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THE LIMITS OF BUCHSTAB'S ITERATION SIEVE

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The limits of Buchstab's iteration sieve*)

by

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ABSTRACT

An analysis is presented leading to explicit equations for the limits of the Buchstab iteration sieve. Moreover, the limits are computed for some values of the relevant parameter κ.

KEY WORDS & PHRASES: *Buchstab's iteration sieve, differential-difference equations.*

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*) This paper will be submitted for publication elsewhere

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1. INTRODUCTION

The purpose of this note is to present an analysis leading to explicit equations for the limits of the Buchstab iteration sieve and to compute the limits for some values of the relevant parameter κ. The approximations given in Tables 1, 2 and 3 have been computed on a CDC 6600 CYBER 73/173 computer system at the Mathematical Centre in Amsterdam. Six years ago H. Diamond and W.B. Jurkat made numerical computations for the limits of 10 iterations (unpublished). We used their results to test the solutions of our equations. The same equations have been obtained in the meantime by D. Rawsthorne.

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2. ASSUMPTIONS

We shall be using the notations of Halberstam and Richert [7]. Let $\mathcal{A}$ be a finite sequence of integers and $\mathcal{P}$ a set of primes. For a real number $z \geq 2$ let

$$P(z) = \prod_{p<z, p\in\mathcal{P}} p$$

and

$$S(\mathcal{A},\mathcal{P},z) = |\{a \in \mathcal{A};\ (a,P(z)) = 1\}|,$$

where $|\{.\}|$ denotes the cardinality of the set $\{.\}$. The fundamental problem of sieve theory is to give a lower bound and an upper bound for $S(\mathcal{A},\mathcal{P},z)$ for various values of z. The trivial, but useful, bound

$$S(\mathcal{A},\mathcal{P},z) \geq 0$$

holds without any restriction. In order to get nontrivial estimates one must impose on our sequences $\mathcal{A}$ and $\mathcal{P}$ some regularity conditions. A very elegant and, in practice, fruitful set of conditions has been elaborated by Halberstam and Richert in a series of papers [4],[5] and [6].

For every $d|P(z)$ denote

$$\mathcal{A}_d = \{a \in \mathcal{A};\ a \equiv 0 \pmod d\},$$

and let $|\mathcal{A}_d|$ represent the cardinality of $\mathcal{A}_d$, i.e. the number of those elements in $\mathcal{A}$ which are divisible by d.

<u>ASSUMPTIONS:</u>

(A_1) every $|\mathcal{A}_d|$ can be written in the form

$$|\mathcal{A}_d| = \frac{\omega(d)}{d} X + r(\mathcal{A},d)$$

where X is a positive parameter independent of d;

(A_2) $\omega(d)$ is a multiplicative function such that $0 \le \omega(p) \le (1 - \frac{1}{A_1})p$ and

$$\sum_{\substack{w \le p < z \\ p \in \mathcal{P}}} \frac{\omega(p)}{p} \log p \le \kappa \log \frac{z}{w} + A_2,$$

for all $z > w \ge 2$ with some constants $A_1 > 1$, $A_2 > 1$ and $\kappa \ge 0$;

(A_3) there exists $\eta > 0$ such that

$$\sum_{\substack{d < X^\eta \\ d|P(X)}} c_1^{\Omega(d)} |r(\mathcal{A},d)| \le X(\log X)^{-c_2}$$

for any $c_1 > 0$ and $c_2 > 0$, provided $X > X(\eta,c_1,c_2)$; here, $\Omega(d)$ denotes the number of prime factors of d.

Several papers on sieve methods, for example [3], [1] and especially the fundamental paper of SELBERG [12], revealed that under such assumptions the sieve problem reduces itself to a search for functions F(s) and f(s) for which

$$XV(z)(f(s) - \varepsilon) \le S(\mathcal{A},\mathcal{P},z) \le XV(z)(F(s) + \varepsilon), \tag{1}$$

where

$$V(z) = \prod_{p|P(z)} (1 - \frac{\omega(p)}{p}),$$

$s = \eta \log X/\log z$, ε is any positive constant and $X > X(\varepsilon,\eta,\kappa,c_1,c_2)$. The functions $F(s)$ and $f(s)$ are universal in the sense that they may depend at most on the parameter κ but not on the sequences $\mathcal{A}$ and $\mathcal{P}$. Different sieves yield different pairs of admissible functions $F(s)$ and $f(s)$. The problem of finding the best possible functions $F(s)$ and $f(s)$ has not been solved effectively in general. Brun's sieve method yields functions which are very good for large s. We need

(2) $$F(s) = 1 + O(e^{-s}) \quad \text{and} \quad f(s) = 1 + O(e^{-s}), \qquad \text{as } s \to \infty,$$

but much sharper estimates can be derived (see [6]).

3. ITERATION SIEVE OF BUCHSTAB

In 1938 A.A. BUCHSTAB [3] had a beautiful idea of improving sieve results by means of the following elementary relation

(3) $$S(\mathcal{A},\mathcal{P},z) = S(\mathcal{A},\mathcal{P},w) - \sum_{\substack{w\le p<z \\ p\in\mathcal{P}}} S(\mathcal{A}_p,\mathcal{P},p),$$

which holds for every $z > w \ge 2$. From a given pair of admissible functions $F_0(s)$ and $f_0(s)$ satisfying (1) and (2) he obtains from (3) a new pair

$$F_1(s) = 1 - s^{-\kappa} \int_s^\infty (f_0(t-1) - 1)dt^\kappa$$

$$f_1(s) = 1 - s^{-\kappa} \int_s^\infty (F_0(t-1) - 1)dt^\kappa,$$

which for some values of the variable s may turn out to be better than the original pair. One can then repeat the Buchstab procedure with the new pair of admissible functions

$$\min(F_0(s),F_1(s)) \quad \text{and} \quad \max(f_0(s),f_1(s)),$$

thus getting further improvements. If we used Brun's results to initiate this process, then in the limit we would arrive at the Rosser sieve. A detailed exposition of Rosser's sieve, based on a different idea however, can be found in [8].

4. STATEMENT OF THE PROBLEM

In 1950 A. SELBERG [11] discovered a very powerful upper bound sieve method which gives

$$S(\mathcal{A},\mathcal{P},z) \le XV(z)\left(\frac{1}{\sigma(s)} + \varepsilon\right), \tag{4}$$

where $\sigma(s)$ is the continuous solution of the differential-difference equation

$$\begin{cases} s^{-\kappa}\sigma(s) = A^{-1} & \text{if } 0 < s \le 2 \\ (s^{-\kappa}\sigma(s))' = -\kappa s^{-\kappa-1}\sigma(s-2) & \text{if } \quad s > 2, \end{cases} \tag{5}$$

with $A = 2^{\kappa}e^{\kappa\gamma}\Gamma(\kappa+1)$, γ the Euler constant (see [7], p.194). Selberg's sieve is very strong for large values of κ and small values of s. If $\kappa > 1$ and $s \le 2$ then the upper bound (4) is even stronger than that obtained by Rosser's sieve. Consequently, this might lead to better limit functions of the Buchstab iterations. The first iteration has been carried out by ANKENY and ONISHI [1] and the second iteration by PORTER [9]. It is not difficult to write down a system of equations which should be satisfied by the limit functions F(s) and f(s). There must exist two numbers $\alpha \ge 1$ and $\beta \ge 1$ (sieving limits) such that

$$\begin{aligned} F(s) &= 1/\sigma(s) && \text{if } s \le \alpha, \\ f(s) &= 0 && \text{if } s \le \beta, \\ (s^{\kappa}F(s))' &= \kappa s^{\kappa-1}f(s-1) && \text{if } s > \alpha, \\ (s^{\kappa}f(s))' &= \kappa s^{\kappa-1}F(s-1) && \text{if } s > \beta. \end{aligned} \tag{6}$$

Hence one can compute $F(s)$ and $f(s)$ by the method of steps. It remains to determine two unknown constants α and β. They should be obtained from the asymptotic behaviour (2). The aim of this note is just to solve the differential-difference problem (6) effectively and to compute α and β for special values of κ with $1 < \kappa \leq 2.0$. Our arguments are much the same as those used in [8] for the Rosser sieve. We would like to draw the reader's attention to a very elegant and almost forgotten doctoral thesis of J.J.A. BEENAKKER [2] in which the author develops a theory of the differential-difference equation

$$\alpha x f'(x) + f(x-1) = 0.$$

This equation is a special case of those which have been investigated in [8] independently but later.

5. SOME DIFFERENTIAL-DIFFERENCE EQUATIONS

Here we collect some results of [8] concerning the differential-difference equation

$$sG'(s) = -aG(s) - bG(s-1), \qquad s > \alpha,\ \alpha \geq 1. \tag{7}$$

It is often convenient to study such an equation together with its adjoint equation

$$(sg(s))' = ag(s) + bg(s+1). \tag{8}$$

For any real numbers a, b there exists a solution $g(s)$ of (8) which is regular on the half-plane $\mathrm{Re}\, s > 0$ and satisfies

$$g(s) \sim s^{a+b-1} \quad \text{as} \quad s \to \infty,\ s \text{ real}.$$

If $a+b < 1$ we have the surprisingly simple formula

$$g(s) = \frac{1}{\Gamma(1-a-b)} \int_0^\infty \exp\left(-sz + b \int_0^z \frac{1-e^{-u}}{u}\, du\right) \frac{dz}{z^{a+b}}. \tag{9}$$

If $a+b \geq 1$ one should take the analytic continuation of $g(s)$ with respect to a and b. To this end, expand the function

$$R(z) = \exp(b \int_0^z \frac{1-e^{-u}}{u} du)$$

into a Taylor series

$$R(z) = R(0) + R'(0)z + \ldots + R^{(n)}(0) \frac{z^n}{n!} + R_n(z),$$

say, and integrate termwise getting

$$g(s) = \sum_{\ell=1}^{n} (-1)^{\ell} R^{(\ell)}(0) \binom{a+b-1}{\ell} s^{a+b-1-\ell}$$
$$+ \frac{1}{\Gamma(1-a-b)} \int_0^{\infty} e^{-sz} R_n(z) z^{-a-b} dz.$$

This formula defines $g(s)$ for $\mathrm{Re}(a+b) < n+2$. The idea of solving problem (6) rests on the observation that the "inner product"

$$\langle G,g\rangle = sG(s)g(s) - b \int_{s-1}^{s} G(x)g(x+1)dx$$

is constant for $s \geq \alpha$.

6. EQUATIONS FOR THE SIEVING LIMITS

Letting

$$P(s) = F(s) + f(s) \quad \text{and} \quad Q(s) = F(s) - f(s),$$

by (2) we get

$$(10) \qquad P(s) = 2 + O(e^{-s}) \quad \text{and} \quad Q(s) = O(e^{-s}) \quad \text{as } s \to \infty.$$

As we will see later, we have $\alpha \geq \beta$, which we henceforth assume for simplicity. Therefore, by (6) one can easily deduce that

$$sP'(s) = -\kappa P(s) + \kappa P(s-1) \quad \text{if } s > \alpha,$$

$$sQ'(s) = -\kappa Q(s) - \kappa Q(s-1) \quad \text{if } s > \alpha.$$

The corresponding adjoint equations take the form

$$(11) \qquad \begin{aligned} (sp(s))' &= \kappa p(s) - \kappa p(s+1), \\ (sq(s))' &= \kappa q(s) + \kappa q(s+1). \end{aligned}$$

By (9) we obtain

$$(12) \qquad p(s) = \int_0^\infty \exp\left(-sz - \kappa \int_0^z \frac{1-e^{-u}}{u}\, du\right) dz.$$

The formula for q(s) is slightly more complicated. For all $\kappa < 2$ we have

$$(13) \qquad q(s) = s^{2\kappa-1} - \kappa(2\kappa-1)s^{2\kappa-2} + \tfrac{1}{2}\kappa(\kappa-1)(2\kappa-1)^2 s^{2\kappa-3}$$
$$+ \frac{1}{\Gamma(1-2k)} \int_0^\infty e^{-sz}\left[\exp\left(\kappa \int_0^z \frac{1-e^{-u}}{u}\, du\right) - 1 - kz - \tfrac{1}{4}\kappa(2\kappa-1)z^2\right] \frac{dz}{z^{2\kappa}} .$$

We remark that if 2κ is a positive integer then q(s) is a polynomial of degree $2\kappa-1$ with rational coefficients. Since $p(s) \sim s^{-1}$ and $q(s) \sim s^{2\kappa-1}$ as $s \to \infty$, by (10) we derive

$$\langle P,p\rangle = 2 \quad \text{and} \quad \langle Q,q\rangle = 0.$$

Hence, on taking $s = \alpha$ we get

$$(14) \qquad \begin{cases} \alpha P(\alpha)p(\alpha) + \kappa \displaystyle\int_{\alpha-1}^{\alpha} P(x)p(x+1)\,dx = 2, \\[2ex] \alpha Q(\alpha)q(\alpha) - \kappa \displaystyle\int_{\alpha-1}^{\alpha} Q(x)q(x+1)\,dx = 0. \end{cases}$$

Note that

$$(s^{1-\kappa}p(s))' = -\kappa s^{-\kappa}p(s+1),$$

$$(s^{1-\kappa}q(s))' = \kappa s^{-\kappa}q(s+1).$$

Therefore, by partial integration we obtain

$$\kappa\int_{\alpha-1}^{\alpha} f(x)p(x+1)dx = -\alpha f(\alpha)p(\alpha) + (\alpha-1)f(\alpha-1)p(\alpha-1) + \int_{\alpha-1}^{\alpha} x^{1-\kappa}p(x)dx^{\kappa}f(x),$$

and similarly

$$\kappa\int_{\alpha-1}^{\alpha} f(x)q(x+1)dx = \alpha f(\alpha)q(\alpha) - (\alpha-1)f(\alpha-1)q(\alpha-1) - \int_{\alpha-1}^{\alpha} x^{1-\kappa}q(x)dx^{\kappa}f(x).$$

By (6) we have

$$x^{\kappa}f(x) = \begin{cases} \displaystyle\int_{\beta}^{x} \frac{dt^{k}}{\sigma(t-1)} & \text{if } \beta \le x \le \alpha+1, \\ 0 & \text{if } \quad x < \beta, \end{cases}$$

so that, by (14) we finally obtain

$$(15)\qquad \begin{cases} \alpha\,\dfrac{p(\alpha)}{\sigma(\alpha)} + \kappa\displaystyle\int_{\beta-1}^{\alpha}\frac{p(x+1)}{\sigma(x)}\,dx = 2 \\ \alpha\,\dfrac{q(\alpha)}{\sigma(\alpha)} - \kappa\displaystyle\int_{\beta-1}^{\alpha}\frac{q(x+1)}{\sigma(x)}\,dx = 0, \end{cases}$$

provided $\beta \le \alpha \le \beta+1$, and

$$(16)\qquad \begin{cases} \alpha \dfrac{p(\alpha)}{\sigma(\alpha)} + \kappa \displaystyle\int_{\alpha-2}^{\alpha} \dfrac{p(x+1)}{\sigma(x)}\,dx + (\alpha-1)f(\alpha-1)p(\alpha-1) = 2 \\[2ex] \alpha \dfrac{q(\alpha)}{\sigma(\alpha)} - \kappa \displaystyle\int_{\alpha-2}^{\alpha} \dfrac{q(x+1)}{\sigma(x)}\,dx - (\alpha-1)f(\alpha-1)q(\alpha-1) = 0, \end{cases}$$

provided $\alpha \geq \beta+1$. In the latter case we can find one equation with one unknown parameter α:

$$(17)\qquad \frac{\alpha}{\sigma(\alpha)}\{p(\alpha)q(\alpha-1) + q(\alpha)p(\alpha-1)\} + \kappa \int_{\alpha-2}^{\alpha} [q(\alpha-1)p(x+1) - p(\alpha-1)q(x+1)]\frac{dx}{\sigma(x)} = 2q(\alpha-1).$$

Having computed α we can find $f(\alpha-1)$ from any of (16). To get β we use the formula

$$(18)\qquad (\alpha-1)^{\kappa} f(\alpha-1) = \int_{\beta}^{\alpha-1} \frac{dt^{\kappa}}{\sigma(t-1)} .$$

7. NUMERICAL COMPUTATION

On the basis of the above formulas we have found that the critical value for κ with the property $\alpha = \beta+1$ is approximately equal to $\kappa_0 = 1.8344323$ for which we actually have $\alpha = \beta+1 = 4.8819016$. In the ranges $1 < \kappa < \kappa_0$ resp. $\kappa_0 < \kappa \leq 2$ we used (15) resp. (17)-(18) to compute the following approximations (for details, see [10]).

TABLE 1.

κ	α	β
1.1	2.6139542	2.2222008
1.2	2.9707579	2.4440641
1.3	3.2966727	2.6666073
1.4	3.6086127	2.8903541
1.5	3.9114805	3.1158210
1.6	4.2070237	3.3431530
1.7	4.4971333	3.5720603
1.8	4.7837692	3.8023257
1.9	5.0692758	4.0338225
2.0	5.3577276	4.2664498

The cases of rational values of κ may turn out to be useful for future applications. Having this in mind we prepared the following

TABLE 2.

κ	α	β
1.0002	2.0223222	2.0004502
1.001	2.0503075	2.0022478
1.01	2.1652207	2.0223726
4/3	3.4018518	2.7410304
5/3	4.4008963	3.4955984
5/4	3.1360309	2.5552172
7/4	4.6407926	3.6870329
7/6	2.8568941	2.3700618
11/6	4.8787741	3.8793592
8/7	2.7730138	2.3172387
9/7	3.2511596	2.6347537
10/7	3.6959687	2.9545759
9/8	2.7082466	2.2776350
11/8	3.5315852	2.8342762

8. A COMPARISON WITH ROSSER'S SIEVE

In Rosser's sieve the functions $F(s)$ and $f(s)$ are the continuous solutions of the following differential-difference equation

$$\begin{cases} s^{\kappa}F(s) = A_1 & \text{if } s \le \beta_1 \\ s^{\kappa}f(s) = 0 & \text{if } s \le \beta_1 \end{cases}$$

$$\begin{cases} (s^{\kappa}F(s))' = \kappa s^{\kappa-1}f(s-1) & \text{if } s > \beta_1 \\ (s^{\kappa}f(s))' = \kappa s^{\kappa-1}F(s-1) & \text{if } s > \beta_1, \end{cases}$$

such that $F(s) = 1 + O(e^{-s})$ and $f(s) = 1 + O(e^{-s})$ as $s \to \infty$. It turns out that $\beta_1 - 1$ is the greatest real zero of $q(s)$ and

$$A_1 = 2(\beta_1-1)^{\kappa-1}/p(\beta_1-1).$$

On the basis of these formulas we computed

TABLE 3.

κ	β	A_1
1	2	$2e^{\gamma} = 3.56214484$
1.1	2.26057452	4.40840026
1.2	2.52866481	5.51094507
1.3	2.80289152	6.95285156
1.4	3.08226086	8.84647618
1.5	$\frac{5+\sqrt{3}}{2} = 3.36602540$	11.34422212

It has been proved in [8] that

$$\beta_1 = c_1\kappa + O(\kappa^{2/3})$$

where c_1 is the solution of $c \log c = c + 1$, $c_1 = 3.59112147\ldots$. It would be interesting to find analogous asymptotic formulas for our limits α and β given by (16). It is very likely that our β is asymptotically equivalent to the Ankeny and Onishi limit for the first step of Buchstab's iterations. The limit ν is the unique solution of

$$\int_{\nu}^{\infty} \left(\frac{1}{\sigma(t-1)} - 1\right) dt^{\kappa} = \nu^{\kappa}.$$

As $\kappa \to \infty$ they showed that

$$\nu \sim c\kappa$$

where

$$c = \frac{2}{e \log 2} \exp\left(\int_0^{\log 2} \frac{e^u - 1}{u}\, du\right) = 2.44518586\ldots .$$

Hence, the power of Selberg's sieve for large κ is evident.

For some reasons it is interesting to know the local behaviour of sieving limits in the vicinity of $\kappa = 1$. For Rosser's sieve we are able to show that

$$\beta_1 = 2 + (3-2a)(\kappa-1) + O((\kappa-1)^2) \quad \text{as } \kappa \to 1+,$$

where

$$a = \int_0^\infty e^{-z}\left[\exp\left(\int_0^z \frac{1-e^{-u}}{u}\,du\right) - 1 - z\right]\frac{dz}{z^2} = .21892758\ldots\ .$$

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