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Department of Mathematical Statistics Report MS-R8508 September

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On Roads with No Overtaking

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A road which narrows at a bottleneck from an ∞-lane road to a one-lane road is studied with the aid of two independent stochastic processes. Special attention is given to headways. At the bottleneck an equilibrium headway can be viewed as the maximum of a shifted exponential random variable and a minimum headway. After the bottleneck the situation becomes far more complicated. However, at a sufficiently large distance from the bottleneck an equilibrium headway may be approximated by the maximum of a shifted exponential random variable and a minimum headway, with the parameters of the shifted exponential random variable depending on the desired speed of the car. The distance from the bottleneck only affects the location, not the scale. Results are checked by Monte Carlo experiments.

1980 Mathematics Subject Classification: 60K30, 60G35.

Key Words & Phrases : Traffic Flow, Stochastic Processes.

1. Introduction

Stochastic models for traffic flow can be divided into three categories according to the possibility of overtaking they allow: unrestricted, restricted or no overtaking at all.

Up to now the work on unrestricted overtaking has proven to be the most successful. Some important results are given in section 2. The basic assumption is that each car drives at its desired speed, which remains constant over time. However, unrestricted overtaking is only possible in low volume traffic, and consequently the results are not applicable to more dense traffic.

Restricted overtaking is the collective term for all kinds of overtaking between unrestricted and no overtaking. Despite their diversity all the models which allow restricted overtaking (Rényi (1964), Miller (1962), Newell (1966), Morse & Yaffe (1971), Brill (1971)) have in common that strong simplifications cannot prevent complex results. Hence Breiman (1969) concludes that working "along these lines is, at present, virtually useless". At present the situation does not seem much better.

More promising are the models which do not allow overtaking. The basic assumption is that each car drives at its desired speed, unless its headway (the time distance between a car and its predecessor, measured at a fixed point along the road) threatens to become less than the minimum headway, the minimal value the car driver is willing to accept. Then, in order to prevent this the speed is adjusted accordingly. The practical value of these models is not restricted to the one-lane road only: in dense traffic the possibility of changing lanes is nearly absent, and hence in that case a n-lane road can be expected to behave approximately as n one-lane roads.

The early work of Miller (1965) and Hodgson (1968) on no overtaking is not very realistic, with the assumption of zero minimum headways (leading to an infinite road capacity) and Poisson or deterministic arrivals at the beginning of the no overtaking zone. Cowan (1971), (1975) is more realistic, assuming stochastic minimum headways and an arrival process which was founded on empirical observations. However, considering the impact of the assumed arrival process on the results, it seems better to use an arrival process which is derived in a formal way.

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Consider a road with no entrances or exits which leads from $-\infty$ to $+\infty$ along the real axis. On the negative part there are enough lanes to ensure unrestricted overtaking. On the nonnegative part, however, the road consists of one lane only. The origin is a bottleneck.

Let us enumerate the cars on the road by the order in which they pass the origin. Whenever this enumeration is in some way essential, we shall denote a process as a sequence of random variables, otherwise as a set of random variables. For example, we shall call a sequence of random variables a Poisson point process if all the increments are independent and have an identical exponential distribution, and we shall call a set of random variables a Poisson point process if the number of occurrences in disjoint finite intervals are independent and Poisson distributed. Assign to car n two random variables: $\boldsymbol{S}_n$, the minimum headway car n is willing to accept, and $\boldsymbol{V}_n$, the desired speed of car n. $(\boldsymbol{S}_n)_{n=1}^{\infty}$ and $(\boldsymbol{V}_n)_{n=1}^{\infty}$ are assumed to be two sequences of i.i.d. random variables. We shall denote the cumulative distribution function of $\boldsymbol{S}_n$ by G, and of $\boldsymbol{V}_n$ by K.

Although on the negative part of the real axis overtaking is unrestricted, we may expect that close to the origin the observed speed distribution will differ from the desired speed distribution, because drivers are anticipating the situation at the bottleneck. We shall refer to this zone as the merging zone. It seems reasonable to assume that the merging zone is a finite interval. The stretch of road before the merging zone can be treated as a road with unrestricted overtaking.

On the nonnegative part of the real axis we shall make use of the *desired arrival time* $\boldsymbol{D}_n^{(r)}$ and the *real arrival time* $\boldsymbol{A}_n^{(r)}$ of car n at point r. For each n and each r the following inequality holds:

$$\boldsymbol{A}_n^{(r)} \geqslant \boldsymbol{D}_n^{(r)}$$

In this respect the desired arrival process $(\boldsymbol{D}_n^{(r)})_{n=1}^{\infty}$ and the real arrival process $(\boldsymbol{A}_n^{(r)})_{n=1}^{\infty}$ bear, for a fixed point r, resemblance to the arrival and departure processes in queueing theory. Of course, before the merging zone, the processes $(\boldsymbol{A}_n^{(r)})_{n=1}^{\infty}$ and $(\boldsymbol{D}_n^{(r)})_{n=1}^{\infty}$ coincide. In the merging zone these processes start to differ.

On the nonnegative part of the real axis the cars are travelling in the order by which they passed the origin, while each car maintains at least a minimum time headway. Thus we can write

$$\boldsymbol{A}_n^{(r)} = \max(\boldsymbol{D}_n^{(r)}, \boldsymbol{A}_{n-1}^{(r)} + \boldsymbol{S}_n) \tag{1.1}$$

Given the sequences $(\boldsymbol{D}_n^{(r)})_{n=1}^{\infty}$ and $(\boldsymbol{S}_n)_{n=1}^{\infty}$, the knowledge of only one real arrival time is sufficient to completely determine the sequence $(\boldsymbol{A}_n^{(r)})_{n=1}^{\infty}$. We only need to construct the process $(\boldsymbol{D}_n^{(r)})_{n=1}^{\infty}$ in order to obtain the real arrival process.

For example, starting from the assumption that the desired arrival process at the bottleneck $(\boldsymbol{D}_n^{(r)})_{n=1}^{\infty}$ is a Poisson point process, which is made plausible in section 2, we derive in section 3 the real arrival process at the bottleneck $(\boldsymbol{A}_n^{(0)})_{n=1}^{\infty}$. The last process is used to obtain an interpretation of an equilibrium headway at the bottleneck as the maximum of a minimum headway and a shifted exponential random variable, which gives rise to an approximation of $(\boldsymbol{A}_n^{(0)})_{n=1}^{\infty}$ by a certain renewal process.

This approximation is used in section 4 to construct an approximate desired arrival process for an internal point of the no overtaking zone, which in turn is used to derive an approximate real arrival process for such a point. Furthermore, we derive for a point at a sufficiently large distance from the bottleneck an approximation of an equilibrium headway as the maximum of a minimum headway and a shifted exponential random variable with the parameters of the distribution of the shifted exponential random variable depending on the desired speed of the car.

Sections 3 and 4 provide the main body of this report. Section 3.1 is straightforward queueing theory, and section 4.1 is comparable to the work of Cowan (1971),(1975). The other results are by the author, as is the explicit use of desired and real arrival processes.

2. Before the Merging Zone

The stretch of road before the merging zone can be treated as a road with unrestricted overtaking: each car drives at his desired speed, which remains constant over time. A major result for such a road was obtained by Breiman (1963), extended by Thedéen (1964), and later put in a more general context by Kallenberg (1978). They considered the following situation:

Let $\{X_n : n=1,2,...\}$ be a point process on $(-\infty,x_0)$, and $(V_n)_{n=1}^{\infty}$ a sequence of i.i.d. random variables, independent of $\{X_n : n=1,2,...\}$, with common distribution K. Define:

$$X_n^{(t)} = X_n + t\, V_n \tag{2.1}$$

and let $N_t(I)$ be the number of $X_n^{(t)}$ in a finite interval I. Assume

(a) $\lim_{x\to-\infty} \dfrac{N_0([x,x_0])}{x_0 - x} = \sigma$ w.p. 1

(b) There is a bounded function M such that $EN_0(I) \leqslant M(|I|)$ for every finite interval I, where $|I|$ is the length of I.

(c) $K(v) = \int_0^v k(u)\, du$ where $k(u)$ is almost everywhere continuous with respect to a Lebesgue measure, and bounded on every finite interval.

Theorem 2.1 (Breiman (1963)) *Under (a), (b), and (c) above, for fixed I, j*

$$\lim_{t\to\infty} P(N_t(I) = j) = \frac{\lambda^j}{j!} e^{-\lambda} \tag{2.2}$$

where $\lambda = \sigma|I|$.

Theorem 2.2 (Thedéen (1964)) *Let $I_1, I_2,..., I_n$ be n disjoint but otherwise arbitrary intervals on the real line. Under (a), (b), and (c) above*

$$\lim_{t\to\infty} P(N_t(I_\nu) = j_\nu,\ \nu = 1,2,...,n) = \prod_{\nu=1}^{n} \frac{\lambda^{j_\nu}}{j_\nu!} e^{-\lambda_\nu} \tag{2.3}$$

where $\lambda_\nu = \sigma|I_\nu|$.

Interpret $X_n^{(t)}$ as the position along the road of car n at time t. Under the mild conditions of Theorem 2.2 the process $\{X_n^{(t)} : n=1,2,...\}$ tends to a Poisson point process, independent of the "initial state" $\{X_n : n=1,2,...\}$. Thus, it seems reasonable to assume that the spatial distribution of traffic with unrestricted overtaking constitutes a Poisson point process. Relevant properties of a Poisson point process are given in Theorem 2.3 and Theorem 2.4.

Theorem 2.3 (Ryll-Nardzewski (1954)) *Let $\{Q_n : n=1,2,...\}$ be a Poisson point process on $(-\infty,\infty)$ with intensity λ, and $(R_n)_{n=1}^{\infty}$ a sequence of exchangeable random variables, independent of $\{Q_n : n=1,2,...\}$, then $\{Q_n+R_n : n=1,2,...\}$ is also a Poisson point process on $(-\infty,\infty)$ with the same intensity λ.*

Corollary (Time invariance) If $\{X_n^{(t)} : n=1,2,...\}$ is a Poisson point process on $(-\infty,\infty)$ for a certain t, then $\{X_n^{(t)} : n=1,2,...\}$ is a Poisson point process on $(-\infty,\infty)$ for all t.

Theorem 2.3 is derived for $(-\infty,\infty)$, but we are interested in the stretch of road before the merging zone, which is of the form $(-\infty,x_0)$ or $(-\infty,x_0]$. For convenience take it as $(-\infty,x_0)$.

Corollary (Time Invariance) If $\{X_n : n=1,2,...\}$ is a Poisson process on $(-\infty,x_0)$, and $(V_n)_{n=1}^{\infty}$

an independent sequence of i.i.d. positive random variables, then $\{X_n^{(t)} : X_n^{(t)} < x_0\}$ is a Poisson process on $(-\infty, x_0)$ for all $t \geq 0$.

The process $\{X_n^{(t)} : n=1,2,...\}$ gives the positions of the cars along the road at a certain time t. However, the process $\{A_n^{(r)} : n=1,2,...\}$ is of more interest. Before the merging zone $A_n^{(r)}$ equals $D_n^{(r)}$, and by using (2.1) we get that $D_n^{(r)}$ can be computed from:

$$X_n^{(t)} - r = V_n \, [D_n^{(r)} - t] \tag{2.4}$$

for $r \leq 0$.

THEOREM 2.4 (RÉNYI (1964)) *Let $\{Q_n : n=1,2,...\}$ be a Poisson point process on $(0,\infty)$ with intensity λ, and $(R_n)_{n=1}^{\infty}$ a sequence of i.i.d. positive random variables with $E[R_n]^{-1} < \infty$. Then $\{Q_n R_n : n=1,2,...\}$ is also a Poisson point process on $(0,\infty)$ but with intensity $\lambda \, E[R_n]^{-1}$.*

COROLLARY. If $\{X_n^{(t)} : n=1,2,...\}$ is a Poisson point process on $(-\infty, r)$, and $E[V_n] < \infty$, then $\{D_n^{(r)} : n=1,2,...\}$ is a Poisson point process on (t,∞).

COROLLARY. If $\{D_n^{(r)} : n=1,2,...\}$ is a Poisson point process on (t,∞), and $E[V_n]^{-1} < \infty$, then $\{X_n^{(t)} : n=1,2,...\}$ is a Poisson point process on $(-\infty, r)$.

Unrestricted overtaking is not a necessary condition for a Poisson tendency to occur, e.g., Unkelbach (1979) also established such a tendency for the model of Newell (1966).

3. THE BOTTLENECK

3.1. Delays.

Let $W_n = A_n^{(0)} - D_n^{(0)}$ be the delay imposed on car n by the bottleneck. From equation (1.1) it follows that:

$$\begin{aligned} W_n &= \max(0, A_{n-1}^{(0)} - D_n^{(0)} + S_n) \\ &= \max(0, W_{n-1} - (D_n^{(0)} - D_{n-1}^{(0)}) + S_n) \end{aligned} \tag{3.1}$$

The random variable W_n resembles the waiting time in an M/G/1 queue.

THEOREM 3.1 (LINDLEY (1952)) *Let $(U_n)_{n=1}^{\infty}$ be a sequence of i.i.d. random variables, and $(W_n)_{n=1}^{\infty}$ a sequence given by $W_n = \max(0, W_{n-1} - U_{n-1})$. A necessary and sufficient condition for the distribution of W_n to tend to a non-degenerate limit as $n \to \infty$, is that $0 < E[U_n] < \infty$ or $U_n = 0$ certainly. The limiting distribution does not dependent on the distribution of W_1. If $E[U_n] \leq 0$ and $P(U_n \neq 0) > 0$, then $\lim_{n\to\infty} P(W_n < w) = 0$ for any w.*

From section 2 we learned that it is reasonable to assume that $\{D_n^{(0)} : n=1,2,...\}$ is a Poisson point process. Let us make the further assumption that the behaviour of the car drivers in the merging zone is such that the real arrivals at the bottleneck are in the same order as the desired arrivals, then $(D_n^{(r)})_{n=1}^{\infty}$ is also a Poisson point process, and consequently the increments $T_n = D_n^{(0)} - D_{n-1}^{(0)}$ are exponentially distributed with parameter $\lambda > 0$. Thus, if we apply Lindley's theorem with $U_{n-1} = S_n - T_n$ we get that an equilibrium distribution H for the delays imposed by the bottleneck exists if and only if the value of ρ defined by:

$$\rho = \lambda \int_0^{\infty} x \, dG(x) \tag{3.2}$$

is less than 1, where G is, as previously defined, the distribution of minimum headways. If H exists, it is continuous on $(0,\infty)$, and defined to be right-continuous at 0. Assume $\rho < 1$, and let W_{n-1} be a random variable with distribution H, independent of T_n and S_n. Then $\max(0, W_{n-1} - T_n + S_n)$ must also have distribution H, so the following equation of Wiener-Hopf type (cf. Spitzer (1959)) must hold:

$$H(w) = \int_{[0,\infty)} \int_0^{\infty} G(w-y+t)\,\lambda e^{-\lambda t}\,dt\,dH(y)$$

For $w > 0$ we can rewrite $H(w)$ as:

$$H(w) = e^{\lambda w} \int_{[0,\infty)} \int_w^{\infty} G(t-y)\,\lambda e^{-\lambda t}\,dt\,dH(y)$$

Then differentiating gives:

$$h(w) = \lambda\,H(w) - \lambda \int_0^{\infty} G(w-y)\,dH(y) \tag{3.3}$$

This formula is identical to the integrodifferential equation of Takacs for queues in equilibrium (Cooper (1981),p. 227). Now let H^* and G^* be the Laplace-transforms of the distributions of W_n and S_n respectively.

Equation (3.3) implies:

$$\begin{aligned} H^*(s) &= \int_{\{0\}} e^{-sw}\,dH(w) + \int_0^{\infty} e^{-sw}\,h(w)\,dw \\ &= H(0) + \lambda \int_0^{\infty} e^{-sw}\,H(w)\,dw - \lambda \int_0^{\infty} \int_{[0,\infty)} e^{-sw}\,G(w-y)\,dH(y)\,dw \\ &= H(0) + \lambda\,\frac{H^*(s)}{s} - \lambda\,\frac{G^*(s)H^*(s)}{s} \end{aligned}$$

which has as solution:

$$H^*(s) = \frac{sH(0)}{s-\lambda+\lambda G^*(s)}$$

From $H^*(0) = 1$ we get:

$$\begin{aligned} H(0) &= \lim_{s \downarrow 0}\,[1 - \lambda\,\frac{1-G^*(s)}{s}] \\ &= \lim_{s \downarrow 0}\,[1 - \lambda \int_0^{\infty} \frac{1-e^{-sx}}{s}\,dG(x)] \\ &= 1 - \lambda \int_0^{\infty} x\,dG(x) \\ &= 1 - \rho \end{aligned} \tag{3.4}$$

Note that $H(0)$ equals the fraction of undelayed cars. Hence ρ can be interpreted as the fraction of delayed cars.

The formula for $H^*(s)$ now becomes:

$$H^*(s) = \frac{s(1-\lambda \int_0^{\infty} x\,dG(x))}{s+\lambda G^*(s)-\lambda} \tag{3.5}$$

This formula is identical to the well known Pollaczek-Khinchin formula from queueing theory.

3.2. Headways.
For the headway of car n at the origin $Y_n^{(0)} = A_n^{(0)} - A_{n-1}^{(0)}$ we obtain from equation (1.1):

$$\begin{aligned} Y_n^{(0)} &= \max (D_n^{(0)} - A_{n-1}^{(0)}, S_n) \\ &= \max ((D_n^{(0)} - D_{n-1}^{(0)}) - W_{n-1}, S_n) \\ &= \max (T_n - W_{n-1}, S_n) \end{aligned} \tag{3.6}$$

The random variable W_{n-1} is independent of T_n, because it depends only on the sequences $(D_k^{(0)} - D_{k-1}^{(0)})_{k<n}$ and $(S_k)_{k<n}$. Thus, for $t > 0$ the density of $T_n - W_{n-1}$ is equal to $\lambda e^{-\lambda t} H^*(\lambda)$, and hence the equilibrium distribution F of $Y_n^{(0)}$ is given by:

$$F(y) = (1 - H^*(\lambda)\, e^{-\lambda y})\, G(y) \tag{3.7}$$

where, by (3.5):

$$H^*(\lambda) = \left[\int_0^\infty e^{-\lambda x} dG(x) \right]^{-1} \left[1 - \int_0^\infty x\, dG(x) \right] \tag{3.8}$$

One could interpret an equilibrium headway at the bottleneck as a maximum of a shifted exponential time and a minimum headway. Such a maximum would have as distribution:

$$F_\theta(y) = (1 - e^{-\lambda(y-\theta)})\, G(y) \tag{3.9}$$

and as expectation:

$$\begin{aligned} \int_0^\infty [1 - (1 - e^{-\lambda(y-\theta)})\, G(y)]\, dy &= \int_0^\infty [1 - G(y)]\, dy + e^{\lambda\theta} \int_0^\infty e^{-\lambda y}\, G(y)\, dy \\ &= \int_0^\infty x\, dG(x) + \frac{e^{\lambda\theta}}{\lambda} \int_0^\infty e^{-\lambda x} dG(x) \end{aligned} \tag{3.10}$$

Note that for

$$\begin{aligned} \theta &= \frac{1}{\lambda} \ln H^*(\lambda) \\ &= \frac{1}{\lambda} \left\{ \ln \left[1 - \int_0^\infty x\, dG(x) \right] - \ln \left[\int_0^\infty e^{-\lambda x} dG(x) \right] \right\} \end{aligned} \tag{3.11}$$

the expectation of the headway equals $\frac{1}{\lambda}$, which means that the flow of the real arrival process equals the flow of the desired arrival process. Furthermore, because $H^*(\lambda)$ is a Laplace-transform of a probability distribution, evaluated at a certain point $\lambda > 0$, $H^*(\lambda)$ must lie between 0 and 1. Therefore the value of θ must be negative.

Let us now distinguish between headways which are minimum headways (*following* headways) and headways which are not (*non-following* or *leading* headways). By (3.9) the distribution of leading headways in the equilibrium situation is equal to the distribution of $(T_n - \theta \mid T_n - \theta > S_n)$:

$$F_L(y) = \left[\int_0^\infty e^{-\lambda x} dG(x) \right]^{-1} \int_0^y \lambda e^{-\lambda t}\, G(t)\, dt \tag{3.12}$$

and the distribution of following headways is in that case equal to the distribution of $(S_n \mid S_n > T_n - \theta)$:

$$F_F(y) = \frac{1}{\rho} \left\{ G(y) - (1-\rho) \left[\int_0^\infty e^{-\lambda x} dG(x) \right]^{-1} \int_0^y e^{-\lambda t}\, dG(t) \right\} \tag{3.13}$$

Note that, though all following headways are minimum headways, $F_F(y)$ is in general not equal to G(y). Large minimum headways have an increased chance to be following headways.

Formula (3.7) can now be rewritten as

$$F_{BN}(y) = p\ F_F(y) + (1-p)\ F_L(y) \tag{3.14}$$

with

$$p = \lambda \int_0^\infty x\ dG(x) \tag{3.15}$$

as fraction following headways.

When the variables S_n take a value $\tau > 0$ with probability 1, our model becomes:

$$F_\tau(y) = \begin{cases} 0 & \text{if } y < \tau \\ 1 - (1-p)\ e^{-\lambda(y-\tau)} & \text{if } y \geqslant \tau \end{cases} \tag{3.16}$$

with

$$p = \lambda\tau \tag{3.17}$$

which is identical to a model proposed by Tanner (1961).

Some traffic theorists have proposed general traffic models which can be viewed as generalizations of the Tanner model, obtained by simply plugging in a distribution for τ, and lifting the restriction on p. The way the distribution for τ is plugged in depends on the probabilistic interpretation of the term $e^{-\lambda(y-\tau)}$ in the Tanner model. If one interprets this term as $P(T_n > y \mid T_n > \tau)$, then one obtains the *Semi Poisson* model (Buckley (1968)), given by:

$$F_{SP}(y) = p\ G(y) + (1-p) \left[\int_0^\infty e^{-\lambda x} dG(x) \right]^{-1} \int_0^y \lambda e^{-\lambda t}\ G(t)\ dt \tag{3.18}$$

An interpretation of $e^{-\lambda(y-\tau)}$ as $P(T_n + \tau > y)$ leads to the *Generalized Queueing* model given independently by Cowan (1975) and Branston (1976):

$$F_{GQ}(y) = p\ G(y) + (1-p) \int_0^y G(y-t)\ \lambda e^{-\lambda t}\ dt \tag{3.19}$$

By now it must be clear that, despite what Cowan (1975) claims for the Generalized Queueing model, both this model and the Semi Poisson model can in general not be valid for a bottleneck as considered in this section. In the next section we shall see that the desired velocity distribution enters the headway distribution at an interior point of the no overtaking zone. Thus the Semi Poisson model and the Generalized Queueing model will in general also not be valid for an interior point of a no overtaking zone. Presumably, the same will hold for a restricted overtaking zone.

4. An Internal Point of the No Overtaking Zone

4.1. *Approximate distribution of the actual journey times.*

Let us now consider a point at distance r downstream of the bottleneck. In order to derive an arrival process $(A_n^{(r)})_{n=1}^\infty$ for such a point we have to define a desired arrival process $(D_n^{(r)})_{n=1}^\infty$. Making use of the desired speeds it seems reasonable to assume:

$$D_n^{(r)} = A_n^{(0)} + r / V_n \tag{4.1}$$

Substituting into (1.1) gives

$$A_n^{(r)} = \max\left(A_n^{(0)} + r / V_n,\ A_{n-1}^{(r)} + S_n\right) \tag{4.2}$$

Thus for the *actual journey time* $Z_n^{(r)} = A_n^{(r)} - A_n^{(0)}$ follows:

$$Z_n^{(r)} = \max\,(r / V_n,\ Z_{n-1}^{(r)} - (A_n^{(0)} - A_{n-1}^{(0)}) + S_n)$$
$$= \max\,(r / V_n,\ Z_{n-1}^{(r)} - Y_n^{(0)} + S_n) \tag{4.3}$$

In this formula both $Y_n^{(0)}$ and $Z_{n-1}^{(r)}$ appear. Unfortunately, both variables are dependent on $Y_{n-1}^{(0)}$ and therefore will in general not be independent. In order to obtain at least approximate solutions for the distributions of $Z_n^{(r)}$ and $Y_n^{(r)}$, we will substitute $\max\,(T_n - \theta,\ S_n)$ for $Y_n^{(0)}$ with θ as given in (3.11). We may expect close approximations when $(Y_n^{(0)})_{n=1}^{\infty}$ is a sequence of nearly independent variables, which occurs when ρ is either close to 0 or close to but not exceeding 1. Furthermore, if r is large the traffic behavior will be dominated by the cars with low desired speed, and thus the speed distribution will be the most important parameter.

For $(Z_n^{(r)})_{n=1}^{\infty}$ follows:

$$Z_n^{(r)} = \max\,(r / V_n,\ Z_{n-1}^{(r)} + \min\,(0,\ S_n - T_n + \theta)) \tag{4.4}$$

To prove that there exists an equilibrium distribution $\Omega^{(r)}$ for $Z_n^{(r)}$ we need an extension of theorem 3.1.

THEOREM 4.1 (HELLAND & NILSEN (1976)) *Let* $(U_n)_{n=1}^{\infty}$ *and* $(R_n)_{n=1}^{\infty}$ *be two independent sequences of i.i.d. random variables, and* $(Z_n)_{n=1}^{\infty}$ *a sequence given by* $Z_n = \max\,(R_n,\ Z_{n-1} - U_n)$. *Assume* $E\,|R_n| < \infty$. *A sufficient condition for the distribution of* Z_n *to tend to a non-degenerate limit is that* $0 < E[U_n] < \infty$. *The limiting distribution does not depend on the distribution of* Z_1. *If* $E[U_n] \leqslant 0$ *and* $P(U_n \neq 0) > 0$ *then* $\lim_{n\to\infty} P(Z_n < z) = 0$ *for any z.*

Assume $E[V_n]^{-1} < \infty$, and $\rho < 1$, then $\Omega^{(r)}$ exists by Theorem 4.1. Now let $Z_{n-1}^{(r)}$ be a random variable with distribution function $\Omega^{(r)}$, and independent of r / V_n, S_n, and T_n. Then $Z_n^{(r)}$, given by (4.4), must also have distribution function $\Omega^{(r)}$, from which we have:

$$\Omega^{(r)}(u) = P(Z_n^{(r)} \leqslant u \text{ and } Z_{n-1}^{(r)} \leqslant u) + P(Z_n^{(r)} \leqslant u \text{ and } Z_{n-1}^{(r)} > u)$$
$$= \Phi^{(r)}(u)\left\{\Omega^{(r)}(u) + \int_u^{\infty} P(S_n - T_n \leqslant u - z - \theta)\, d\Omega^{(r)}(z)\right\}$$
$$= \Phi^{(r)}(u)\left\{\Omega^{(r)}(u) + \int_u^{\infty}\int_{\theta+z-u}^{\infty} G(t - \theta - z + u)\, \lambda e^{-\lambda t}\, dt\, d\Omega^{(r)}(z)\right\}$$
$$= \Phi^{(r)}(u)\left\{\Omega^{(r)}(u) + e^{-\lambda\theta}\int_0^{\infty} G(t)\, \lambda e^{-\lambda t}\, dt \int_u^{\infty} e^{-\lambda(z-u)} d\Omega^{(r)}(z)\right\}$$
$$= \Phi^{(r)}(u)\left\{\Omega^{(r)}(u) + \left[1 - \lambda\int_0^{\infty} x\, dG(x)\right]\int_u^{\infty} e^{-\lambda(z-u)} d\Omega^{(r)}(z)\right\}$$
$$= \Phi^{(r)}(u)\left\{\Omega^{(r)}(u) + (1-\rho)\int_u^{\infty} e^{-\lambda(z-u)} d\Omega^{(r)}(z)\right\}$$

where $\Phi^{(r)}(u) = 1 - K(r / u)$ is the distribution function of r / V_n. Rearranging this equation gives:

$$\Omega^{(r)}(u) = (1-\rho)\,\frac{\Phi^{(r)}(u)}{1 - \Phi^{(r)}(u)}\int_u^{\infty} e^{-\lambda(z-u)} d\Omega^{(r)}(z) \tag{4.5}$$

Differentiating with respect to u leads to a differential equation with the following solution under the boundary condition $F(\infty) = 1$:

$$\Omega^{(r)}(u) = \frac{(1-\rho)\Phi^{(r)}(u)}{1-\rho\Phi^{(r)}(u)} \exp\left\{-\lambda \int_u^\infty \frac{1-\Phi^{(r)}(t)}{1-\rho\Phi^{(r)}(t)}\, dt\right\} \tag{4.6}$$

which can also be written as:

$$\Omega^{(r)}(u) = \Psi^{(r)}(u) \exp\left\{-\lambda \int_u^\infty (1 - \Psi^{(r)}(t))dt\right\} \tag{4.7}$$

with

$$\Psi^{(r)}(u) = \frac{(1-\rho)\Phi^{(r)}(u)}{1-\rho\Phi^{(r)}(u)} \tag{4.8}$$

Equations (4.7) and (4.8) imply that $\Omega^{(r)}(u) < \Psi^{(r)}(u) < \Phi^{(r)}(u)$ if $\Phi^{(r)}(u) < 1$, and $\Omega^{(r)}(u) = \Psi^{(r)}(u) = 1$ if $\Phi^{(r)}(u) = 1$. Thus, if $\Phi^{(r)}$ is non-degenerate then $Z_n^{(r)}$ is stochastically larger than r / V_n.

The bottleneck model derived in section 3 is not the only model leading to equation (4.6). For example, Cowan (1975) obtained this equation by assuming the Generalized Queueing model at the bottleneck. In general, every bottleneck model which implies that $(Y_n^{(0)} - S_n \mid Y_n^{(0)} > S_n)$ has an exponential distribution will lead to (4.6). All the models mentioned in section 3 have this property.

Let us now compute $P(Z_{n-1}^{(r)} - T_n \leqslant u)$ in order to gain some probabilistic insight into equation (4.7):

$$\begin{aligned}
P(Z_{n-1}^{(r)} - T_n \leqslant u) &= \int_0^\infty \Omega^{(r)}(t+u)\, \lambda e^{-\lambda t}\, dt \\
&= \Omega^{(r)}(u) + \int_0^\infty \int_u^{t+u} d\Omega^{(r)}(z)\, \lambda e^{-\lambda t}\, dt \\
&= \Omega^{(r)}(u) + \int_u^\infty \int_{z-u}^\infty \lambda e^{-\lambda t}\, dt\, d\Omega^{(r)}(z) \\
&= \Omega^{(r)}(u) + \int_u^\infty e^{-\lambda(z-u)} d\Omega^{(r)}(z)
\end{aligned}$$

The last term appears at the right-hand side of formula (4.5). Substitution gives:

$$\begin{aligned}
P(Z_{n-1}^{(r)} - T_n \leqslant u) &= \Omega^{(r)}(u) + \frac{1-\Phi^{(r)}(u)}{(1-\rho)\Phi^{(r)}(u)}\, \Omega^{(r)}(u) \\
&= \frac{1-\rho\Phi^{(r)}(u)}{(1-\rho)\Phi^{(r)}(u)}\, \Omega^{(r)}(u) \\
&= \exp\left\{-\lambda \int_u^\infty \frac{1-\Phi^{(r)}(t)}{1-\rho\Phi^{(r)}(t)}\, dt\right\}
\end{aligned} \tag{4.9}$$

and thus

$$\begin{aligned}
P(Z_{n-1}^{(r)} \leqslant u \mid Z_{n-1}^{(r)} - T_n \leqslant u) &= \frac{P(Z_{n-1}^{(r)} \leqslant u)}{P(Z_{n-1}^{(r)} - T_n \leqslant u)} \\
&= \Psi^{(r)}(u)
\end{aligned} \tag{4.10}$$

which yields an interpretation of $\Psi^{(r)}(u)$ as a conditional probability.

4.2. Headways at a large distance from the bottleneck

From formula (4.2) we have for $Y_n^{(r)} = A_n^{(r)} - A_{n-1}^{(r)}$

$$Y_n^{(r)} = \max\left((A_n^{(0)} - A_{n-1}^{(0)}) - (A_{n-1}^{(r)} - A_{n-1}^{(0)}) + r / V_n,\ S_n\right)$$
$$= \max\left(Y_n^{(0)} - Z_{n-1}^{(r)} + r / V_n,\ S_n\right) \tag{4.11}$$

Again substituting $\max(T_n - \theta, S_n)$ for $Y_n^{(0)}$ we arrive at:

$$Y_n^{(r)} = \max\left(\max(T_n - \theta, S_n) - Z_{n-1}^{(r)} + r / V_n,\ S_n\right)$$
$$= \max\left(T_n - \theta - Z_{n-1}^{(r)} + r / V_n,\ S_n - Z_{n-1}^{(r)} + r / V_n,\ S_n\right) \tag{4.12}$$

The distribution of $Y_n^{(r)}$, being a maximum of three dependent variables, can be quite complicated. It might be useful to have an easier to handle approximation to $Y_n^{(r)}$, such as e.g.

$$\tilde{Y}_n^{(r)} = \max\left(T_n - \theta - Z_{n-1}^{(r)} + r / V_n,\ S_n\right) \tag{4.13}$$

which is the maximum of two independent variables. To evaluate the quality of $\tilde{Y}_n^{(r)}$ as an approximation to $Y_n^{(r)}$ let us compute the probability that $\tilde{Y}_n^{(r)}$ differs from $Y_n^{(r)}$. First note that $\tilde{Y}_n^{(r)} \neq Y_n^{(r)}$ if and only if the following two independent events:

$$\{T_n - \theta < S_n\} \text{ and } \{r / V_n > Z_{n-1}^{(r)}\}$$

occur simultaneously. The probability of the first event is simply equal to ρ, the probability of being a follower at the bottleneck. The probability of the second event is:

$$P(r / V_n - Z_{n-1}^{(r)} > 0) = \int_0^\infty \Omega^{(r)}(u)\, d\Phi^{(r)}(u)$$
$$= \int_0^\infty \Psi^{(r)}(u) \exp\left\{-\lambda \int_u^\infty [1 - \Psi^{(r)}(t)]\, dt\right\} d\Phi^{(r)}(u)$$

Combining the two probabilities, and expressing both $\Phi^{(r)}$ and $\Psi^{(r)}$ in K, the desired speed distribution function, we finally have:

$$P(\tilde{Y}_n^{(r)} \neq Y_n^{(r)}) = \rho(1-\rho) \int_0^\infty \frac{1-K(v)}{(1-\rho)+\rho K(v)} \exp\left\{-\lambda r \int_0^v \frac{1}{s^2} \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds\right\} dK(v)$$

Now take an a such that $K(a)>0$. Then

$$P(\tilde{Y}_n^{(r)} \neq Y_n^{(r)}) \leqslant K(a) + \exp\left\{-\lambda r \int_0^a \frac{1}{s^2} \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds\right\}$$

and by letting r tend to infinity:

$$\lim_{r\to\infty} P(\tilde{Y}_n^{(r)} \neq Y_n^{(r)}) \leqslant K(\epsilon)$$

Assume K is continuous. Then, by letting a tend to the left end-point of the support of K:

$$\lim_{r\to\infty} P(\tilde{Y}_n^{(r)} \neq Y_n^{(r)}) = 0 \tag{4.14}$$

Furthermore, it is obvious that

$$P(\tilde{Y}_n^{(0)} \neq Y_n^{(0)}) = 0$$

so $\tilde{Y}_n^{(r)}$ may be considered a good approximation if r is either large or small.

We will now determine the asymptotic behavior of the distribution of $\tilde{Y}_n^{(r)}$, and hence also of $Y_n^{(r)}$, as r tends to infinity. From (4.13) we obtain:

$$P(\tilde{Y}_n^{(r)} \leqslant y) = P(T_n - \theta - Z_{n-1}^{(r)} + r/V_n \leqslant y)\, G(y)$$

$$= \left[1 - \int_0^\infty P(Z_{n-1}^{(r)} - T_n \leqslant u-\theta-y)\, d\Phi^{(r)}(u)\right] G(y)$$

By (4.9) we can write:

$$\int_0^\infty P(Z_{n-1}^{(r)} - T_n \leqslant u-\theta-y)\, d\Phi^{(r)}(u) = \int_0^\infty \exp\left\{-\lambda \int_{u-\theta-y}^\infty \frac{1-\Phi^{(r)}(t)}{1-\rho\Phi^{(r)}(t)}\, dt\right\} d\Phi^{(r)}(u)$$

$$= \int_0^\infty \exp\left\{-\lambda r \int_0^{\frac{v}{1-v(y+\theta)/r}} \frac{1}{s^2}\, \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds\right\} dK(v)$$

Now assume $K(v) = \int_0^v k(u)\, du$ with k bounded. Using the Taylor expansion:

$$\int_0^{\frac{v}{1-vz}} \frac{1}{s^2}\, \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds = \int_0^v \frac{1}{s^2}\, \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds + \frac{K(v)}{(1-\rho)+\rho K(v)}\, z + o(z)$$

for $z \to 0$, we get for fixed v:

$$\int_0^{\frac{v}{1-v(y+\theta)/r}} \frac{1}{s^2}\, \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds = \int_0^v \frac{1}{s^2}\, \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds +$$

$$+ \frac{K(v)}{(1-\rho)+\rho K(v)}\, \frac{y+\theta}{r} + o(1/r)$$

for $r \to \infty$. Dominated convergence shows that $P(\tilde{Y}_n^{(r)} \leqslant y)$, and hence $P(Y_n^{(r)} \leqslant y)$, is asymptotically equivalent to

$$\int_0^\infty (1 - e^{-\gamma(v)(y - \tau(v))})\, G(y)\, dK(v) \tag{4.15}$$

with

$$\gamma(v) = \lambda\, \frac{K(v)}{(1-\rho)+\rho K(v)} \tag{4.16}$$

and

$$\tau(v) = -\left[\theta + r\frac{(1-\rho)+\rho K(v)}{K(v)} \int_0^v \frac{1}{s^2}\, \frac{K(s)}{(1-\rho)+\rho K(s)}\, ds\right] \tag{4.17}$$

This result is quite intriguing, and seems to suggest that for large r a headway is approximately the maximum of a shifted exponential random variable and a minimum headway. Both the location and the scale of the shifted exponential random variable depend on the desired speed. We note that the scale parameter $\gamma(v)$ does not depend on r, and that the location parameter $\tau(v)$ tends to $-\infty$, as r tends to ∞.

5. Monte Carlo Experiments

The results of section 3 are known to be exact. However, the results of section 4 are only valid approximately. In order to check the validity of these results, Monte Carlo experiments were performed, using a FORTRAN 77 program which generates minimum headways and desired speeds according to two Beta distributions:

$$P(S_n \leq s) = (s_{\max} - s_{\min})^{-1} \int_{s_{\min}}^{s} (s_{\max} - t)^{\alpha_s - 1} (t - s_{\min})^{\beta_s - 1} \, dt \tag{5.1}$$

$$P(V_n \leq v) = (v_{\max} - v_{\min})^{-1} \int_{v_{\min}}^{v} (v_{\max} - t)^{\alpha_v - 1} (t - v_{\min})^{\beta_v - 1} \, dt \tag{5.2}$$

Given a value of ρ, desired arrivals at the bottleneck are generated according to a Poisson process with intensity

$$\lambda = \frac{\rho}{s_{\max} - s_{\min}} \frac{\beta_s}{\alpha_s + \beta_s} \tag{5.3}$$

The parameters $s_{\max}$, $s_{\min}$ (measured in seconds), $v_{\max}$, $v_{\min}$ (measured in m/sec), α_s, β_s, α_v, and β_v are assigned values, which seem to correspond to real traffic data.

Minimum headways, desired speeds, and desired arrivals at the bottleneck are used to compute the real arrivals at the bottleneck and at five points further down the road (500m, 1000m, 1500m, 2000m, 2500m), in accordance with formulas (1.1) and (4.2). To reach equilibrium conditions 500 cars are generated, but not saved. Then, the data of 1000 cars are filed. The essential part of the program and some of its output are given in respectively Appendix 1 and Appendix 2.

The validity of the distribution of actual journey times given by (4.6) was checked by Q-Q plots, as was recommended by Wilk & Gnanadesikan (1968). These plots are reproduced in Appendix 3.

The computation of quantiles of the distribution given by (4.15) was considered to be too time-consuming. Therefore, we had to confine ourselves to less formal ways to check the validity of (4.15) instead of using ordinary Q-Q plots. In Appendix 4 headways are plotted versus desired speeds. In Appendix 5, for each of 8 desired speed classes of 125 cars, and for each of the five points along the road, the quantiles of the observed headways are plotted versus the quantiles of a standard exponential distribution. We saw that we can view a headway as the maximum of a minimum headway and a shifted exponential random variable with parameters depending on the desired speed. If the desired speed classes are fine enough, we can consider these parameters fixed within such a class. As the minimum headways are bounded, the distribution of headways within a desired speed class should be approximately exponential in the tail, and hence there should be a linear piece in each Q-Q plot of Appendix 5. Furthermore, the scale and the location of the shifted exponential random variable are respectively given by the slope and the intersection with the X-axis of the extension of this linear piece.

Except for large values of ρ, the Q-Q plots of actual journey times show the desired straight lines. For the largest values of ρ the Q-Q plots tend to deviate from straight lines. This is, however, not in as much due to inadequacy of the theoretical distributions, as to the strong dependence between two consecutive actual journey times, which affects the speed of convergence of the empirical distribution towards the theoretical distribution.

The plots of headways versus desired speeds clearly show the dependence between headways and desired speeds. The following vehicles are displayed as a dark horizontal cloud. The fraction of non-following vehicles is decreasing as the distance from the bottleneck tends to ∞.

The Q-Q plots of headways show that for ρ increasing from 0 to 1, the headway distributions for the highest desired speed classes depart from exponentiality first. Furthermore, the X-coordinates of the intersections described above increase. For a fixed value of ρ, these coordinates decrease as the distance from the bottleneck tends to ∞ or as the desired speed tends to its minimum value. In the plots there is also some evidence for the convergence of the slopes of the linear pieces towards a fixed value as r tends to ∞.

Resuming, the results of the Monte Carlo experiments seem to be in accordance with the theory developed in section 4.

6. Future Research

Fundamental in the treatment of the no overtaking zone is equation (4.4), which can be rewritten as:

$$Z_n = \max(R_n, Z_{n-1} - U_n) \tag{6.1}$$

(cf. Theorem 4.1). It can be thought of as a generalization of the much studied equation:

$$W_n = \max(0, W_{n-1} - U_{n-1}) \tag{6.2}$$

(cf. Theorem 3.1) which arises in the study of the single server queue with independent interarrival and service times. Equation (6.1) deserves the same amount of attention: it not only arises here, but also in the random exchange model (Helland & Nilsen (1976)), and in the study of the single server queue with weakly dependent inputs (Kingman (1965)).

The type of road studied in this report is of little practical value. One could enhance the practical value e.g. by considering an ∞-lane road which narrows at a bottleneck to an n-lane road, with no lane changing after the bottleneck. However, in making this small step towards a true general model one can expect to encounter the same problems as involved in the step from the M/G/1 queue to the M/G/n queue. Up to now no exact general solutions are known for the M/G/n queue (cf. Cohen (1982)), only approximate general solutions (Köllerström (1974)). Hence, it would be better to focus future research on obtaining an approximate general model in a direct manner. Promising in this respect are the so-called kinematic models, adopted from statistical mechanics (Prigogine & Herman (1971), Phillips (1979), Michalopoulos, Beskos & Yamauchi (1984), Kühne (1984)).

The model of Prigogine & Herman (1971) can be viewed as a convex combination of the unrestricted overtaking model and a no overtaking model. Surprisingly, this model fails for high traffic volumes, which raises doubt about the accuracy of the no overtaking model they used. By deriving the speed distribution at a given point in the no overtaking zone from (4.4), it must be possible to formulate a more accurate kinematic no overtaking model. Then, data should decide whether or not the convex combination of this model and the no overtaking model is a satisfactory general model.

Acknowledgements

This research was supported by the Netherlands Foundation for the Technical Sciences (STW) through grant CWI33.0508. The author wishes to thank P. Groeneboom, J.A.C. van Toorenburg, and R.D. Gill.

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Appendix 1

Computer Simulation

The lines on the next page constitute the essential part of the FORTRAN 77 program used to simulate the situation at the bottleneck and at NR points further along the road at a distance DR of each other. The real functions G05DBF and G05DLF are random number generators from the NAG-library. The variable S contains the minimum headway, W the delay imposed by the bottleneck, Y0 the headway at the bottleneck, and V0 the desired velocity. The arrays Z, Y, and V contain respectively the actual journey times, the headways, and the actual speeds for each of the NR points along the road. The variable W and the elements of the array Z are assumed to be set to some initial value (presumably 0). First, ND iterations are used to reach equilibrium, then the program outputs the data of N cars.

To illustrate the program a little more, its output for parameter values ND=500, N=10, NR=1, DR=500, $\rho=0.50$, $v_{\min}=15.0$, $v_{\max}=30.0$, $\alpha_v=3.0$, $\beta_v=3.0$, $s_{\min}=0.0$, $s_{\max}=3.0$, $\alpha_s=3.0$, and $\beta_s=1.5$ is also reproduced.

```
      DO 10 I=1-ND,N
         YO=GO5DBF(ET)
         S=SMIN+SB*GO5DLF(ALFAS,BETAS,IFAIL)
         VO=VMIN+VB*GO5DLF(ALFAV,BETAV,IFAIL)
         IF(YO.GT.W+S)THEN
            YO=YO-W
            W=0.
         ELSE
            W=W+(X-YO)
            YO=S
         ENDIF
         DRVO=DR/VO
         DO 20 J=1,NR
            H=Z(J)-YO
            RVO=DRVO*J
            IF(RVO.GT.H+S)THEN
               Y(J)=RVO-H
               Z(J)=RVO
               V(J)=VO
            ELSE
               Y(J)=S
               Z(J)=H+S
            ENDIF
20       CONTINUE
         IF(I.GT.O)THEN
            WRITE(6,30)S,W,YO,VO,(Z(J),Y(J),V(J),J=1,NR)
30          FORMAT(15F9.3)
         ENDIF
10    CONTINUE
```

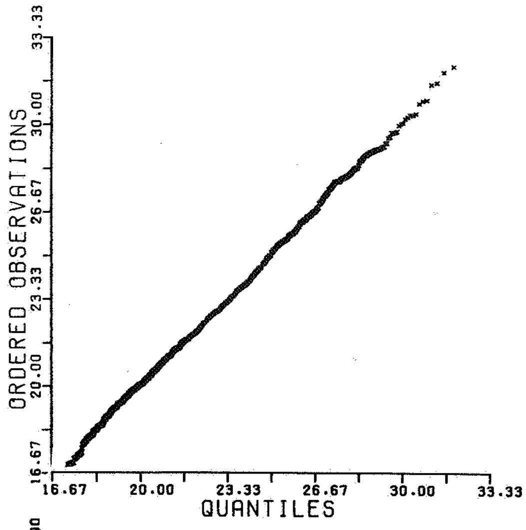
S	W	YO	VO	Z(1)	Y(1)	V(1)
2.538	3.108	2.538	22.006	24.972	2.538	20.022
2.696	0.434	2.696	26.450	24.972	2.696	20.022
2.634	0.000	8.161	26.359	19.446	2.634	20.022
2.289	0.000	4.803	20.248	24.694	10.051	20.248
1.356	0.557	1.356	21.189	24.694	1.356	20.248
2.716	1.329	2.716	25.502	24.694	2.716	20.248
2.020	0.000	2.037	26.572	24.677	2.020	20.248
1.736	0.509	1.736	23.820	24.677	1.736	20.248
1.709	0.000	7.247	20.831	24.003	6.573	20.831
2.171	2.058	2.171	25.152	24.003	2.171	20.831

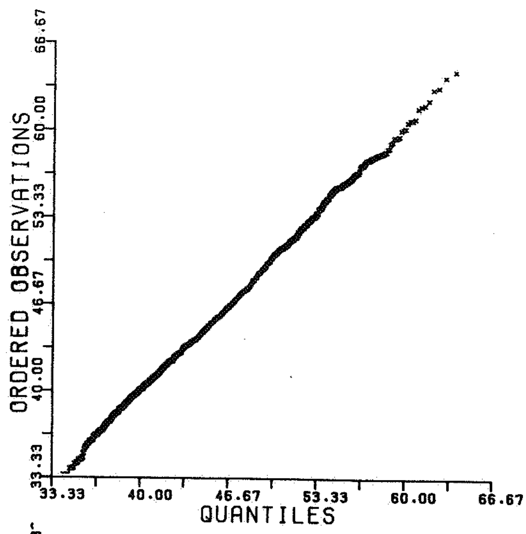
Appendix 2

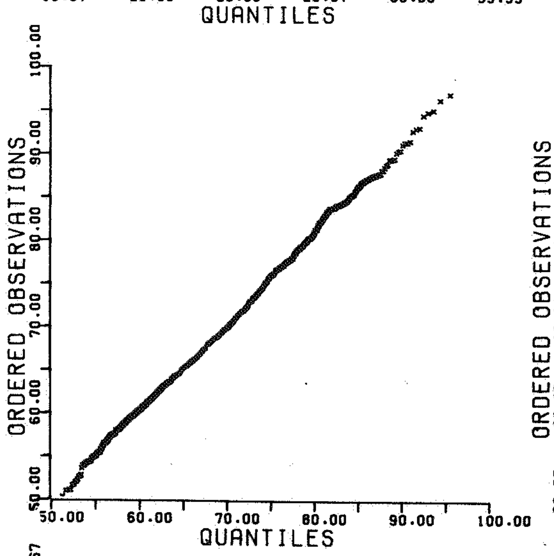
Q-Q Plots of Actual Journey Times

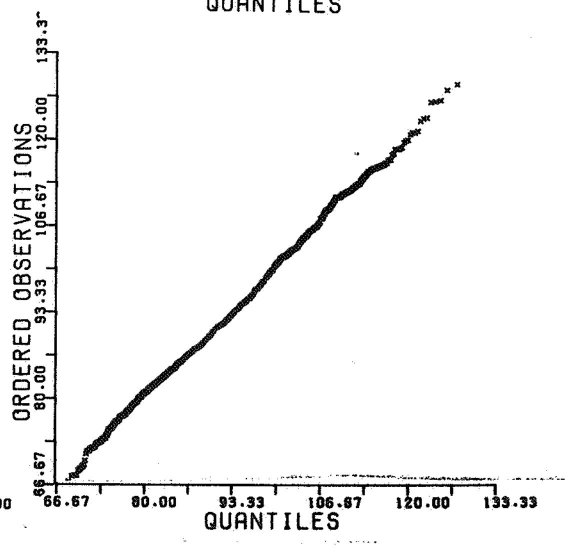
On the next pages Q-Q plots are given of the ordered observed values of $Z_n^{(r)}$ against the quantiles of the distribution given in (4.16). These plots are given for the following values of r:

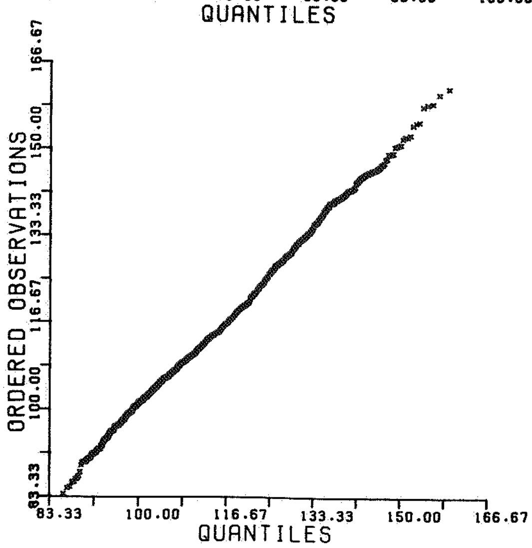
500m	1000m
1500m	2000m
2500m	











$\rho = 0.05$

$v_{\min} = 15.0$

$v_{\max} = 30.0$

$\alpha_v = 3.0$

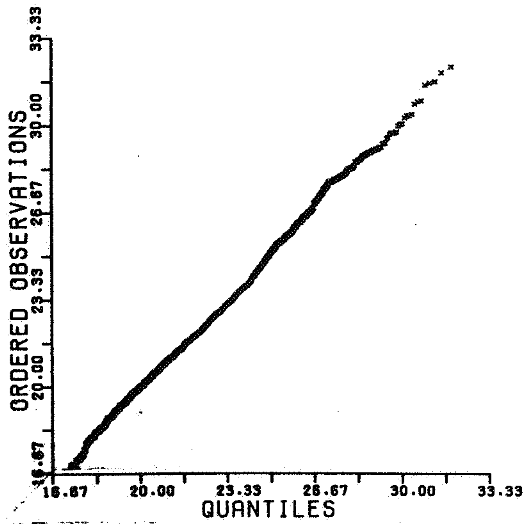
$\beta_v = 3.0$

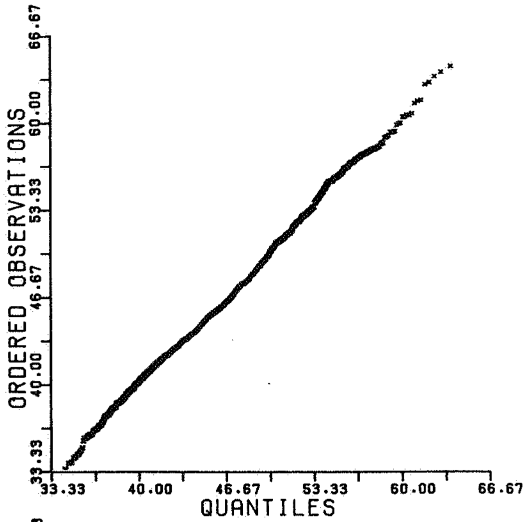
$s_{\min} = 0.0$

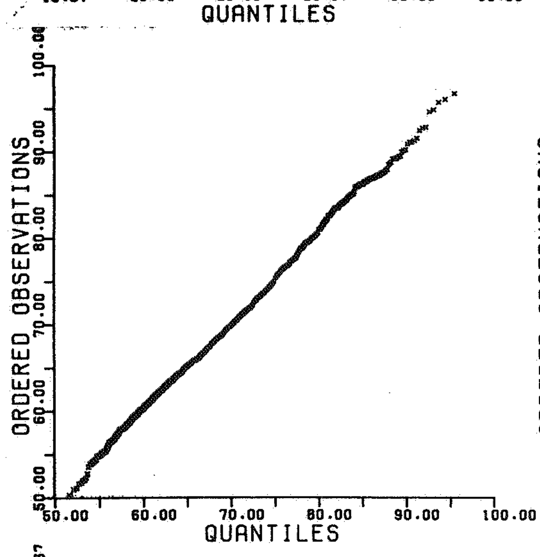
$s_{\max} = 3.0$

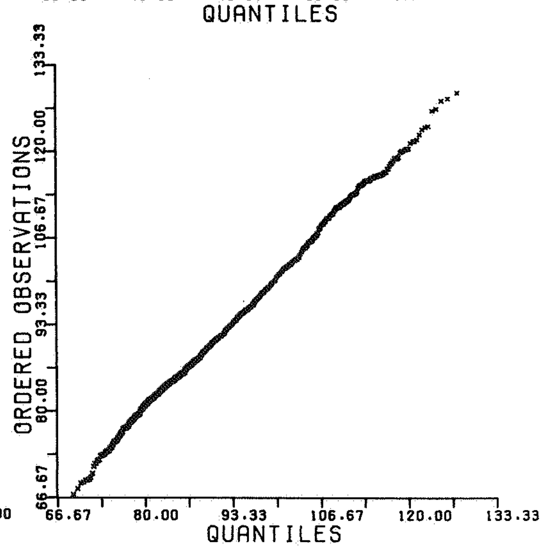
$\alpha_s = 3.0$

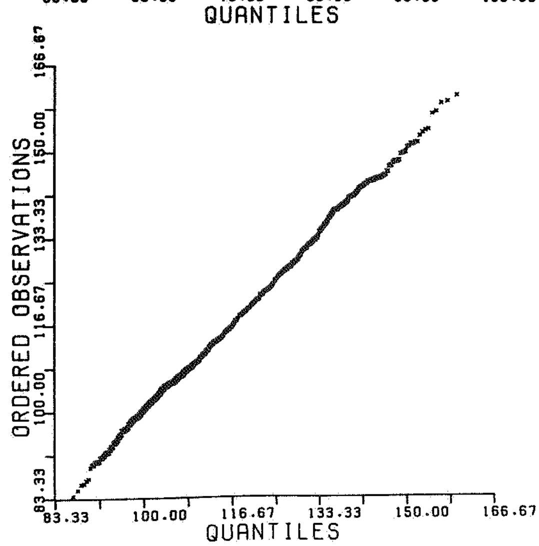
$\beta_s = 1.5$











$\rho = 0.10$

$v_{\min} = 15.0$

$v_{\max} = 30.0$

$\alpha_v = 3.0$

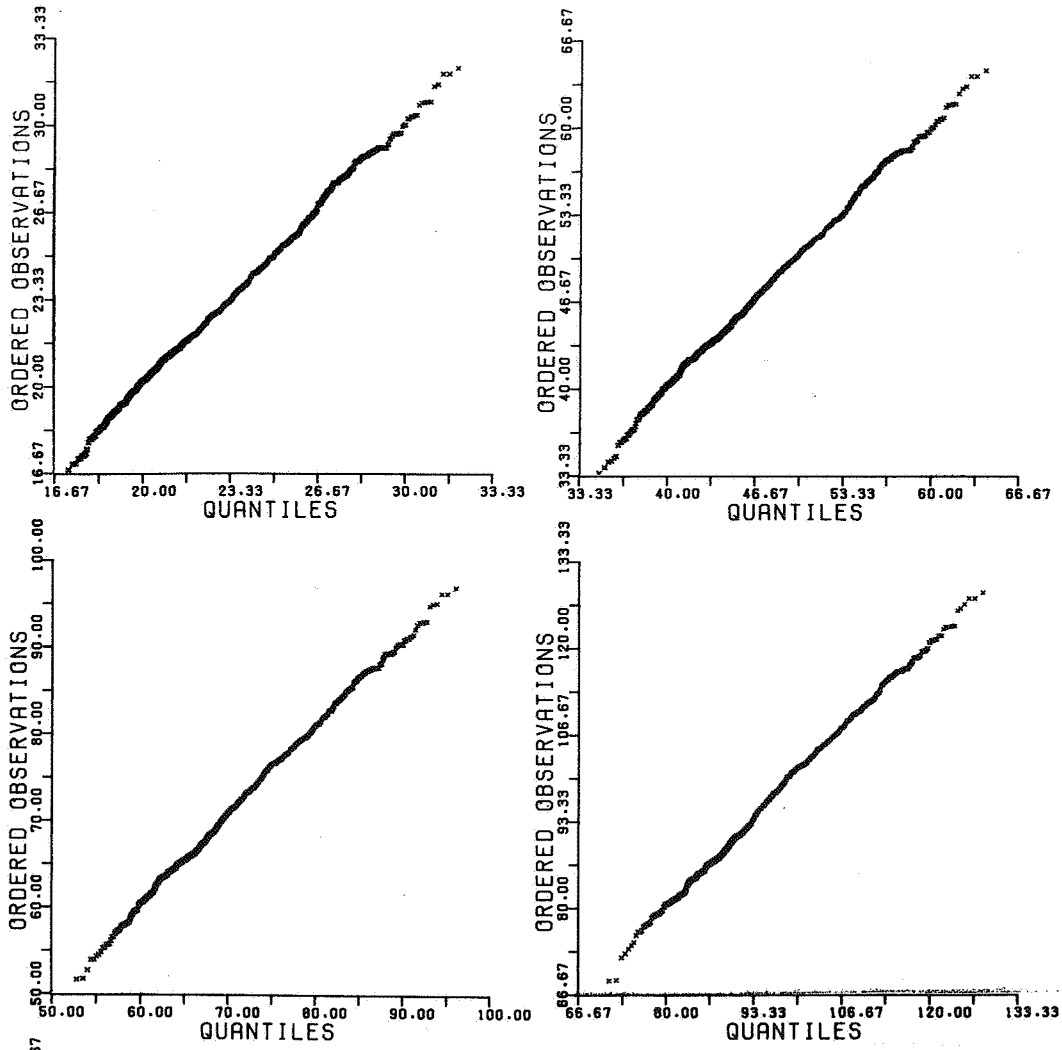
$\beta_v = 3.0$

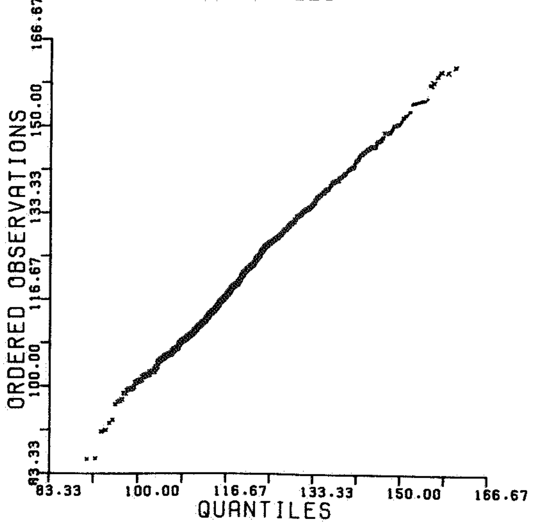
$s_{\min} = 0.0$

$s_{\max} = 3.0$

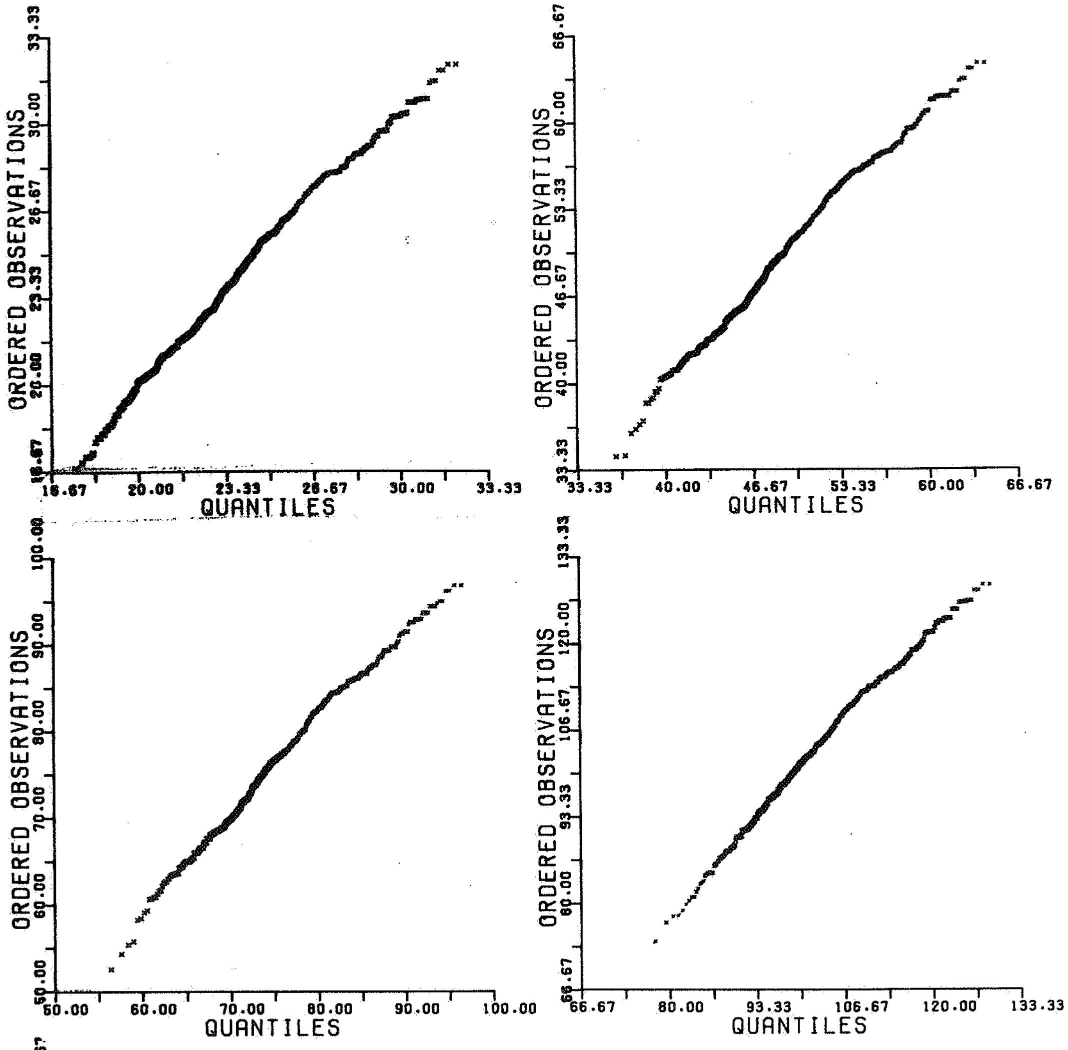
$\alpha_s = 3.0$

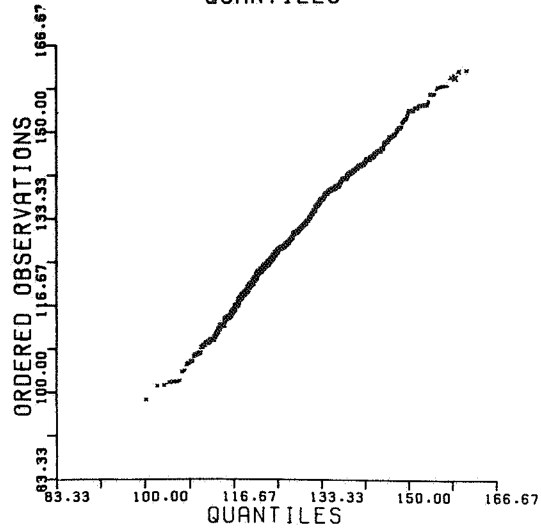
$\beta_s = 1.5$



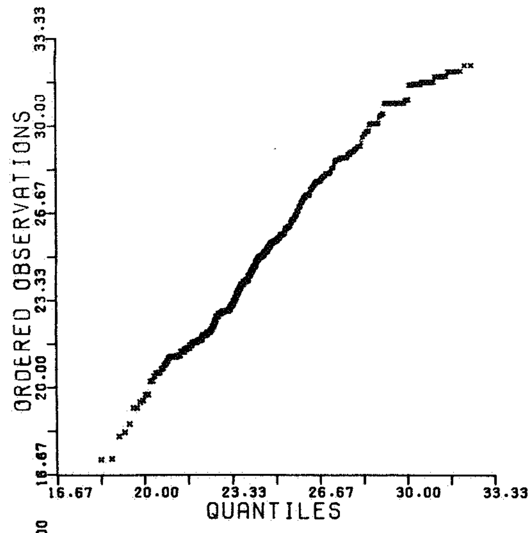


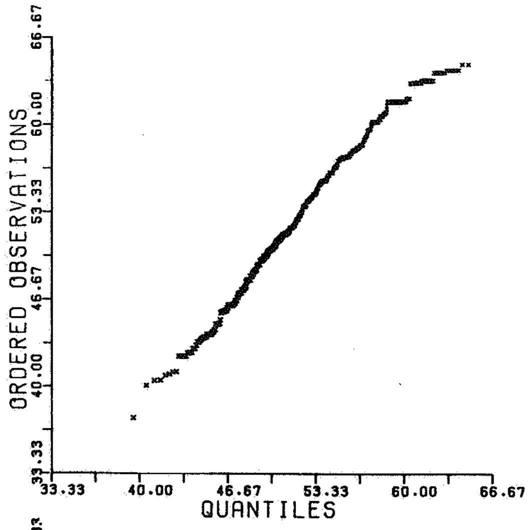
$\rho = 0.25$
$v_{\min} = 15.0$
$v_{\max} = 30.0$
$\alpha_v = 3.0$
$\beta_v = 3.0$
$s_{\min} = 0.0$
$s_{\max} = 3.0$
$\alpha_s = 3.0$
$\beta_s = 1.5$

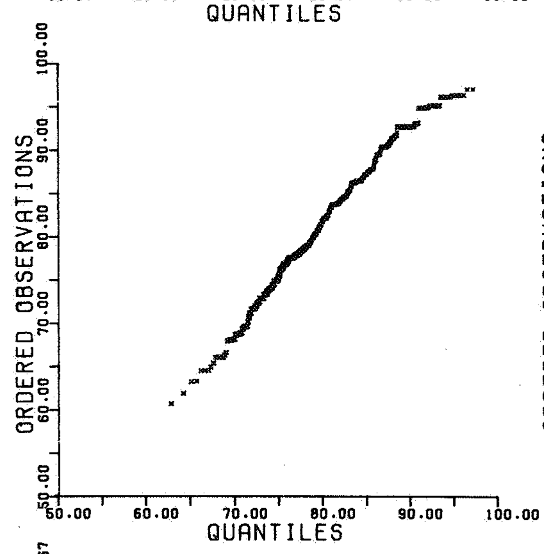


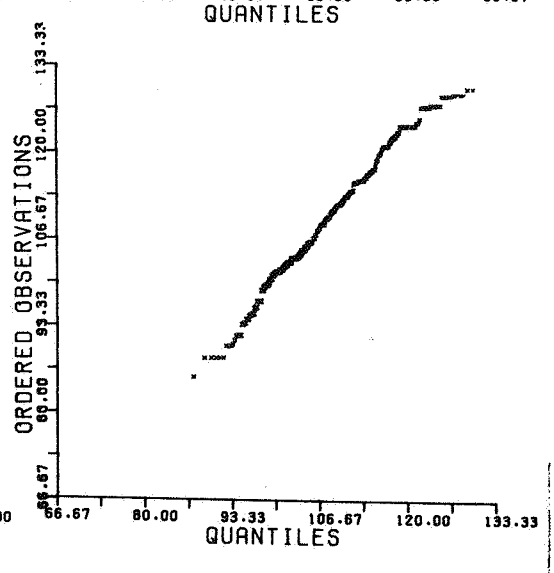


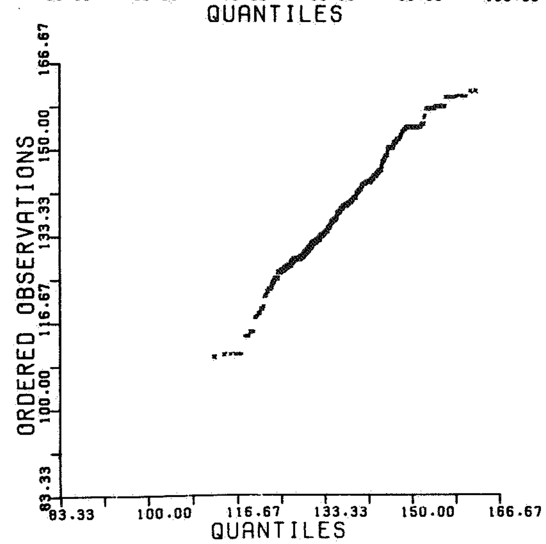
$$
\begin{aligned}
\rho &= 0.50\\
v_{\min} &= 15.0\\
v_{\max} &= 30.0\\
\alpha_v &= 3.0\\
\beta_v &= 3.0\\
s_{\min} &= 0.0\\
s_{\max} &= 3.0\\
\alpha_s &= 3.0\\
\beta_s &= 1.5
\end{aligned}
$$



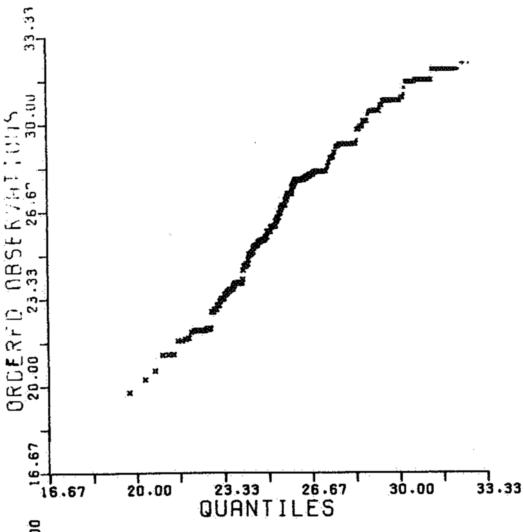


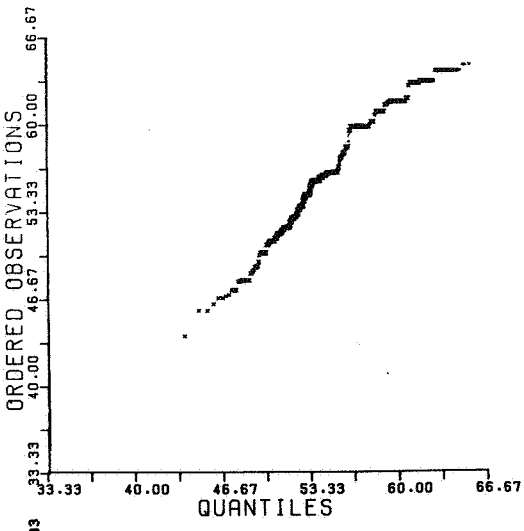


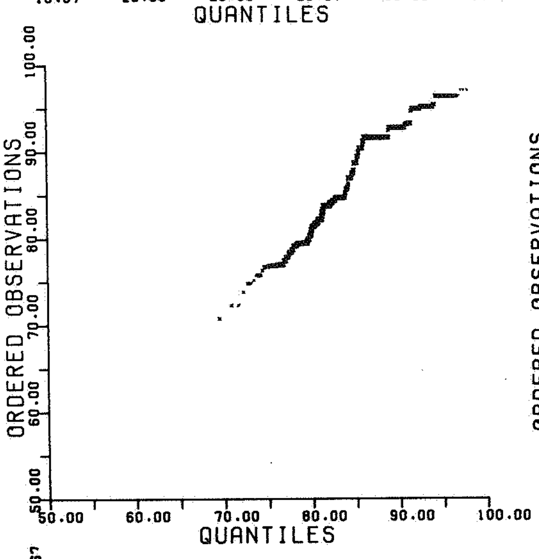


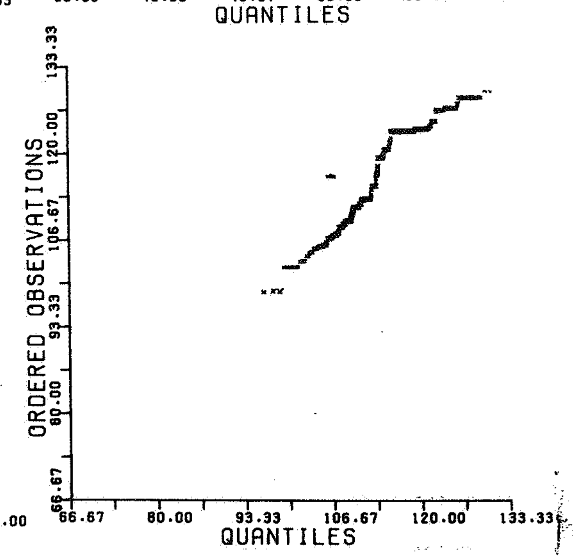


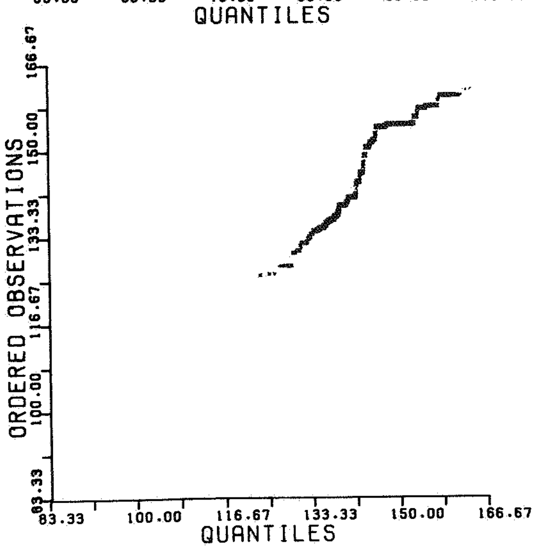
$\rho = 0.75$
$v_{\min} = 15.0$
$v_{\max} = 30.0$
$\alpha_v = 3.0$
$\beta_v = 3.0$
$s_{\min} = 0.0$
$s_{\max} = 3.0$
$\alpha_s = 3.0$
$\beta_s = 1.5$











$\rho = 0.90$

$v_{\min} = 15.0$

$v_{\max} = 30.0$

$\alpha_v = 3.0$

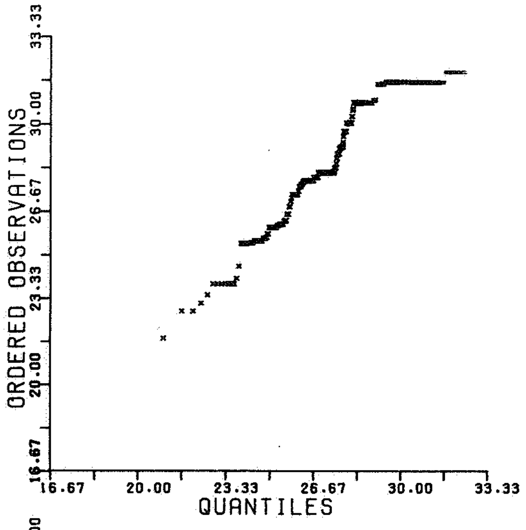
$\beta_v = 3.0$

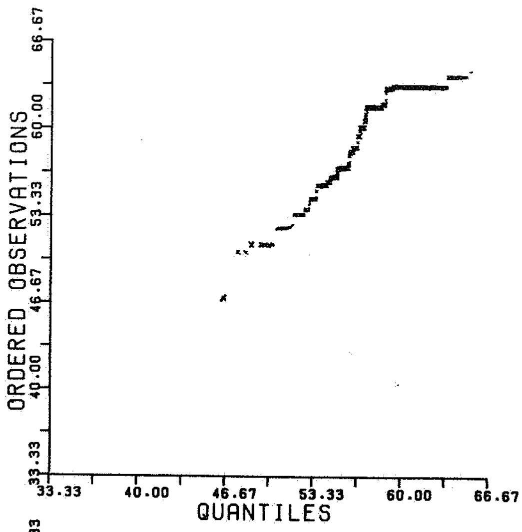
$s_{\min} = 0.0$

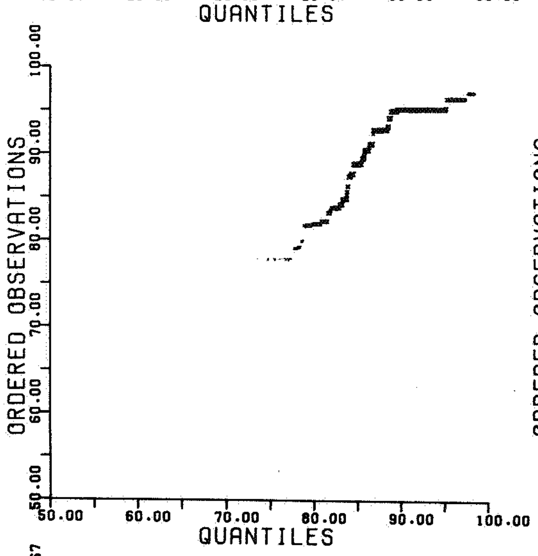
$s_{\max} = 3.0$

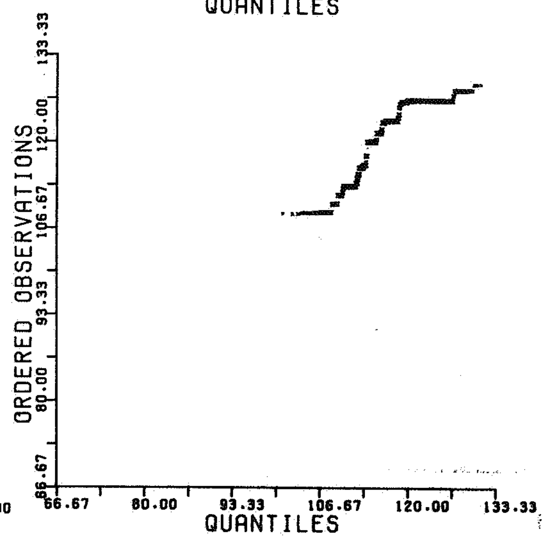
$\alpha_s = 3.0$

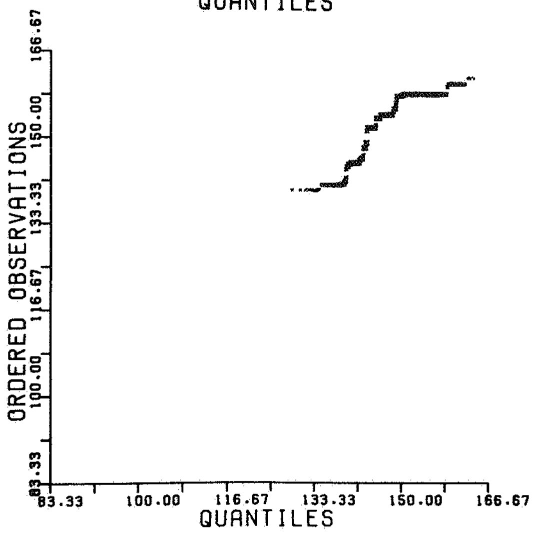
$\beta_s = 1.5$











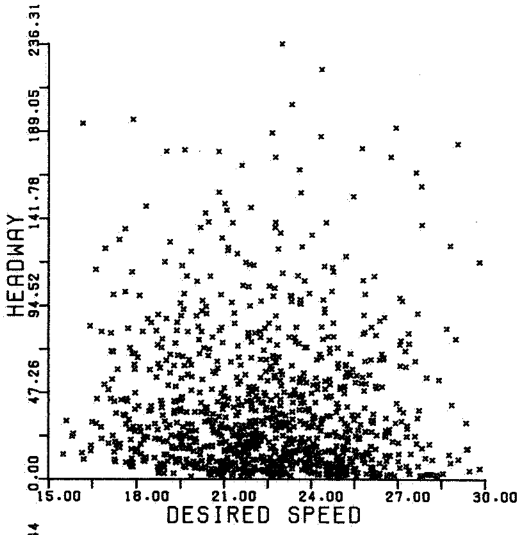
$\rho = 0.95$
$v_{min} = 15.0$
$v_{max} = 30.0$
$\alpha_v = 3.0$
$\beta_v = 3.0$
$s_{min} = 0.0$
$s_{max} = 3.0$
$\alpha_s = 3.0$
$\beta_s = 1.5$

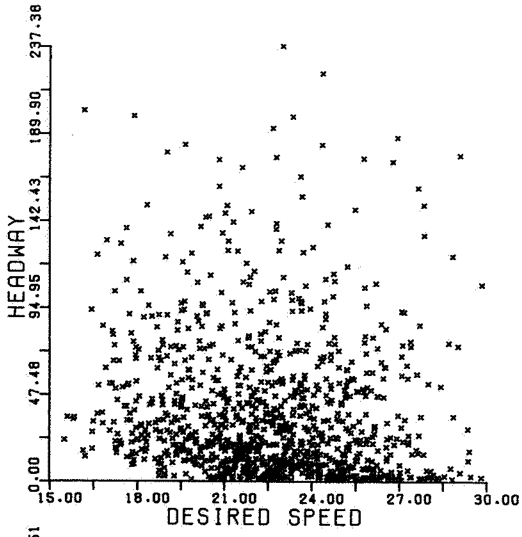
Appendix 3

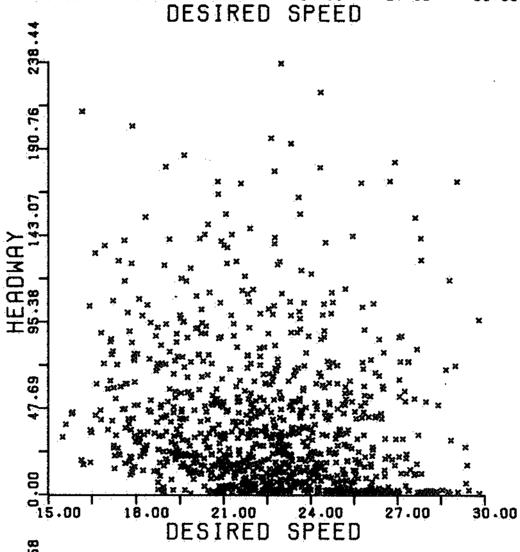
Plots of Headways versus Desired Speeds

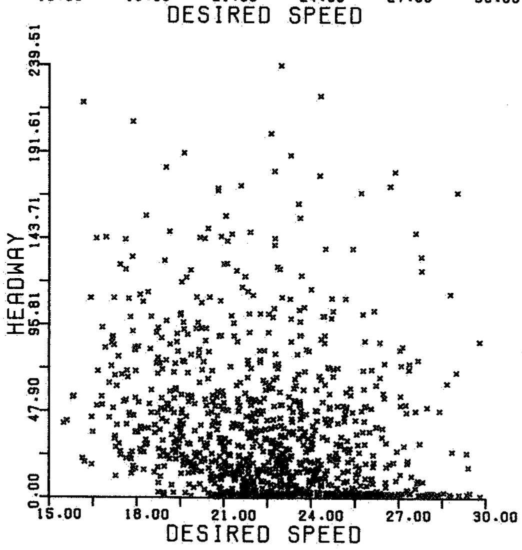
The plots are given for the following values of r:

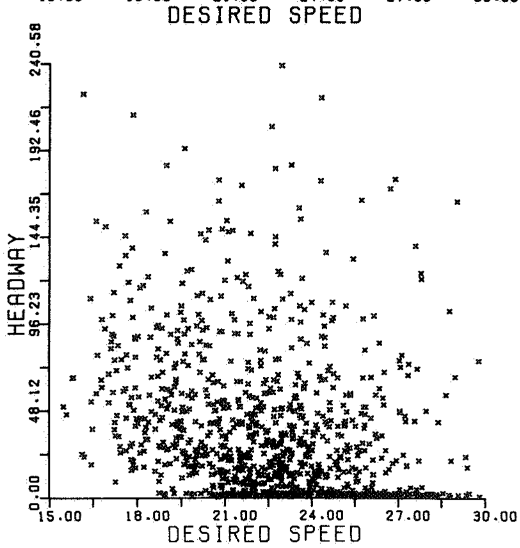
500m	1000m
1500m	2000m
2500m	



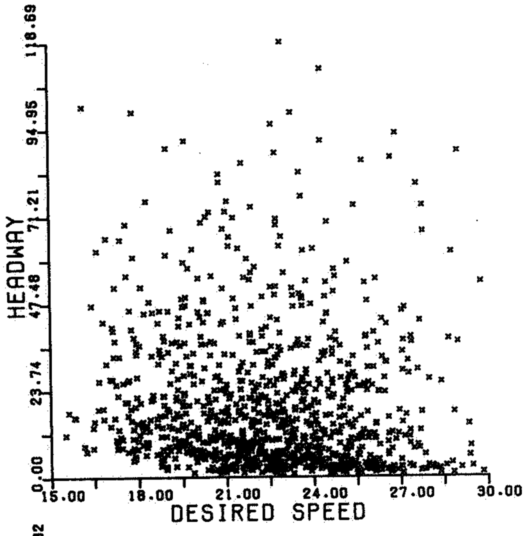


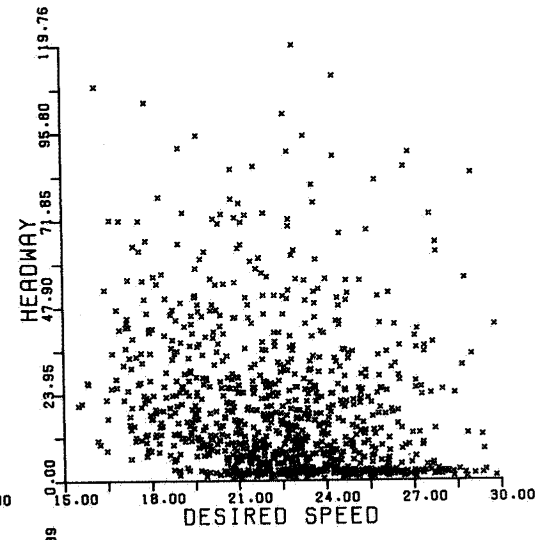


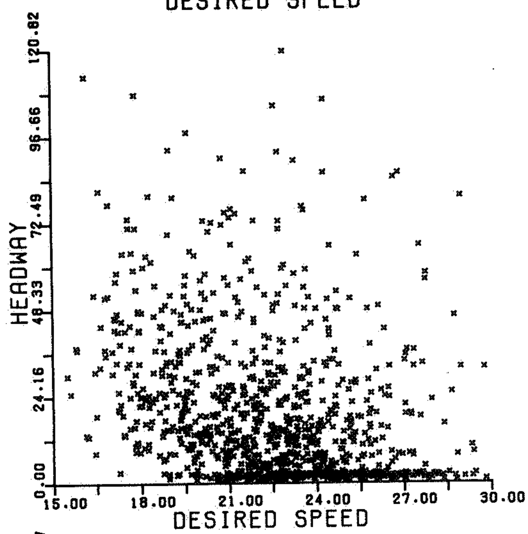


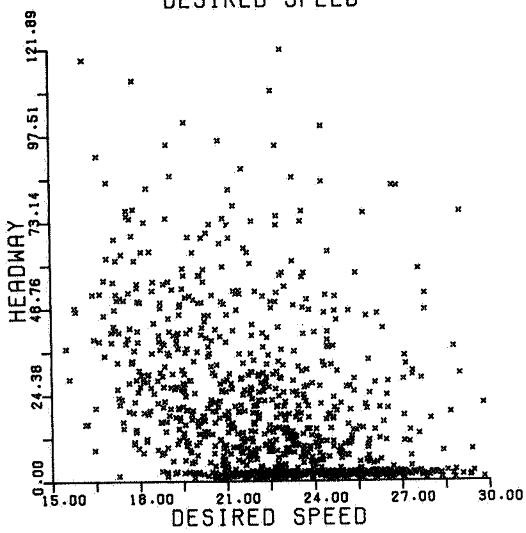


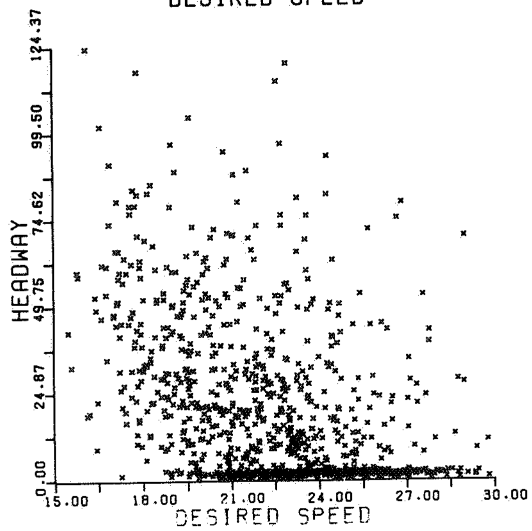
$$
\begin{aligned}
\rho &= 0.05\\
v_{\min} &= 15.0\\
v_{\max} &= 30.0\\
\alpha_v &= 3.0\\
\beta_v &= 3.0\\
s_{\min} &= 0.0\\
s_{\max} &= 3.0\\
\alpha_s &= 3.0\\
\beta_s &= 1.5
\end{aligned}
$$



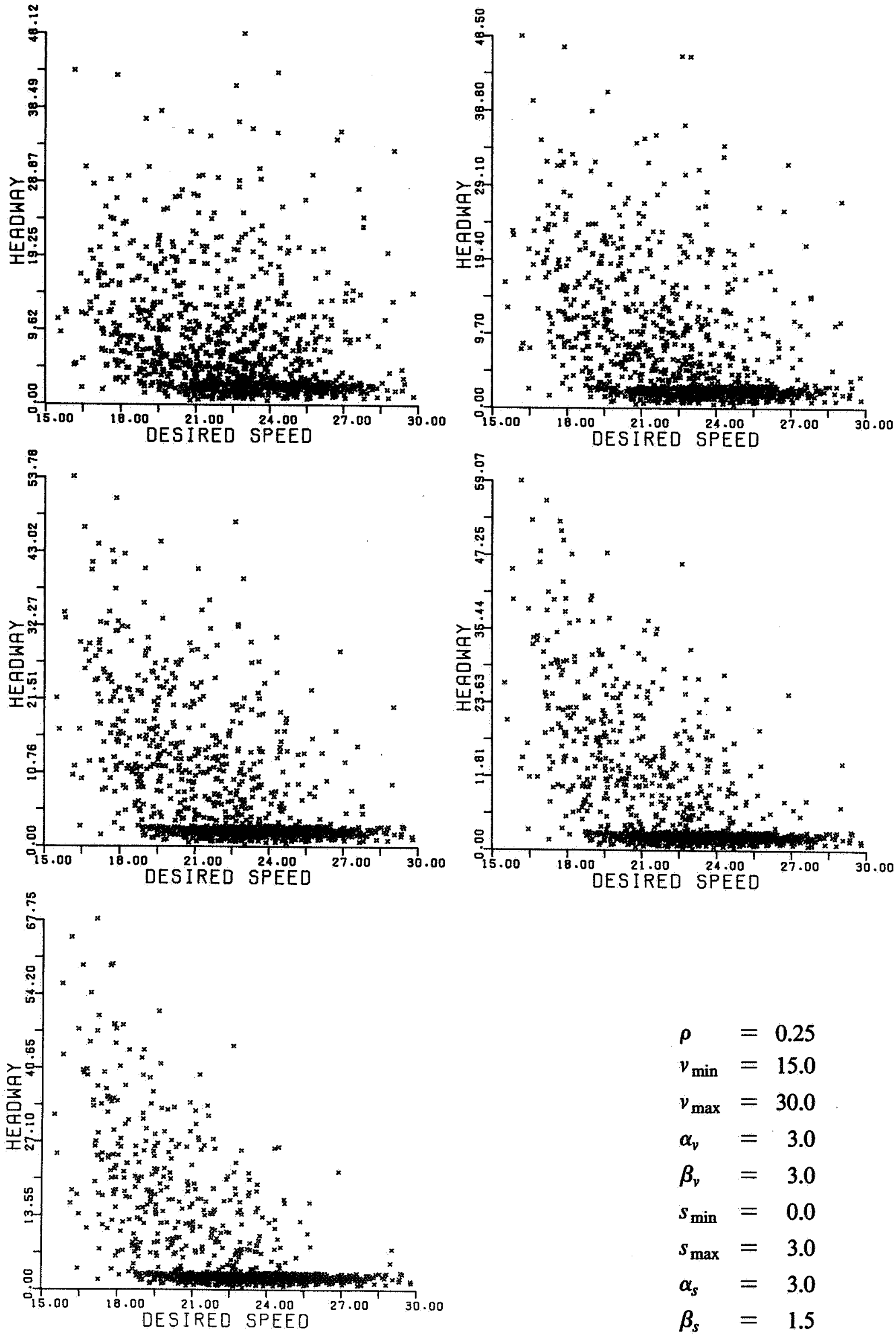




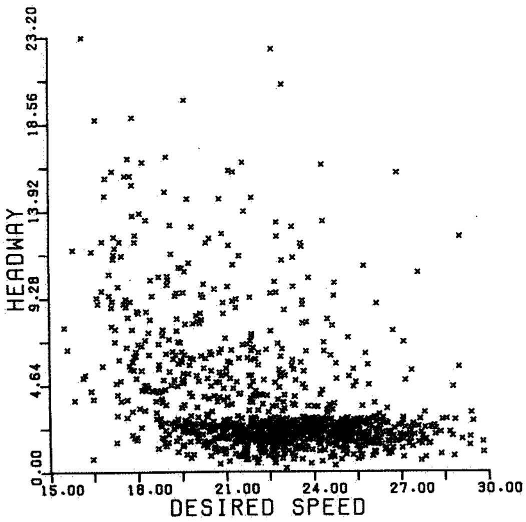


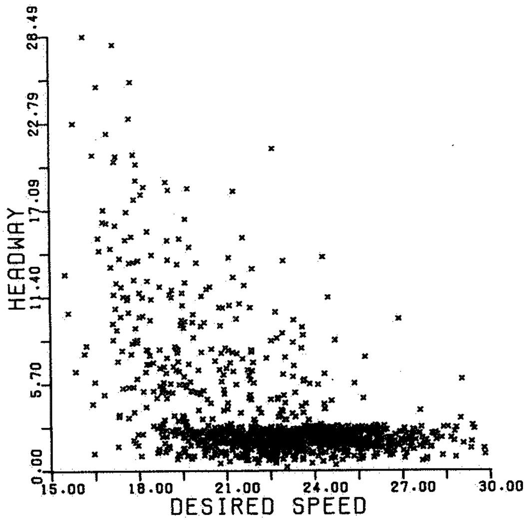


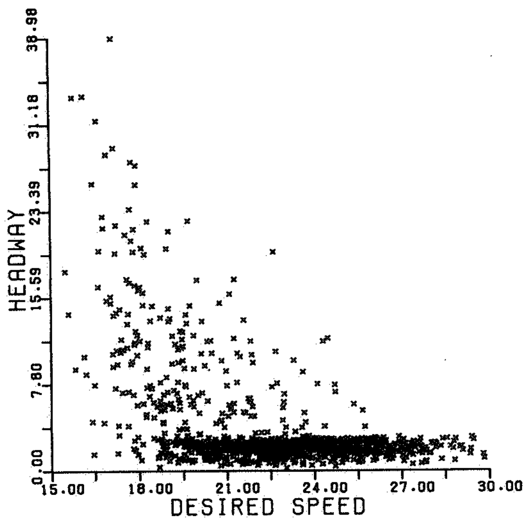
$$
\begin{aligned}
\rho &= 0.10\\
v_{\min} &= 15.0\\
v_{\max} &= 30.0\\
\alpha_v &= 3.0\\
\beta_v &= 3.0\\
s_{\min} &= 0.0\\
s_{\max} &= 3.0\\
\alpha_s &= 3.0\\
\beta_s &= 1.5
\end{aligned}
$$

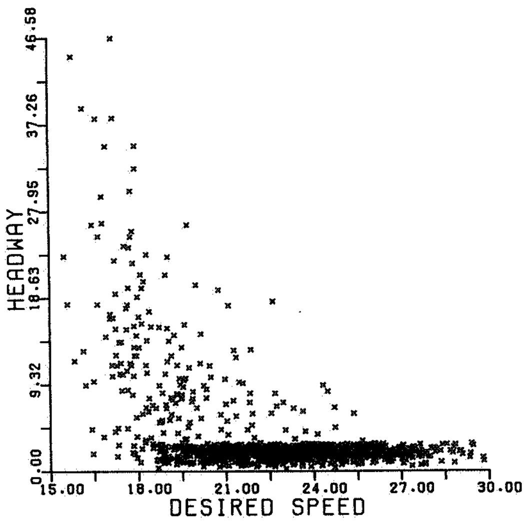


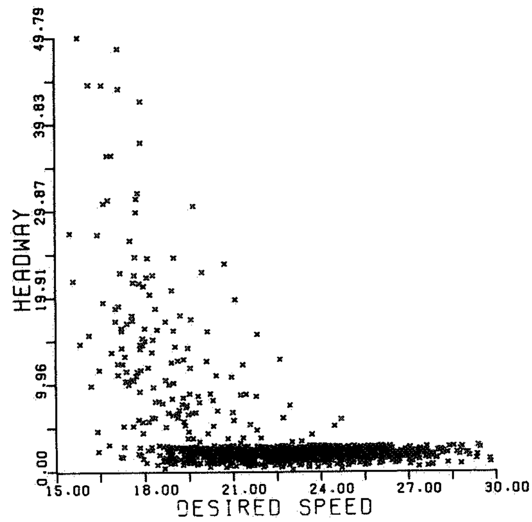
$$
\begin{aligned}
\rho &= 0.25 \\
v_{\min} &= 15.0 \\
v_{\max} &= 30.0 \\
\alpha_v &= 3.0 \\
\beta_v &= 3.0 \\
s_{\min} &= 0.0 \\
s_{\max} &= 3.0 \\
\alpha_s &= 3.0 \\
\beta_s &= 1.5
\end{aligned}
$$











$\rho = 0.50$

$v_{min} = 15.0$

$v_{max} = 30.0$

$\alpha_v = 3.0$

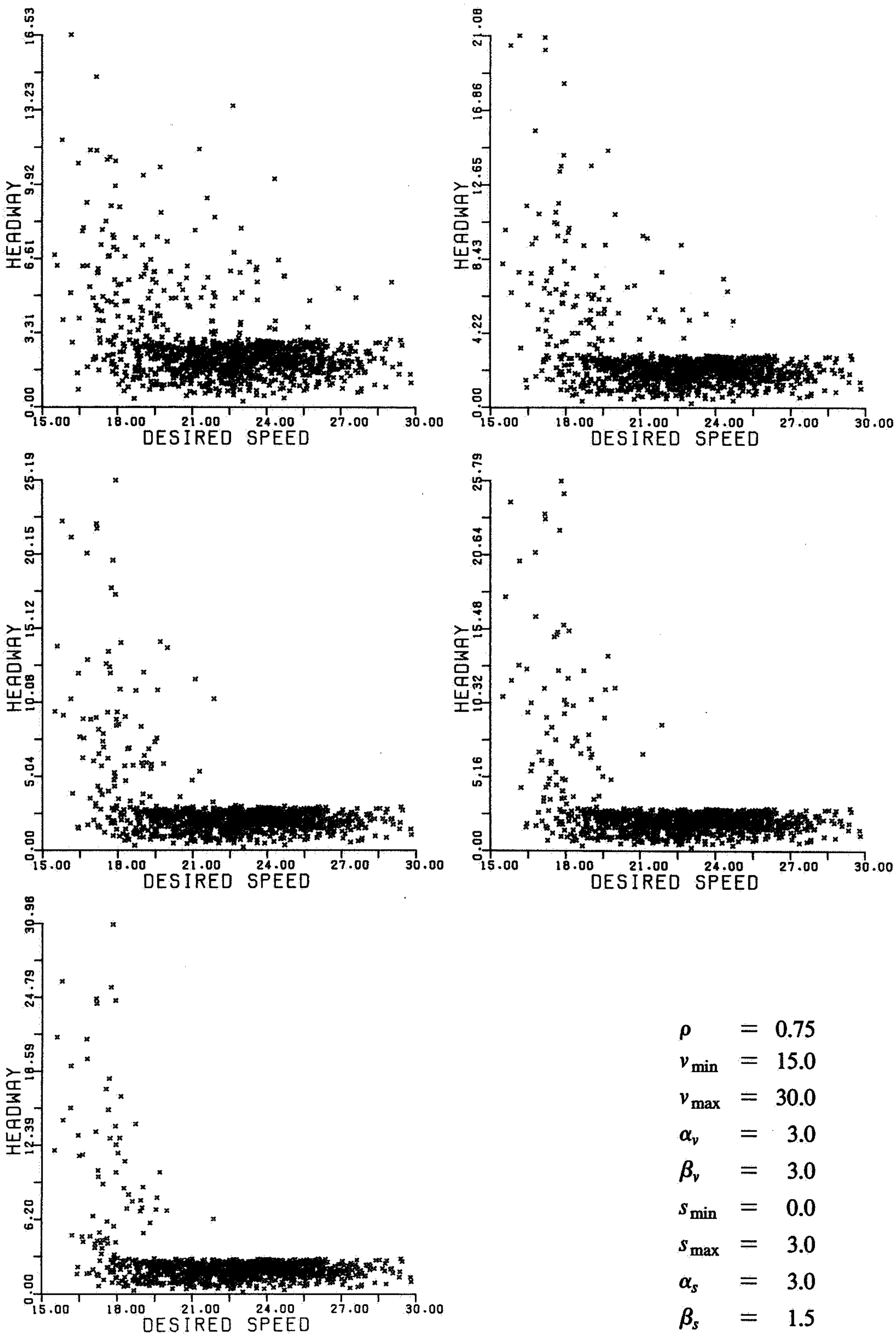
$\beta_v = 3.0$

$s_{min} = 0.0$

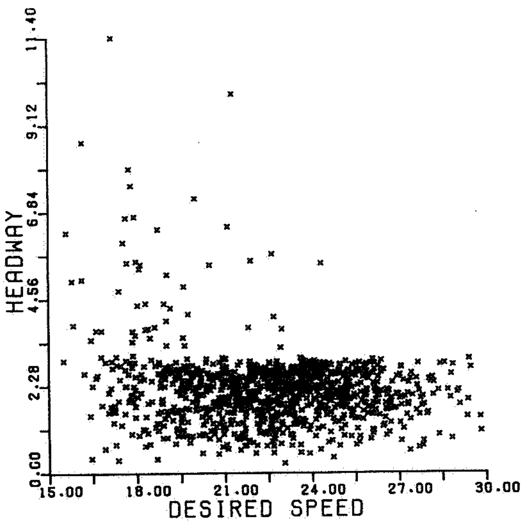
$s_{max} = 3.0$

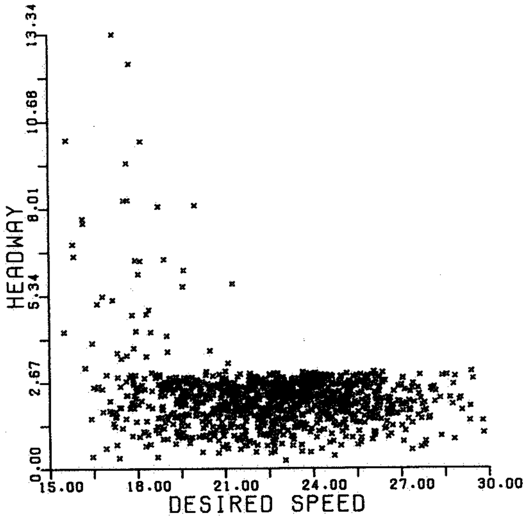
$\alpha_s = 3.0$

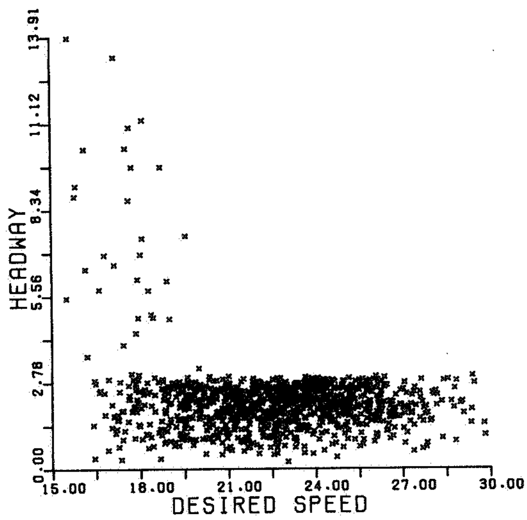
$\beta_s = 1.5$

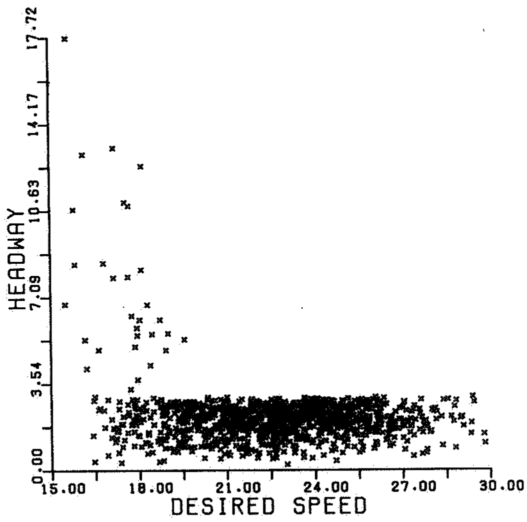


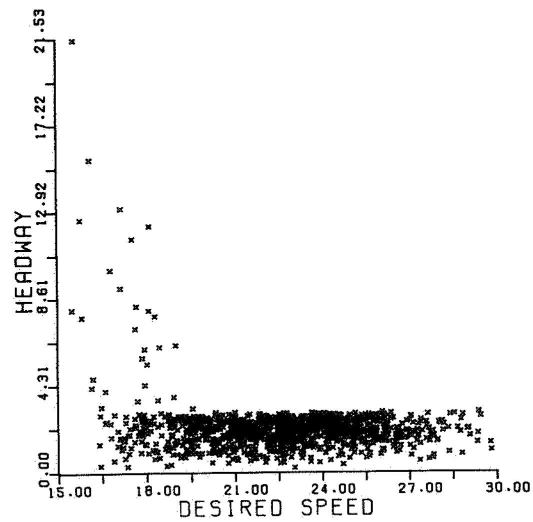
$$
\begin{aligned}
\rho &= 0.75\\
v_{\text{min}} &= 15.0\\
v_{\text{max}} &= 30.0\\
\alpha_v &= 3.0\\
\beta_v &= 3.0\\
s_{\text{min}} &= 0.0\\
s_{\text{max}} &= 3.0\\
\alpha_s &= 3.0\\
\beta_s &= 1.5
\end{aligned}
$$



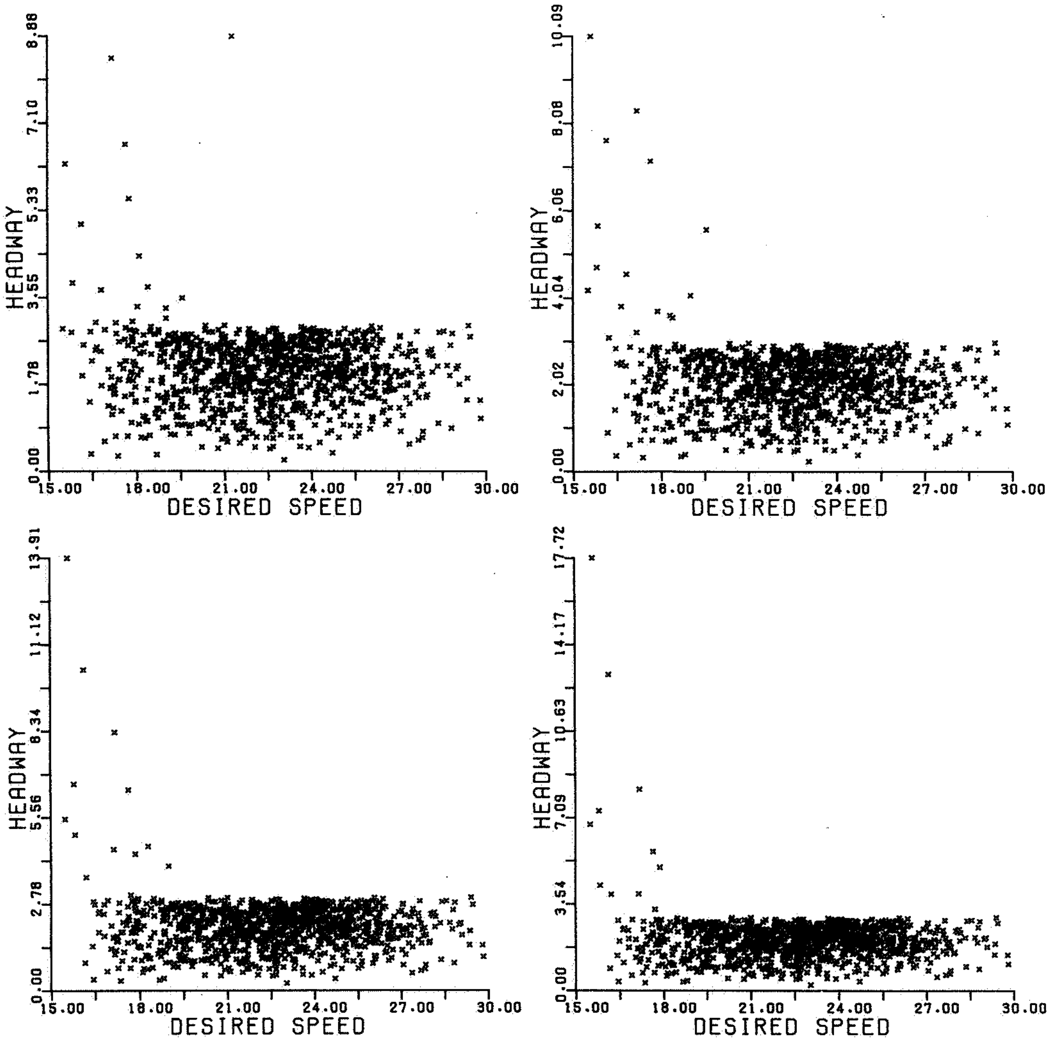


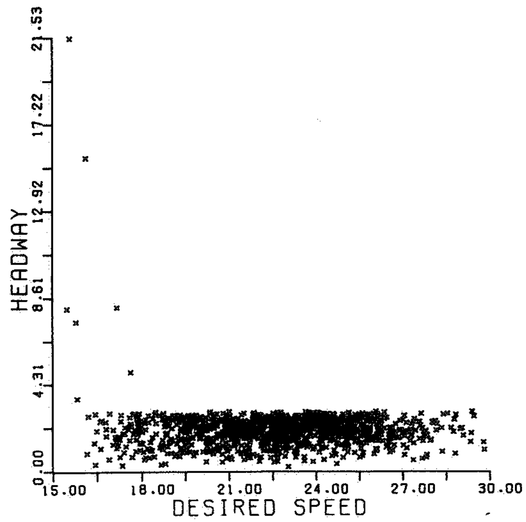






ρ	=	0.90
$v_{\min}$	=	15.0
$v_{\max}$	=	30.0
α_v	=	3.0
β_v	=	3.0
$s_{\min}$	=	0.0
$s_{\max}$	=	3.0
α_s	=	3.0
β_s	=	1.5



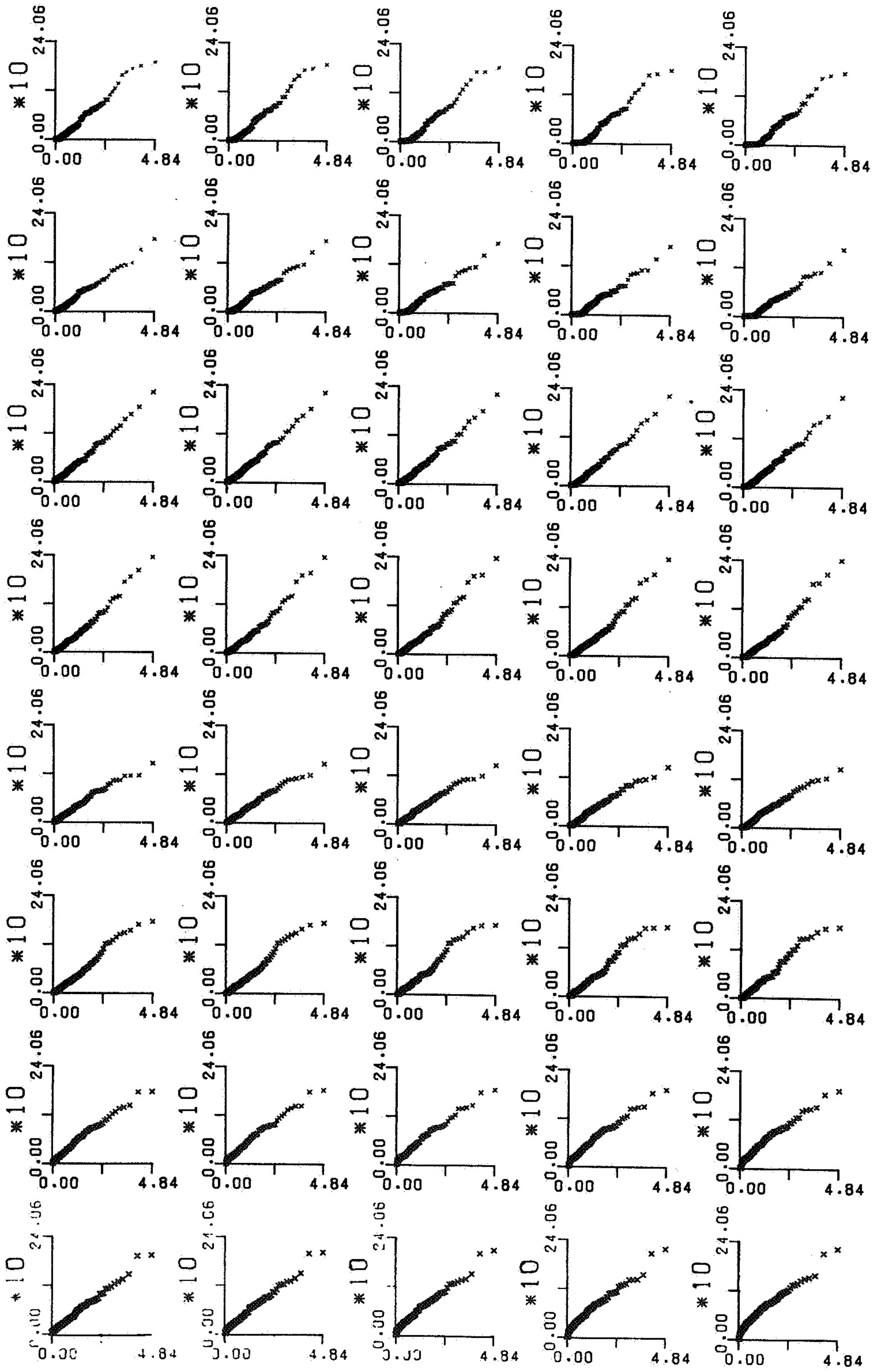


$$\rho = 0.95$$
$$v_{\min} = 15.0$$
$$v_{\max} = 30.0$$
$$\alpha_v = 3.0$$
$$\beta_v = 3.0$$
$$s_{\min} = 0.0$$
$$s_{\max} = 3.0$$
$$\alpha_s = 3.0$$
$$\beta_s = 1.5$$

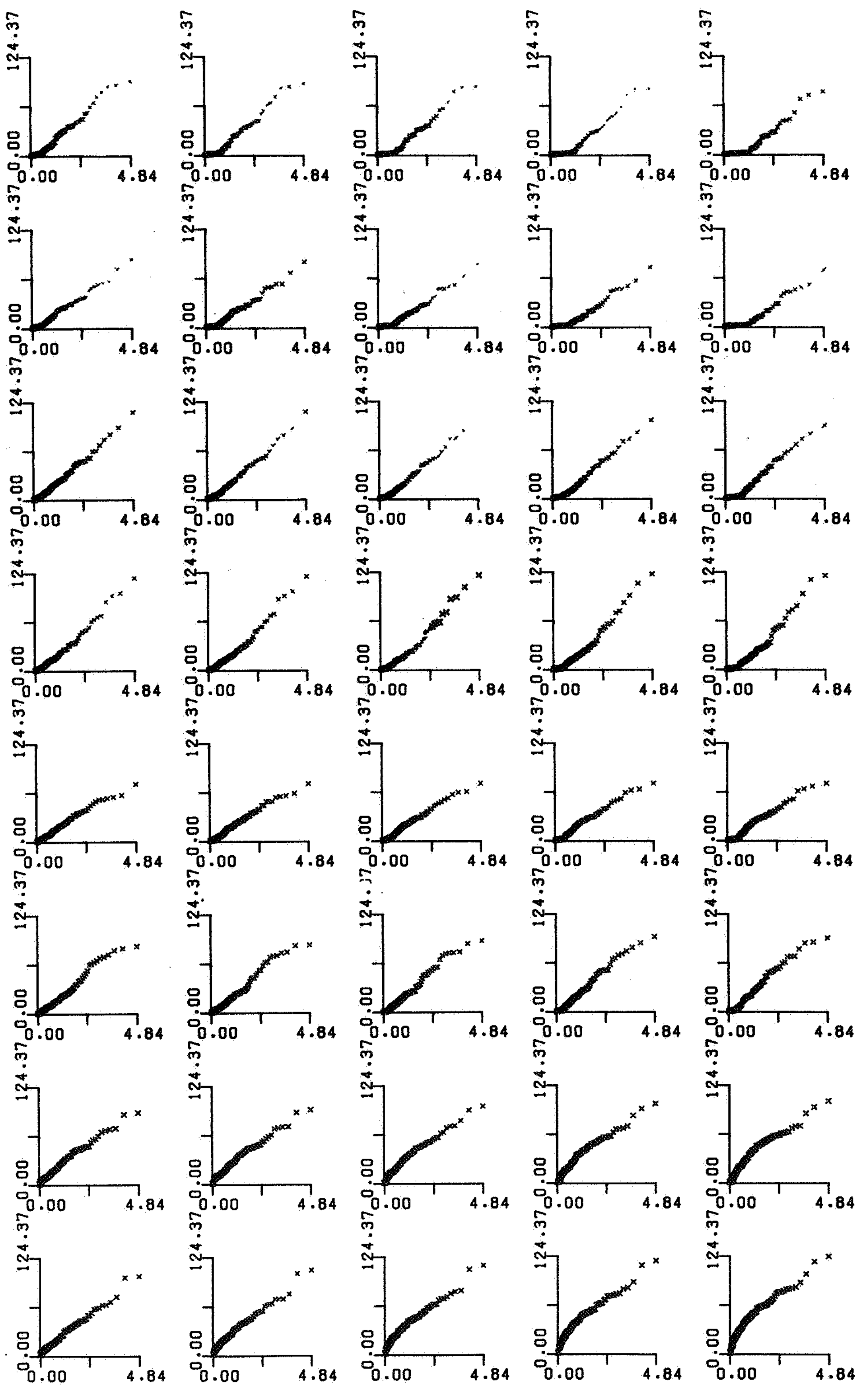
Appendix 4

Q-Q Plots of Headways

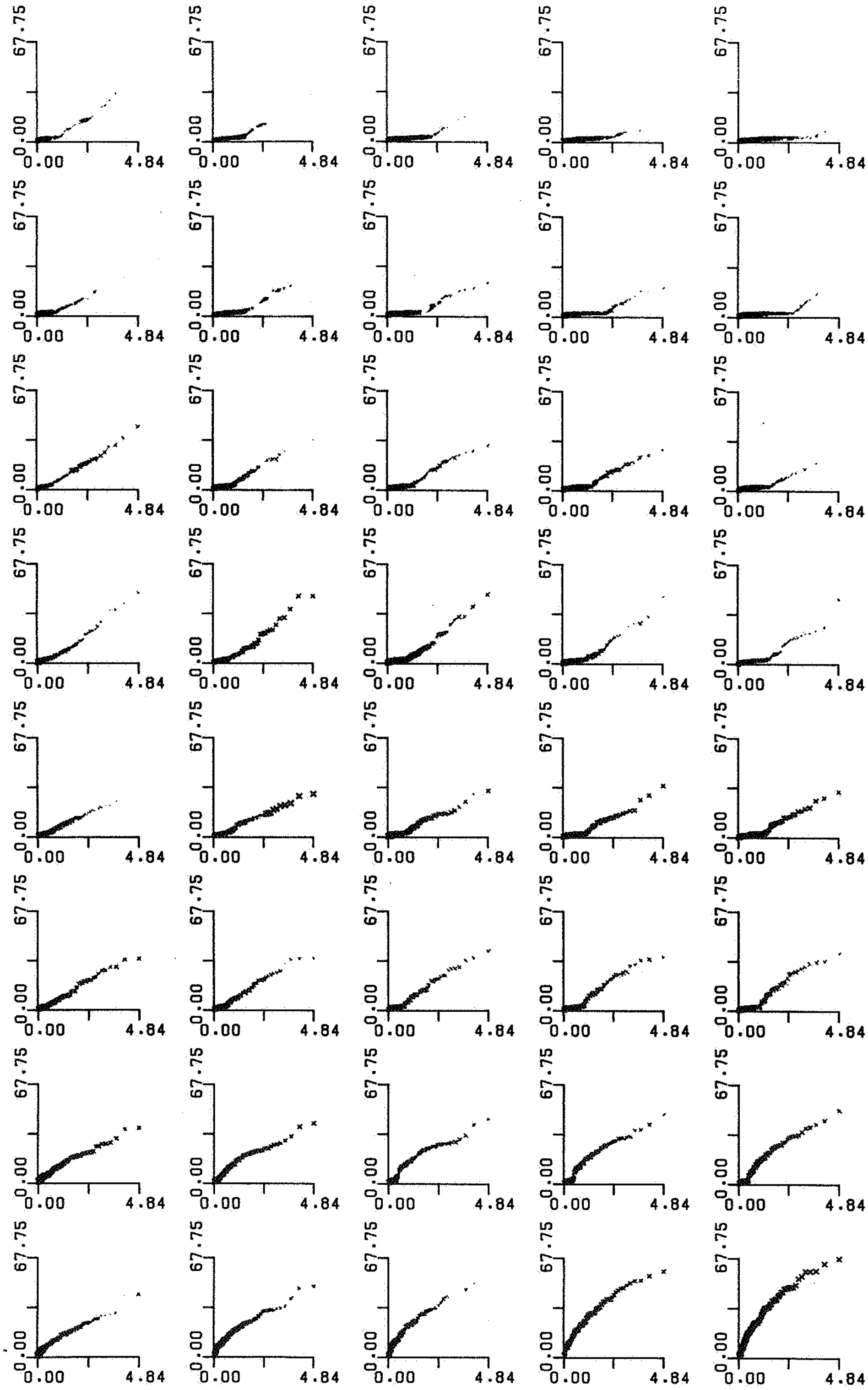
For each of the 8 desired speed classes of 125 cars, Q-Q plots are given for the points at respectively 500m, 1000m, 1500m, 2000m, and 2500m from the bottleneck. The first row of plots is concerned with the highest desired speed class, the last row with the lowest desired speed class. In section 5 the Q-Q plots are explained.



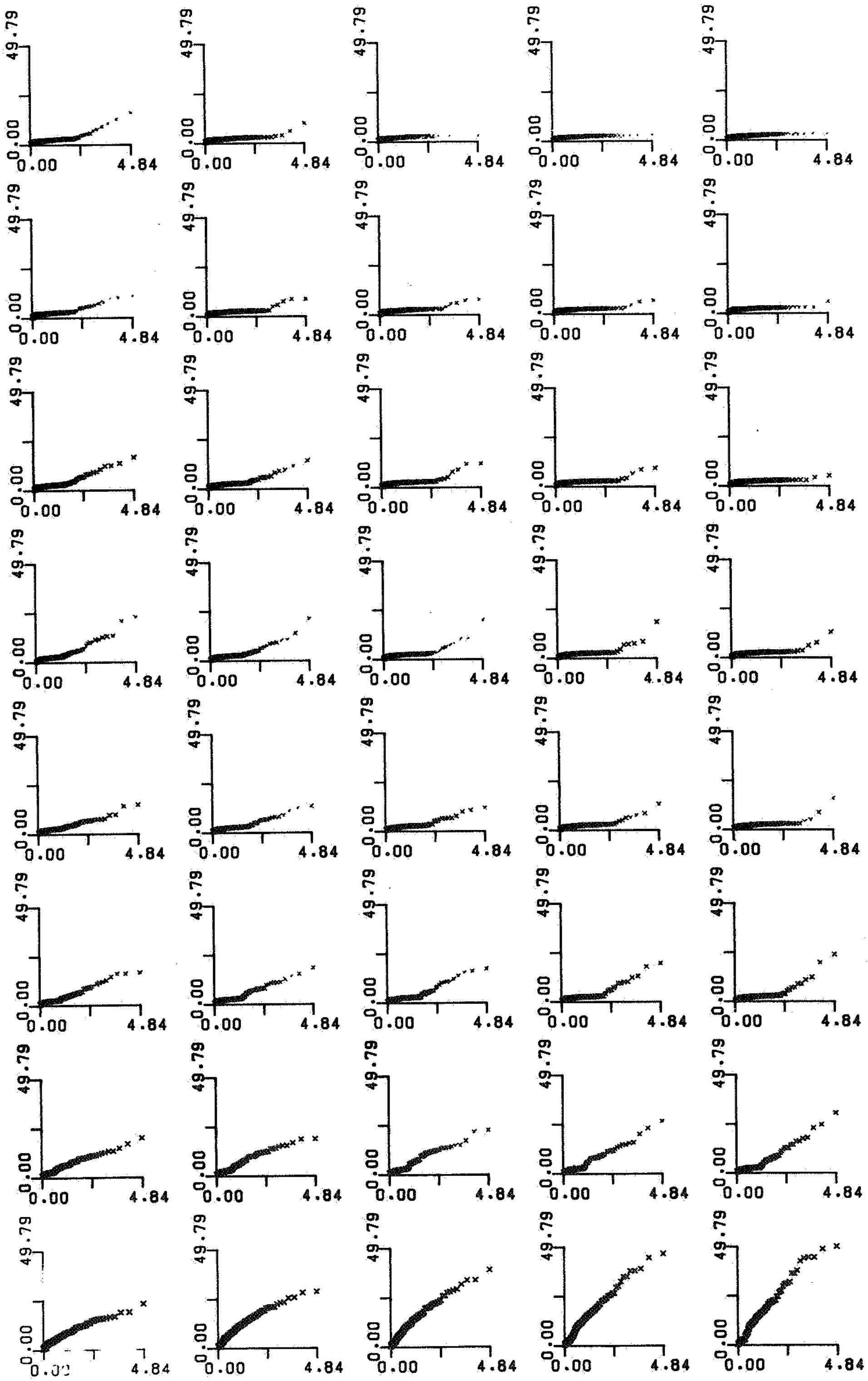
ρ	=	0.05	α_v	=	3.0	$s_{\max}$	=	3.0
$v_{\min}$	=	15.0	β_v	=	3.0	α_s	=	3.0
$v_{\max}$	=	30.0	$s_{\min}$	=	0.0	β_s	=	1.5



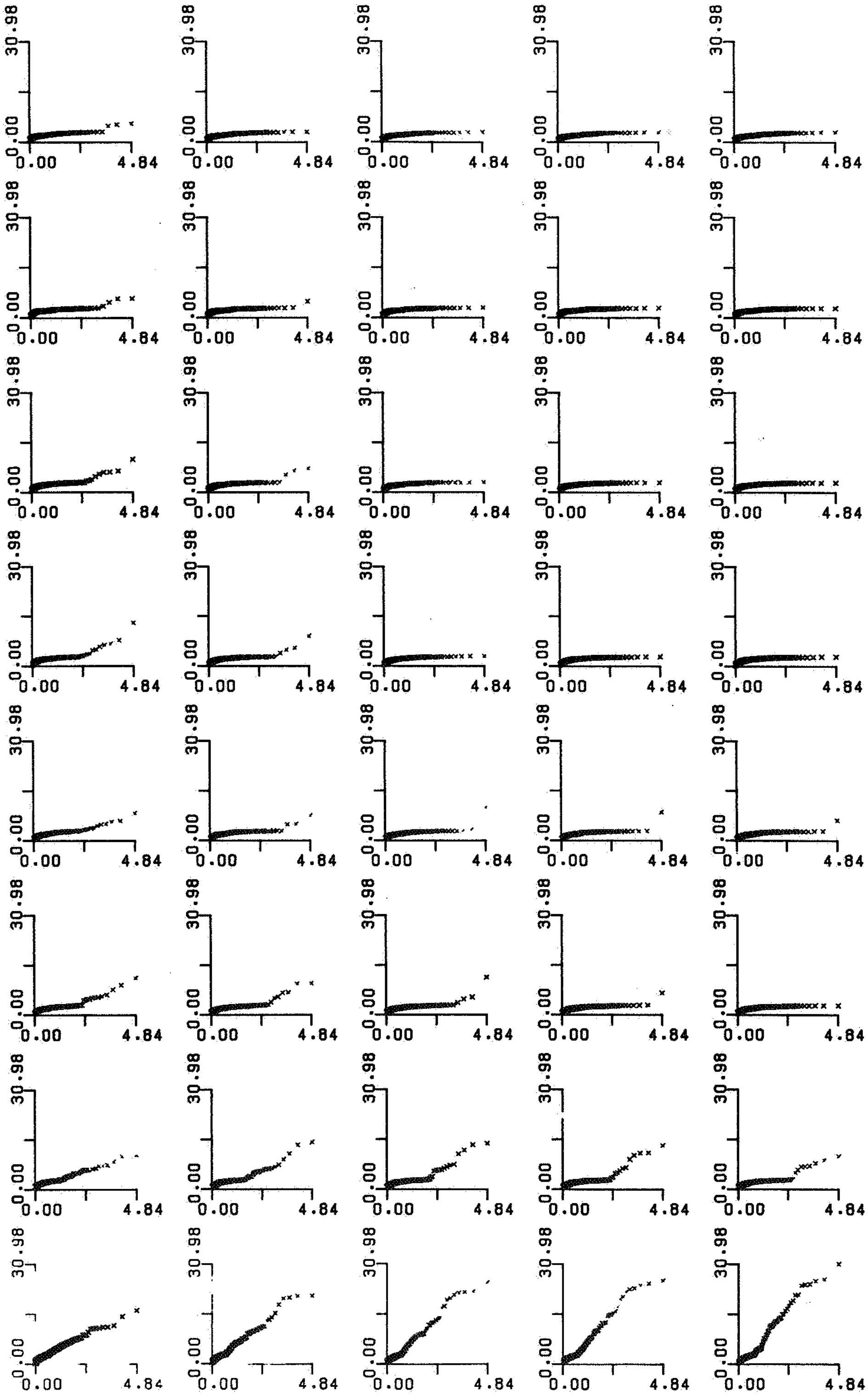
ρ	=	0.10	α_v	=	3.0	$s_{\max}$	=	3.0
$v_{\min}$	=	15.0	β_v	=	3.0	α_s	=	3.0
$v_{\max}$	=	30.0	$s_{\min}$	=	0.0	β_s	=	1.5



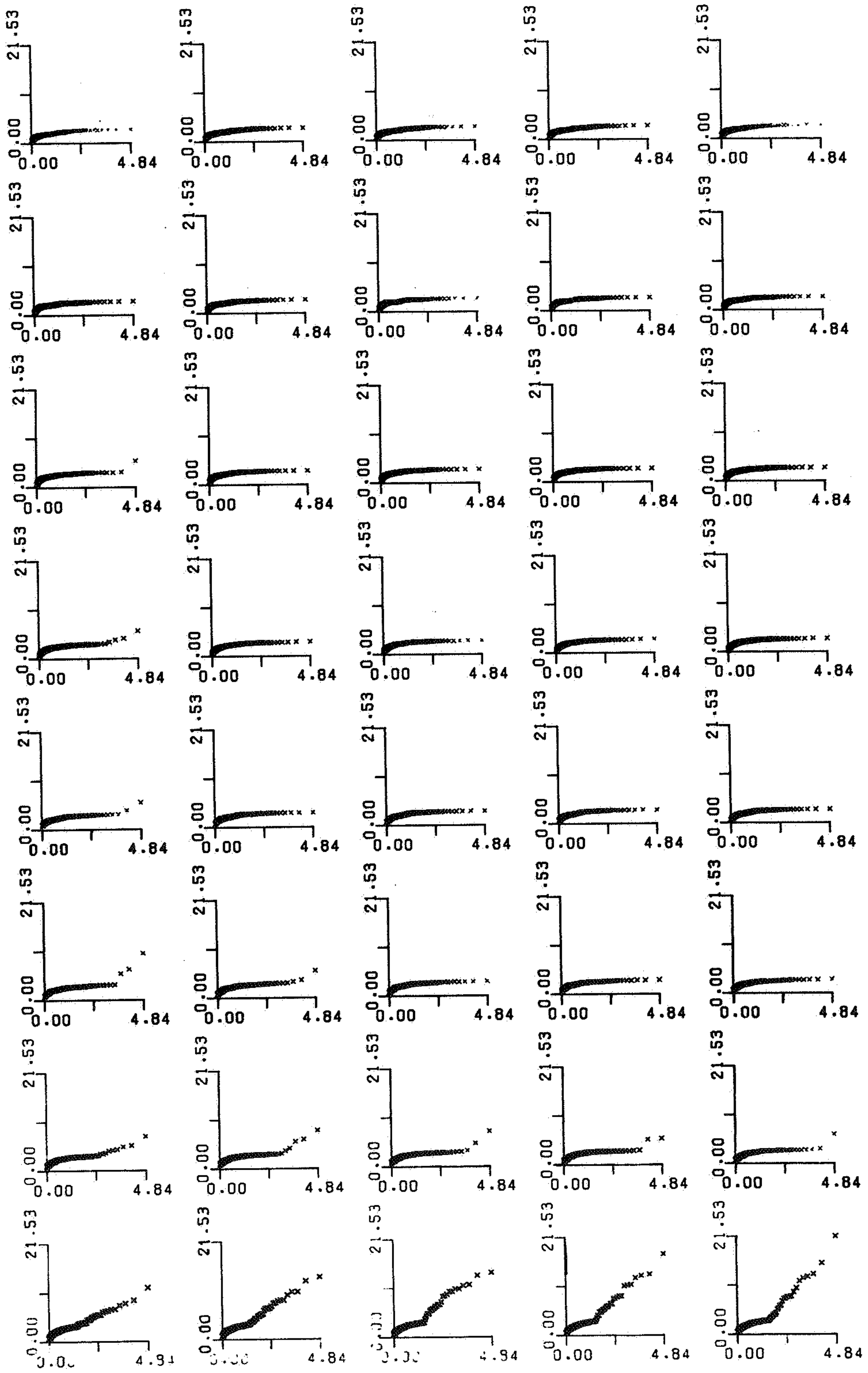
$\rho = 0.25$	$\alpha_v = 3.0$	$s_{\max} = 3.0$
$v_{\min} = 15.0$	$\beta_v = 3.0$	$\alpha_s = 3.0$
$v_{\max} = 30.0$	$s_{\min} = 0.0$	$\beta_s = 1.5$



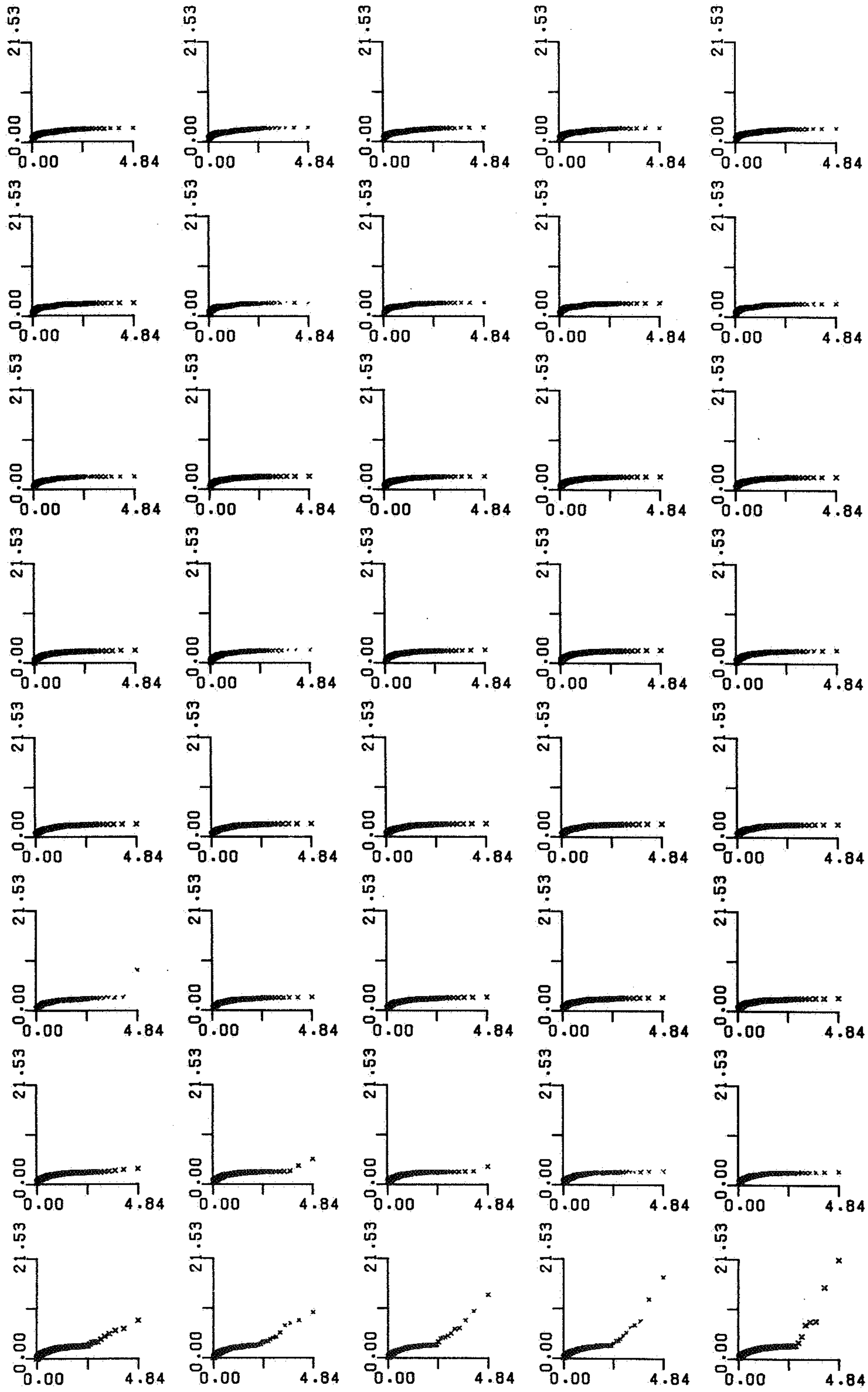
ρ	=	0.50	α_v	= 3.0	$s_{\max}$	=	3.0
$v_{\min}$	=	15.0	β_v	= 3.0	α_s	=	3.0
$v_{\max}$	=	30.0	$s_{\min}$	= 0.0	β_s	=	1.5



$\rho = 0.75$	$\alpha_v = 3.0$	$s_{\max} = 3.0$
$v_{\min} = 15.0$	$\beta_v = 3.0$	$\alpha_s = 3.0$
$v_{\max} = 30.0$	$s_{\min} = 0.0$	$\beta_s = 1.5$



ρ	= 0.90		α_v	= 3.0		$s_{\max}$	= 3.0
$v_{\min}$	= 15.0		β_v	= 3.0		α_s	= 3.0
$v_{\max}$	= 30.0		$s_{\min}$	= 0.0		β_s	= 1.5



ρ	=	0.95	α_v	=	3.0	$s_{\max}$	=	3.0
$v_{\min}$	=	15.0	β_v	=	3.0	α_s	=	3.0
$v_{\max}$	=	30.0	$s_{\min}$	=	0.0	β_s	=	1.5

Appendix 5

List of the Most Important Symbols

Symbol	Meaning
ρ	Fraction followers at the bottleneck
λ	Traffic intensity before the bottleneck
$A_n^{(r)}$	Real arrival time of car n at point r
$D_n^{(r)}$	Desired arrival time of car n at point r
S_n	Minimum headway of car n
T_n	Random variable distributed exp(λ)
V_n	Desired speed of car n
W_n	Delay imposed on car n by the bottleneck
X_n	Position along the real axis of car n at time 0
$X_n^{(t)}$	Position along the real axis of car n at time t
$Y_n^{(r)}$	Headway of car n at point r
$\tilde{Y}_n^{(r)}$	Approximation to $Y_n^{(r)}$
$Z_n^{(r)}$	Actual journey time of car n at point r
G	Distribution of S_n
H	Distribution of W_n
K	Distribution of V_n
$\Phi^{(r)}$	Distribution of r / V_n
$\Omega^{(r)}$	Distribution of $Z_n^{(r)}$

Contents