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The influence of a homogeneous wind upon an infinitely wide North Sea

by

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§ 1 Introduction*)

In this report some aspects of the work carried out by the applied mathematics division of the Amsterdam Mathematical Centre in connection with the problem of the motion of the North Sea are considered.

The general problem of the hydrodynamic behaviour of a shallow sea subjected to a storm can be attacked only by means of considerable simplifications: linearisation of the hydrodynamic equations, neglection of vertical motions, etc.

The following equations will be used

$$\frac{\partial u}{\partial t} + \lambda u - \Omega v + g\frac{\partial \zeta}{\partial x} = \frac{U}{\rho h}$$
$$\frac{\partial v}{\partial t} + \lambda v + \Omega u + g\frac{\partial \zeta}{\partial y} = \frac{V}{\rho h} \qquad 1.1$$
$$\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} + \frac{\partial \zeta}{\partial t} = 0.$$

Here g is the acceleration of gravity, λ a friction coefficient, Ω the coefficient of Coriolis, h the depth, u and v averages over a vertical of the horizontal components of the velocity, ζ the elevation of the sealevel above the undisturbed level, U and V the components of the tangential stress on the surface of the sea due to the wind.

The North Sea is usually considered as a rectangle $|x| < a$, $0 < y < b$ where $x = \pm a$, $y=0$ represent coasts and $y=b$ the open end at the ocean. The depth is taken constant. The ocean may be considered infinitely deep and the following boundary conditions hold

$$x = \pm a, \quad u=0 \ ; \quad y=0, \ v=0 \ ; \quad y=b, \ \zeta=0.$$

The numerical values are approximately

$$b=850 \text{ km}, \quad h=65 \text{ m (harmonic average)}, \quad \lambda=0{,}09 \text{ h}^{-1}, \Omega=0{,}48 \text{ h}^{-1}.$$

Next, if W represents the velocity of the wind at sealevel

$$\frac{\sqrt{U^2+V^2}}{\rho} = 3.5 \times 10^{-6}\, W^2$$

in corresponding units.

In this paper, however, a simpler model will be discussed. We shall consider a uniformly time-dependent wind upon an infinitely

*) Research carried out under the direction of Prof.Dr D.van Dantzig.

wide sea, i.e. the strip $-\infty < x < \infty$, $0 < y < b$. Moreover only wind will be considered, the direction of which is perpendicular to the coast, i.e. U=0, V=V(t). Under these conditions there are no variations in the stream and the elevation in the x-direction and hence the terms in 1.1 with $\frac{\partial}{\partial x}$ disappear. If next the following units are introduced in 1.1

t	$b/2\pi\sqrt{gh}$	h	;	y	$b/2\pi$	km
u,v	$\sqrt{gh}$	km/h	;	ζ	h	m

the equations 1.1 become

$$\begin{aligned} \left(\frac{\partial}{\partial t}+\lambda\right) u - \Omega v &= 0 \\ \left(\frac{\partial}{\partial t}+\lambda\right) v + \Omega u + \frac{\partial \zeta}{\partial y} &= V \qquad 1.2 \\ \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} &= 0 , \end{aligned}$$

with the boundary conditions

$$y=0 \quad v=0 \quad ; \quad y=2\pi \quad \zeta=0. \qquad 1.3$$

In the numerical case the units are as follows

t	1.5 h	u,v	91 km/h
y	135 km	ζ	65 m

and $\lambda = 0.14$, $\Omega = 0.71$.

If W is in m/sec, then $|V| = 1.1 \times 10^{-5}\ W^2$.

§2. General solution

If upon the equation 1.2 Laplace transformation is applied,

$$\bar{\zeta}(y,p) = \int_{-\infty}^{\infty} e^{-pt}\, \zeta(y,t)dt \quad \text{etc.} \qquad 2.1$$

we obtain the equations

$$\begin{aligned} (p+\lambda)\,\bar{u} - \Omega\,\bar{v} &= 0 \\ (p+\lambda)\,\bar{v} + \Omega\,\bar{u} + \frac{\partial \bar{\zeta}}{\partial y} &= \bar{V} \qquad 2.2 \\ \frac{\partial \bar{v}}{\partial y} + p\,\bar{\zeta} &= 0. \end{aligned}$$

The boundary conditions are obviously

$$y=0 \qquad \bar{v}=0 \quad ; \quad y=2\pi \qquad \bar{\zeta}=0.$$

From the system 2.2 we may derive

$$\frac{\partial^2\bar{\zeta}}{\partial y^2} - q^2\bar{\zeta} = 0 \qquad 2.3$$

where
$$q^2 = p(p+\lambda) + \Omega^2 \frac{p}{p+\lambda}. \qquad 2.4$$

The boundary conditions of 2.3 are

$$y=0 \qquad \frac{\partial\bar{\zeta}}{\partial y} = \bar{V}$$

$$y=2\pi \qquad \bar{\zeta} = 0.$$

The solution of 2.3 satisfying both boundary conditions is

$$\bar{\zeta}(y,p) = -\bar{V}\,\frac{\operatorname{sh}(2\pi-y)q}{q \operatorname{ch} 2\pi q}. \qquad 2.5$$

By means of the inversion theorem of the Laplace transformation we have

$$\zeta(y,t) = -\frac{1}{2\pi i}\int_L e^{pt}\,\frac{\operatorname{sh}(2\pi-y)q}{q \operatorname{ch} 2\pi q}\,\bar{V}\,dp \qquad 2.6$$

where L is a vertical $(\sigma - i\infty,\ \sigma + i\infty)$ with Re $\sigma > 0$.

In the following sections the following cases will be studied.

a free motions

b a constant wind starting at t=0

$$V=0 \quad t<0 \;; \qquad V=-1 \quad t>0.$$

c periodic motions of the form

$$\zeta(y,t) = Z(y)e^{i\omega t}$$

d a periodic wind starting at t=0

$$V=0 \quad t<0 \;; \qquad V=-\sin\omega t \quad t>0.$$

§3 Free motions

The free motions satisfy the equations 1.2 with V=0. For a free motion where u, v, ζ contain the time factor e^{pt} the value of p should satisfy ch $2\pi q=0$ (cf. 2.5). Thus $\zeta(y,t)$ is of the form

$$\zeta(y,t) = e^{pt} \cos\left(\frac{k}{2} + \frac{1}{4}\right)y \;, \qquad k=0,1,2\ldots \qquad 3.1$$

where p is determined by

$$q^2 = -\left(\frac{k}{2} + \frac{1}{4}\right)^2 , \qquad k=0,1,2\ldots .$$

For each k a triplet of eigenvalues is obtained which are the roots of

$$p^3 + 2\lambda p^2 + \left\{ \lambda^2 + \Omega^2 + \left(\frac{k}{2} + \frac{1}{4}\right)^2 \right\} p + \lambda\left(\frac{k}{2} + \frac{1}{4}\right)^2 = 0. \qquad 3.2$$

In the numerical case $\lambda^2 = 0.02$, $\Omega^2 = 0.5$ the first few eigenvalues are

k= 0	-0.0153 ,	-0.134 $\pm$ i 0.749
1	-0.074 ,	-0.104 $\pm$ i 1.028
2	-0.107 ,	-0.088 $\pm$ i 1.434
3	-0.122 ,	-0.080 $\pm$ i 1.886 .

The real roots form a decreasing sequence converging to $-\lambda$.
The complex roots are situated left from the line Re $p = -\frac{\lambda}{2}$.
The real parts converge to $-\frac{\lambda}{2}$, the imaginary parts are approximately $\pm\left(\frac{k}{2} + \frac{1}{4}\right)$.
The lowest real eigenvalue α_0 may be approximated by

$$\alpha_0 \backsim \frac{-\lambda}{1+16\Omega^2} , \qquad 3.3$$

or better

$$-\frac{\lambda}{1+16\Omega^2} < \alpha_0 < -\frac{\lambda}{1+16(\Omega^2+\lambda^2)} .$$

The main free motion is

$$\begin{cases} u = \dfrac{1+16\lambda\alpha_0+16\alpha_0^2}{4\Omega} e^{\alpha_0 t} \sin\dfrac{y}{4} \\ v = -4\alpha_0 e^{\alpha_0 t} \cos\dfrac{y}{4} \\ \zeta = e^{\alpha_0 t} \cos\dfrac{y}{4} . \end{cases} \qquad 3.4$$

With the values given above we have always

$$\frac{v}{u} = 0.18 .$$

Thus the mean stream makes the constant angle of about 10° with the coast.

§ 4 Elevation at the coast due to a stepfunction wind

If at t=0 the sea is at rest and from t=0 onwards a constant wind V=-1 blows we have for the elevation $\zeta(y,t)$ according to 2.6

$$\zeta(y,t) = \frac{1}{2\pi i} \int_L e^{pt} \frac{\mathrm{sh}(2\pi-y)q}{pq\ \mathrm{ch}\ 2\pi q}\, dp . \qquad 4.1$$

The right-hand side may be evaluated by means of the calculus of residues. The poles are p=0 and the roots of 3.2. The residue at p=0 gives the stationary solution which is reached for $t \to \infty$

$$\zeta(y,\infty) = 2\pi - y. \qquad 4.2$$

For large values of t the main contribution is that from the pole at α_0. If for α_0 the approximation 3.4 is taken we obtain

$$\zeta(y,t) \backsim (2\pi-y) - \frac{256\Omega^2}{\pi(1+16\Omega^2)} \exp - \frac{\lambda t}{1+16\Omega^2} \cos \frac{y}{4} . \qquad 4.3$$

In the numerical case $\lambda^2=0.02$, $\Omega^2=0.5$ we obtain in particular for the elevation at the coast

$$\zeta(0,t) = 2\pi - 4.56\ e^{-0.0153t} - 0.28\ e^{-0.074t} \ldots$$
$$- (0.52 \cos 0.749t - 0.01 \sin 0.749t)e^{-0.134t} \ldots \qquad 4.4$$

Another way of obtaining $\zeta(0,t)$ is as follows.
We may expand $\bar{\zeta}$ into the series

$$\bar{\zeta} = \frac{1}{pq} \sum_0^\infty (-1)^j\ \varepsilon_j e^{-4j\pi q} \qquad 4.5$$

with $\varepsilon_0=1$, $\varepsilon_j=2$ $j \geq 1$.
The terms in this series represent the successive reflections at ocean and coast.
Since for large p

$$q = p + \frac{\lambda}{2} + \frac{4\Omega^2-\lambda^2}{8p}$$

the j^{th} term of the series 4.5 corresponds to an original vanishing for $t < 4j\pi$.
It is easily seen that $\zeta(0,t)$ is continuous but that at $t=4j\pi$, $j=1,2\ldots$ its derivative makes a jump of $\varepsilon_j \exp -2j\lambda\pi$.
For $0 < t < 4\pi$ we have in particular

$$\zeta(0,t) \doteq \frac{(p+\lambda)^{\frac{1}{2}}}{p^{3/2}\left\{(p+\lambda)^2+\Omega^2\right\}^{\frac{1}{2}}} .$$

From the expansion of the right-hand side in negative powers of p an expansion of ζ in rising powers of t results.
We find in this way in the numerical case

$$\zeta(0,t) = t\left\{1 - 0.354\ t - 0.0404\ t^2 + 0.00182\ t^3 \ldots\right\}. \qquad 4.6$$

By means of 4.4 and 4.6 $\zeta(0,t)$ has been computed for a number of t values which are given in table 1 and plotted in figure 1.
We see that the elevation at the coast tends very slowly towards its ultimate value of 2π.
At t=30 the elevation is still at slightly more than half its ultimate value.
A wind of 30 m/sec causes an ultimate elevation of 4 meter.

Table 1

Elevation at the coast due to a stepfunction windfield

t = 0	$\zeta(0,t)$ = 0
1	0.928
2	1.608
3	1.993
4	2.077
5	2.014
6	1.969
8	1.891
10	2.170
12.5	2.496
15	2.537
17.5	2.664
20	2.884
25	3.105
30	3.377
35	3.588
40	3.794
50	4.153
∞	6.283 = 2π

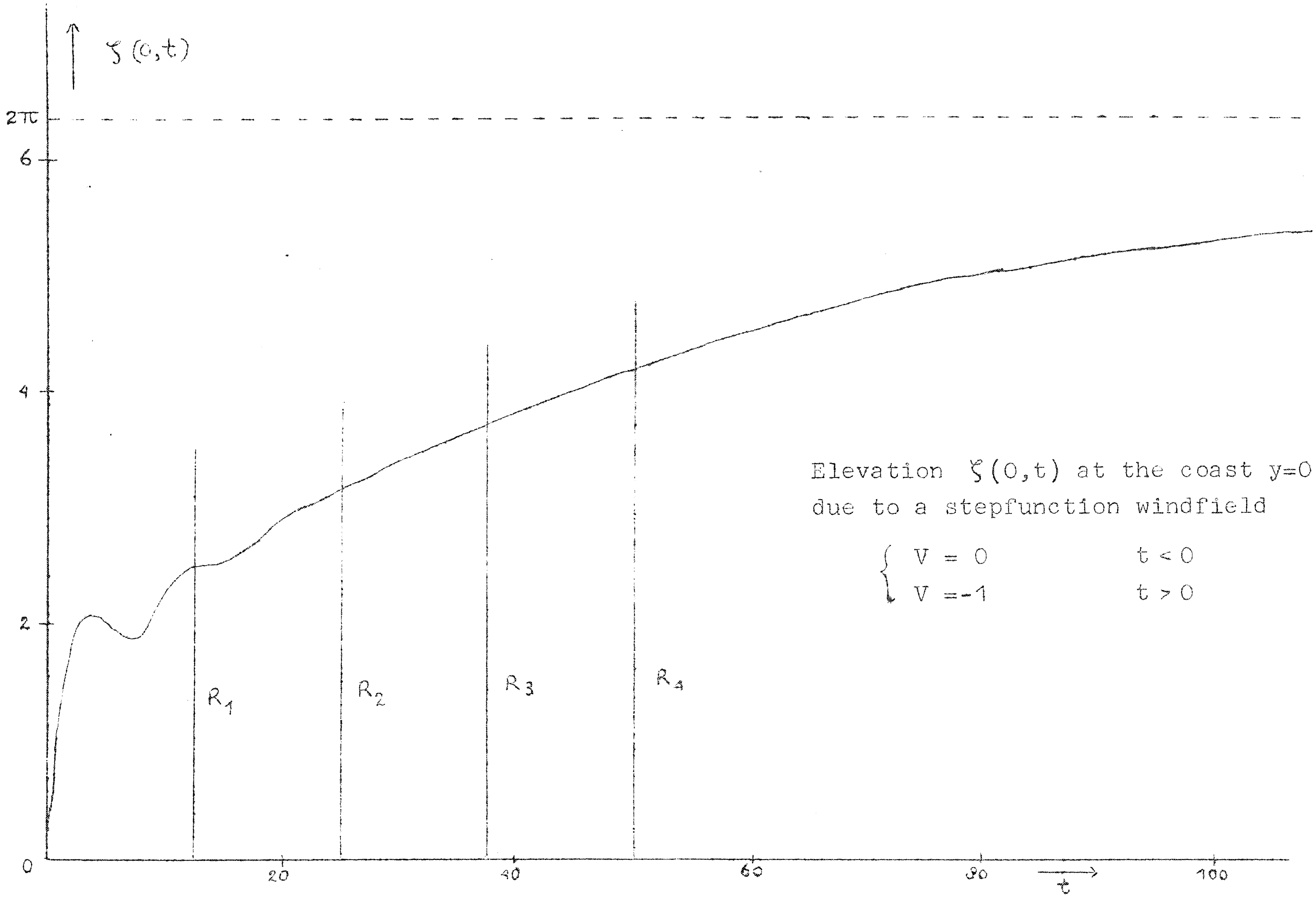


figure 1 R_j ($j=1,2\ldots$) j^{th} reflection at $t = 4j\pi$

§ 5 Periodic motions

If V=-sin ω t for all t there is a solution of the form

$$\zeta(y,t) = \mathrm{Im}\left\{ Z(y)\, e^{i\omega t} \right\}. \qquad 5.1$$

According to 2.6 we have

$$Z(y) = \frac{\mathrm{sh}(2\pi - y)q}{q\ \mathrm{ch}\ 2\pi q},$$

where

$$q^2 = \omega i(\lambda + \omega i) + \Omega^2 \frac{\omega i}{\lambda + \omega i}.$$

The maximum elevation at the coast is given by

$$M(\omega) = \left| \frac{\mathrm{th}\ 2\pi q}{q} \right|. \qquad 5.2$$

In the numerical case λ^2=0.02, Ω^2=0.5 we have computed Z(0), $M(\omega) = |Z(0)|$ and arg Z(0) for a number of ω values. See table 2 and figures 2,3,4.

$M(\omega)$ appears to have an absolute maximum 2π at ω=0 and a secondary maximum at ω=0.69 where it is only 2.34. This represents the resonance with the first eigenfunction which has the period 0.75 and the Coriolis effect for which Ω =0.71. In the case of the North Sea the dangerous storms extend over a period of about two days, so that ω should be of the order 0.1. Since $M(\omega)$ is some measure of the effect of the storm upon the coast we see that storms of longer duration are more dangerous.

A typical case of a long storm is given by ω=0.1. This storm extends over a period of 10π i.e. about two days. In this case we have

$$\zeta(y,t) = M(y)\sin(\omega t - \Theta(y)) \qquad 5.3$$

where M(y) and $\Theta(y)$ are given in table 3 and figure 5.

Table 2

A periodic windfield ; y = 0.

ω	Z	$\|Z\|$	arg Z
0	2π	2π	0
0.01	4.91 - i 2.15	5.36	-24°
0.02	3.38 - i 2.29	4.08	-34°
0.05	2.01 - i 1.46	2.48	-36°
0.1	1.62 - i 0.90	1.85	-29°
0.2	1.49 - i 0.57	1.59	-21°
0.3	1.51 - i 0.50	1.59	-18°
0.4	1.59 - i 0.53	1.68	-18°
0.5	1.72 - i 0.68	1.85	-22°
0.6	1.84 - i 1.06	2.12	-30°
0.7	1.56 - i 1.72	2.32	-48°
0.8	0.67 - i 1.67	1.80	-68°
0.9	0.70 - i 1.30	1.48	-62°
1.0	0.52 - i 1.77	1.85	-74°
1.1	-0.30 - i 0.48	0.56	-123°

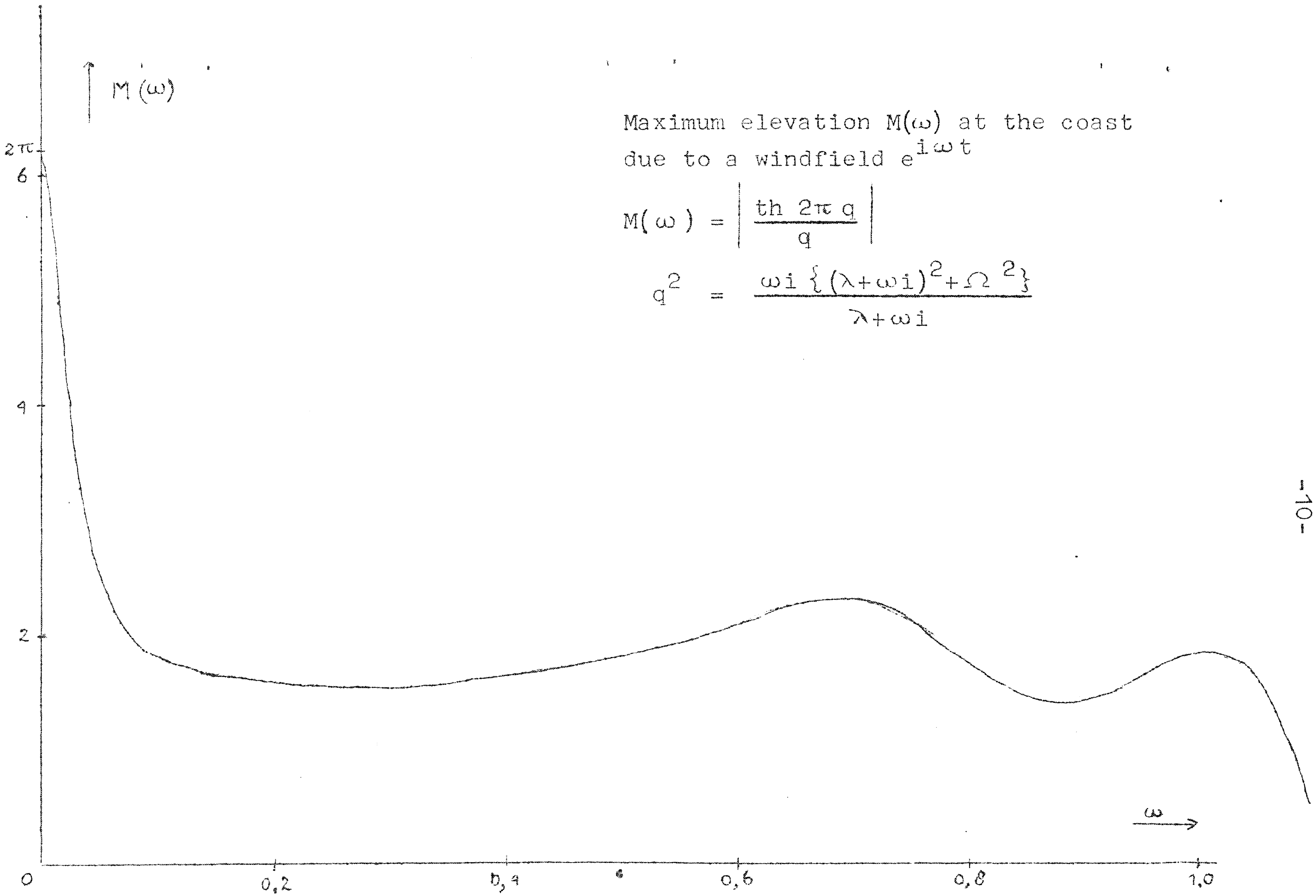

$M(\omega)$
Maximum elevation $M(\omega)$ at the coast due to a windfield $e^{i\omega t}$
$$M(\omega) = \left| \frac{\text{th } 2\pi q}{q} \right|$$
$$q^2 = \frac{\omega i \{(\lambda+\omega i)^2 + \Omega^2\}}{\lambda + \omega i}$$
2π
6
4
2
0
0,2
0,4
0,6
0,8
1,0
ω

figure 2

Phaseshift of the elevation at the coast arg $Z(0,\omega)$ with respect to a periodic windfield of period ω

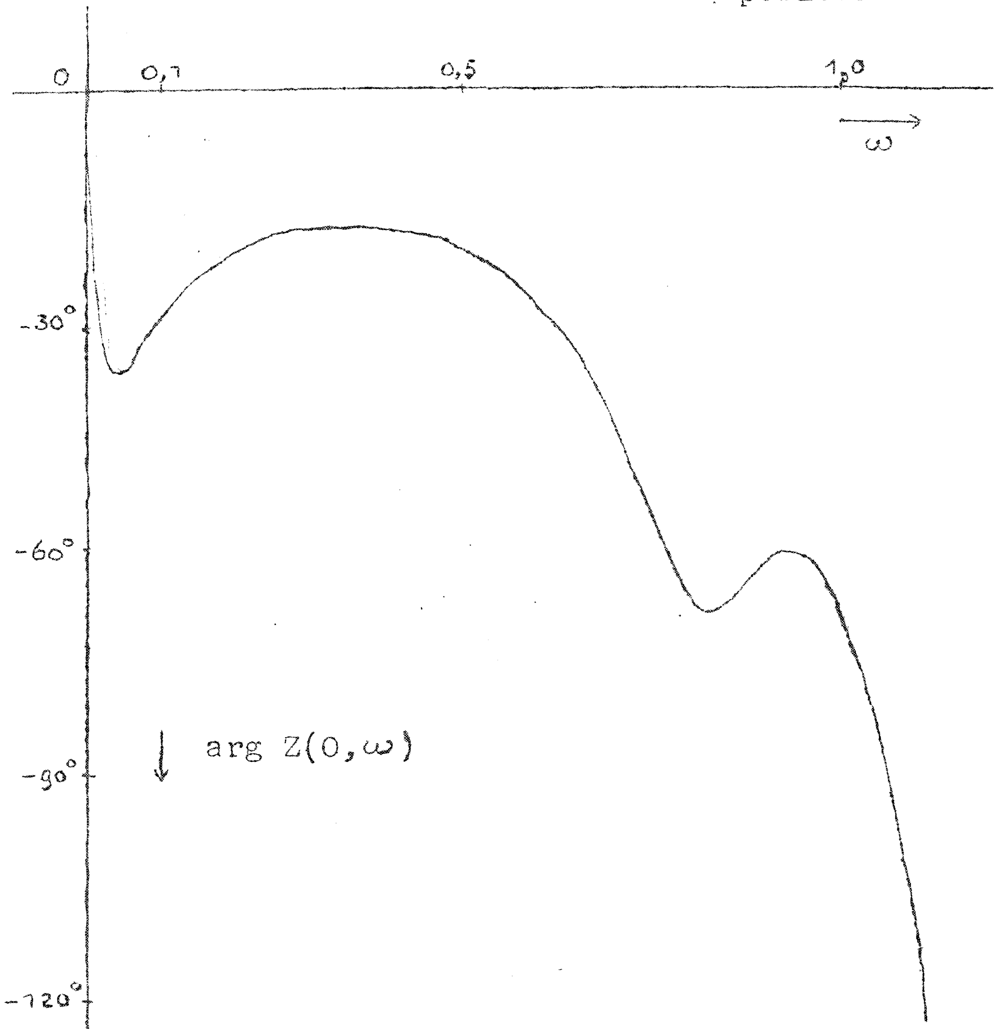
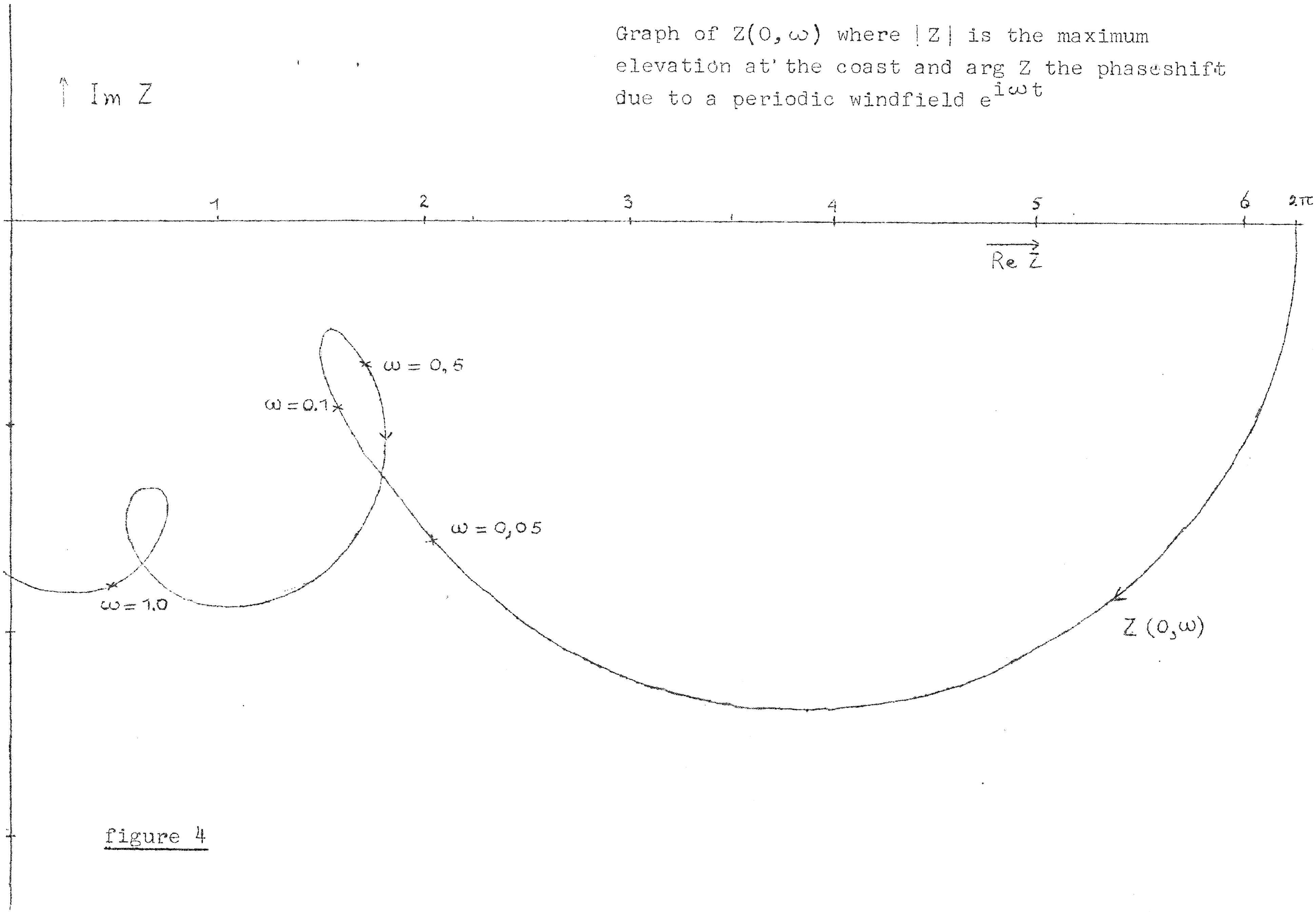


figure 3



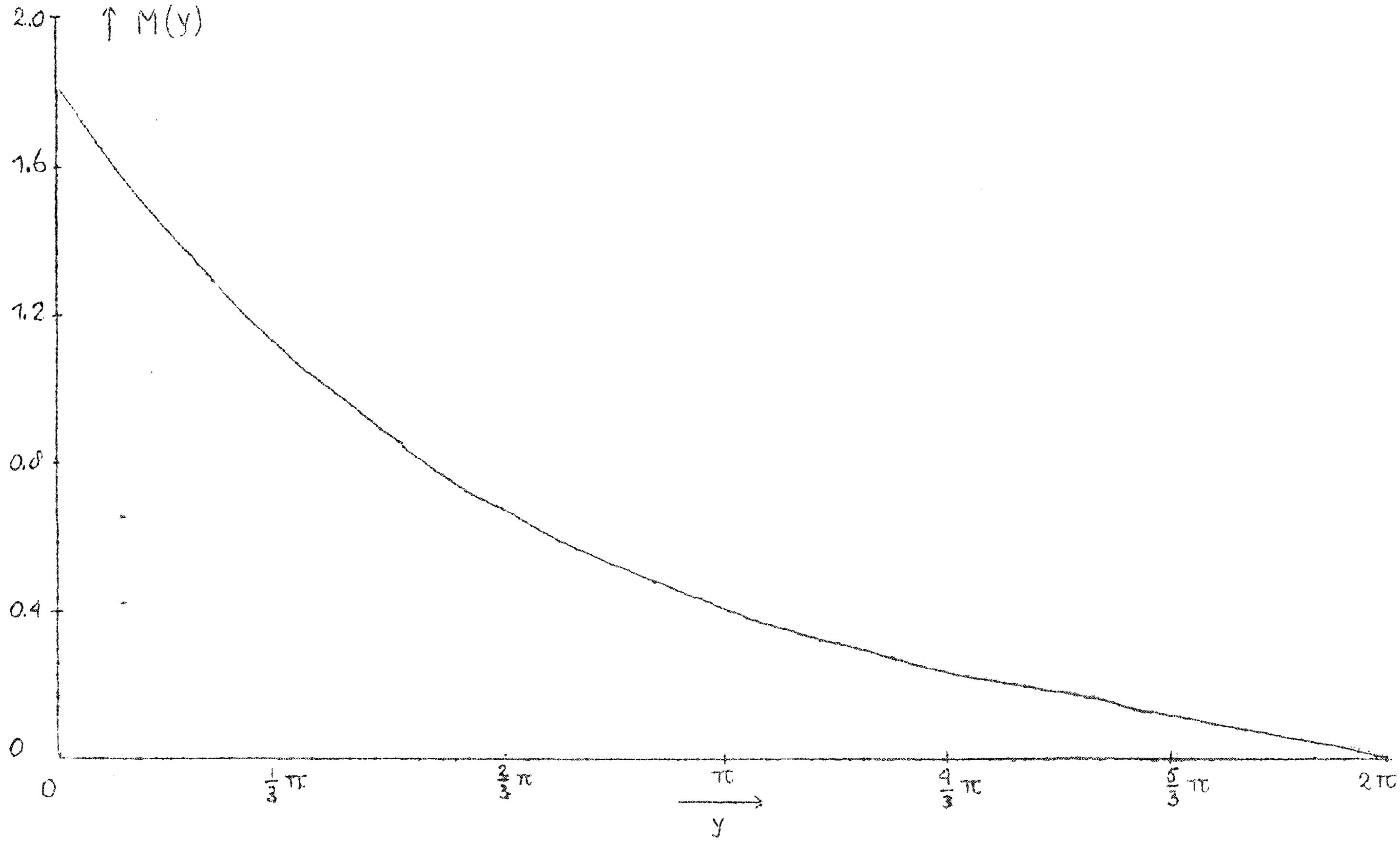
Graph of Z(0, ω) where |Z| is the maximum elevation at the coast and arg Z the phaseshift due to a periodic windfield $e^{i\omega t}$

figure 4

Table 3

Elevation due to periodic windfield - sin 0.1 t.

y	M	Θ
0	1.85	-29°
$\frac{1}{3}\pi$	1.13	-45°
$\frac{2}{3}\pi$	0.69	-60°
π	0.42	-73°
$\frac{4}{3}\pi$	0.24	-85°
$\frac{5}{3}\pi$	0.11	-89°
2π	0	-95°



Maximum elevation at the sea
$M(y) = \text{Max}\ \zeta(y,t)$ due to the
periodic windfield $-\sin 0.1\ t$

figure 5

§ 6 Elevation at the coast due to a suddenly starting periodic wind

If initially the sea is at rest and if at t=0 a sinusoidal wind starts

$$V = - \sin \omega t \qquad 6.1$$

the elevation $\zeta(y,t)$ is determined by

$$\zeta(y,t) = \frac{1}{2\pi i} \int_L e^{pt} \frac{\omega}{p^2+\omega^2} \frac{\operatorname{sh}(2\pi-y)q}{q \operatorname{ch} 2\pi q} dp \, . \qquad 6.2$$

The right-hand side may be evaluated by means of the calculus of residues. The poles at $p=\pm i\omega$ give the quasi-stationary solution which is reached for $t \to \infty$

$$\zeta(y,t) \backsim \operatorname{Im}\left\{ Z(y)e^{i\omega t} \right\} .$$

This is the periodic solution which has been considered in the preceding section. For large values of t we have an appreciable contribution from the pole at α_o only.
If the following approximation is used

$$\alpha_o \backsim \frac{-\lambda}{1+16\Omega^2} ,$$

we have

$$\zeta(y,t) \backsim \operatorname{Im}\left\{ Z(y)e^{i\omega t} \right\} + \frac{256\lambda\Omega^2}{\pi(1+16\Omega^2)^2} \frac{\omega}{\omega^2+\alpha_o^2} e^{-\frac{\lambda t}{1+16\Omega^2}} \cos \frac{y}{4}. \qquad 6.3$$

In the numerical case $\lambda^2=0.02$, $\Omega^2=0.5$ we have $\alpha_o=-0.0153$ and accordingly

$$\zeta(y,t) \backsim \operatorname{Im}\left\{ Z(y)e^{i\omega t} \right\} + \frac{0.0693}{\omega} e^{-0.0153t} , \qquad 6.4$$

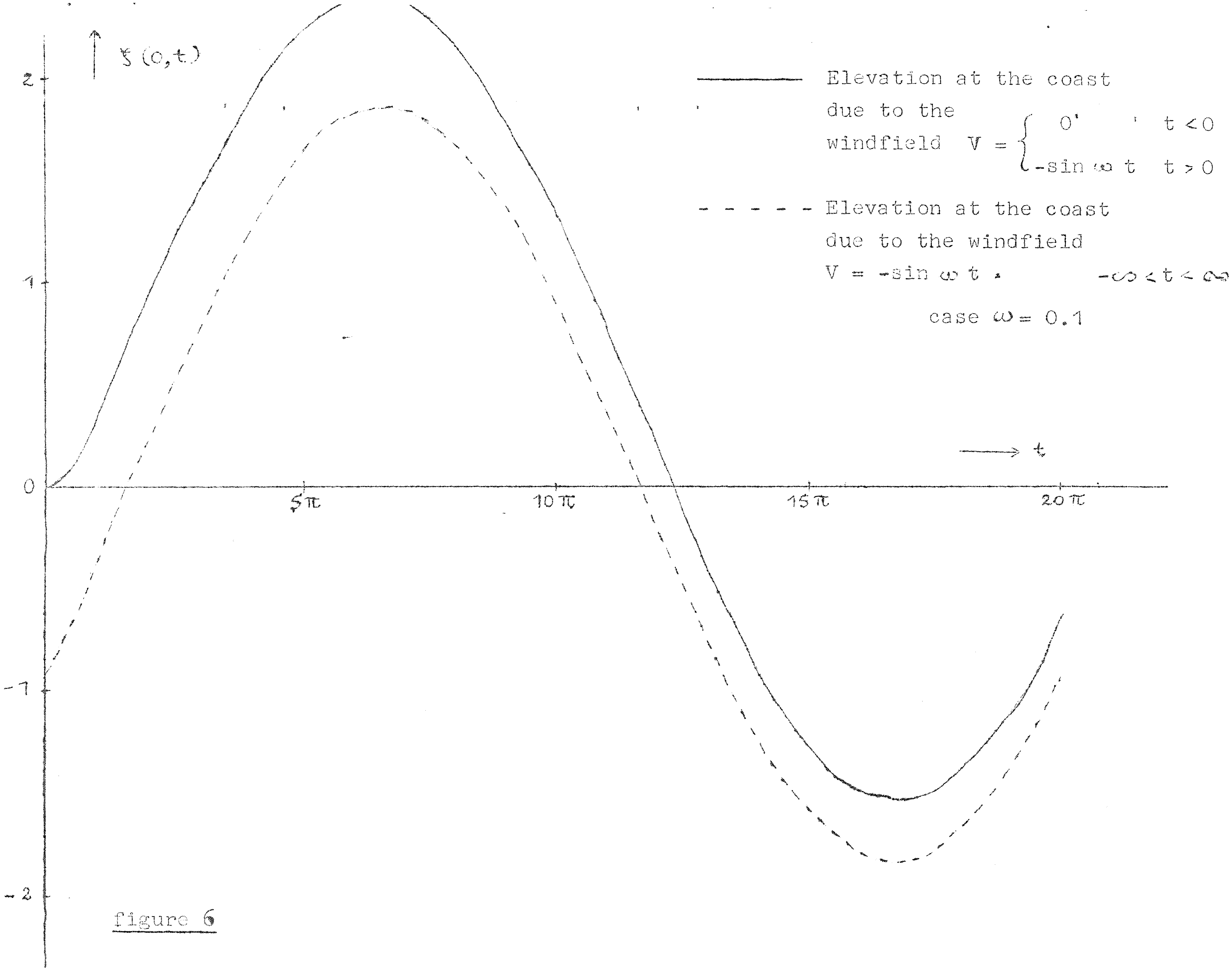
provided $\omega >> 0.01$.

In the case $\omega=0.1$ the following computations have been carried out. The values of $\zeta(0,t)$ are given in table 4. In figure 6 both $\zeta(0,t)$ and the quasi-stationary motion $\operatorname{Im}\left\{ Z(0)\, e^{i\omega t} \right\}$ are given.
It appears that the second term in the right-hand side of 6.4 accounts for a rather considerable and slowly decreasing contribution.

Table 4

Elevation at the coast due to a sinusoidal windfield, $\omega = 0.1$

	$\mathrm{Im}\,(Ze^{i\omega t})$	ζ	diff.
t = 0	-0.897	0	0.897
1	-0.732	0.047	0.779
2	-0.557	0.175	0.732
π	-0.353	0.363	0.716
2π	0.226	0.960	0.734
3π	0.783	1.443	0.660
4π	1.263	1.891	0.628
5π	1.619	2.209	0.590
6π	1.817	2.359	0.542
7π	1.837	2.353	0.516
8π	1.678	2.166	0.488
9π	1.353	1.811	0.458
10π	0.897	1.333	0.436
11π	0.353	0.766	0.413
12π	-0.226	0.165	0.391
13π	-0.783	-0.412	0.371
14π	-1.263	-0.910	0.353
15π	-1.619	-1.284	0.335
16π	-1.817	-1.498	0.319
17π	-1.837	-1.533	0.304
18π	-1.678	-1.389	0.289
19π	-1.353	-1.078	0.275
20π	-0.897	-0.635	0.262



ξ (0,t)
2
1
0
-1
-2
5π
10π
15π
20π
t
Elevation at the coast due to the windfield V = { 0 t < 0 ; -sin ω t t > 0 }
Elevation at the coast due to the windfield V = -sin ω t , -∞ < t < ∞
case ω = 0.1

figure 6

The maximum value of $\zeta(0,t)$ is reached at about $t=6.5\pi$, i.e. 1.5π later than the moment of maximum wind intensity. The shape of the actual motion follows very closely that of the quasi-stationary motion.

For a storm with a peak value of 35 m/sec we obtain in this way a maximum elevation of 2.05 m occurring about 7 hours later than the time of maximum wind intensity. The quasi-stationary motion would have given a maximum elevation of 2.55 m.

§ 7. A half-plane sea

In order to appreciate the influence of the ocean a comparison should be made with the model of a half-plane sea $y > 0$. In this case the windfield is assumed to extend over the whole area. The equations are as in § 2. But here we have the solution

$$\zeta(y,t) = -\bar{V}\,\frac{e^{-qy}}{q} \qquad 7.1$$

with q given by 2.4.

The quasi-stationary motion generated by the windfield $V=-\sin\omega t$ becomes

$$\zeta(0,t) = \mathrm{Im}\,(Ze^{i\omega t}) \qquad 7.2$$

with

$$Z = \frac{(\lambda+\omega i)^{\frac{1}{2}}}{(\omega i)^{\frac{1}{2}}\left\{(\lambda+\omega i)^2+\Omega^2\right\}^{\frac{1}{2}}} \quad . \qquad 7.3$$

For a number of ω values the values of $|Z|$ are given below together with the corresponding values of the ocean case

	$\lvert Z\rvert$ halfplane	$\lvert Z\rvert$ strip
ω = 0	2π	2π
0.1	1.84	1.85
0.2	1.59	1.64
0.5	1.85	1.85
0.7	2.26	2.32
0.9	1.62	1.48

We observe here the same resonance at about $\omega = \Omega$.

If ω is of the order 0.1 the results from the two models differ only very slightly. Thus for long storms the influence of the ocean may be expected to be very small.

If the sea is initially at rest and if at t=0 the windfield $-\sin \omega t$ starts we obtain for the particular case $\omega = 0.1$

	$\zeta(0,t)$ halfplane	$\zeta(0,t)$ strip
$t = 0$	0	0
2π	0.96	0.96
4π	1.89	1.89
6π	2.37	2.36
8π	2.18	2.17
10π	1.37	1.33

This confirms the assertion given above.

§ 8 The case $\Omega = 0$

a For $\Omega = 0$ the solution 2.6 reduces to

$$\zeta(y,t) = -\frac{1}{2\pi i} \int_L e^{pt}\, \overline{V}(p)\, \frac{\operatorname{sh}(2\pi - y)q}{q \operatorname{ch} 2\pi q}\, dp \,, \qquad 8.1$$

where now

$$q^2 = p(p+\lambda).$$

The eigenvalues are determined by

$$p(p+\lambda) + \left(\frac{k}{2} + \frac{1}{4}\right)^2 = 0 \,, \qquad k=0,1,2\ldots$$

so that for each k a pair of conjugate complex eigenvalues

$$p_k = -\frac{\lambda}{2} \pm \frac{i}{2} \sqrt{(k+\tfrac{1}{2})^2 - \lambda^2} \qquad 8.2$$

are obtained.
For $\lambda^2 = 0.02$ the first few eigenvalues are

$k = 0$	-0.0707	$\pm\, i\ 0.240$
$= 1$	-0.0707	$\pm\, i\ 0.747$
$= 2$	-0.0707	$\pm\, i\ 1.248$
$= 3$	-0.0707	$\pm\, i\ 1.748$.

The corresponding eigenfunctions are again

$$e^{pt} \cos\left(\frac{k}{2} + \frac{1}{4}\right) y \,.$$

b If at t=0 the sea is at rest the windfield $V=-1,\ t > 0$ causes the elevation

$$\zeta(y,t) = \frac{1}{2\pi i} \int_L e^{pt} \frac{sh(2\pi-y)q}{pq\, ch\, 2\pi q} dp.$$

The stationary solution is obviously

$$\zeta(y,\infty) = 2\pi - y.$$

Table 5

Elevation at the coast due to a stepfunction windfield.
$\Omega = 0$.

t	$\zeta(0,t)$
0	0
1	0.964
2	1.867
3	2.712
4	3.505
5	4.252
6	4.958
8	6.253
10	7.423
15	8.007
20	6.600
25	5.343
30	5.814
35	6.355
40	6.585
50	6.157
∞	2π

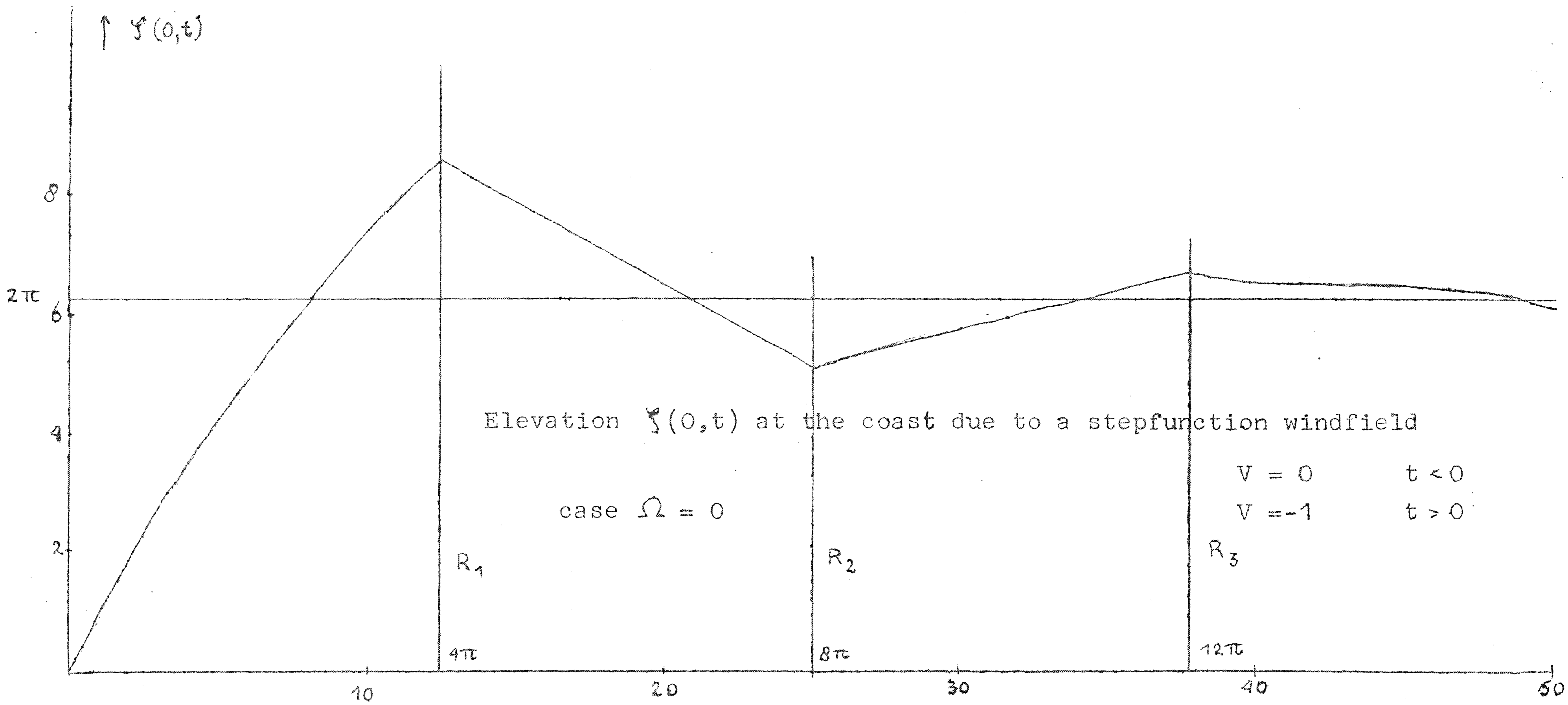


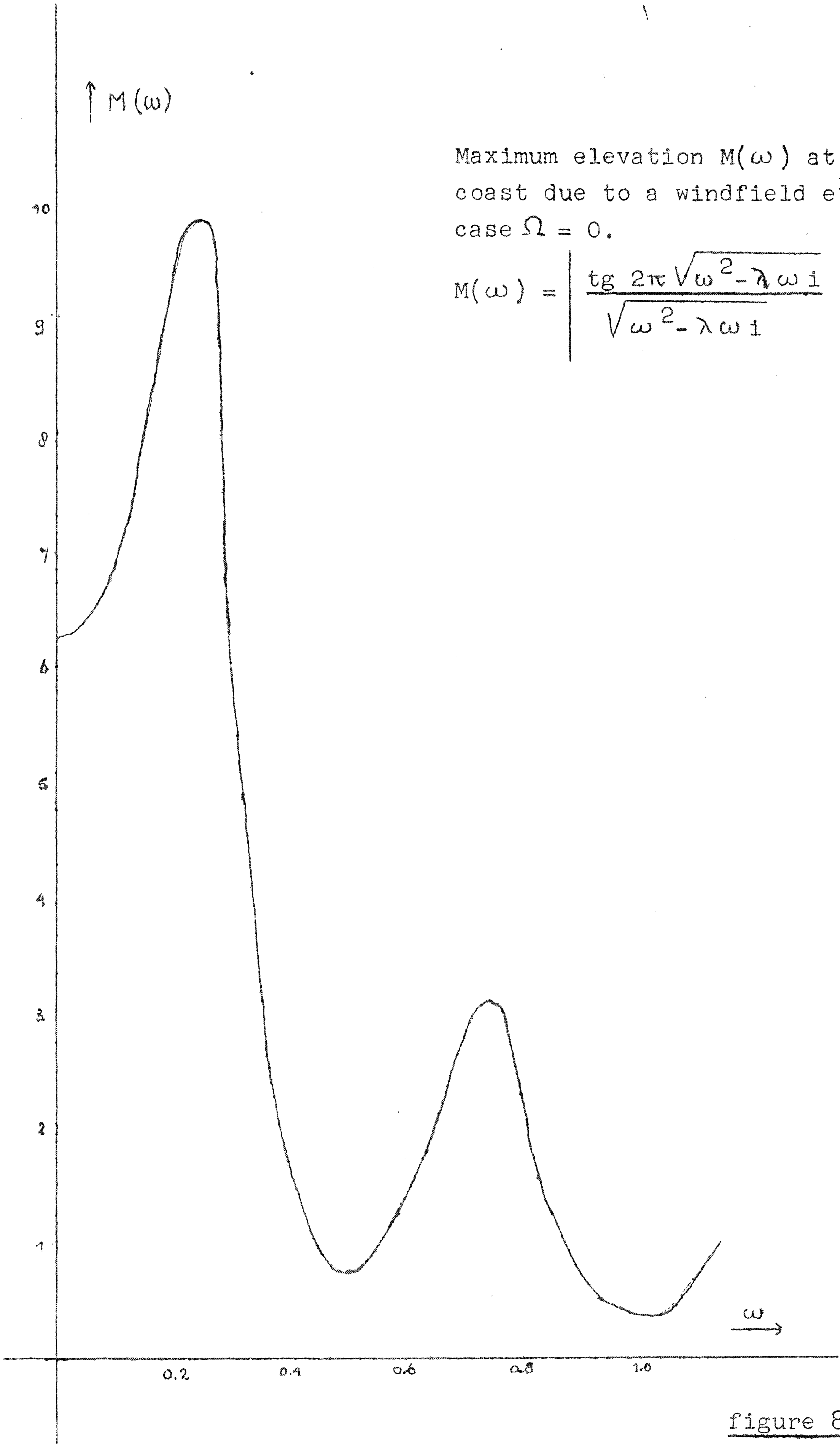
figure 7

R_j (j=1,2...) j^{th} reflection at $t=4j\pi$

Table 6

A periodic windfield ; y=0 , Ω =0.

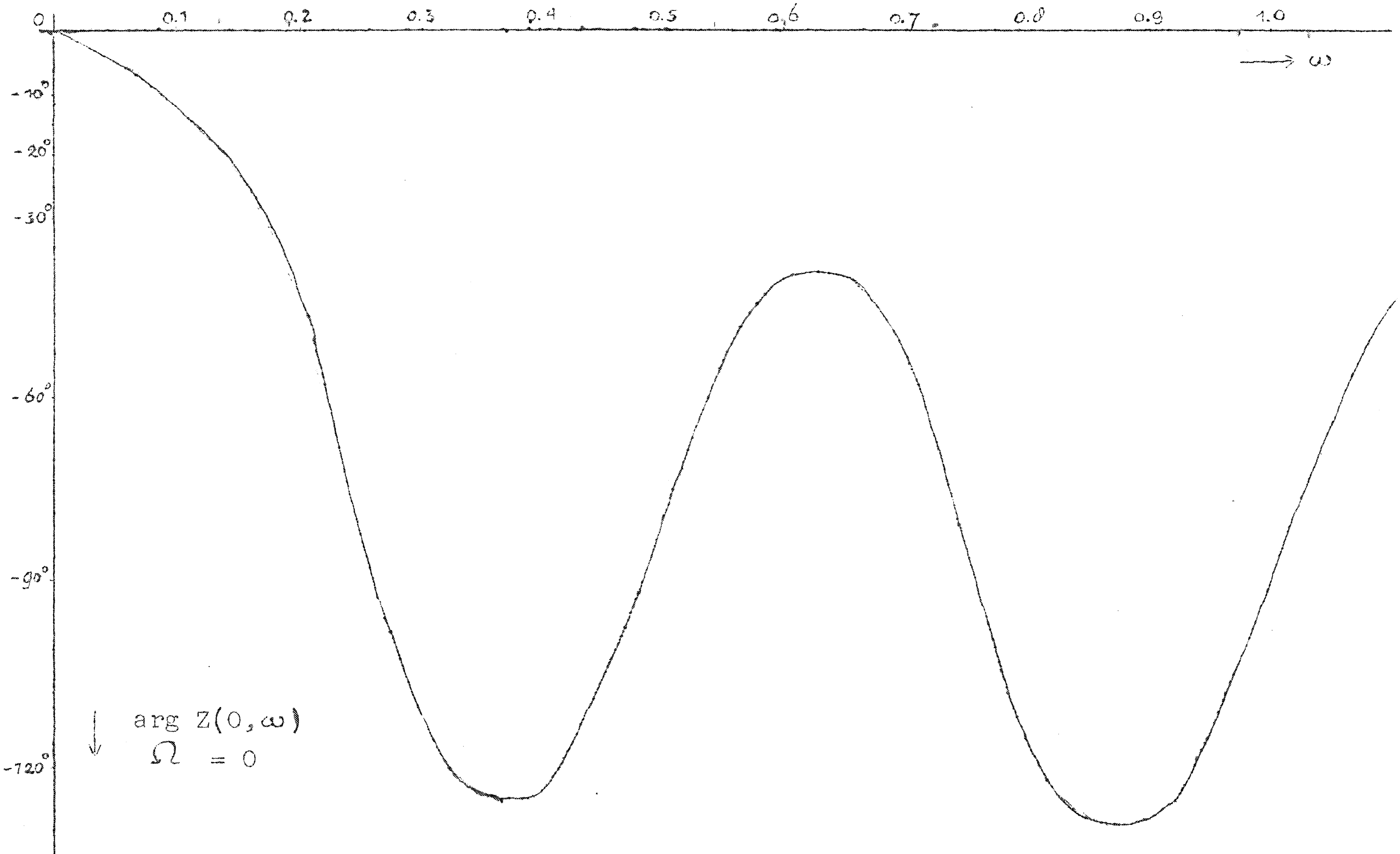
ω	Z	|Z|	arg Z
0	2π	2π	0
0.01	6.29- i 0.12	6.29	-1°
0.02	6.32- i 0.23	6.32	-2°
0.05	6.43- i 0.62	6.46	-5°
0.1	6.87- i 1.54	7.18	-13°
0.2	6.71- i 6.93	9.65	-46°
0.3	-2.12- i 5.35	5.76	-112°
0.4	-1.01- i 1.54	1.84	-123°
0.5	0.16- i 0.80	0.81	-79°
0.6	1.05- i 0.87	1.37	-40°
0.7	1.57- i 2.29	2.77	-55°
0.8	-1.05- i 2.07	2.32	-117°
0.9	-0.54- i 0.68	0.87	-128°
1.0	0.02- i 0.42	0.41	-87°
1.1	0.54- i 0.50	0.74	-43°



Maximum elevation M(ω) at the coast due to a windfield $e^{i\omega t}$ case $\Omega = 0$.

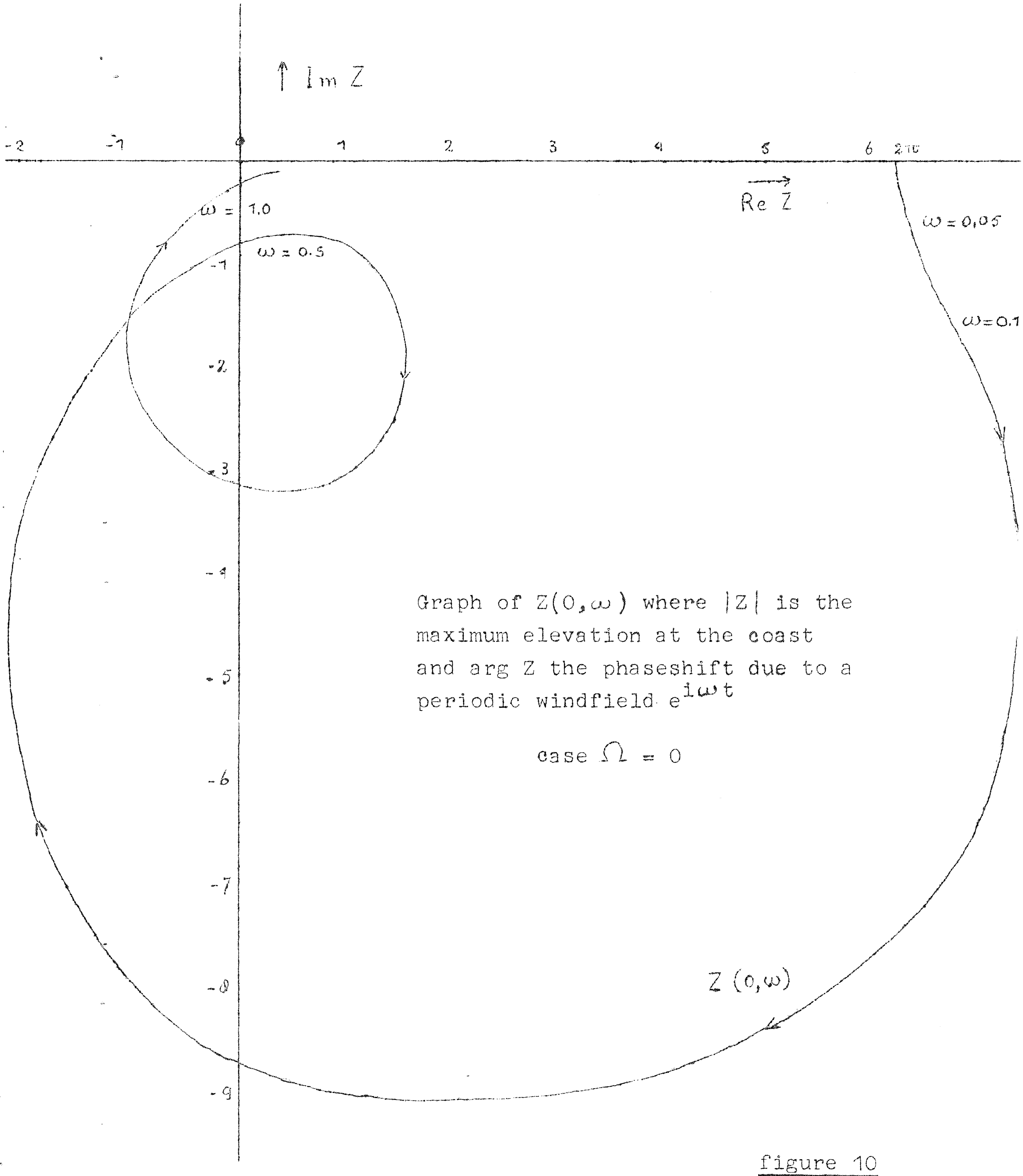
$$M(\omega) = \left| \frac{\mathrm{tg}\ 2\pi\sqrt{\omega^2 - \lambda\omega i}}{\sqrt{\omega^2 - \lambda\omega i}} \right|$$

figure 8



Phaseshift of the elevation at the coast arg $Z(0,\omega)$ with respect to a periodic windfield of period ω. case $\Omega = 0$.

<u>figure 9</u>



Graph of Z(0,ω) where |Z| is the maximum elevation at the coast and arg Z the phaseshift due to a periodic windfield $e^{i\omega t}$

case $\Omega = 0$

figure 10

For arbitrary y and t we find by means of the calculus of residues

$$\zeta(y,t)=(2\pi-y)-\frac{e^{-\frac{\lambda}{2}t}}{\pi}\sum_{k=0}^{\infty}\frac{\cos(\frac{k}{2}+\frac{1}{4})y}{(\frac{k}{2}+\frac{1}{4})^2}\left\{\cos v_k t+\frac{\lambda}{2v_k}\sin v_k t\right\}, \qquad 8.3$$

where $$v_k=\tfrac{1}{2}\sqrt{(k+\tfrac{1}{2})^2-\lambda^2}.$$

As in section 4 we have also the alternative expansion

$$\zeta(0,t)=\sum_{0}(-1)^j\,\varepsilon_j\int_{4j\pi}^{t}e^{-\frac{\lambda}{2}}\,I_0\left(\tfrac{\lambda}{2}\sqrt{\tau^2-16j^2\pi^2}\right)d\tau, \qquad 8.4$$

where the summation breaks off at $j=\left[\frac{t}{4\pi}\right]$.

In the numerical case $\lambda^2=0.2$ $\zeta(0,t)$ is given by table 5 and figure 7.

c The periodic solutions which are generated by the windfield $V=-\sin\omega t$ are of the form 5.1 with q given by

$$q^2=-\omega^2+\lambda\omega i.$$

At the coast y=0 we have in particular

$$\zeta(0,t)=\mathrm{Im}\left\{\frac{\mathrm{tg}\;2\pi\sqrt{\omega^2-\lambda\omega i}}{\sqrt{\omega^2-\lambda\omega i}}\,e^{i\omega t}\right\} \qquad 8.5$$

or

$$\zeta(0,t)=M\sin(\omega t-\Theta).$$

In the numerical case $\lambda^2=0.02$ M and Θ have been computed for a number of ω values. See table 6 and figures 8,9 and 10.

Table 7

Elevation at the coast due to a sinuoidal windfield.
$\omega = 0.1$, $\Omega = 0.$

	$Im\,(Ze^{i\omega t})$	ζ	diff.
t = 0	-1.54	0	1.54
π	0.65	0.55	-0.10
2π	2.78	2.67	-0.11
3π	4.64	4.51	-0.13
4π	6.04	5.96	-0.08
5π	6.85	6.85	-0.00
6π	7.00	7.04	0.04
7π	6.45	6.50	0.05
8π	5.28	5.31	0.03
9π	3.58	3.59	0.01
10π	1.54	1.53	-0.01
11π	-0.65	-0.67	-0.02
12π	-2.78	-2.79	-0.01
13π	-4.64	-4.64	-0.00
14π	-6.04	-6.04	+0.00
15π	-6.85	-6.84	0.01
16π	-7.00	-6.99	0.01
17π	-6.45	-6.45	-
18π	-5.28	-5.28	-
19π	-3.58	-3.58	-
20π	-1.54	-1.54	-

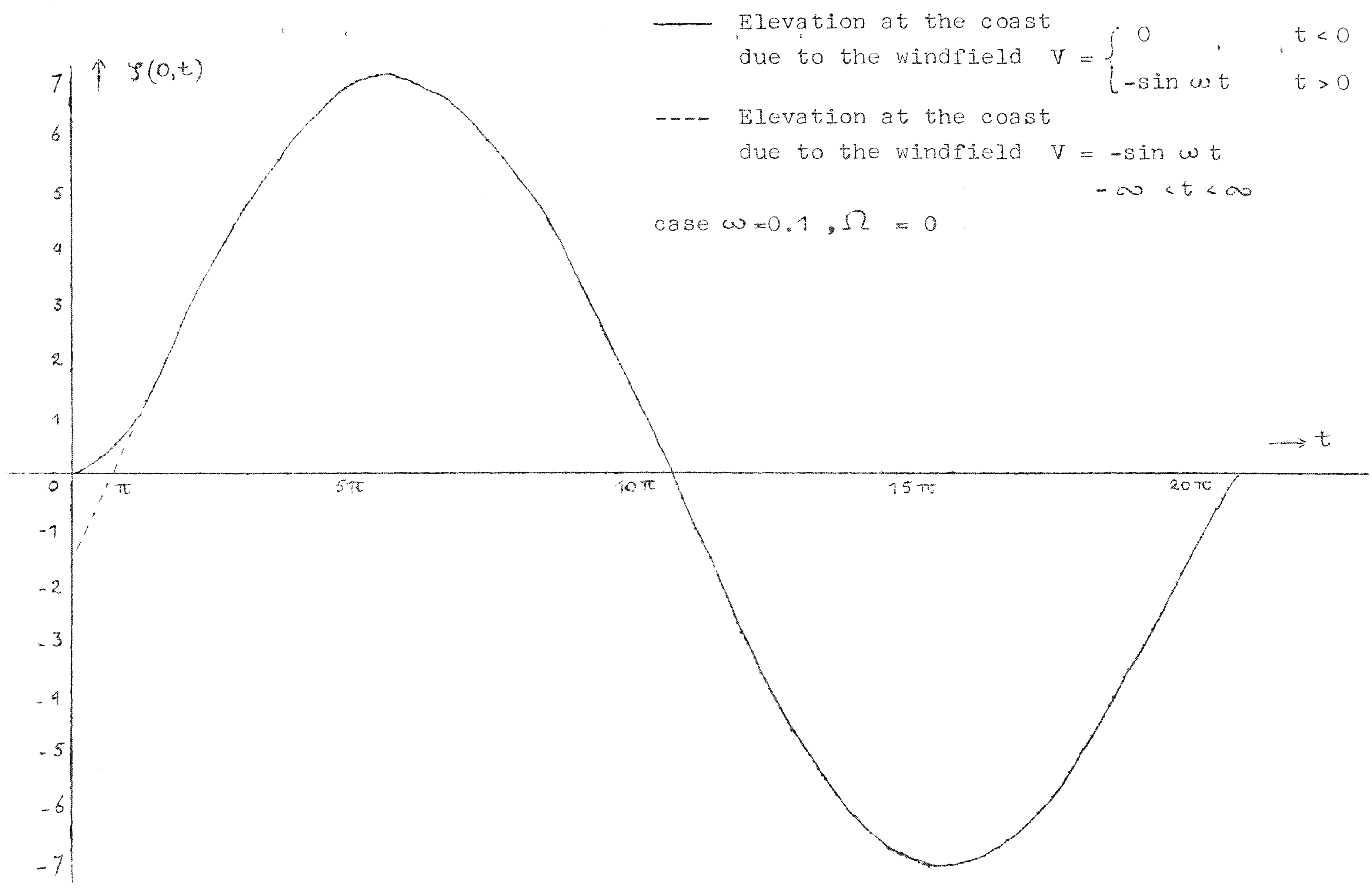

Elevation at the coast
due to the windfield V = { 0 , t < 0 ; -sin ω t , t > 0 }
Elevation at the coast
due to the windfield V = -sin ω t
-∞ < t < ∞
case ω = 0.1 , Ω = 0
ζ(0,t)
t
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
π
5π
10π
15π
20π

figure 11

§ 9 The case $\lambda = 0$

We shall show below that the absence of a damping coefficient gives rise to a quite anomalous behaviour of the elevation. Since there is no dissipation of energy any disturbance from the wind causes an infinitely repeated oscillation of the surface of the sea. We have

$$\zeta(y,t) \doteq -\bar{V}\ \frac{\operatorname{sh}(2\pi-y)\sqrt{p^2+\Omega^2}}{\sqrt{p^2+\Omega^2}\ \operatorname{ch}2\pi\sqrt{p^2+\Omega^2}}. \qquad 9.1$$

For a unit disturbance $V=-\delta(t)$ we have for the elevation at the coast

$$\zeta(0,t) \doteq \frac{\operatorname{th}\ 2\pi\sqrt{p^2+\Omega^2}}{\sqrt{p^2+\Omega^2}} \qquad 9.2$$

so that

$$\zeta(0,t) = \sum_0 (-1)^k\ \varepsilon_k\ J_o(\Omega\sqrt{t^2-16k^2\pi^2}). \qquad 9.3$$

Thus we observe at the k^{th} reflection always a jump of $|\Delta\zeta|=2$. For the case $V=-1$, $t>0$ we find

$$\zeta(0,t) = \sum_0 (-1)^k\ \varepsilon_k \int_{4k\pi}^{t} J_o(\Omega\sqrt{\tau^2-16k^2\pi^2})\,d\tau. \qquad 9.4$$

The values of ζ for a number of t values are given in table 8. They show clearly the irregular behaviour of ζ.
The eigenfunctions in this case are

$$\exp\left\{ti\sqrt{\left(\frac{k}{2}+\frac{1}{4}\right)^2+\Omega^2}\right\}\cos\left(\frac{k}{2}+\frac{1}{4}\right)y\ , \qquad 9.5$$

i.e. purely oscillatory.

Table 8

Elevation at the coast due to a stepfunction windfield. $\lambda=0$.

t	ζ	t	ζ
0	0	26	1.18
1	0.96	28	2.15
2	1.69	30	1.49
3	2.05	32	1.29
4	2.02	34	1.30
5	1.71	36	1.33
6	1.32	38	1.90
7	1.02	40	1.36
8	0.95	42	0.83
9	1.10	44	1.23
10	1.38	46	1.86
12	1.79	48	1.15
14	1.74	50	0.91
16	1.20		
18	0.78		
20	1.97		
22	1.89		
24	0.91		

§ 10 Conclusions and final remarks

This report is only a first step towards the solution of the much more complicate problem of the determination of the influence of an arbitrary windfield upon the elevation of the rectangular North Sea model $-a < x < a$, $0 < y < b$.
In this report the influence of the long sides $x=\pm a$ has been left out of consideration.

With regard to the influence of the friction term λ we observe that it results in the general damping factor

$$e^{-\frac{\lambda t}{2}} .$$

In spite of the smallness of λ this influence is considerable. For $t=16$ (one day) its value is about 1/3, for $t=32$ (two days) it is about 1/9.

In the rectangular North Sea model a similar effect may be expected. However, a peculiar effect is caused by the main free motion with the exceptionally small damping factor of about

$$\exp - \frac{\lambda t}{1+16\Omega^2} .$$

For $t=32$ its value is still 0.62.

The force of Coriolis appears to enhance the inertia of the system. We point out the striking difference in the response of the sea to a step-function, figs.1 and 7, in the cases $\Omega^2=0.5$ and $\Omega=0$.

In the case of the rectangular model of the North Sea this effect is certainly less pronounced since the long sides suppress the mean stream in the x-direction.

As regards the influence of a periodical storm of frequency ω a striking difference exists between the cases $\Omega \neq 0$ and $\Omega=0$. If $\Omega^2=0.5$ a resonance occurs at $\omega=0$ and the next one at about $\omega=\Omega$ which, however, is of minor importance. In this case the presence of an ocean boundary has hardly any influence.

If $\Omega=0$ there is an important resonance at $\omega=0.25$ and a less important one at $\omega=0.75$ (see fig.8). Here the presence of the ocean boundary is essential.

For the rectangular model this does not help very much. However, this model will be investigated in a future report.