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On the compactness operator in general topology

by

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Introduction

This report contains a collection of results obtained by the authors during the spring of 1966. It is a pre-publication of an extensive paper bearing the same title by J. de Groot, E. Wattel, G. Strecker, and H. Herrlich. It also uses a number of results by J. de Groot who initiated the investigation in this area.

The purpose of the report is to investigate the notion of compactness from a basic set-theoretical standpoint. First, compactness is defined without using the definition of a topological space. Later a slight, but natural, generalization of the notion of a topological space is employed.

In the last chapter a study is made of the class of all pairs of spaces for which the collection of closed sets of each member of a given pair is precisely the collection of compact sets of the other member of the pair.

§1. The compactness operator

1.1. Definition:

A collection of arbitrary sets $\mathcal{P}$ is called centered in a set H iff $\{P \cap H \mid P \in \mathcal{P}\}$ has f.i.p. (i.e. the property that each finite intersection of members of the collection is non-empty).

1.2. Definition:

Let X be a set, and let $\mathcal{K}$ be an arbitrary collection of subsets of X. A set $H \subset X$ is called compact relative to $\mathcal{K}$ provided that every subcollection of $\mathcal{K}$ which is centered in H has a non-empty intersection in H.

1.3. Notation convention:

Let X be a set, and $\mathcal{K}$ a family of subsets; then we denote by $\rho\mathcal{K}$ the family of all subsets of X which are compact relative to $\mathcal{K}$, and by $\varepsilon\mathcal{K}$ the family of all finite unions of arbitrary intersections of members of $\mathcal{K}$. Clearly ρ and ε are operators and their domain is the collection of all families of subsets of X.

A member of $\rho\rho\mathcal{K}$ is compact relative to $\rho\mathcal{K}$, and we call it a <u>square-compact relative to $\mathcal{K}$</u>. (For $\rho\rho$ we can also write ρ^2 etc.)
The following two relations involving ε and ρ are well known.

i) $\varepsilon\varepsilon = \varepsilon$.

ii) $\rho\varepsilon = \rho$. This is a restatement of Alexander's subbase theorem (cf. [1]).

1.4. <u>Lemma</u>:

Let X be a set, and let $\mathcal{K}$ be an arbitrary collection of subsets of X. Let $C \in \rho\mathcal{K}$ and $S \in \rho\rho\mathcal{K}$. Then $C \cap S \in (\rho\mathcal{K}) \cap (\rho^2\mathcal{K})$.

<u>Proof</u>: i) $C \cap S \in \rho^2\mathcal{K}$

Let $\mathcal{C}' \subset \rho\mathcal{K}$ be such that $\mathcal{C}'$ is centered in $C \cap S$; then $\mathcal{C}' \cup \{C\}$ is centered in S. $\mathcal{C}' \cup \{C\} \subset \rho\mathcal{K}$, and from $S \in \rho^2\mathcal{K}$ it follows that $\bigcap [\mathcal{C}' \cup \{C\}] \cap S \neq \phi$. Hence $\bigcap(\mathcal{C}') \cap (C \cap S) \neq \phi$ and we conclude that $C \cap S \in \rho^2\mathcal{K}$.

ii) $C \cap S \in \rho\mathcal{K}$

Let $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'$ is centered in $C \cap S$.
Then the collection $\mathcal{K}'' = \{C \cap K \mid K \in \mathcal{K}'\}$ is a subcollection of $\rho\mathcal{K}$, and it is centered in $C \cap S$. But $C \cap S \in \rho^2\mathcal{K}$ and hence $\bigcap(\mathcal{K}'') \cap (C \cap S) \neq \phi$. $\bigcap(\mathcal{K}'') = \bigcap(\mathcal{K}') \cap C$;
$\bigcap(\mathcal{K}') \cap (C \cap S) \neq \phi$, and so $C \cap S \in \rho\mathcal{K}$

1.5. <u>Theorem</u>:

For every set X and every collection of subsets $\mathcal{K}$, the collection $\rho^2\mathcal{K}$ is closed under finite unions and arbitrary intersections ($\rho^2\mathcal{K} = \varepsilon\rho^2\mathcal{K}$).

<u>Proof</u>: The fact that $\rho^2\mathcal{K}$ is closed under finite unions is obvious from the definition of $\rho^2\mathcal{K}$ (i.e. the collection of compact sets relative to $\rho\mathcal{K}$).

So we only have to prove that $\rho^2\mathcal{K}$ is closed under arbitrary intersections. Let $\mathcal{S}'$ be a collection of members of $\rho^2\mathcal{K}$, and suppose $\bigcap \mathcal{S}' = S_0 \neq \phi$. We have to prove that $S_0 \in \rho^2\mathcal{K}$

Let $\mathcal{C}' \subset \rho\mathcal{K}$ be centered in S_0.

Pick and fix some member $S' \in \mathcal{S}'$ and consider the system

$\mathcal{C}'' = \{C \cap S \cap S' \mid C \in \mathcal{C}', S \in \mathcal{S}'\}$.

From the lemma it follows that every member of $\mathcal{C}''$ is a member of $\rho\mathcal{K}$; and $\mathcal{C}''$ is centered in S'; hence it has a non-empty intersection in S'; but the intersection is a subset of S_0. Hence: $(\cap\mathcal{C}')\cap S_0 = \cap(\mathcal{C}'')\cap S_0 = \cap(\mathcal{C}'')\cap S' \neq \phi$, and this proves that $S_0 \in \rho^2\mathcal{K}$, which proves the theorem.

1.6. Lemma:

Let X be a set, and $\mathcal{K}$ a collection of subsets. Then each member of $\rho\mathcal{K}$ is a member of $\rho^3\mathcal{K}$, $(\rho\mathcal{K}\subset\rho^3\mathcal{K})$.

Proof: Let $C\in\rho\mathcal{K}$; and let $\mathcal{S}'\subset\rho^2\mathcal{K}$ such that $\mathcal{S}'$ is centered in C. Pick and fix some $S'\in\mathcal{S}'$ and consider the collection $\mathcal{S}'' = \{S\cap C\cap S' \mid S\in\mathcal{S}'\}$. From lemma 1.4 it follows that every member of $\mathcal{S}''$ is a member of $\rho\mathcal{K}$. So we have a system $\mathcal{S}''$ that is centered in S'; and hence $\cap(\mathcal{S}')\cap C = \cap(\mathcal{S}'')\cap S' \neq \phi$. Thus $C\in\rho^3\mathcal{K}$

1.7. Theorem:

For every set X and every collection of subsets $\mathcal{K}$, $\rho^2\mathcal{K} = \rho^4\mathcal{K}$.

Proof: By lemma 1.6 $\rho\mathcal{K}\subset\rho^3\mathcal{K}$ and thus $\rho^2\mathcal{K}\subset\rho^4\mathcal{K}$.

On the other hand, every member of $\rho^4\mathcal{K}$ is compact relative to $\rho^3\mathcal{K}$ and hence it is compact relative to $\rho\mathcal{K}$; so $\rho^4\mathcal{K}\subset\rho^2\mathcal{K}$

1.8. Proposition:

If for every $K\in\mathcal{K}$ and $C\in\rho\mathcal{K}$; $K\cap C\in\mathcal{K}$; then $\mathcal{K}\subset\rho^2\mathcal{K}$ and $\rho\mathcal{K} = \rho^3\mathcal{K}$.

Proof: Let K be a member of $\mathcal{K}$. We must show first that K is a member of $\rho^2\mathcal{K}$

Assume that $\mathcal{C}'\subset\rho\mathcal{K}$ such that $\mathcal{C}'$ is centered in K. The collection $\mathcal{C}'' = \{C\cap K \mid C\in\mathcal{C}'\}$ is by assumption a system of members of $\mathcal{K}$, that is centered in each element of $\mathcal{C}''$. From the fact, that every member of $\mathcal{C}''$ is a member of $\rho\mathcal{K}$ it follows that $(\cap\mathcal{C}')\cap K = \cap\mathcal{C}'' \neq \phi$ and hence $K\in\rho^2\mathcal{K}$. The proof that $\rho\mathcal{K} = \rho^3\mathcal{K}$ is similar to the proof of 1.7.

1.9. Remark:

From 1.3, 1.5, and 1.7 we can derive the following relations between the operators ρ and ε.

(1) $\varepsilon\varepsilon = \varepsilon$

(2) $\rho\varepsilon = \rho$

(3) $\varepsilon\rho^2 = \rho^2$

(4) $\rho^4 = \rho^2$.

These relations define the structure of a semigroup with the following multiplication table.

(ϕ)	ε	ρ	ρ^2	ρ^3	$\varepsilon\rho$
ε	ε	$\varepsilon\rho$	ρ^2	ρ^3	$\varepsilon\rho$
ρ	ρ	ρ^2	ρ^3	ρ^2	ρ^2
ρ^2	ρ^2	ρ^3	ρ^2	ρ^3	ρ^3
ρ^3	ρ^3	ρ^2	ρ^3	ρ^2	ρ^2
$\varepsilon\rho$	$\varepsilon\rho$	ρ^2	ρ^3	ρ^2	ρ^2

§2. T_ε-spaces and superconnectedness

2.1. Definition:

A set X together with a collection of its subsets $\mathcal{G}$ will be called a T_ε-space provided that $\varepsilon\mathcal{G} = \mathcal{G}$; i.e. $\mathcal{G}$ is closed under the formation of finite unions and arbitrary intersections. $\mathcal{G}$ will be called the collection of closed subsets of the space, and $\mathcal{T} = \{X \setminus G \mid G \in \mathcal{G}\}$ will be called the collection of open subsets of the space.

2.2. Remark:

Thus for T_ε-spaces there is no mention of whether or not ϕ and X are open or closed. Indeed, in the usual definition of topological space there seems to be no compelling reason to force ϕ and X to be closed, except that this was done during the historical development of the theory.

The definition above poses no difficulties for the class of topological spaces usually studied, for if a T_ε-space merely possesses two disjoint closed sets and two disjoint open sets, then both ϕ and X will automatically be open and closed.

The only apparant drawback is that if the closure of a set is defined to be the set together with all of its limit points, then it is not necessarily true that the closure of a dense set is closed. Similarly the interior of a nowhere dense set is not necessarily open. On the other hand, the definition of T_-space is useful for this paper, e.g. 1.5 can be restated in the form: For every family of subsets $\mathcal{K}$ of a set X, $(X,\rho^2\mathcal{K})$ is a T_-space. In what follows, "space" will mean T_-space.
It is clear that every result stated below can be easily restated in terms of usual topological spaces.

2.3. Definition:

A space is called super-connected provided that every open set is connected (cf. [2]).

2.4. Proposition:

For any T_-space $(X,\mathcal{G})$ the following are equivalent:

a) The space is super-connected.

b) Every non-empty open set is dense.

c) X is not the union of two proper closed subsets.

Proof:

a) ==> b). Let U and V be two non-empty open subsets. $U \cup V$ is connected because it is open, and hence $U \cap V \neq \phi$. This implies that U is dense.

b) ==> c). Let F and G be proper closed subsets. Then $X \setminus F$ and $X \setminus G$ are open and non-empty $(X \setminus F) \cap (X \setminus G)$ is dense. Thus $X \neq F \cup G$.

c) ==> a). If X is not the union of two proper closed sets, then no non-empty open set can be contained in a proper closed subset. Thus every open set is connected.

§3. Antispaces

3.1. Introduction:

In chapters 1 and 2 we have found the tools for a description of pairs of T_-spaces in which the compact sets of the first space are precisely the closed sets of the second space, and conversely.

Such a pair of spaces is called an antipair and a member of an antipair is called an antispace (cf. [2]).
By definition an antipair can be formed by a pair of T_1-spaces $(X,\mathcal{G})$ and $(X,\mathcal{K})$ iff $\mathcal{K} = \rho\mathcal{G}$ and $\mathcal{G} = \rho\mathcal{K}$ and hence $\mathcal{G} = \rho^2\mathcal{G}$. On the other hand, $(X,\mathcal{G})$ is an antispace iff $((X,\mathcal{G}),(X,\rho\mathcal{G}))$ is an antipair and $\mathcal{G} = \rho^2\mathcal{G}$. We are going to study first, what sorts of spaces can occur as anti-spaces. Note that each member of an antipair determines the other member uniquely, and that every property of one member corresponds to an adjoint property of the other member of the anti-pair.

3.2. Theorem:

If $(X,\mathcal{G})$ is an arbitrary T_1-space, then $(X,\rho^2\mathcal{G})$ is an antispace and $(X,\rho^2\mathcal{G}),(X,\rho^3\mathcal{G})$ form an antipair.
Proof: By theorem 1.7. $\rho^4\mathcal{G} = \rho^2\mathcal{G}$. Hence $\rho^2(\rho^2\mathcal{G}) = \rho^2\mathcal{G}$; and the theorem follows from 3.1.

3.3. Definition:

A space is called a CC-space provided that every compact subset is closed.

A space is called a C-space provided that a subset A is closed if and only if each intersection of A with a closed compact subset is compact (A is closed $\Longleftrightarrow \forall C$; C compact closed; $C \cap A$ compact, cf. [2]). (Note that every C-space is a topological space.)
A space is called a C^*-space iff it is a compact antispace.

Remark: The definition of C-spaces is very closely related to the definition of k-spaces (cf. [1]). In particular a space is a C-space if and only if it is a CC-space and a k-space.

3.4. Theorem:

For every T_1-space $(X,\mathcal{G})$ the following statements are equivalent:

(i) $(X,\mathcal{G})$ is a C-space.

(ii) $(X,\mathcal{G})$ is a CC-antispace.

(iii) There is a C^*-space $(X,\mathcal{K})$ such that $\rho\mathcal{K} = \mathcal{G}$.

Proof: (i) ==> (ii). Assume that $(X,\mathcal{G})$ is a C-space. The intersection of a compact set with a closed compact set is always compact. By definition every compact set is closed, and $(X,\mathcal{G})$ is a CC-space.

Now we prove that $\mathcal{G}=\rho^2\mathcal{G}$. Let $G\in\mathcal{G}$. Let $\mathcal{C}'\subset\rho\mathcal{G}$ be centered in G. Then if $C_0\in\mathcal{C}'$; $G\cap C_0\in\rho\mathcal{G}$ and $G\cap C\in\mathcal{G}$ for every $C\in\mathcal{C}'$ and this system $\{G\cap C \mid C\in\mathcal{C}'\}$ is centered in $G\cap C_0$ and hence $\cap\{G\cap C \mid C\in\mathcal{C}'\} = (\cap\mathcal{C}')\cap G\neq\phi$. Thus $G\in\rho^2\mathcal{G}$; so $\mathcal{G}\subset\rho^2\mathcal{G}$.

Let $A\in\rho^2\mathcal{G}$. By lemma 1.4., for every $C\in\rho\mathcal{G}$, $A\cap C\in\rho\mathcal{G}$ and hence A is closed by the definition of a C-space. Thus $\rho^2\mathcal{G}\subset\mathcal{G}$; $\rho^2\mathcal{G}=\mathcal{G}$.

This proves that $(X,\mathcal{G})$ is an antispace.

(ii) ==> (iii). $(X,\mathcal{G})$ is an antispace, and hence $(X,\mathcal{G}),(X,\rho\mathcal{G})$ is an antipair; and $(X,\rho\mathcal{G})$ is an antispace. In $(X,\mathcal{G})$ every compact set is closed, and hence in $(X,\rho\mathcal{G})$ every closed set is compact. This implies that $(X,\rho\mathcal{G})$ is a compact antispace, and $\mathcal{G}=\rho(\rho\mathcal{G})$. Hence $(X,\rho\mathcal{G})$ is the C^*-space with the required property.

(iii) ==> (ii). $(X,\mathcal{K})$ is a C^*-space, and hence $(X,\rho\mathcal{K})$ is a CC-space and an antispace.

(ii) ==> (i). $(X,\mathcal{G})$ is a CC-antispace, and hence $\rho^2\mathcal{G}=\mathcal{G}$. Let $A\in\mathcal{G}$ then $A\cap C$ is compact for every $C\in\rho\mathcal{G}$.

Let A be a set such that $A\cap C$ is compact for every compact closed set C. Let $\mathcal{C}'$ be an arbitrary system of compact sets centered in A. $\mathcal{C}=\{C\cap A \mid C\in\mathcal{C}'\}$ is a collection of closed compact sets centered in any one of its members. Thus $\cap\mathcal{C}=\cap\mathcal{C}'\cap A\neq\phi$. Hence by definition $A\in\rho^2\mathcal{G}$ and so $A\in\mathcal{G}$. This proves that $(X,\mathcal{G})$ is a C-space.

Corollary: $(X,\mathcal{G})$ is a C-space, if and only if $(X,\rho\mathcal{G})$ is a C^*-space, and conversely.

3.5. Proposition:

For every collection of subsets $\mathcal{K}$ of a set X, $\rho\mathcal{K}=\rho(\mathcal{K}\cup\rho^2\mathcal{K})$.

Proof: $(\mathcal{K}\cup\rho^2\mathcal{K})\supset\mathcal{K}$ and hence $\rho\mathcal{K}\supset\rho(\mathcal{K}\cup\rho^2\mathcal{K})$. Let $C\in\rho\mathcal{K}$ and let $\mathcal{K}'\subset(\mathcal{K}\cup\rho^2\mathcal{K})$ be a system that is centered in C. Then $\mathcal{K}'$ can be split up into two parts; $\mathcal{K}^0\subset\mathcal{K}$ and $\mathcal{K}^2\subset\rho^2\mathcal{K}$; $\mathcal{K}^0\cup\mathcal{K}^2=\mathcal{K}'$ and both $\mathcal{K}^0$ and $\mathcal{K}^2$ are centered in C. Hence $C^0=(\cap\mathcal{K}^0)\cap C\neq\phi$ and is an

element of $\rho\mathcal{K}$. We must prove now that $(\cap\mathcal{K}') = (\cap\mathcal{K}^2)\cap(\cap\mathcal{K}^0)\cap C =$
$= (\cap\mathcal{K}^2)\cap C^0 \neq \phi$. Suppose that $(\cap\mathcal{K}^2)\cap C^0 = \phi$. Then there exists a finite subcollection $\{\mathcal{K}_i^2\}_{i=1}^n$ of $\mathcal{K}^2$ such that $(\bigcap_{i=1}^n \mathcal{K}_i^2)\cap C^0 = \phi$.
From lemma 1.4 and theorem 1.5 it follows that
$C^2 = (\bigcap_{i=1}^n \mathcal{K}_i^2)\cap C \neq \phi$ and is an element of $\rho\mathcal{K}$; but $C^2\cap(\cap\mathcal{K}^0) = \phi$.
By definition there must be a finite subcollection $\{\mathcal{K}_j^0\}_{j=1}^m$ such that $(\bigcap_{j=1}^m \mathcal{K}_j^0)\cap C^2 = \phi$ and hence $(\bigcap_{i=1}^n \mathcal{K}_i^2)\cap(\bigcap_{j=1}^m \mathcal{K}_j^0)\cap C = \phi$ and this contradicts the assumption that $\mathcal{K}'$ is centered in C.
Now it follows that $(\cap\mathcal{K}')\cap C \neq \phi$ and hence $C\in\rho(\mathcal{K}\cup\rho^2\mathcal{K})$; thus $\rho\mathcal{K}\subset\rho(\mathcal{K}\cup\rho^2\mathcal{K})$; so $\rho\mathcal{K} = \rho(\mathcal{K}\cup\rho^2\mathcal{K})$.

Remark: From Alexander's subbase theorem it follows that $\rho\varepsilon(\mathcal{K}\cup\rho^2\mathcal{K}) = \rho\mathcal{K}$. We will denote $\varepsilon(\mathcal{K}\cup\rho^2\mathcal{K})$ by $\mathcal{K}^+$.

3.6. Theorem:

For every T[illegible]-space $(X,\mathcal{G})$ the following statements are equivalent.

(i) $(X,\rho^2\mathcal{G})$ is a C-space.

(ii) $(X,\rho\mathcal{G})$ is a C^*-space.

(iii) $(X,\rho\mathcal{G})$ is a compact space.

(iv) $(X,\rho^2\mathcal{G})$ is a CC-space.

(v) $X\in\rho^2\mathcal{G}$.

(vi) $\rho\mathcal{G}\subset\rho^2\mathcal{G}$.

Proof: The pattern of proof is: (i) ==> (ii); (ii) ==> (iii); (iii) ==> (iv); (iv) ==> (i); (iii) <==> (v); (v) <==> (vi).
(i) ==> (ii). Let $(X,\rho^2\mathcal{G})$ be a C-space. ($\mathcal{G}^+ = (\mathcal{G}\cup\rho^2\mathcal{G})$) and hence $\mathcal{G}^+\supset\rho^2\mathcal{G}$. Because $(X,\rho^2\mathcal{G})$ is a CC-space (cf. 3.4), $(X,\mathcal{G}^+)$ must be a CC-space. We will prove first that $\mathcal{G}^+\subset\rho^2\mathcal{G}$.
Let $G\in\mathcal{G}^+$. Let $\mathcal{C}'$ be a subsystem of $\rho\mathcal{G}(=\rho\mathcal{G}^+$, cf. 3.5) that is centered in G. Let C_0 be an arbitrary element of $\mathcal{C}'$. Then $G\cap C_0\in\rho\mathcal{G}^+ = \rho\mathcal{G}$. Every element of $\mathcal{C}'$ is a member of $\mathcal{G}^+$ because $(X,\mathcal{G}^+)$ is a CC-space. $\mathcal{C}'$ is centered in $G\cap C_0$ and therefore $G\cap(\cap\mathcal{C}') = (\cap\mathcal{C}')\cap C_0\cap G \neq \phi$. Thus $G\in\rho^2\mathcal{G}$ and $\mathcal{G}^+\subset\rho^2\mathcal{G}$. Hence $\mathcal{G}^+ = \rho^2\mathcal{G}$; $(X,\mathcal{G}^+)$ is a C-space. And now it follows from (3.4) that $(X,\rho\mathcal{G}^+)$ is a C^*-space.
From (3.5) it follows, that $(X,\rho\mathcal{G})$ is a C^*-space.

(ii) ==> (iii). Follows from the definition of a C^*-space.

(iii) ==> (iv). Trivial.

(iv) ==> (i). Follows from (3.4) and (3.2).

(iii) <==> (v). Trivial.

(v) ==> (vi). Follows from (1.4).

(vi) ==> (v). Trivial.

3.7. Theorem:

If $(X,\mathcal{G})$ is a CC-space, then $(X,\rho\mathcal{G})$ is a C^*-space.

Proof: $\rho\mathcal{G}\subset\mathcal{G}$, by definition. $\mathcal{G}$ has as compact sets exactly $\rho\mathcal{G}$. From the inclusion it follows, that $\rho^2(\mathcal{G})\supset\rho(\mathcal{G})$ and the rest follows from 3.6.

3.8. Theorem:

If $(X,\mathcal{G})$ is an antispace, then $(X,\varepsilon(\mathcal{G}\cup\rho\mathcal{G}))$ is a CC-space; $(X,\mathcal{G}\cap\rho\mathcal{G})$ is a C^*-space and $\rho\varepsilon(\mathcal{G}\cup\rho\mathcal{G}) = \mathcal{G}\cap\rho\mathcal{G}$.

Proof: $(\mathcal{G}\cup\rho\mathcal{G})\supset\mathcal{G}$, thus $\rho(\mathcal{G}\cup\rho\mathcal{G})\subset\rho\mathcal{G}\subset(\mathcal{G}\cup\rho\mathcal{G})$ and hence $(X,\varepsilon(\mathcal{G}\cup\rho\mathcal{G}))$ is a CC-space.

Now we only have to prove that $\rho(\mathcal{G}\cup\rho\mathcal{G}) = \mathcal{G}\cap\rho\mathcal{G}$ and then the theorem follows from 3.7.

If $A\in\rho(\mathcal{G}\cup\rho\mathcal{G})$, then $A\in\rho\mathcal{G}$ and $A\in\rho^2\mathcal{G} = \mathcal{G}$. (Note that $\rho(\mathcal{G}\cup\rho\mathcal{G})\subset\rho\mathcal{G}$ and $\rho(\mathcal{G}\cup\rho\mathcal{G})\subset\rho^2\mathcal{G}$.) Hence $A\in\mathcal{G}\cap\rho\mathcal{G}$; this implies that $\rho(\mathcal{G}\cup\rho\mathcal{G})\subset\mathcal{G}\cap\rho\mathcal{G}$.

Now fix an arbitrary set $B\in\mathcal{G}\cap\rho\mathcal{G} = \rho^2\mathcal{G}\cap\rho\mathcal{G}$.

For every subset C of B we can find by lemma 1.4 that $C\in\rho\mathcal{G}$ if and only if $C\in\rho^2\mathcal{G} = \mathcal{G}$. This implies, that the T spaces $(B,\hat{\mathcal{G}})$ and $(B,\rho\hat{\mathcal{G}})$ coincide where $\hat{\mathcal{G}} = \{G\cap B \mid G\in\mathcal{G}\}$. From the fact that $B\in\rho\mathcal{G}$ it follows that $B\in\rho(\mathcal{G}\cup\rho\mathcal{G})$, and this proves that $\mathcal{G}\cap\rho\mathcal{G}\subset\rho(\mathcal{G}\cup\rho\mathcal{G})$; thus $\rho(\mathcal{G}\cup\rho\mathcal{G}) = \mathcal{G}\cap\rho\mathcal{G}$, and hence $(X,\mathcal{G}\cap\rho\mathcal{G})$ is a C^*-space.

Corollary: For every T space $(X,\mathcal{G})$, the space $(X,\rho(\rho\mathcal{G}\cap\rho^2\mathcal{G}))$ is a C-space.

3.9. Remark: It should be noticed that there exist compact C-spaces and in that case the compact sets and the closed sets coincide. These spaces are anti-spaces, and the "adjoint" member of their antipair is identical with the space itself. These spaces are exactly the maximal compact spaces; i.e. spaces such that there is no finer topology that leaves the space compact. This notion may serve as a generalization of compact Hausdorff.

A more profound treatment of C-spaces and compact C-spaces can be found in [2].

We conclude with some results that express a partial duality between compactness and super-connectedness in antispaces.

3.10. Proposition:

Every antispace, which is not a C-space is super-connected.

Proof: From 3.6 we can conclude for any antispace $(X,\mathcal{G})$ that $(X,\mathcal{G})$ is a C-space iff $X\in\mathcal{G}$. When $(X,\mathcal{G})$ is not a C-space, X is not closed, and hence ϕ is not open, thus any two non-empty open subsets intersect.

Corollaries: If no member of an antipair is a C-space, then both members are super-connected.

Every non-compact antispace has an adjoint space that is super-connected.

Every non-compact C-space determines, and is determined by some compact super-connected space.

References:

[1] J.L. Kelley: General Topology. (1955), Van Nostrand.

[2] J. de Groot: An isomorphism criterium in General Topology. (to be published).