length $q-1$ and a single orbit (with representative $\{1,-1\}$) of length $(q-1)/2$ if q is odd, and $(q-2)/2$ orbits of length $q-1$ if q is even.

If $\{0,1\}$ is adjacent to γ, then we must have $\Gamma = J(q+1,2)$, by definition of the Johnson graph $J(q+1,2)$ (cf. 1.2 of [5]), and so G must have a known rank 3 representation. Here $G = P\Sigma L(2,8)$ appears with $H = Frob_{7\cdot 6}$.

More generally, let i be such that $X_\gamma = \Gamma_i(\gamma)$; then, since $J(q+1,2)$ has diameter 2, we have $diam\Gamma \le 2i$. We fix a neighbor $\delta = \{1,\alpha\}$ of γ in Γ. Applying w and a suitable power of h to δ, we obtain

$\{\eta,\eta\alpha^{-1}\}\in\Gamma_1(\gamma)\subseteq\Gamma_{\le 2}(\delta)$. Transforming δ to γ by means of

$$\begin{bmatrix}-1 & \alpha\\ 1 & -1\end{bmatrix} \text{ we find } \{\frac{\alpha-\eta}{\eta-1},\frac{\alpha-\eta\alpha^{-1}}{\eta\alpha^{-1}-1}\}\in\Gamma_{\le 2}(\gamma),$$

Taking $\eta=\alpha^2$, we get $\{-\alpha/(\alpha+1),0\}\in\Gamma_{\le 2}(\gamma)\}$. If $\alpha\ne -1$, it follows that $X_\gamma=\Gamma_2(\gamma)$, and so, by the above remark, $diam\Gamma\le 4$, as required. Therefore, suppose $\alpha=-1$ and p is odd. Taking $\eta\ne 1,-1$, we get $\{1,(\frac{\eta-1}{\eta+1})^2\}\in\Gamma_{\le 2}(\gamma)\subseteq\ \Gamma_{\le 3}(\delta)$. Taking $\eta=2$, we see $\{1,9\}\in\ \Gamma_{\le 3}(\delta)$. If $diam\Gamma>6$, this forces $9\equiv -1 \bmod p$, whence $p=5$. But then q is a nontrivial power of 5 and an $\eta\in\mathbf{F}_q\setminus\mathbf{F}_p$ can be found such that $(\frac{\eta-1}{\eta+1})^2\ne -1$; applying the same argument once more leads to a contradiction.

Consequently, $diam\Gamma\le 6$. We show that q must be small. From the above, we see at least the H-orbits X_γ, the one containing $\{1,-1\}$, and at least $(q-3)/2a$ further orbits, so $2+(q-3)/2a\le\ diam\Gamma\le 6$. This shows that $a\le 3$ if $p=3$ and $a=1$, $q\le 11$ if $p\ge 5$. If $q=9$, then soc G is an alternating group so Γ is known (cf. 5) and if $q=7,11$, there are at least two suborbits of size at most 13, so $k\le 13$ by Lemma 2.7 and Γ is known (cf. §1.5 of [5]). Since $q>5$, only the case $q=3^3$ remains. Then, there is a unique suborbit of size 13 and one of size 52, while the remaining 4 suborbits all have length 78. Since $k\ne 52$ (because Γ is not a Johnson graph) it follows that $k=13$, contrary to the assumption $k\ge 14$.

This establishes that $diam\Gamma\le 4$. Then, by the same argument as above, $2+(q-3)/2a\le 4$ if p is odd, and $1+(q-2)/2a\le 4$ if $p=2$. The only new cases to consider arise when $p=2$, so let $q=2^a$. Then $q\le 32$. If $q=32$, then all nontrivial suborbits distinct from X_γ have size 5x31, and so $k=k_2=155$, contradicting Lemma 2.7 [5]. If $q=16$, then the suborbits have sizes 1,15,30,30, and 60. Taking into account that $k_2=30$, we find that $k=15$, $k_3=60$, and $k_4=30$. But it is readily seen that there is no corresponding feasible intersection array. We have seen above that for $q=8$ we find the Johnson graph $J(9,2)$. Since $q>5$, this ends the proof of (ia).

(ib). Now let $\varepsilon=-1$. We shall view X as the group $PSU(2,q)$, so elements are (projectively) represented by matrices x with $x^{\mathsf T}=x^{-\sigma}$, where $\mathsf T$ stands for transposed and σ for the standard Frobenius of order 2 of $\mathbf{F}_{q^2}$. The group X preserves the hermitean form $\langle(\alpha_1,\beta_1),(\alpha_2,\beta_2)\rangle=\ \alpha_1\alpha_2^q+\beta_1\beta_2^q$ on $\mathbf{F}_{q^2}^2$. (cf. [23] for details). Take ξ to be a generator of $\mathbf{F}_{q^2}^*$, and put $\zeta=\xi^{q-1}$. Then the elements h, w, described by the same matrices as in (ia), generate $H_1:=H\cap PGU(2,q)$. Denote by Ω the set of projective points over $\mathbf{F}_{q^2}$, and identify $\alpha\in\mathbf{F}_{q^2}$ with the 1-space containing $(\alpha,1)$. Then G leaves invariant the subset Δ (of size $q+1$) of points represented by vectors (α,β) with $\langle(\alpha,\beta),\ (\alpha,\beta)\rangle=0$, and for every point of $\Omega\setminus\Delta$ represented by (α,β), there is a unique orthogonal point $(\beta^q,-\alpha^q)$. Now H is the stabilizer of the orthogonal pair of points related to the standard basis, so $V\Gamma$ may be identified with the set of all orthogonal pairs $\{\alpha,-\alpha^{-1}\}$ with $\alpha\in\mathbf{F}_{q^2}$, $\alpha^{1+q}\ne -1$. Since h preserves α^{1+q} for $\alpha\in\mathbf{F}_{q^2}$, the 'double' value $\alpha^{\pm(1+q)}\in\mathbf{F}_q$ parametrizes $\langle h\rangle$-orbits. It readily follows from this description that on $V\Gamma$, the subgroup H_1 has $(q-2)/2$ orbits of length $q+1$ if q is even, and $(q-3)/2$ orbits of length $q+1$ and a single orbit of length $(q+1)/2$ (containing 1) if q is odd. The H-orbit structure will be completely determined if we know the Frobenius action; but this is also clear from the above picture. For instance, if q is odd, then, among the H_1-orbits of length $q+1$, there are precisely $(p-3)/2$ invariant under the Frobenius of order a. Then $a>1$ implies there are orbits of length $>(q+1)$, so by Lemma 2.7 of [5] there are at most 2 orbits of length $q+1$. Thus $(p-3)/2\le 2$, i.e., $p\le 7$. Let e be the number of divisors of a (including 1 and a). By Lemma 2.7 [5], and the orbit lengths, we must have $k_{e+1}\le k_e$ if q is even and $k_{e+2}\le k_{e+1}$ if q is odd, so $d\le 3e$ if q even and $d\le 3e+3$ if q odd. But H has at least $(q-2)/2a$ orbits if q is even and at least $1+(q-p)/2a+(p-3)/2$ if q is odd, so $2^a=q\le 6ae+2$ if q even and $p^a=q\le\ 6ae+4a+3$ if q is odd. Using that $k\ge 14$, we also have $q\ge 13$, so that q is one of 16, 32, 64, 27, 81, 25, 13. Inspection of the subdegrees in these specific cases shows that no feasible intersection array exists.

(ii). Y is a Borel subgroup of X. Then G acts doubly transitive on $V\Gamma$ and so Γ is a clique.

(iii). soc $Y\cong \mathrm{Alt}_5$ and $p\ne 2,5$. We may view $V\Gamma$ as the class of X-conjugates of Y. Thus $v=q(q^2-1)/120$ and $|H|=120$ or 60 (as H is a maximal subgroup of G and there are precisely two conjugacy classes of Alt_5 in $L(2,q)$ which fuse in $PGL(2,q)$).

Let x be an element of order 5 and let $\varepsilon_5\in\{\pm 1\}$ be such that 5 divides $q-\varepsilon_5$. There are $q(q+\varepsilon_5)/2$ groups of order 5 in X, all conjugate to $\langle x\rangle$, and $N_X(\langle x\rangle)$ is a dihedral group of order $5(4,q-\varepsilon_5)$. Therefore, the number of X-classes of dihedral subgroups of order 10 is $(4,q-\varepsilon_5)/2$, each class has size $q(q+\varepsilon_5)/2$, and

x belongs to a unique member of each class. Now Y contains 6 dihedral groups of order 10 from a single class, D say, so a member of D belongs to $6v/q(q+\varepsilon_5)/2 = (q-\varepsilon_5)/10$ vertices of Γ. Consequently, there are $6(q-\varepsilon_5-10)/10$ vertices Y_1 of Γ meeting Y in a subgroup of order 10. As the stabilizer in Y of the unique 5-group in $Y \cap Y_1$ is a member of D and hence also contained in Y_1, we see that the Y-orbit of Y_1 has size 6. The conclusion is that the number of Y-orbits of size 6 in $V\Gamma$ is $(q-\varepsilon_5-10)/10$.

Suppose first that q is a prime, so that $Y = H$. If $q \geq 19$, there are at least 2 H-orbits of size 6, so that $k = 6$, contrary to the assumptions. Thus $q \leq 19$ and we are done by a straightforward check using the ATLAS [7]. Next suppose, q is not a prime. Then, by maximality of H, it must be the square of a prime, and by LIEBECK [20] $q = 9$ or 49. Since in the first case the theorem is readily seen to hold, we may assume q=49. But then $\varepsilon_5 = -1$ and there are 4 Y-orbits of size 6, whence at least 2 H-orbits of size at most 12, forcing $k \leq 12$. This establishes the theorem in case $Y \cong \mathrm{Alt}_5$ and $p \neq 2,5$.

(iv) Let $p > 3$, and $Y \cong \mathrm{Alt}_4$ (with $q \equiv 3$ or 5 mod 8) or Sym_4 (with $q \equiv \pm 1$ mod 8). Then q is a prime number and, if $q \equiv \pm 1$ mod 8, there are two conjugacy classes of subgroups of X isomorphic to Sym_4 which fuse in $PGL(2,q)$ so $h = |H| = 12$, or 24. But h=12 implies $k \leq 12$, in which case there is nothing left to prove. Thus h=24 and $\Gamma_1(\gamma)$ is a regular H-orbit.

If d=2, the complement of Γ is distance-transitive with the same group G, so we may assume $k_2 = k$ so $v = 1+24+24=49$ and $v = q(q^2-1)/48$ or $q(q^2-1)/24$, contradicting that q is a prime. If $d > 2$, we get $k = k_2 = 24$ and we are done by Lemma 2.7 [5].

(v) $Y = PSL(2,r)$ where $q = r^m$ and m is an odd prime number. There is a unique X-class, so $v = q(q^2-1)/(r(r^2-1))$. Recall that $q = p^a$ so that $r = p^{a/m}$. Now by multiplicity freeness, H has at most two orbits on $P(\mathbf{F}_q^2)$; but we see one Y-orbit of length r+1 and other Y-orbits are regular of length $r(r^2-1)/(2,p-1)$, so we must have $q+1 = (r+1) + br(r^2-1)/(2,p-1)$, where b divides $|G/X|$, so $b \,|\, (2,p-1)m$. It follows that $(r^{m-1}-1)/(r^2-1) = (q-r)/(r(r^2-1)) = b/(2,p-1) \leq m$. Consequently, either $m \leq 3$ or $r = 2$ and $m = 5$. In the latter case H contains a torus and so is dealt with in (i).

Therefore, we have $m = 3$, and $(2,p-1) \,|\, b$, so $H \geq PGL(2,q)$.

Now $v = r^2(r^4 + r^2 + 1)$. Let $\varepsilon \in \{1,-1\}$. There are $r(r+\varepsilon)/2$ tori (i.e., abelian subgroups consisting entirely of semi-simple elements) of order $r-\varepsilon$ in $H_1 = PGL(2,r)$, and similarly with q instead of r, whence each torus of $PGL(2,q)$ of order $r-\varepsilon$ is contained in $vr(r+\varepsilon)/(q(q+\varepsilon)) = r^2 + \varepsilon r + 1$ conjugates of H_1. Thus there are $(r(r+\varepsilon)/2)(r^2+\varepsilon r) = r^2(r+\varepsilon)^2/2$ vertices of $V\Gamma$ meeting H in a torus of order $r-\varepsilon$. The same computation can be done for dihedral subgroups of order $2(r-\varepsilon)$, showing that any two conjugates whose intersection contains a torus of order $r-\varepsilon$, meet in a dihedral subgroup of order $2(r-\varepsilon)$.

Fix a dihedral D of order $2(r-1)$ in H_1 and denote by T its normal cyclic subgroup (a torus) of order $r-1$. Then T normalizes two root subgroups of H_1, say U_1, U_2, which are interchanged by D. Let H_2 be a conjugate of H_1 with $H_1 \cap H_2 = D$. We scrutinize the H_1-orbit containing H_2. There are precisely two root groups Z_i (i=1,2) of G normalized by T (and interchanged by D). Choose notation so that $Z_i \leq C_G(U_i)$. Now H_2 meets each Z_i in a root subgroup S_i of H_2 normalized by T. There are $r(r+1)$ subgroups of Z_1 distinct from U_1 normalized by T. If S'_1 is such a subgroup, then $\langle S'_1, D\rangle$ is a subgroup of G conjugate to H_1. This accounts for all $r(r+1)$ conjugates of H_1 meeting H_1 in T. It follows that there are precisely $r(r+1)$ H_1-orbits in $V\Gamma$ of vertices meeting H_1 in a dihedral subgroup of order $2(r-1)$. They can be parametrized by the $\mathbf{F}_r^*$-orbits on $\mathbf{F}_q \setminus \mathbf{F}_r$.

Suppose $diam\Gamma \geq 5$. Then, by [5], Lemma 2.6, we may assume that $G = P\Gamma L(2,q)$. If e denotes the number of divisors of a (including 1 and a), then, since the H-orbit sizes of vertices meeting H in a dihedral $2(q-1)$ only depend on the order of the Galois automorphism, the number of H-orbits of vertices meeting H in a dihedral $2(r-1)$ is at least $r(r+1)/ea$. On the other hand, there are orbits of size bigger than that, for instance those containing H^x, where x corresponds to the matrix

$$\begin{bmatrix} 1 & b \\ -b^{-1} & 0 \end{bmatrix}$$

where $b \in \mathbf{F}_q \setminus \mathbf{F}_r$. Thus, by [5], Lemma 2.7 we have $r(r+1)/ea \leq 2$. This implies that q is one of 8,27,64. A straightforward check of subdegrees against feasible intersection arrays shows that the theorem holds for these values of q.

Finally, suppose $diam\Gamma \leq 4$. Then the number of nontrivial H-orbits is 4. On the other hand, by the same argument as above, it is at least $r(r-1)/a$, so $r = 2$ and $q = 8$. But then H is not maximal, and we are done.

(vi). soc $Y \cong PSL(2,r)$ where $q = r^2$. By Lemma 2.6(i) of [5], we may assume $G \geq P\Sigma L(2,q)$. By maximality of H, and observing that if q is odd, there are two classes of subgroups isomorphic to $PSL(2,r)$, we have $G = P\Sigma L(2,q)$ and $H = P\Gamma L(2,r)\cdot\langle\gamma\rangle$, where γ is the standard Frobenius automorphism of $PSL(2,q)$ of order 2. Furthermore, as a G-set, $V\Gamma$ can be identified with the $L(2,q)$-class of γ. Thus, Proposition 2.5 of [5] applies. Clearly, cases (i) and (ii) of its conclusion do not hold.

Suppose q is odd. First consider the case where $\delta \in \Gamma$ is adjacent to γ if δ and γ commute. Then the product of any two noncommuting involutions in Y has the same order. But any element in a torus of Y order $(r \pm 1)/2$ arises as such a product, so (as r is odd) it follows that $(r-1)/2=2$, and $q=25$. The resulting graph has been found by J.I. HALL [11] in his determination of locally Petersen graphs.

It remains to study the case where γ and $\delta \in \Gamma(\gamma)$ do not commute. Then case 2.5 (iii) of [5] is at hand, so $\gamma\delta$ has order 2 iff $\delta \in \Gamma_d(\gamma)$. Also, no two involutions in $V\Gamma$ have a product of order 4, so (by consideration of involutions in Γ commuting with σ) $r \equiv 3,5 \bmod 8$.

To finish, we shall use another interpretation of $V\Gamma$. Since $G = P\Sigma L(2,r^2) \cong P\Sigma O^-(4,r)$, we can view $H \cong P\Sigma O(3,r).2$ as the stabilizer of a nonisotropic vector in elliptic projective 3-space. (The two choices of points according to square or non-square norm if q is odd correspond to the two classes of $PSL(2,r)$ in $PSL(2,q)$.) We can thus view $V\Gamma$ as the set of nonisotropic points with square norm.
Suppose q is odd. Then, from this picture it is readily seen that, if γ and δ are vertices of Γ, there is $g \in H = G_\gamma$ such that δ and δg are orthogonal (consider the projection of δ on the orthoplement of $\langle\gamma\rangle$). This yields that commuting involutions in the earlier picture occur at distance 2, whence $d \leq 2$, a contradiction.
Suppose q is even. Then, a direct computation (cf. [6], Chapter 12) shows that vertices corresponding to orthogonal points can be found at distance at most 3, regardless of the choice of adjacency, so $d \leq 3$, and $q=16$, yielding the Doro graph. This ends the proof for $n=2$.

3. Proof for $n \geq 3$; structure preserving vertex stabilizers.

The following result is essentially due to SAXL [22], cf. the remark following [5], Lemma 2.1. Recall that, for $d \leq n/2$, the Grassmann graph $G(n,d,q)$ has vertex set $VG(n,d,q)$ the collection of d-dimensional subspaces of $\mathbf{F}_q^n$.

3.1. Lemma. *Let G, Γ and H be as above and suppose G acts multiplicity-freely on $V\Gamma$.*

(i) *If τ_n is the number of involutions in* Sym$_n$, *then*

$$|P\Gamma L(n,q) \cap H| \geq (1+\tau_n)^{-1}[G : G \cap P\Gamma L(n,q)]^{-1}\prod_{i=2}^{n}\frac{q^i-1}{q-1}.$$

(ii) *If n is even, the group G acts multiplicity-freely on $VG(n,n/2,q)$ with rank $n/2+1$. Consequently, the number of H-orbits on $VG(n,n/2,q)$ is at most $n/2+1$.*

For dimension $n \leq 5$, the subgroups of $L(n,q)$ have been determined, cf. KANTOR & LIEBLER [17] for references and details. Nevertheless, we start with the same approach for finding all multiplicity-free permutation representations as used by INGLIS, LIEBECK & SAXL [13] namely to apply Aschbacher's division of cases for a skew-linear group $H_0 = P\Gamma L(V) \cap H$ (a normal subgroup of H of index at most 2) acting projectively on a module V over $\mathbf{F}_q$ of dimension n. ASCHBACHER [2] discerns 8 cases (C1),...,(C8) in which H preserves a certain structure on V. We shall go over the various possibilities now. Denote by ϕ the natural projection map $\Gamma L(n,q) \rightarrow P\Gamma L(n,q)$.

(C1) and (C2). *Y stabilizes a subspace.* We are as in one of (i),(ii),(iii) or (iv) of INGLIS, LIEBECK & SAXL [13]. There are no changes with respect INGLIS et al. (i.e. this leads to the Grassmann graphs), except that for $n = 3$ generalized hexagons of order $(q, 1)$ occur (they are distance-transitive as polarities exist) and for $n = 3$ and $q = 2$, the Coxeter graph arises.

(C3). *There is an extension field of order $r=q^m$, for some prime $m \mid n$, and $\mathbf{F}_r H_0$-module W such that V is the module obtained from W by restriction of scalars to $\mathbf{F}_q$.* There is a torus, L say, in $SL(n,q)$ of order $q^{m-1} + q^{m-2}+...+1$ such that $H = N_G(\phi L)$. As all such tori are conjugate, we may take $V\Gamma$ to be the set of conjugates of L. Similarly to case (v) of the proof of Theorem 3.2 in COHEN & VAN BON [5], one can show that if L_1 is a conjugate of L which commutes with L, then $L_1 \in \Gamma_d(L)$. Let $N \in \Gamma(L)$. Then, according to

Lemma 2.7(ii),(iii) of [5], $N_H(\phi N)$ is the unique one of maximal order among all $N_H(\phi M)$ for $M \in V\Gamma$ such M and L do not commute. In other words, $k = [H : N_H(\phi N)]$ is minimal among all conjugates of L not commuting with L.

As here $n \le 7$, we have either $m = n$ or one of $(m,n) = (2,6),(3,6),(2,4)$.
Case $m=n$. In view of maximality of H, we have that n is a prime; in particular, $n \in \{3,5,7\}$. All nontrivial orbit sizes of $H_0 := \phi^{-1} H \cap \Gamma L(n,q)$ on $V\Gamma$ are multiples of $|L|/(n,|L|)$ (for the centralizer in L of a conjugate L_1 of L distinct from it is trivial and the normalizer interchanges the n distinct characters of L_1 on $V \otimes \mathbb{F}_{q^n}$). Thus, there are at most $e([H : N_G(\phi L_1)]) = e(2na(n,q^{n-1} + \cdots +1))$ different nontrivial H-orbit sizes, where $e(x)$ stands for the number of divisors of x. By Lemma 2.7.(vi) of [5], this yields $diam\Gamma \le 3e(2na(n,q^{n-1} + \cdots +1))$. On the other hand, we have $v \le 1 + diam\Gamma \cdot |H|$, so

$$v = \frac{1}{m} q^{n^2(m-1)/2m}(q^n-1)(q^{n-2}-1)...(q-1)/(q^n-1)(q^{n-m}-1)...(q^m-1) \le$$

$$1+6e(2na(n,q^{n-1} + \cdots +1))an(q^n-1)/(q-1).$$

This gives that we have one of $(n,q) = (3,2),(3,3),(3,4)$. In the first case, we find the projective line of order 7 on which $PGL(2,7) \cong \mathrm{aut}\,L(3,2)$ acts doubly transitively, so $V\Gamma$ is a clique, a contradiction. In the cases $q=3$ and $q = 4$, we get graphs on 144 and 960 vertices, respectively, which, by closer inspection of possible intersection arrays, are readily seen not to provide distance-transitive graphs.

From now on, we may assume that m is a proper divisor of n.

Suppose $m = 2$, so $n = 4$ or 6, and L is a torus of order $q+1$.
The case $n = 4$ can be done by geometry, using the isomorphisms $L(2,q^2) \cong P S\Omega^-(4,q)$ and $L(4,q) \cong PS\Omega^+(6,q)$. Thus, we can (and shall) view $V\Gamma$ as the set of elliptic lines in the hyperbolic geometry $O^+(6,q)$. Fix a line $l \in V\Gamma$. Any line $m \in V\Gamma$ belongs to one of the sets V_i $(1 \le i \le 6)$ given below:

V_i	$\lvert V_i \rvert$	description of V_i
V_1	$(q^2-1)(q^2+1)$	$\langle l,m \rangle$ degenerate, $l \cap m \ne \varnothing$
V_2	$q(q^2+1)(q+1)(q-2)/2$	$\langle l,m \rangle$ nondegenerate, $l \cap m \ne \varnothing$
V_3	$q(q^2+1)(q^2-1)(q+1)(q-2)/2$	$\langle l,m \rangle$ degenerate, $l \cap m = \varnothing$
V_4	$q^3(q^2+1)(q^2-1)(q-1)/4$	$\langle l,m \rangle$ elliptic, $l \cap m = \varnothing$
V_5	$q^2(q^2+1)(q^2-1)(q-1)(q-2)/4$	$\langle l,m \rangle$ hyperbolic, $l \cap m = \varnothing$, $m \notin l^\perp$
V_6	$q^2(q^2+1)/2$	$\langle l,m \rangle$ hyperbolic, $l \cap m = \varnothing$, $m \in l^\perp$

If $q = 2$, then $V_i = \varnothing$ for $i = 2,3,5$, and the Johnson graph $J(8,3)$ appears. Otherwise, $diam\Gamma \ge 6$, so, by Lemma 2.6 of [5], we may assume that H acts transitively on the set of nonisotropic points of $O^+(6,q)$. Now V_6 is a single orbit corresponding to L_1 (the commuting conjugate of L) so $\Gamma_d(l) = V_6$. On the other hand, a straightforward check shows that an H-orbit off V_6 of minimal length lies in V_2 (and has size $(q+1)(q^2+1)q/2$) if q is odd, and lies in V_1 (and has size $(q^2-1)(q^2+1)$) if q is even. In both cases, it is easily seen that there are members of $V_6 = \Gamma_d(l)$ in $\Gamma_{\le 4}(l)$, contradicting that $d \ge 6$.

Suppose $n = 6$. Take l such that $L = \langle l \rangle$, and let $K = \langle k \rangle \in V\Gamma \setminus \Gamma_d(L)$ be such that $l^{-1}k$ has 4-dimensional fixed space and $\langle l,k \rangle \cong SL(2,q)$ stabilizes a 2-dimensional complement of this fixed space. The H-orbit size of K is certainly not maximal. So the number of such orbits is bounded by 2. Also $N_H(\phi L, \phi K) \le C_G(\phi\langle L,K \rangle)$. Now the n=2 case gives that the number of such orbits (varying K over the conjugates of L in $\langle L,K \rangle$) is at least $(p-3)/2$. Since this number is bounded by 2, we get $p \le 7$. If p is odd, then H is centralized by an involution in $PGL(6,q)$ and so by Lemma 2.6 of [5], we may take $PGL(n,q) \le G$ and H is the centralizer of an involution in $N_G(\phi L)$; but then there are pairs of involutions from this class with products of order 4 (from the $PGL(2,q)$ picture), so we are done by [5]. It remains to consider the case where $p = 2$.

Suppose $q = 2$. Then direct computation (we used CAYLEY) shows that the H-orbits on $V\Gamma$ have sizes 1, 336, 5040, 201060, 25920, 315, 3780, in the respective cases where $\langle L,N \rangle$ is a group of type $\mathbb{Z}_3$, $\mathbb{Z}_3^2$, [36], Alt_5, $L(2,8)$, Alt_4, [24]. Thus, $d = 6$, and $\Gamma(L)$ must be the H-orbit of size 315. But then, there is a subspace decomposition $V = V_1 \oplus V_2$ with $\dim V_i = 2i$ such that L and N coincide on V_1, and generate a subgroup isomorphic to Alt_4 in the subgroup A of G normalizing V_1 and V_2. Now A acts on L^A as

$SL(V_2) \cong \mathrm{Alt}_8$ on its set of groups of order 3 fixing 5 points, and the above adjacency leads to an isomorphism of the subgraph of Γ induced on L^A with $J(8,3)$. In particular, commuting pairs occur at distance 3, so $d \le 3$, a contradiction.

Now $q \ge 4$, q even. From the geometry it is readily seen that there are at least three H-orbits of the same length consisting of $\langle K \rangle$ such that $\langle L,K \rangle \cong SL(4,q)$, a contradiction.

Finally, suppose $m = 3$. Then $n = 6$. Now $|H_0| \le (q^3-1)(q-1)|PGL(2,q^3)|\cdot 3a$, so Lemma 3.1 yields $q \le 4$. For $q = 3,4$, direct check reveals that the number of H-orbits on the set of maximal flags in $\mathbf{F}_q^6$ exceeds $\tau_n+1 = 76$, contradicting the remark after Lemma 2.1 of [5]. If $q = 2$, it can be verified that too many subdegrees are equal for the graph Γ to be distance-transitive.

(C4) and (C7). There is a Y-invariant tensor product decomposition $V = V_1 \otimes \cdots \otimes V_j$ with $j>1$ and $\dim V_i > 1$ for all i $(1 \le i \le j)$. Then, as $n \le 7$, we have $j = 2$ and $(\dim V_1, \dim V_2) =$ (2,2) or (2,3).

First consider $\dim V_1 = 2$, and $\dim V_2 = 3$, so n=6. Then by Lemma 3.1

$$q(q^2-1)q^3(q^3-1)(q^2-1)(q-1)a \ge |H| \ge \frac{1}{2}\frac{1}{76}\prod_{i=1}^{6}\frac{q^i-1}{q-1}$$

implying $q^4 a \ge \frac{1}{152}(q^6-1)(q^5-1)(q^2+1)/(q-1)^2$, which is absurd.

Thus, assume $\dim V_1 = \dim V_2 = 2$. Then H is an orthogonal group and will be dealt with in (C8).

(C5). *There is a divisor m of a such that, with $q = r^m$, the subgroup H_0 is conjugate to a subgroup normalizing $PSL(n,r)$.*

Lemma. *If σ is the standard Frobenius $\xi \mapsto \xi^r$ of order m. Then $H = C_G(\sigma)$, and the permutation character of G on H is multiplicity-free if and only if $m = 2$.*

If $m = 2$, the statement follows from [10].

Suppose for the remainder of the proof of this lemma that $m > 2$. Denote by P, S the set projective points of $\mathbf{F}_q^n$, $\mathbf{F}_r^n$, respectively. Then P partitions into the three H invariant sets S, $S_1 = \{p \in P\setminus S \mid |pp^\sigma \cap S| \ne \varnothing\}$, and $S_2 = \{p \in P\setminus S \mid |pp^\sigma \cap S| = \varnothing\}$, where pp^σ denotes the projective line of P on p and $p\sigma$. Since these three sets are nonempty and G is doubly transitive on P, we are done unless G contains a duality (i.e., graph automorphism) δ. Also, H cannot have 4 or more orbits on P. Consider $p \in S_1$ and denote by l the unique line pp^σ on p meeting S. Then $H_x \le H_l$, and, as S_1 must be a single H-orbit, the group H_l acts transitively on the $r(r^{m-1}-1)$ points of $l\setminus S$, so $r(r^{m-1}-1) \mid r(r^2-1)m$. Hence either $m = 5$ and $r = 2$, or $m = 3$. In the first case, we obtain a contradiction with Lemma 3.1, so from now on we may assume $m = 3$.

Now consider the H-invariant sets of incident point, hyperplane pairs $\{s,t\}$, for $s \in S_i$, $t \in \delta S_j$ $(0 \le i,j \le 2)$. If $n > 3$, all 6 of them are nonempty and if $n = 3$, there are 5 nonempty sets among them. Since G acts multiplicity-freely on the set of all incident point,hyperplane pairs with rank 5 and 4 in the respective cases, this leads to a contradiction with the multiplicity-freeness of G on $V\Gamma$, and so finishes the proof of the lemma. □

Due to the lemma, we only need consider the case where $m = 2$. Then H is the centralizer of the involution σ and, in view of the proof of Theorem 3.2 Case (vii) [5], we may assume $\sigma \in G$, $V\Gamma = \sigma^G$, $\Gamma(\sigma) \le H$, $H \cap \sigma^G$ is a class of s-transpositions for some prime s, and $n \le 4$. According to [1] and [9], $n = 4$ and $r \in \{2,3\}$.

If $r = 2$, then $\Gamma(\sigma)$ is isomorphic to the complement of the Johnson graph $J(8,2)$, so Γ contains a quadrangle, $k = 28$, $a_1 = 6$, and by Terwilliger [24] Γ has diameter at most 7, a contradiction as the permutation rank exceeds 8 (cf. Gow [10]).

If $r = 3$ then $\Gamma(\sigma)$ is the graph of nonisotropics in $O^+(6,3)$, so Γ contains a quadrangle, $k = 117$ and $a_1 = 36$, leading to the same contradiction as for $r = 2$.

(C6). *There is a prime $r \ne p$ such that $r^m = n$ for some m, and an r-group R acting irreducibly on V and normalized by H_0, such that $R/Z(R)$ has order r^{2m} and $Z(R)$ has order at least 3 (and dividing $q-1$). Furthermore, a is odd and equals the order of p in the group of units of the integers modulo $|Z(R)|$.* Now

$|H \cap P\Gamma L(n,q)| \le r^{2m}|Sp(2m,r)|a = r^{2m+m^2}\prod_{i=1}^{m}(r^{2i}-1)a$, so, by Lemma 3.1,

$$r^{2m+m^2}\prod_{i=1}^{m}(r^{2i}-1) \geq \frac{1}{2}(1+\tau_n)^{-1}\prod_{i=2}^{n}\frac{(q^i-1)}{(q-1)} .$$

Using that $|Z(R)|$ divides $(q-1)$ and $2 < r^m = n \leq 7$, and that $|Z(R)$ is either odd or divisible by 4, we see that the only possible values for the triple (m,r,q) are (3,1,4), (3,1,7), (2,2,5). In the first case, we have the example on 280 vertices described in Table 1. In the second case, a look at the character table of aut$L(3,7)$ (cf. ATLAS [7]) immediately gives a contradiction with multiplicity freeness. Finally, let $(m,r,q) = (2,2,5)$. Then, by use of the isomorphism $L(4,5) \cong PS\Omega^+(6,5)$, the vertex set $V\Gamma$ may be viewed as the stabilizer of an orthonormal frame (6 nonisotropic 1-spaces that are mutually ortogonal), say $\{\mathbf{F}_5 v_i\}_{1\leq i\leq 6}$ in $O^+(6,5)$. Now v_1+2v_2, $v_1+v_2+2v_3+2v_4$, $v_1+v_2+v_3+2v_4+2v_5+2v_6$, $v_1+v_2+v_3+v_4+v_5$ are clearly representatives of distinct H-orbits, whose 1-space are isotropic, showing that H has at least 4 orbits of isotropic points. This implies that it cannot be multiplicity-free (cf. the remark following Lemma 2.1 of [5]).

(C8). *There is a nondegenerate H_0-invariant quadratic, symplectic, or hermitean form on V.* If the form is symplectic or hermitean, then H is the centralizer of an involution, and we proceed as in [5]. First, consider the case of a symplectic form. Then $m=2$ in view of [5]. Using the isomorphisms $PSp(4,q) \cong PS\Omega(5,q)$ and $L(4,q) \cong PS\Omega^+(6,q)$, we can view $V\Gamma$ as the set of projective points $\langle x\rangle$ with $Q(x) = 1$, for $x \in W = \mathbf{F}_q^6$ and Q a fixed nondegenerate quadratic form on W of Witt index 3, and $G \cap L(4,q)$ as the simple socle of the group fixing Q. From this picture, it is straightforward that $V\Gamma$ cannot be distance-transitive, unless $q = 2$ or 3, in which cases there are distance-transitive graph structures on Γ as listed in Table 1 (on 28 and 117 vertices, respectively).

Now consider the case where H_0 fixes a hermitean form. Then, according to [5], there are involutions $x,y \in V\Gamma$ such that xy has order 4, so $\Gamma(x)$ coincides with a class of r-transpositions for some prime number r, and by FISCHER [9] and ASCHBACHER [1], either $(n,q) = (4,9)$ or $q = 4$. In the first case we get that Γ satisfies $k = 126$, $a_1 = 45$ and contains quadrangles, so that, by TERWILLIGER [24], *diam*$\Gamma \leq 5$, less than the number of H-orbits (cf. Gow [10]), a contradiction. Therefore assume $q = 4$. For $n = 3$, we get an example, the graph Γ from Table 1 on 280 vertices, so assume $n \geq 4$. Then the same argument as given at the end of the proof of Theorem 3.2 in [5] applies.

It remains to discuss the case where H_0 stabilizes an quadratic form. By maximality of H in G, we take q to be odd.

Suppose n is odd. If $G \leq P\Gamma L(n,q)$, then the permutation rank of G on $VG(n, 2,q)$ is 3 or 2 according as $n \geq 5$ or $n = 3$, whereas H has 4, respectively 3 orbits on this set. Consequently, G is not multiplicity-free on $V\Gamma$, a contradiction. Hence G contains a graph automorphism. Now G has permutation rank 5 on the set of incident point,hyperplane pairs, whereas $H \cap P\Gamma L(n,q)$ has at least 7 orbits on this set, again a contradiction with multiplicity freeness.

Thus $n = 2m$ is even. First, suppose the Witt index of the form is maximal (i.e., equal to m). Then G has permutation rank $m+1$ on the set of m-spaces, but there are at least $m+2$ H-orbits on this set (if $n = 4$, there are elliptic, hyperbolic, tangent and isotropic lines, and if n=6, there are totally isotropic, degenerate with 2-dimensional radical, degenerate with hyperbolic quotient, degenerate with elliptic quotient, nondegenerate).

Finally, let the Witt index be smaller than m. Then it is $m-1$. If $G \leq P\Gamma L(n,q)$, then G has permutation rank 2 on the set of 1-spaces, and H has 3 orbits on this set (observe that if $n \geq 4$, no outer automorphism can be realized in $P\Gamma L(n,q)$), so again G cannot be multiplicity-free on $V\Gamma$. Thus G contains a diagram automorphism. Now $H \cap P\Gamma L(n,q)$ has 3 orbits on the set of 1-spaces, and from this it readily follows that there are at least 6 orbits on the set of incident point, hyperplane pairs. Since G has permutation rank 5 on the latter set, we have a contradiction with multiplicity freeness, and we are done.

4. Proof for $n \geq 3$; irreducible groups with simple socle.

We retain the notation $\phi:\Gamma L(n,q) \to P\Gamma L(n,q)$, $V = \mathbf{F}_q^n$, $H_0 = \phi^{-1}(H \cap P\Gamma L(n,q))$. In this section, we deal with the case where H_0 is not as described in one of (C1),...,(C8). Then, according to ASCHBACHER [2], the socle Z of H is a nonabelian simple group acting absolutely irreducibly on $\mathbf{F}_q$. Moreover, we have $H = N_G(Z)$, and $C_G(\phi Z) = 1$, so H embeds in aut$Z$. The resulting upper bound $|\mathrm{aut}Z|$ on H will be frequently applied in conjunction with Lemma 3.1. We further divide this case into four subcases, viz. (i) Z

is a simple Chevalley group of characteristic p; (ii) Z is a simple Chevalley group of characteristic $r \neq p$ and cannot be viewed as a simple Chevalley group of characteristic p; (iii) Z is an alternating group Alt_m with $m \geq 7$, $m \neq 8$; (iv) Z is sporadic group.

(i) From known literature (e.g.[8, 17, 19]) we derive

Lemma. *Let Z be a simple Chevalley group of characteristic p (including the derived groups $PSp(4,2)'$, $G_2(2)'$, $G_2(3)'$, ${}^2F_4(2)'$) that is a subgroup of $L(n,q)$ for which* (C1),...,(C8) *does not hold. Then either $Z \cong PSp(4,2)'$ and $q = 4$, or $Z \cong L(2,r)$ for some power $r = p^m$ of p.*

The case $Z \cong PSp(4,2)'$ leads to the graph on 56 vertices mentioned in Table 1. Therefore, we assume $Z \cong L(2,r)$. By a result of Donkin, cf. LIEBECK [19], $n \geq 2^{m/(m,a)}$. As $n \leq 7$, we have $m/(m,a) \leq 2$. Suppose $m = (m,a)$. Then $m = a$, for otherwise (C5) would hold. By Lemma 3.1, we have

$$q(q^2-1)a \geq \frac{1}{2}(1+\tau_n)\prod_{i=2}^{n}\frac{q^i-1}{q-1},$$

whence $n = 3$. But then $Z = PS\Omega(3,q)$ and belongs to (C8), a contradiction.
Therefore $x = (m,a)$ satisfies $m = 2x$ and there is an odd number k such that $a = kx$. Set $s = p^x$. Then Lemma 3.1 gives

$$s^2(s^4-1)m \geq \frac{1}{2}(1+\tau_n)\prod_{i=2}^{n}\frac{s^{ik}-1}{s^k-1},$$

leading to $k = 1$ (recall that $n \geq 2^2$), and either $n=5$ and $q=2$, or $n = 4$.
If $(n,q) = (5,2)$, a look at the ATLAS [7] shows that $H = N_G(Z)$ is nonmaximal, again a contradiction. Consequently, $n = 4$, and we are in case (C3) (cf. [17]), a contradiction.

(ii) From known literature (e.g. [18]) we derive

Lemma. *Let Z be a Chevalley group of characteristic $r \neq p$ acting projectively and irreducibly on the $\mathbf{F}_q$-vector space V of dimension at most 7. Denote by μ the minimal dimension of such a module. Then Z is isomorphic to one of $L(2,4)$ ($\mu = 2$), $L(2,8)$ ($\mu = 6$), $L(2,7)$ ($\mu = 3$), $L(2,9)$ ($\mu = 3$), $L(2,11)$ ($\mu = 5$), $L(2,13)$ ($\mu = 6$), $L(3,4)$ ($\mu = 4$), $L(4,2)$ ($\mu = 7$), $PSp(6,2)$ ($\mu = 7$), $PSU(4,2)$ ($\mu = 4$), $PSU(3,3)$ ($\mu = 6$), $PSU(4,3)$ ($\mu = 6$).*

Suppose $n = 3$. Then an absolutely irreducible embedding of each of the three groups listed in the table with $\mu \leq 3$ defies (C8).
So let $n \geq 4$. Each of $PSp(6,2)$, $L(4,2)$, $L(2,13)$, $L(2,8)$ fails in view of Lemma 3.1. We check the remaining possibilities for Z in their order of appearance in the lemma.
$Z \cong L(2,4)$ or $L(2,7)$. Lemma 3.1 yields $n = 4$ and $q \leq 3$, so $q = 3$. Now, in the former case, we obtain a contradiction with the maximality of H, and in the latter case is absurd as $L(2,7)$ does not embed in $L(4,3)$.
Suppose $Z \cong L(2,9)$. As $Z \cong PSp(4,2)'$, we may also assume $p \neq 2$. But then Lemma 3.1 yields a contradiction.
Suppose $Z \cong L(2,11)$. Then Lemma 3.1 (and $\mu \geq 5$) gives $n = 5$ and $q = 2$, which is absurd as 11 does not divide $|\mathrm{aut}L(5,2)|$.
Let $Z \cong L(3,4)$. If $n \geq 6$, we get a contradiction with Lemma 3.1. By [17], we must have $n = 4$ and $q = 9$, in which case, X embeds via $PSU(4,3)$, a contradiction with the maximality of H.
If $Z \cong PSU(4,2)$, then we may assume $p \neq 2,3$. Lemma 3.1 then yields $n = 4$ and $q = 5,7$, whence, by the requirement $q \equiv 1 \bmod 3$ (cf. [17]) $q = 7$. In order to study the action of Z on V, we present Z as the group generated by the following matrices (they are given here as the matrices presented in [20] are in error).

$$\begin{bmatrix}4&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&4&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&2&1&1\\0&1&2&1\\0&1&1&2\end{bmatrix}, \begin{bmatrix}2&0&1&6\\0&1&0&0\\1&0&2&6\\6&0&6&2\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&2\end{bmatrix}.$$

Straightforward computation shows that there are 2 orbits, say S and T on the set of projective points (as stated in [20]) with length 40 and 360, respectively, and that there are 240 (projective lines) containing pre-

cisely 2 points from S, 90 lines having precisely 4 points of S, 1440 lines having precisely 1 point of S, and 1080 lines entirely contained in T. Consequently, the permutation rank of G on the set of lines (being 3) exceeds the number of H-orbits of lines, a contradiction with multiplicity freeness of G on H.
Suppose $Z \cong PSU(3,3)$. Lemma 3.1 gives $n = 6$ and $q = 2$, but in view of $Z \cong G_2(2)'$, we may assume $p \neq 2$, and we are done.
Finally, suppose $Z \cong PSU(4,3)$. Now either $n = 6$, and $q \in \{2,4\}$ or $n = 7$ and $q = 2$. As the possibility $q = 2$ fails by Lagrange, we have $n = 6$ and $q = 4$. But then Z embeds in $PSU(6,2)$ and hence H is not maximal in G. This ends the proof of case (ii).

(iii) By well-known results $Z \cong \mathrm{Alt}_m$ and $n = \dim V \leq 7$ gives $m \leq 9$.
Let $m = 7$. Then Lemma 3.1 gives that either $n = 5$ and $q = 2$, or $n \leq 4$. In the former case, H is nonmaximal (cf. [7]), so assume $n \leq 4$.
If $n = 3$, then Lemma 3.1 gives $q \leq 25$. In view of [7], we must have $p \leq 7$, and by Lagrange and [7], $q = 25$ remains. But then Z is contained in $PSU(3,5)$, yielding a contradiction with the maximality of H.
Now suppose $n = 4$. If $p = 2$, then $q = 2$ and G is doubly transitive on $V\Gamma$, leading to a contradiction with $diam\Gamma > 1$, so $p \geq 3$. Lemma 3.1 gives $q = 3,5$ contradicting Lagrange.

Finally, let $m = 9$. Then p divides $m!$ (as $n \leq 7$). If $p \neq 2$, then, by consideration of the subgroup Alt_8, $n = 7$, contradicting Lemma 3.1. So $p = 2$, forcing $n \geq 8$, a contradiction.

(iv) It is well known (cf. LIEBECK [20]) that the only sporadic groups having a projective representation of degree at most 7 are among $M_{11}, M_{12}, M_{22}, J_1, J_2$. If p does not divide $|Z|$, then by the ATLAS [7] we have $Z = J_2$, $n = 6$, and $\phi^{-1}Z = 2{\cdot}J_2$. Since p is odd, there is a symplectic form left invariant by Z, and so $H = N_G(Z)$ is nonmaximal.

From now on, suppose p divides $|Z|$. We proceed with a case by case analysis.
Let $Z \cong M_{11}$. By JAMES [16], the only irreducible projective modular characters for Z of dimension at most 7 occurs for $p = 3$ and $n = 5$. If $G \leq PGL(5,3)$, then Lemma 3.1 yields $|H| \geq 9680$. But $|H| = |Z| = M_{11} = 7920$, a contradiction. Hence G contains graph automorphisms, and by maximality of H, we have that there is a graph automorphism σ normalizing Z. Since $\mathrm{out}M_{11} = 1$, we must have $H \leq C_Z(\sigma)$, a classical group, conflicting with maximality of H in G.

$Z \cong M_{12}$. If the representation has no multiplier, then, by JAMES [16], we have $n \geq 10$, which is absurd, so we may assume $\phi^{-1}H$ contains a subgroup $Z \cong 2{\cdot}M_{12}$. Now n must be even, and, in view of Lemma 3.1, either $n = 6$ and $q = 2$ or $n = 4$ and $q \leq 13$. But 11 must divide $|L(n,q)|$ whence $n = 4$ and $q = 11$. Since $|L(4,11)|$ is not a multiple of 3^3, this is impossible.

$Z \cong M_{22}$. Applying [17] gives $n \geq 6$. Lemma 3.1 then gives $q = 2$, contradicting Lagrange.

$Z \cong J_1$. Consider a Frobenius subgroup F of order $7{\cdot}6$. Suppose $p \neq 7$. Then $n \geq 6$ for a faithful representation of F, and by Lemma 3.1 we get $q = 2$, again contradicting Lagrange. Thus $p = 7$. By [17], $n \geq 6$, contradicting Lemma 3.1.

$Z \cong J_2$. If p=3, then $n = 4$ from Lemma 3.1. But consideration of the subgroup isomorphic to $5^2{:}D_{12}$ shows that $n \geq 6$. Then $q \leq 3$, contradicting Lagrange. This ends the proof of case (iv) and hence Theorem 1.1. □

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