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# On LAN for Counting Processes

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Shiryayev's result concerning LAN binary experiments is discussed, with applications to counting processes. It is shown, in particular, that the convergence in probability of the Hellinger process to a continuous nondecreasing function ensures the LAN property of the likelihood ratio process, provided that a related Lindeberg-type condition also holds.

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## 1. Introduction : LAN binary experiments

*1.1.* In this paper we review the methods for establishing the local asymptotically normal ( LAN ) property of the likelihood ratio process associated with a pair of counting processes.

We begin, however, with the general setting of the problem in terms of a sequence of binary experiments, and discuss its elegant solution due to Shiryayev (1985); see Subsection 1.2. I learned of it from Shiryayev during his visit to the Centre for Mathematics and Computer Science, Amsterdam, Spring 1985. I owe thanks to him for making available preprints of his research concerning this subject.

In Section 2 we restrict our attention to the special case of binary experiments generated by marked point processes (for a justification of such a restriction see Subsection 1.3), and discuss an alternative way of establishing LAN, slightly different to that of Shiryayev (1985) mentioned above. Such a modification is motivated exclusively by the convenience of its further application[1] - in deriving Corollary 4.2, for instance.

It should be noted that in most applications encountered in practice observations come from a finite dimensional counting processes, so that the associated mark space consists of finite number of points. Therefore Sections 3 and 4 are devoted to this special case : in Section 3 the likelihood ratio process is characterized, and in Section 4 the LAN property is established ( Theorem 4.1 and Corollary 4.2 ) by the alternative way discussed in Section 2. In fact, the results are easily extended to the case of a general mark space ( cf. Subsection 2.2 ).

*1.2.* For each n let $\{\Omega^n , \mathcal{F}^n , P^n\}$ be a stochastic basis — a complete probability space equipped with a nondecreasing family $\{\mathcal{F}_t^n , t \geq 0\}$ of right-continuous sub $\sigma$-algebras of $\mathcal{F}^n$ augmented by sets from $\mathcal{F}^n$ of zero probability. For the sake of simplicity, we discuss only the case in which $t \in [0 , 1]$. Let $P^n$ be another probability measure defined on $(\Omega^n , \mathcal{F}^n)$.

In this subsection we discuss shortly the general result due to Shiryayev (1985) concerning the LAN property of a sequence of binary experiments

1. Here we have in mind statistical application in the spirit of, e.g., Gill (1979) or Dzhaparidze (1985). Note that the Sections 3 and 4 below represent a revised version of the corresponding sections of the latter paper.

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$$(\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n_t, 0 \leq t \leq 1\}, P^n, \underline{P}^n), \quad n = 1, 2, \ldots \tag{1.1}$$

Suppose that $\underline{P}^n$ is locally absolutely continuous with respect to $P^n$, $\underline{P}^n \overset{loc}{\ll} P^n$, in the sense that

$$\underline{P}^n_t \ll P^n_t, \quad 0 \leq t \leq 1 \tag{1.2}$$

with $P^n_t = P^n|\mathcal{F}^n_t$ and $\underline{P}^n_t = \underline{P}^n|\mathcal{F}^n_t$, the restrictions of $P^n$ and $\underline{P}^n$ to the σ-algebra $\mathcal{F}^n_t$. Suppose for simplicity that $P^n_0 = \underline{P}^n_0$.

Then, according to Kabanov et al. (1979), there exists a unique (to within $\underline{P}^n$ and $P^n$-indistinguishability) cádlág process $(z^n_t, 0 \leq t \leq 1)$ such that, provided $z_t > 0$,

$$z^n_t \equiv \frac{dP^n_t}{dP^n_t} = \mathcal{E}(m^n)_t \equiv \exp\left\{m^n_t - \frac{1}{2}<m^{nc}>_t + \sum_{s \leq t} \Phi_1(\Delta m^n_s)\right\} \tag{1.3}$$

with $\Phi_1(x) = \ln(1+x)-x$, where $m^n$ is a local martingale with jumps

$$\Delta m^n_t \equiv m^n_t - m^n_{t-} > -1 . \tag{1.4}$$

As usual, $m^n = m^{nc} + m^{nd}$ is the decomposition of $m^n$ into continuous and purely discontinuous components respectively, and $<m^{nc}>$ is the compensator of $(m^{nc})^2$.

Let $\mu^n = \mu^{m^n}$ be the integer valued random measure associated with the jumps of $m^n$ as follows

$$\mu^n((0,t],\Gamma) = \sum_{s \leq t} I(\Delta m^n \in \Gamma), \Gamma \in \mathcal{B}(R_0), R_0 = R \setminus \{0\} \tag{1.5}$$

Let $\nu^n = \nu^{m^n}$ be its compensator ( Kabanov et al. (1979) ).

Introduce the Hellinger process $\mathcal{H}^n$ for the measures $P^n$ and $\underline{P}^n$ :

$$\mathcal{H}^n_t \equiv \mathcal{H}^{m^n}_t \equiv \mathcal{H}_t(P^n, \underline{P}^n) \tag{1.6}$$
$$= \frac{1}{4}<m^{nc}>_t + \int_0^t \int_{R_0} (1-\sqrt{1+x})^2 d\nu^n$$

with $d\nu^n = \nu^n(ds, dx)$ (similarly, $d\mu^n$ will stand for $\mu^n(ds, dx)$). Note that by (1.5) the second term on the right-hand side of (1.6) is the compensator of the process

$$\eta^n_t = \int_0^t \int_{R_0} (1-\sqrt{1+x})^2 d\mu^n = \sum_{s \leq t} (1-\sqrt{1+\Delta m^n_s})^2 \tag{1.7}$$

Finally, introduce the limiting object - a continuous Gaussian martingale $W$ the quadratic variation of which is a continuous nondicreasing deterministic function $<W>$. Observe that the Hellinger process associated with $W$ is defined as

$$\mathcal{H}^W \equiv \mathcal{H}(P, \underline{P}) = \frac{1}{4}<W> \tag{1.8}$$

where $P$ and $\underline{P}$ are the "limiting" measures such that

$$\frac{dP}{d\underline{P}} = \mathcal{E}(W) = \exp\left\{W - \frac{1}{2}<W>\right\}; \tag{1.9}$$

see Gill and Johansen (1986).

The next theorem is due to Shiryayev (1985).

Theorem 1.1. *Let the following Conditions S hold for each t,* $0 \leq t \leq 1$ *:*

*S.I. For each* $\epsilon > 0$

$$\tilde{\eta}^n_{(>\epsilon)t} \equiv \int_0^t \int_{|x|>\epsilon} (1-\sqrt{1+x})^2 d\nu^n \to 0$$

*in $P^n$ probability as $n \to \infty$*

*S.II.* $\mathcal{K}^n \to \mathcal{K}^W$ *in $P^n$ probability as $n \to \infty$ ; see (1.6) and (1.8).*

*Then, for each t, $0 \leq t \leq 1$ the following weak convergence in* $\mathcal{D}([0,1])$ *takes place with respect to $P^n$ and $\underline{P}^n$ (in the sense of* GREENWOOD *and* SHIRYAYEV *(1985), § 2.2 ):*

(i)

$$z^n = \mathcal{E}(m^n) \xrightarrow{d(P^n)} \mathcal{E}(W) = \exp\left\{W - \frac{1}{2}<W>\right\};$$

*see* (1.2) *and* (1.8).

(ii)

$$z^n \xrightarrow{d(\underline{P}^n)} \exp\left\{W + \frac{1}{2}<W>\right\}$$

REMARK 1.1. Assertion (ii) is deduced from Assertion (i) by arguments in GREENWOOD and SHIRYAYEV (1985) proving Statement 3 of THEOREM 8 on p. 99.

REMARK 1.2. As for Assertion (i), for its proof Shiryayev (1985) utilizes the following representation of the likelihood ratio process :

$$z = \exp\left\{M^n_{(\leq\epsilon)} - 2\mathcal{K}^W + \sum_{i=1}^{3} R^{n,i}\right\} \tag{1.10}$$

where

$$M^n_{(\leq\epsilon)t} = \int_0^t \int_{|x|\leq\epsilon} (x - 2(1 - \sqrt{1+x})^2) d(\mu^n - \nu^n) + m_t^{nc}, \tag{1.11}$$

while

$$R_t^{n,1} \equiv R^{n,1}_{(>\epsilon)t} = \int_0^t \int_{|x|>\epsilon} (x - 2(1 - \sqrt{1+x})^2) d(\mu^n - \nu^n), \tag{1.12}$$

$$R^{n,2} = 2(\mathcal{K}^W - \mathcal{K}^n) \tag{1.13}$$

and

$$R^{n,3} = \frac{1}{2}\int_0^t \int_{R_0} \Phi_2(\sqrt{1+x} - 1) d\mu^n \tag{1.14}$$

$$= \frac{1}{2}\sum_{s\leq t} \Phi_2(\sqrt{1 + \Delta m_s^n} - 1)$$

with $\Phi_2(x) = \ln(1 + x) - x + \frac{1}{2}x^2$.

The representation (1.10) is easily derived from (1.3) by taking into consideration (1.6), (1.7), (1.11) and (1.12) which yield

$$M^n_{(\leq\epsilon)} + R^{n,1}_{(>\epsilon)} = m^n - 2(\eta^n - \mathcal{K}^n + \frac{1}{4}<m^{nc}>).$$

The proof of THEOREM 1.1 is the direct consequence of the following two assertions :

1.*A*. The process (1.11) is a local square integrable martingale such that

$$M^n_{(\leqslant\epsilon)} \overset{d(P^n)}{\rightarrow} W$$

1.*B*. The processes (1.12) - (1.14) are asymptotically negligible - for each $t$, $0\leqslant t\leqslant 1$,

$$\sup_{s\leqslant t}|R_s^{n,i}|\rightarrow 0\ ,\ i\ =\ 1,2,3 \qquad (1.15)$$

in $P^n$ probability as $n\rightarrow\infty$.

We do not discuss the proof of Assertion 1.A here. We remark only that it is a consequence of the functional CLT in Liptser and Shiryayev (1980).

As for Assertion 1.B for $i = 2$, it follows directly[1] from Condition *S.II*.

Next, for each $\epsilon$, $0<\epsilon$ a constant $C>0$ can be chosen such that $|x-2(1-\sqrt{1+x})|^2\leqslant C|x|$ whenever $|x|>\epsilon$. Using this one can easily dominate the left-hand side of (1.15) for $i = 1$ by a quantity which under Condition *S.I* tends to zero in $P^n$ probability as $n\rightarrow\infty$. So one gets (1.15) for $i = 1$. Finally, by analoguous arguments one can dominate also

$$\sup_{s\leqslant t}|\int_0^s\int_{|x|>\epsilon}\Phi_2(\sqrt{1+x}-1)d\mu^n|$$

by a quantity that vanishes in $P^n$ probability as $n\rightarrow\infty$. Consequently, to prove Assertion 2.B for $i = 3$, one has to verify that

$$\sup_{s\leqslant t}|\int_0^s\int_{|x|\leqslant\epsilon}\Phi_2(\sqrt{1+x}-1)d\mu^n|\leqslant C\epsilon\int_0^t\int_{|x|\leqslant\epsilon}x^2d\mu^n \qquad (1.16)$$

(here the constant $C$ arises by taking into consideration that for sufficiently small values of $x$ $\Phi_2(x) = O(x^3)$), and also that under the Conditions $S$ the right-hand side of the inequality (1.16) can be made arbitrarily small. So, one arrives at (1.15) for $i = 3$.

*1.3.* It should be noted that the presence of the continuous component $m^{nc}$ do not affect the course of the proof just outlined much : it enters only in the expression (1.11), and all that is required of it is the convergence of its quadratic characteristic so as to meet Assertion 1.*A*. Because of its minor significance, and because of its absence in the special case of our prior interest, this component will be neglected in the remainder of this paper.

## 2. Applications to marked processes

*2.1.* In this subsection we continue with the brief discussion of the alternative approach to the proof of Theorem 1.1. In doing this we adopt the setting of § 12 in Kabanov et al. (1980) in terms of marked point processes.

This approach is fully realized in Section 4 where Theorem 4.2 is proved, which establishes LAN for counting processes - processes with finite dementional mark spaces. The last restriction is not essential - it is motivated exclusively by practical considerations, as the results of Section 4 are aimed at applications to statistical inference about counting processes (cf. the footnote[1)] on p. 1). A reader interested in the extention of the precise results of Section 4 to arbitrary marked processes might consult the next subsection.

Consider again a sequence (1.1) of binary experiments. For each $n$ let the entries in (1.1) be as in Kabanov et al. (1980), § 12.

1. In fact, by Lemma 1 of Mc Leish (1979), p. 146 from Condition *S.II* and the continuity of $\mathcal{K}^W$ follows $\sup_{s\leqslant t}|\mathcal{K}_s^n-\mathcal{K}_s^W|\rightarrow 0$ and, in particular $\sup_{s\leqslant t}|\Delta\mathcal{K}_s^n|\rightarrow 0$ in $P^n$ probability as $n\rightarrow\infty$ ; cf. Greenwood and Shiryayev (1985), p. 105.

Consider a marked process

$$X^n = \{(T_j^n, X_j^n), P^n\} \tag{2.1}$$

where $T_j^n$ are stopping times, $T_1^n > 0$ and $T_{j+1}^n > T_j^n$, while $X_j^n$ are $\mathcal{F}_{T_j}^n$-measurable random elements taking values from a mark space, $(E^n, \mathcal{E}^n)$; see KABANOV et al. (1980), § 12.

To the process (2.1) associate the integer valued random measure $p^n = p^{X^n}$ on $(0, 1] \times E$ with

$$p^n((0, t], \Gamma) = \sum_{j \geqslant 1} I(T_j^n \leqslant t) I(X_j^n \in \Gamma), \ \Gamma \in \mathcal{E}^n . \tag{2.2}$$

Consider also another marked process

$$\underline{X}^n = \{(T_j^n, X_j^n), \underline{P}^n\} \tag{2.3}$$

with the similarly associated integer valued random measure $\underline{p}^n = p^{\underline{X}^n}$

Let $\tilde{p}^n$ and $\tilde{\underline{p}}^n$ be the compensators of $p^n$ and $\underline{p}^n$ respectively, setisfying the following conditions $K$:

$K$. 1. $a_t^n = 1$ implies $\underline{a}_t^n = 1$ $\underline{P}^n$a.s. with

$$a_t^n = \tilde{p}^n(\{ t \}, E^n) = \int\int_{(t,t]E^n} d\tilde{p}^n, \ \underline{a}_t^n = \tilde{\underline{p}}^n(\{ t \}, E^n) \tag{2.4}$$

$K$. 2. $\tilde{\underline{p}}^n$ is dominated by $\tilde{p}^n$ $\underline{P}^n$a.s. and the associated Hellinger process $\mathcal{H}^n$ is bounded : $\underline{P}^n$a.s.

$$\mathcal{H}_t^n = \int_0^t\int_{E^n} (\sqrt{\lambda^n} - 1)^2 + \sum_{s \leqslant t} (\sqrt{1 - \underline{a}_s^n} - \sqrt{1 - a_s^n})^2 < \infty \tag{2.5}$$

where $\lambda^n = \lambda^n(t,x)$ is defined by the relation $d\tilde{\underline{p}}^n = \lambda^n d\tilde{p}^n$ $P^n$a.s.

Now, THEOREM 20 in KABANOV et al. (1980), p. 48 tells us that under the Conditions $K$ the relation (1.3) holds for the local martingale $m^n$ defined as follows[1] (of course $m^{nc} = 0$ here) :

$$m_t^n = \int_0^t\int_{E^n} (\lambda^n - 1) dq^n \equiv \mathfrak{m}\, (\lambda^n - 1)_t \tag{2.6}$$

with the "martingale measure"

$$q^n((0, t], \Gamma) = (p^n - \tilde{p}^n)((0, t], \Gamma) + \sum_{s \leqslant t} \frac{(p^n - \tilde{p}^n)(\{s\}, E^n)}{1 - a_s^n} \tilde{p}^n(\{s\}, \Gamma), \ \Gamma \in \mathcal{E}^n . \tag{2.7}$$

(cf. THEOREM 3.1 below). Observe that

$$\Delta m_t^n = \int_{E^n} (\lambda^n(t,x) - 1) p^n(\{t\}, dx) + \tag{2.8}$$

$$+ (1 - p^n(\{t\}, E^n)) \left[ \frac{1 - \underline{a}_t^n}{1 - a_t^n} - 1 \right] > -1$$

(cf. KABANOV et al. (1979), LEMMA 7, p. 676) where, by (2.2),

$$p^n(\{t\}, \Gamma) = \sum_{j \geqslant 1} I(T_j^n = t) I(X_j^n \in \Gamma), \ \Gamma \in \mathcal{E}^n .$$

Hence, in this particular case the process $\eta^n$ defined by (1.7) has the following form :

$$\eta_t^n = \int_0^t\int_{E^n} \{u^n(s,x)\}^2 p^n(ds,dx) + \sum_{s \leqslant t} (1 - p^n(\{s\}, E^n)) \left[ \sqrt{\frac{1 - \underline{a}_s^n}{1 - a_s^n}} - 1 \right]^2 \tag{2.9}$$

with $u^n = \sqrt{\lambda^n} - 1$. Of course, its compensator $\tilde{\eta}^n$ coincides with the Hellinger process which

1. In contrast with KABANOV et al. (1980) we assume here, for simplicity, that $d\underline{P}_0^n / dP_0^n = 1$ and $a^n < 1$.

appeared in (2.5), $\tilde{\eta}^n = \mathcal{H}^n$.

Analogously, for $m^n$ given by (2.6)

$$\eta^n_{(>\epsilon)t} \equiv \int_0^t \int_{|x|>\epsilon} (1-\sqrt{1+x})^2 d\mu^{m^n} \tag{2.10}$$

$$= \sum_{s\leq t}(1-\sqrt{1+\Delta m^n I(|\Delta m^n|>\epsilon)})^2$$

$$= \int_0^t\int_{E^n} I(|\lambda^n(s,x)-1|>\epsilon)\{u^n(s,x)\}^2 p^n(ds,dx)$$

$$+ \sum_{s\leq t}\left[1-p^n(\{s\},E^n)\right] I\left[|\underline{a}^n_s - a^n_s|>\epsilon(1-a^n_s)\right]\left[\sqrt{\frac{1-\underline{a}^n_s}{1-a^n_s}}-1\right]^2,$$

because by (2.8)

$$I(|\Delta m^n_t|>\epsilon)\Delta m^n_t = \int_{E^n} I(|\lambda^n(t,x)-1|>\epsilon)(\lambda^n(t,x)-1)p^n(\{t\},dx) \tag{2.11}$$

$$+ (1-p^n(\{t\},E^n))I(|\underline{a}^n_t - a^n_t|>\epsilon(1-a^n_t))\left[\frac{1-\underline{a}^n_t}{1-a^n_t}-1\right]$$

The process $\eta^n_{(>\epsilon)}$ in (2.10) has the following compensator

$$\tilde{\eta}^n_{(>\epsilon)t} = \int_0^t\int_{E^n} I(|\lambda^n(s,x)-1|>\epsilon)\{u^n(s,x)\}^2\tilde{p}^n(ds,dx) \tag{2.12}$$

$$+ \sum_{s\leq t} I(|\underline{a}^n_s - a^n_s|>\epsilon(1-a^n_s))(\sqrt{1-\underline{a}^n_s}-\sqrt{1-a^n_s})^2 .$$

Let $\tilde{\eta}^n_{(>\epsilon)}$ and $\mathcal{H}^n$ of form (2.5) and (2.12) respectively satisfy the Conditions $S$ of THEOREM 1.1. Then, as has been indicated in the privious section, the conclusion of the theorem holds. Here we will follow an alternative way of establishing this fact outlined in the following steps.

1) First the equivalence of the Conditions $S$ with the following Conditions $S'$ is shown :

*S.I'.* For each $t$, $0\leq t\leq 1$ and $\epsilon$, $0<\epsilon<1$

$$\int_0^t\int_{E^n} I(|u^n|>\epsilon)(u^n)^2 d\tilde{p}^n + \sum_{s\leq t} I(|b^n_s|>\epsilon)(b^n_s)^2(1-a^n_s)\to 0$$

in $P^n$ probability as $n\to\infty$, where

$$b^n_t = \int_{E^n} u^n(t,x)d\tilde{p}^n(\{t\},dx) .$$

*S.II'.* For each $t$, $0\leq t\leq 1$

$$<\mathrm{m}(u^n)>_t = \int_0^t\int_{E^n}(u^n)^2 d\tilde{p}^n + \sum_{s\leq t}(1-a^n_s)(b^n_s)^2 \to \tfrac{1}{4}<W>_t$$

in $P^n$ probability as $n\to\infty$, where $\mathrm{m}(u^n)$ is given by (2.6) with $u^n$ instead of $\lambda^n - 1$.

2) Next, $\mathrm{m}(u^n)$ is a local square integrable martingale, and by the Conditions $S'$ one can apply COROLLARY 2 of LIPTSER and SHIRYAYEV (1980), p. 671 to arrive at the conclusion that

$$\mathrm{m}(u^n) \overset{d(P^n)}{\rightarrow} W/2$$

3) From this last relation and (2.6) with $\lambda^n - 1 = 2u^n + (u^n)^2$ one gets

$$m^n \overset{d(P^n)}{\rightarrow} W ,$$

provided that the following relation holds :

$$\mathrm{m}((u^n)^2) \rightarrow 0$$

in $P^n$ probability as $n \rightarrow \infty$.
4) Finally, by (1.3) with $m^{nc} = 0$, it remains to show that

$$\frac{1}{2}\sum_{s \leq t} \Phi_1(\Delta m_s^n) = \sum_{s \leq t} \{\Phi_2(\sqrt{1+\Delta m_s^n} - 1) - (\sqrt{1+\Delta m_s^n} - 1)^2\}$$
$$= 2R_t^{n,3} - \eta_t^n$$

(here we use the elementary equality

$$2\Phi_2(\sqrt{x} - 1) = \Phi_1(x-1) + 2(\sqrt{x} - 1)^2$$

and then (1.7) and (1.14) ) has the same limit in $P^n$ probability as the compensator of the second term, $-\tilde{\eta}^n = -\mathcal{K}^n$. This limit is $-\mathcal{K}^W$, by Condition *S.II*. Thus, along with Assertion 1.*B* for $i = 3$ one has to ensure the asymptotic negligability of the local martingale $\eta^n - \tilde{\eta}^n$.

*2.2.* Consider the special case in which for each $n$ the mark space $E^n$ consists of $r_n$ points, say $x_1^n, \ldots, x_{r_n}^n$. If we define

$$N_t^{in} = \sum_{j \geq 1} I(T_j^n \leq t) I(X_j^n = x_j^n)$$

then we obtain an $r_n$-variate counting process $\mathbb{N}^n = \mathrm{col}\{N^{in}, i = 1, \ldots, r_n\}$.
Under this restriction the Conditions $K$ of the privious subsection are reformulated as Condition I and II given in THEOREM 3.1, for in this special case

$$\int_0^t \int_{E^n} (\sqrt{\lambda^n} - 1)^2 d\tilde{p}^n = \int_0^t \sum_{i=1}^{r_n} (\sqrt{\lambda^{in}} - 1)^2 dA^{in}$$

(cp. (2.5) and (4.5) where $\Delta \overline{A}^n$ and $\Delta \overline{A}^n$ stand for $a^n$ and $a^n$, respectively).
Similarly, the Conditions $S$ are reformulated as the Conditions $\mathcal{K}$ in Section 4, since the process (2.12) is specified here as in (4.9).Thus THEOREM 4.1 below is a version of Shiryayev's THEOREM 1.1.
Next, compare the Conditions $S'$ with the Conditions $\mathbb{U}$ stipulated in Proposition 4.1 below to observe that steps 1) and 2) of the previous subsection are taken in this proposition.
Finally, check that in the present special case $\mathrm{m}((u^n)^2)$ in step 3) coincides with $R^{(1)}$ of (4.22), while $R^{n,3}$ and $\eta^n - \tilde{\eta}^n$ in step 4) coincide with $R^{(3)}$ of (4.25) and $R^{(1)} - R^{(2)}$ of (4.23), (4.25), respectively. Hence the conjectures in these steps are proved in the LEMMAS 4.1 - 4.3.

## 3. THE LIKELIHOOD RATIO FOR COUNTING PROCESSES

*3.1.* Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t, 0 \leq t \leq 1\}$ satisfying the usual conditions. Let $\mathbb{N} = \{\mathbb{N}_t, \mathcal{F}_t, P; 0 \leq t \leq 1\}$ be a multivariate ($r$-variate) counting process: $\mathbb{N} = col\{N^1, \ldots, N^r\}$. Consider its Doob-Meyer decomposition $\mathbb{N} = \mathbb{M} + \mathbb{A}$ where $\mathbb{M} = \{\mathbb{M}_t, \mathcal{F}_t, P; 0 \leq t \leq 1\}$ is a local square integrable martingale, and $\mathbb{A} = \{\mathbb{A}_t, \mathcal{F}_t, P; 0 \leq t \leq 1\}$ a predictable compensator.

LEMMA 3.1. *The quadratic variation and quadratic characteristic of* $\mathbb{M}$ *are given by the following*

*relations:*
1) $[\mathbb{M}] = diag\mathbb{N} - [\mathbb{A}] - [\mathbb{M},\mathbb{A}] - [\mathbb{A},\mathbb{M}]$
2) $<\mathbb{M}> = diag\mathbb{A} - [\mathbb{A}]$

PROOF. By definition $[\mathbb{N}] = diag\mathbb{N}$, and this gives 1). To get 2) take the compensator of both sides of 1). □

REMARK 3.1. Denote $\bar{N} = N^1 + ... + N^r, \bar{N} = \bar{M} + \bar{A}$. From 2) it follows that

$$<\bar{M}>_t = \bar{A}_t - [\bar{A}]_t = \int_0^t (1-\Delta\bar{A})d\bar{A}, \quad \Delta<\bar{M}> = (1-\Delta\bar{A})\Delta\bar{A} \ ,$$

hence $0 \leq \Delta\bar{A} \leq 1$. For simplicity assume $\Delta\bar{A} < 1$ (in fact one can easily dispence with this restriction; see e.g. KABANOV et al. (1975) or (1980)).

REMARK 3.2. Consider $V_t = I_r - \Delta\mathbb{A}_t \otimes \mathbb{1}_r$ and $V_t^{-1} = I_r + (1-\Delta\bar{A}_t)^{-1}\Delta\mathbb{A}_t \otimes \mathbb{1}_r$ with $\mathbb{1}_r = col\{1, \ldots, 1\}$ and $I_r = diag\mathbb{1}_r$. Then

$$\begin{aligned} <\mathbb{M}>_t &= \int_0^t V diag d\mathbb{A} \\ &= \int_0^t diag d\mathbb{A}\, V^T \\ &= \int_0^t V^{1/2} diag d\mathbb{A}\, V^{1/2T} \end{aligned}$$

with

$$V^{1/2} = I_r - (1-\sqrt{1-\Delta\bar{A}})\Delta\mathbb{A}/\Delta\bar{A} \otimes \mathbb{1}_r \cdot I(\Delta\bar{A} > 0)$$

(satisfying $(V^{1/2})^2 = V$, of course), and

$$\begin{aligned} diag\mathbb{A}_t &= \int_0^t V^{-1} d<\mathbb{M}> \\ &= \int_0^t d<\mathbb{M}> V^{-1T} \\ &= \int_0^t V^{-1/2} d<\mathbb{M}> V^{-1/2T} \end{aligned}$$

with

$$V^{-1/2} = I_r + \frac{1-\sqrt{1-\Delta\bar{A}}}{\sqrt{1-\Delta\bar{A}}} \frac{\Delta\mathbb{A}}{\Delta\bar{A}} \otimes \mathbb{1}_r \cdot I(\Delta\bar{A} > 0)$$

LEMMA 3.2.
*Let* $\mathcal{Q}_t = \int_0^t V^{-1} d\mathbb{A} = \int_0^t (1-\Delta\bar{A})^{-1} d\mathbb{A}$ and $\mathfrak{M}_t = \int_0^t V^{-1} d\mathbb{M} = \mathbb{M}_t + [\mathcal{Q}, \bar{M}]_t = \mathbb{M}_t + [\mathbb{A}, \bar{\mathfrak{M}}]_t$ *where* $\bar{\mathfrak{M}}$ *is the sum of the component of* $\mathfrak{M}$. *Then*

1) $[\mathfrak{M}]_t = diag\mathbb{N}_t + \int_0^t (1-\Delta\bar{N}) d[\mathcal{Q}]$ ,

2) $<\mathfrak{M}> = diag\mathbb{A} + [\mathcal{C},\mathbb{A}]$.

PROOF. As $\Delta\mathbb{N}^{\otimes 2} = diag\Delta\mathbb{N}$, $(1-\Delta\bar{N})^2 = (1-\Delta\bar{N})$ and $\Delta\mathbb{N}(1-\Delta\bar{N}) = 0$, 1) follows from

$$\Delta\mathfrak{M} = \Delta\mathbb{M} + \Delta\mathcal{C}\Delta\bar{M} = \Delta\mathbb{N} - \Delta\mathcal{C}(1-\Delta\bar{N}). \tag{3.1}$$

□

To get 2) take the compensators of both sides of 1).

*3.2.* Suppose that a probability measure $\underline{P}$ in addition to the probability measure $P$ is given on a measurable space $(\Omega,\mathfrak{F})$ with a filtration of special form $\mathfrak{F}_t = \omega\{\mathbb{N}_s{:}s\leq t\}, 0\leq t\leq 1$. Along with $\mathbb{N} = (\mathbb{N}_t,\mathfrak{F}_t,P)$, consider the counting process $\underline{\mathbb{N}} = (\mathbb{N}_t,\mathfrak{F}_t,\underline{P})$ with compensator $\underline{\mathbb{A}} = (\underline{\mathbb{A}}_t,\mathfrak{F}_t,\underline{P})$.

THEOREM 3.1. *( KABANOV et al. (1980)). 1) For absolute continuity of $\underline{P}$ with respect to $P$ $(\underline{P}\ll P)$ the following conditions are necessary and sufficient: $\underline{P}$-a.s.*

I. $\Delta\bar{A} = 1$ *implies* $\Delta\underline{\bar{A}} = 1$.

II. *The components $\underline{A}^i$ and $A^i$, $i = 1, \ldots, r$ of $\underline{\mathbb{A}}$ and $\mathbb{A}$ are related as $\underline{A}_t^i = \int_0^t \lambda^i dA^i$ where $col\{\lambda^1, \ldots, \lambda^r\} = \Lambda = \{\Lambda_t,\mathfrak{F}_t\}$ is a nonnegative predictable process such that the associated Hellinger process is bounded:* $\mathcal{K}_t = \int_0^t \sum_{i=1}^r (\sqrt{d\underline{A}^i} - \sqrt{dA^i})^2 + \sum_{\substack{s\leq t \\ 0<\Delta\bar{A}_s<1}} (\sqrt{1-\Delta\underline{\bar{A}}_s} - \sqrt{1-\Delta\bar{A}_s})^2 < \infty.$

2) *Assume $\underline{P}\ll P$, and denote $z_t$ a right-continuous modification of the martingale $E(d\underline{P}/dP|\mathfrak{F}_t)$ $0\leq t\leq 1$. Then $z_t = \exp\{m_t + \sum_{s\leq t}\Phi_1(\Delta m_s)\}$ where $\Phi_1(x) = \ln(1+x)-x$, and*

$$m_t = \int_0^t (\Lambda - \mathbb{I}_r)^T d\mathfrak{M} \tag{3.2}$$

REMARK 3.3. The process $z = (z_t,\mathfrak{F}_t,P)$, being a nonnegative supermartingale with $(z|P) = 1$ as well as a local martingale, is a solution of the Doleans-Dade equation $z_t = 1 + \int_0^t z_{s-}dm_s$, $0\leq t\leq 1$ (LIPTSER and SHIRYAYEV (1978), p. 288, or GILL and JOHANSON (1986) ), hence $z_t = \mathcal{E}(m)_t$.

REMARK 3.4. By (3.1) and (3.2)

$$\Delta m = (\Lambda - \mathbb{I}_r)^T\Delta\mathbb{N} + (1-\Delta\bar{N})\left[\frac{1-\Delta\underline{\bar{A}}}{1-\Delta\bar{A}} - 1\right] \tag{3.3}$$

and

$$\Phi_1(\Delta m) = \Phi_1^T(\Lambda - \mathbb{I}_r)\Delta\mathbb{N} + (1-\Delta\bar{N})\Phi_1\left[\frac{1-\Delta\underline{\bar{A}}}{1-\Delta\bar{A}} - 1\right] \tag{3.4}$$

with $\Phi_1(x) = \text{col}\{\Phi_1(x^i), i = 1, \ldots, r\}$ for $x = \text{col}\{x^1, \ldots, x^r\}$. Hence
$z_t = \exp\{\int_0^t \ln^T \Lambda d\mathbb{N} - \underline{\bar{A}}_t^c + \bar{A}_t^c + \sum_{s\leq t}(1-\Delta\bar{N}_s)\ln\frac{1-\Delta\underline{\bar{A}}_s}{1-\Delta\bar{A}_s}\}$ (cf. LIPSTER and SHIRYAYEV (1978), p. 312).

REMARK 3.5. By (3.3)

$$\eta_t \equiv \sum_{s\leq t}(1-\sqrt{1+\Delta m_s})^2 \tag{3.5}$$

$$= \int_0^t \mathbb{U}^T \text{diag} d\mathbb{N}\mathbb{U} + \sum_{s\leq t}(1-\Delta\bar{N}_s)\left[\sqrt{\frac{1-\Delta\underline{\bar{A}}_s}{1-\Delta\bar{A}_s}} - 1\right]^2$$

with $\mathbb{U} = \mathrm{col}\{\sqrt{\lambda^i}-1 = \sqrt{dA^i/dA^i}-1, i = 1,\dots,r\}$. The compensator of this process coincide with the Hellinger process, $\tilde{\eta} = \mathcal{H}$.

REMARK 3.6. It is interesting to note that the class of "alternative" compensators $\mathbb{A}$ is restricted to those for which $\underline{\mathbb{A}}-\mathbb{A}$ is dominated by $<\mathbb{M}>$ in the sense that for a certain $r$-vertor valued predictable process $\mathbb{H}$

$$\underline{\mathbb{A}}_t-\mathbb{A}_t = \int_0^t \mathbb{H}^T d<\mathbb{M}> .$$

If $\underline{P} \ll P$ then $z = \mathcal{E}(m)$ with $m_t = \int_0^t \mathbb{H}^T d\mathbb{M}$. Obviously, $\mathbb{H} = V^{-1T}(\Lambda-\mathbb{I})$ and $\underline{\mathbb{A}}-\mathbb{A} = <\mathbb{M},m>$.

*3.3.*. Here we give a useful representation for the likelihood ratio process, to be used in the next section.

LEMMA 3.3. *Let $\underline{P} \ll P$. Then*

$$z = \exp\{2\mathrm{m}(\mathbb{U})-2\mathcal{H} + R\} \tag{3.6}$$

*where*

$$\mathrm{m}(\mathbb{U})_t = \int_0^t \mathbb{U}^T d\mathfrak{M} \tag{3.7}$$

*is a local square integrable martingale with*

$$<\mathrm{m}(\mathbb{U})>_t = \int_0^t \mathbb{U}^T \mathrm{diag} d<\mathfrak{M}>\mathbb{U} < \infty \quad P\text{a.s.}, \tag{3.8}$$

*while*

$$R_t = 2\sum_{s\leqslant t}\Phi_2(\sqrt{1+\Delta m_s}-1) + 2[\mathfrak{M},\mathcal{H}]_t - \int_0^t \mathbb{U}^T \mathrm{diag} d\mathfrak{M}\mathbb{U} \tag{3.9}$$

*with*

$$\Phi_2(x) = \ln(1+x)-x+\frac{1}{2}x^2 .$$

PROOF. By (3.2)

$$m_t = 2\int_0^t \mathbb{U}^T d\mathfrak{M} + \int_0^t \mathbb{U}^T \mathrm{diag} d\mathfrak{M}\mathbb{U} . \tag{3.10}$$

By (3.4) and (3.5)

$$\frac{1}{2}\sum_{s\leqslant t}\Phi_1(\Delta m_s) = \sum_{s\leqslant t}\Phi_2(\sqrt{1+\Delta m_s}-1)-\mathcal{H}_t-(\eta_t-\tilde{\eta}_t) , \tag{3.11}$$

since $\frac{1}{2}\Phi_1(x-1) = \Phi_2(\sqrt{x}-1)-(\sqrt{x}-1)^2$ and $\mathcal{H} = \tilde{\eta}$; obviously,

$$\eta_t-\tilde{\eta}_t = \int_0^t \mathbb{U}^T \mathrm{diag} d\mathbb{M}\mathbb{U} + \sum_{s\leqslant t}\Delta\overline{\mathfrak{M}}(\sqrt{1-\Delta\underline{\overline{A}}}-\sqrt{1-\Delta\overline{A}})^2 . \tag{3.12}$$

Now, (3.6) easily follows from (3.9) - (3.12), taking into account that $\mathfrak{M} = \mathbb{M} + [\mathbb{A},\overline{\mathfrak{M}}]$ by definition.
By Assertion 2) of LEMMA 3.2

$$<\mathfrak{m}(\mathbb{U})>_t = \int_0^t \mathbb{U}^T \mathrm{diag} d\mathbb{A}\mathbb{U} + \sum_{s\leq t} \frac{(\mathbb{U}_s^T \Delta\mathbb{A}_s)^2}{1-\Delta\bar{A}_s} \tag{3.13}$$

$$\leq \int_0^t \mathbb{U}^T \mathrm{diag} d\mathbb{A}\mathbb{U} + \sum_{s\leq t} \frac{\Delta\bar{A}_s}{1-\Delta\bar{A}_s} \mathbb{U}_s^T \mathrm{diag}\Delta\mathbb{A}_s\mathbb{U}_s$$

$$\leq \int_0^t \mathbb{U}^T \mathrm{diag} d\mathbb{A}\mathbb{U} + \sum_{s\leq t} \mathbb{U}_s^T \mathrm{diag}\Delta\mathcal{C}_s\mathbb{U}_s < \infty P \ a.s.$$

Here we first used the Schwartz inequality and then the boundedness of the Hellinger process.[1] Hence (3.8) holds.

□

## 4. LAN FOR COUNTING PROCESSES

*4.1.* Let $\{\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n, 0\leq t\leq 1), P^n)\}$, $n = 1,2,...$ be a sequence of stochastic basis' of the same type as above. Let $\mathbb{N}^n = (\mathbb{N}_t^n, \mathcal{F}_t^n, P^n)$ be an $r_n$-variate counting process with the Doob-Meyer decomposition $\mathbb{N}^n = \mathbb{M}^n + \mathbb{A}^n$, where $r_n$, $n = 1,2,...$ is a nondecreasing sequence of integers.

Define also $\mathfrak{M}_t^n = \int_0^t (V^n)^{-1} d\mathbb{M}^n$ where $V^n = I_{r_n} - \Delta\mathbb{A}^n \otimes \mathbb{I}_{r_n}$.

Let $\mathbb{H}^n = \{\mathbb{H}_t^n, \mathcal{F}_t^n, P^n\}$, $n = 1,2,...$ be a sequence of $r_n$-verctor valued predictable processes such that

$$\mathfrak{m}(\mathbb{H}^n)_t = \int_0^t \mathbb{H}^{nT} d\mathfrak{M}^n \ , \ n = 1,2,... \tag{4.1}$$

is a sequence of local square integrable martingales.

By COROLLARY 2 of LIPTSER and SHIRYAYEV (1980) this sequence is asymptotically normal (see THEOREM 4.1 below) under the following Conditions $\mathbb{H}$ :

$\mathbb{H}$.1. For each $t$, $0\leq t\leq 1$ and $\epsilon$, $0<\epsilon\leq 1$

$$\int_0^t \mathbb{H}_{(>\epsilon)}^{nT} diag d\mathbb{A}^n \mathbb{H}_{(>\epsilon)}^n + \sum_{s\leq t} I(|\mathbb{H}^{nT}\Delta\mathcal{C}^n|>\epsilon)(1-\Delta\bar{A}^n)(\mathbb{H}^{nT}\Delta\mathcal{C}^n)^2 \to 0 \tag{4.2}$$

in $P^n$ probability as $n\to\infty$, where

$$\mathbb{H}_{(>\epsilon)}^n = col\{I(|H^{in}|>\epsilon)H^{in} \ , \ i = 1,\ldots,r_n\} \ , \ \mathbb{H}^n = col\{H^{in} \ , \ i = 1,\ldots,r_n\} \tag{4.3}$$

$\mathbb{H}$.2. For each $t$, $0\leq t\leq 1$

$$<\mathfrak{m}(\mathbb{H}^n)>_t = \int_0^t \mathbb{H}^{nT} diag d\mathbb{A}^n \mathbb{H}^n + \sum_{s\leq t}(1-\Delta\bar{A}^n)(\mathbb{H}^{nT}\Delta\mathcal{C}^n)^2 \to <W> \tag{4.4}$$

where $W = (W_t, \mathcal{F}_t)_{0\leq t\leq 1}$ is a continuous Gaussian martingale with quadratic variation $<W> = [W] = EW^2$, a nondecreasing continuous deterministic function (cf. GREENWOOD and SHIRYAYEV (1985), § 5.2).

1. Use also the following inequalities : $\sum_{s\leq t} I(0<\Delta\bar{A}\leq\frac{1}{2})$ $\mathbb{U}_s^T \mathrm{diag}\Delta\mathcal{C}_s\mathbb{U}_s \leq 2 \sum_{s\leq t} \mathbb{U}_s^T \mathrm{diag}\Delta\mathbb{A}_s\mathbb{U}_s$ and $\sum_{s\leq t} I(\frac{1}{2}\leq\Delta\bar{A}<1)$ $\mathbb{U}_s^T \mathrm{diag}\Delta\mathcal{C}_s\mathbb{U}_s \leq C \sum_{s\leq t} \mathbb{U}_s^T \mathrm{diag}\Delta\mathbb{A}_s\mathbb{U}_s$ with a certain constant $C$ determined by the fact that the number of jumps of $\bar{A}_s$, $s\leq t$, exceeding $\frac{1}{2}$ is finite.

THEOREM 4.1. *Under the Conditions* $\mathbb{H}$

$$\mathfrak{m}(\mathbb{H}^n) \overset{d(P^n)}{\rightarrow} W \tag{4.5}$$

*in the sense of the weak convergence in* $\mathcal{D}([0,1])$ *with resp. to* $P^n$ *(cf. Greenwood and Shiryayev (1985), § 2.2).*

REMARK 4.1. For checking the above statement take into consideration that the integer valued random measure $\mu^n$, associated to $\mathfrak{m}(\mathbb{H}^n)$ by

$$\mu^n((0,t],\Gamma) = \sum_{s\leq t} I(\Delta\mathfrak{m}(\mathbb{H}^n)_s \in \Gamma),\ \Gamma \in \mathcal{B}(R_0),\ R_0 = R\setminus\{0\}$$

with

$$\Delta\mathfrak{m}(\mathbb{H}^n) = \mathbb{H}^{nT}\Delta\mathbb{N}^n - (1-\Delta\overline{N})\mathbb{H}^{nT}\Delta\mathfrak{A}^n\ ,$$

is such that

$$\int_0^t \int_{|x|>\epsilon} x^2\mu^n(ds,dx) = \sum_{s\leq t}\Delta\mathfrak{m}(\mathbb{H}^n)_s^2 I(|\Delta\mathfrak{m}(\mathbb{H}^n)_s|>\epsilon) \tag{4.6}$$

$$= \int_0^t \mathbb{H}^{nT}_{(>\epsilon)} diag d\mathbb{N}^n \mathbb{H}^n_{(>\epsilon)} + \sum_{s\leq t}(1-\Delta\overline{N}^n)(\mathbb{H}^{nT}\Delta\mathfrak{A}^n)^2 I(|\mathbb{H}^{nT}\Delta\mathfrak{A}^n|>\epsilon)$$

Here we have used the following simple relation :

$$I(|\Delta\mathfrak{m}(\mathbb{H}^n)|>\epsilon)\Delta\mathfrak{m}(\mathbb{H}^n) = \mathbb{H}^{nT}_{(>\epsilon)}\Delta\mathbb{N}^n - (1-\Delta\overline{N}^n)I(|\mathbb{H}^{nT}\Delta\mathfrak{A}^n|>\epsilon)\mathbb{H}^{nT}\Delta\mathfrak{A}^n\ . \tag{4.7}$$

Now, we can easily seen that on the left hand side of (4.2) we have the compensator of the expression (4.6). Hence, denoting the compensator of $\mu^n$ by $\nu^n$, one can rewrite (4.2) as follows :

$$\int_0^t \int_{|x|>\epsilon} x^n d\nu^n(ds,dx) \rightarrow 0\ .$$

Below we will need the following simple corollary of theorem 4.1.

COROLLARY 4.1. *Let a sequence* $\mathbb{H}^n$, $n = 1,2,...$ *of* $r_n$*-valued predictable processes satisfy the following Conditions* $\mathbb{H}'$ *: for each* $t$, $0\leq t\leq 1$

$\mathbb{H}'.0.$ $\int_0^t \mathbb{H}^{nT} d\mathbb{A}^n \rightarrow 0$ ;

$\mathbb{H}'.1.$ $\int_0^t \mathbb{H}^{nT}_{(>\epsilon)} diag d\mathbb{A}^n \mathbb{H}^n_{(>\epsilon)} \rightarrow 0$ , $0<\epsilon\leq 1$ ;

$\mathbb{H}'.2.$ $\int_0^t \mathbb{H}^{nT} diag d\mathbb{A}^n \mathbb{H}^n \rightarrow \langle W\rangle_t$

*in* $P^n$ *probability as* $n\rightarrow\infty$. *Then*

$$\int_0^t \mathbb{H}^{nT} d\mathbb{N}^n \overset{d(P^n)}{\rightarrow} W_t\ .$$

*4.2.* Suppose that a probability measure $\underline{P}^n$ in addition to $P^n$ is given on a measurable space $\{\Omega^n, \mathcal{F}^n\}$ of the preceding subsection. Suppose in addition that the filtration $\{\mathcal{F}_t^n, 0 \leqslant t \leqslant 1\}$ is minimal: $\mathcal{F}_t^n = \omega\{\mathbb{N}_s^n : s \leqslant t\}$ where $\mathbb{N}^n = (\mathbb{N}_t^n, \mathcal{F}_t^n, P^n)$ is an $r_n$-variate counting process with the compensator $\mathbb{A}^n = (\mathbb{A}_t^n, \mathcal{F}_t^n, P^n)$. Let $\underline{\mathbb{N}}^n = (\mathbb{N}_t^n, \mathcal{F}_t^n, \underline{P}^n)$ be another counting process with the compensator $\underline{\mathbb{A}}^n = (\underline{\mathbb{A}}_t^n, \mathcal{F}_t^n, P^n)$.

For each, $n$ assume $\underline{P}^n \ll P^n$ and, in accordance with II of THEOREM 3.1, define the Hellinger process

$$\mathcal{H}_t^n = \int_0^t \mathbb{U}^{nT} \mathrm{diag} d\mathbb{A}^n \mathbb{U}^n + \sum_{\substack{s \leqslant t \\ 0<\Delta A_s^n<1}} \left(\sqrt{1-\Delta \underline{\bar{A}}_s^n} - \sqrt{1-\Delta \bar{A}_s^n}\right)^2 \tag{4.8}$$

where

$$\mathbb{U}^n = col\{U^{in} = \sqrt{d\underline{A}^{in}/dA^{in}} - 1, i=1, \ldots, r_n\}.$$

Obviously,

$$\Lambda^n = col\{\lambda^{in} = (U^{in}+1)^2, i=1, \ldots, r_n\}.$$

Let the following Conditions $\mathcal{H}$ be satisfied :

$\mathcal{H}$1. For each $t$, $0 \leqslant t \leqslant 1$ and $\epsilon$, $0 < \epsilon \leqslant 1$

$$\tilde{\eta}^n_{(>\epsilon)t} \equiv \int_0^t \hat{\mathbb{U}}^{nT}_{(>\epsilon)} \mathrm{diag} d\mathbb{A}^n \hat{\mathbb{U}}^n_{(>\epsilon)} \tag{4.9}$$

$$+ \sum_{s \leqslant t} I(|\Delta \underline{\bar{A}}_s^n - \Delta \bar{A}_s^n| > \epsilon(1-\Delta \bar{A}_s^n))\left(\sqrt{1-\Delta \underline{\bar{A}}_s^n} - \sqrt{1-\Delta \bar{A}_s^n}\right)^2 \to 0$$

in $P^n$ probability as $n \to \infty$, where

$$\hat{\mathbb{U}}^n_{(>\epsilon)} = \mathrm{col}\{I(|\lambda^{in}-1| > \epsilon)U^{in}, \; i = 1, \ldots, r_n\} \tag{4.10}$$

$\mathcal{H}$2. For each $t$, $0 \leqslant t \leqslant 1$

$$\mathcal{H}^n \to \frac{1}{4} < W > \tag{4.11}$$

in $P^n$ probability as $n \to \infty$.

PROPOSITION 4.1. *(i) Under the Conditions* $\mathcal{H}$

$$m(\mathbb{U}^n) \overset{d(P^n)}{\to} \frac{1}{4} W \tag{4.12}$$

*(cf. (3.7), (4.1) and (4.5) with* $\mathbb{H} = 2\mathbb{U}$ *)*

*(ii) The Conditions* $\mathcal{H}$ *are equivalent to the Conditions* $\mathbb{U}$ *defined by (4.2) - (4.4) for the special case of* $\mathbb{H} = 2\mathbb{U}$.

REMARK 4.2. As the Conditions $\mathbb{U}$ are those of THEOREM 4.1 for the special case of $\mathbb{H} = 2\mathbb{U}$, the assertion of LEMMA 3.3 concerning the process (3.7) allows us to deduce Assertion (i) of Proposition 4.1 directly from Assertion (ii) and THEOREM 4.1. The Assertion (ii) will be proved below.

REMARK 4.3. Notice the difference between $\hat{\mathbb{U}}^n_{(>\epsilon)}$ given by (4.10) and

$$\mathbb{U}^n_{(>\epsilon)} = \mathrm{col}\{I(|U^{in}| > \epsilon)U^{in}, \; i = 1, \ldots, r_n\} \tag{4.13}$$

(cf. (4.3) ). However using the simple inequalities

$$I(|\sqrt{1+x} - 1| > \epsilon) \leqslant I(|x| > \epsilon)$$

and

$$I(|x-y|>\epsilon)\leq I(|x|>\epsilon/2) + I(|y|>\epsilon/2) .$$

we get

$$\int_0^t \mathbb{U}^{nT}_{(>\epsilon)} \operatorname{diag} d\mathbb{A}^n \mathbb{U}^n_{(>\epsilon)} \leq \int_0^t \hat{\mathbb{U}}^{nT}_{(>\epsilon)} \operatorname{diag} d\mathbb{A}^n \hat{\mathbb{U}}^n_{(>\epsilon)} \tag{4.14}$$

$$\leq \int_0^t \mathbb{U}^{nT}_{(>\epsilon/4)} \operatorname{diag} d\mathbb{A}^n \mathbb{U}^n_{(>\epsilon/4)} + \int_0^t \mathbb{U}^{nT}_{(>\sqrt{\epsilon/2})} \operatorname{diag} d\mathbb{A}^n \mathbb{U}^n_{(>\sqrt{\epsilon/2})} .$$

*Proof of Assertion (ii) of Proposition 4.1.* We proceed in three steps. In step 1) we show that the Conditions $\mathcal{H}$ imply (4.2) with $\mathbb{H} = \mathbb{U}$. In step 2) we show that the Conditions $\mathbb{U}$ imply Condition $\mathcal{H}1$. In conclusion, it is shown in step 3) that the difference between $\mathcal{H}^n$ and the left hand side of (4.4) with $\mathbb{H} = 2\mathbb{U}$ vanishes as $n\to\infty$ under the Conditions $\mathcal{H}$, as well as under the Conditions $\mathbb{U}$.

1) By (4.14), under the Conditions $\mathcal{H}$ the first term on the left hand side of (4.2) with $\mathbb{H} = \mathbb{U}$ tends to zero in $P^n$ probability as $n\to\infty$. We will show that so does the second term, as well. The letter term does not exceed

$$\sum_{s\leq t} I(|(\Lambda^n_s-\mathbb{1}_{r_n})^T\Delta\mathcal{Q}^n_s|>\epsilon)(1-\Delta\overline{A}^n_s)\{(\Lambda^n_s-\mathbb{1}_{r_n})^T\Delta\mathcal{Q}^n_s\}^2 \tag{4.15}$$

$$+ \sum_{s\leq t}(1-\Delta\overline{A}^n_s)(\mathbb{U}^{nT}_s \operatorname{diag}\Delta\mathcal{Q}^n_s\mathbb{U}^n_s)^2 ,$$

as is easily seen by applying the simple inequality

$$|x-y|I(|x-y|>\epsilon)\leq 4|x|^2 I(|x|>\epsilon/2) + 4|y|^2 I(|y|>\epsilon/2) \tag{4.16}$$

(see ANDERSEN and GILL (1982), p. 1107 ) to

$$(\Lambda^n-\mathbb{1}_{r_n})^T\Delta\mathcal{Q}^n = 2\mathbb{U}^{nT}\Delta\mathcal{Q}^n + \mathbb{U}^{nT}\operatorname{diag}\Delta\mathcal{Q}^n\mathbb{U}^n \tag{4.17}$$

$$= 1-\frac{1-\Delta\overline{A}^n}{1-\Delta\overline{\overline{A}}^n} .$$

Since for each $\epsilon>0$ one can choose a constant $C$ that ensures the inequality $|x|\leq C(\sqrt{1+x}-1)^2$ whenever $|x|>\epsilon$ (e.g. via $x^2/(1+|x|)\asymp(\sqrt{1+x}-1)^2$; KABANOV et al. (1979), p. 644), the expression (4.15) in turn does not exceed

$$C\{\sum_{s\leq t} I(|(\Lambda^n_s-\mathbb{1}_{r_n})^T\Delta\mathcal{Q}^n_s|>\epsilon)(1-\Delta\overline{A}^n_s)(\sqrt{1-(\Lambda^n_s-\mathbb{1}_{r_n})^T\Delta\mathcal{Q}^n_s}-1)^2\}^2 \tag{4.18}$$

$$+ \sup_{s\leq t}\mathbb{U}^{nT}_s \operatorname{diag}\Delta\mathbb{A}^n_s\mathbb{U}^n_s\cdot\sum_{s\leq t}\mathbb{U}^{nT}_s \operatorname{diag}\Delta\mathcal{Q}^n_s\mathbb{U}^n_s .$$

By (4.17) and Condition $\mathcal{H}1$, the first term in(4.18) tends to zero in $P^n$ probability as $n\to\infty$. In view of the last inequality in (3.13) and the fact that

$$\sup_{s\leq t}\mathbb{U}^{nT}_s \operatorname{diag}\Delta\mathbb{A}^n_s\mathbb{U}^n_s\leq\sup_{s\leq t}\Delta\mathcal{H}^n_s\to 0 \tag{4.19}$$

in $P^n$ probability as $n\to\infty$ (see the footnote [1]), p. 4), the second term in (4.18) vanishes as well. Thus (4.2) for $\mathbb{H} = \mathbb{U}$ is proved.

2) Let the Conditions $\mathbb{U}$ hold ; By (4.14) again, it suffices to bound the second term of $\tilde{\eta}^n$ (see (4.9) ) and to show that it vanishes as $n\to\infty$. By the simple inequality $|\sqrt{1+x}-1|\leq|x|$, this term does not exceed

$$\sum_{s\leq t} I\left[|(\Lambda^n_s-\mathbb{1}_{r_n})^T\Delta\mathcal{Q}^n_s|>\epsilon\right](1-\Delta\overline{A}^n_s)\left\{(\Lambda^n_s-\mathbb{1}_{r_n})^T\Delta\mathcal{Q}^n_s\right\}^2$$

$$\leq 4\sum_{s\leq t} I\left[|\mathbb{U}_s^{nT}\Delta\mathcal{C}_s^n|>\epsilon/4\right]\{2\mathbb{U}_s^{nT}\Delta\mathcal{C}_s^n\}^2(1-\Delta\overline{A}_s^n)$$

$$+4\sup_{s\leq t}\mathbb{U}_s^{nT} diag\Delta\mathbf{A}_s^n\mathbb{U}_s^n\cdot\sum_{s\leq t}\mathbb{U}_s^{nT} diag\Delta\mathcal{C}_s^n\mathbb{U}_s^n \ ; \qquad (4.20)$$

here we have used (4.16) and (4.17). The second term on the right hand side of (4.20) tends to zero by the same arguments as above (cf. the similar term in (4.18) ) ; so does the first term as well, by (4.2) for $\mathbb{H} = \mathbb{U}$. Thus (4.9) is proved.

3) In view of the assertions proved in steps 1) and 2), all we need is that

$$\sum_{s\leq t} I\left[|(\Lambda_s^n-\mathbb{I}_{r_n})^T\ \Delta\mathcal{C}_s^n|\leq\epsilon\right](1-\Delta\overline{A}_s^n)|\sqrt{1-(\Lambda_s^n-\mathbb{I}_{r_n})^T\Delta\mathcal{C}_s^n}-1)^2- \qquad (4.21)$$

$$-\left\{\frac{1}{2}(\Lambda_s^n-\mathbb{I}_{r_n})^T\Delta\mathcal{C}_s^n\right\}^2|\rightarrow 0$$

in $P^n$ probability as $n\rightarrow\infty$, either under the Conditions $\mathcal{H}$ or $\mathbb{U}$.

Since $1-\sqrt{1-x} = x/2+x^2/8+0(x^3)$ for sufficiently small values of $x$, a constant $C$ can be chosen such that

$$|(1-\sqrt{1-x})^2-(\tfrac{1}{2}x)^2|\leq C|x^3|$$

Applying this inequality to the left-hand side of (4.21), one can see that it does not exceed

$$C\epsilon\sum_{s\leq t}(1-\Delta\overline{A}_s^n)\left\{(\Lambda_s^n-\mathbb{I}_{r_n})^T\Delta\mathcal{C}_s^n\right\}^2\leq$$

$$2C\epsilon\sum_{s\leq t}(1-\Delta\overline{A}_s^n)(2\mathbb{U}_s^{nT}\Delta\mathcal{C}_s^n)^2$$

$$+2C\epsilon\sup_{s\leq t}\mathbb{U}_s^{nT} diag\Delta\mathbf{A}_s^n\mathbb{U}_s^n\cdot\sum_{s\leq t}\mathbb{U}_s^{nT} diag\Delta\mathcal{C}_s^n\mathbb{U}_s^n\ .$$

This and (3.13) imply (4.21). The concluding step 3) is proved.

□

*4.3.*. The next three lemmas establish asymptotic negligability of the remainder term $R$ (see (3.9)) in the representation (3.6).

LEMMA 4.1. *Under the Conditions $\mathcal{H}$, for each $t$, $0\leq t\leq 1$*

$$\sup_{s\leq t} R_s^{(1)}\rightarrow 0\ ,\ R_t^{(1)} = \int_0^t \mathbb{U}^{nT} diag d\mathcal{M}^n\mathbb{U}^n\ , \qquad (4.22)$$

*in $P^n$ probability as $n\rightarrow\infty$.*

PROOF . By Assertion (ii) of Proposition 4.1 and (4.2) with $\mathbb{H} = \mathbb{U}$, the arguments indicated in the footnote [1] on p. 11 lead to

$$\int_0^t \mathbb{U}_{(>\epsilon)}^{nT} diag d\mathcal{C}^n\mathbb{U}_{(>\epsilon)}^n\rightarrow 0$$

in $P^n$ probability as $n\rightarrow\infty$.

Thus, it suffices to prove (4.22) with $\mathbb{U}_{(\leq\epsilon)}^n = \mathbb{U}^n-\mathbb{U}_{(>\epsilon)}^n$ in place of $\mathbb{U}^n$. But this is a direct consequence of Assertion (i) op Proposition 4.1, as

$$\int_0^t \mathbb{U}_{(\leq\epsilon)}^{nT} diag d\mathcal{M}^n\mathbb{U}_{(\leq\epsilon)}^n\leq\epsilon\ m(\mathbb{U}^n)_t$$

□

LEMMA 4.2. *Under the Conditions* $\mathcal{H}$ *for each* $t$, $0 \leqslant t \leqslant 1$

$$\sup_{s \leqslant t} |R_s^{(2)}| \to 0 \, , \; R^{(2)} = [\overline{\mathfrak{M}}^n, \mathcal{H}^n] \tag{4.23}$$

*in* $P^n$ *probability as* $n \to \infty$.

PROOF . By Assertion 2) of LEMMA 3.2, (4.19) and the boundedness of the Hellinger process

$$\begin{aligned} < R^{(2)} >_t &= \sum_{s \leqslant t} (\Delta \mathcal{H}_s^n)^2 \frac{\Delta \overline{A}_s^n}{1 - \Delta \overline{A}_s^n} \\ &\leqslant \sup_{s \leqslant t} \Delta \mathcal{H}_s^n \cdot \sum_{s \leqslant t} \frac{\Delta \mathcal{H}_s^n}{1 - \Delta \overline{A}_s^n} \\ &\leqslant \sup_{s \leqslant t} \Delta \mathcal{H}_s^n \cdot C \mathcal{H}_t^n \to 0 \end{aligned} \tag{4.24}$$

in $P^n$ probability as $n \to \infty$ (a constatnt $C$ is defined by the arguments indicated in the footnote [1)] on p. 11). Obviously, (4.23) is implied by (4.24).

□

LEMMA 4.3. *Under the Conditions* $\mathcal{H}$ *for each* $t$, $0 \leqslant t \leqslant 1$

$$\sup_{s \leqslant t} |R_s^{(3)}| \to 0, \; R_t^{(3)} = \sum_{s \leqslant t} \Phi_2 ( \sqrt{1 + \Delta m_s^n} - 1) \tag{4.25}$$

*in* $P^n$ *probability as* $n \to \infty$.

REMARK 4.3. The last assertion is the special case of Assertion 1.B for $i = 3$, Subsection 1.2. In fact

$$\begin{aligned} R_t^{(3)} &= \int_0^t \Phi_2^T(\mathbb{U}^n) d\mathbb{N}^n + \sum_{s \leqslant t} (1 - \Delta \overline{N}_s^n) \Phi_2 ( \sqrt{\frac{1 - \Delta \overline{A}^n}{1 - \overline{\overline{A}}^n}} - 1) \\ &= \int_0^t \int_{R_0} \Phi_2 ( \sqrt{1 + x} - 1) d\mu^n \end{aligned}$$

with $\mu^n$ defined as in REMARK 4.1 for the particular choice of $\mathbb{H}^n$, namely, $\mathbb{H}^n = \Lambda^n - \mathbb{I}_{r_n}$ (cp. (4.25) with (1.14) and (1.15) ).

*4.4.*. In conclusion, let us formulate the principal results of this paper - THEOREM 4.1 and its COROLLARY 4.2 stating the LAN for counting processes.

THEOREM 4.1. *Under the Conditions* $\mathcal{H}$

(i) $z^n \overset{d(P^n)}{\to} exp\{ W - \frac{1}{2} < W > \}$ ,

(ii) $z^n \overset{d(P^n)}{\rightrightarrows} exp\{ W + \frac{1}{2} < W > \}$ .

PROOF. Assertion (ii) is derived from Assertion (i) by the arguments used in GREENWOOD and SHIRYAYEV (1985) (see the proof of Statement 3 of THEOREM 8 on p. 99).

As for Assertion (i), it follows directly from LEMMA 3.3, Porposition 4.1 and the LEMMAS 4.1 - 4.3.

□

**Corollary 4.2.** *Let the Conditions $\mathcal{K}$ are satisfied. Then the following two statements hold :*

*(i) For each t,* $0 \leqslant t \leqslant 1$

$$z^n = exp\{\mathfrak{m}(2\mathbb{U}^n) - \frac{1}{2} < W > + r^n\}$$

*where a reminder term* $r^n$ *is such that*

$$\sup_{s \leqslant t} |r_s^n| \to 0 \tag{4.26}$$

*both in* $P^n$ *and* $\underline{P}^n$ *probability as* $n \to \infty$*, while the first term* $\mathfrak{m}(2\mathbb{U}^n)$ *is asymptotically normal :*

$$\mathfrak{m}(2\mathbb{U}^n) \overset{d(P^n)}{\to} W \tag{4.27}$$

*and*

$$\mathfrak{m}(2\mathbb{U}^n) \overset{d(\underline{P}^n)}{\to} W + < W > \tag{4.28}$$

*(ii) Let* $\mathbb{S}^n = \{\mathbb{S}_t^n, \mathfrak{F}_t^n, P^n\}$, $n = 1, 2, \ldots$ *be a sequence of* $r_n$*-variate predictable processes such that for some unboundedly increasing sequence of numbers* $k_n$, $n = 1, 2, \ldots$ *it satisfies the Conditions* $\mathbb{H}$ *with* $\mathbb{H}^n = k_n^{-\frac{1}{2}} \mathbb{S}^n$*. Besides, for each t,* $0 \leqslant t \leqslant 1$

$$\int_0^t (\mathbb{U}^n - k_n^{-\frac{1}{2}} \mathbb{S}^n)^T diagd\mathbb{A}^n(\mathbb{U}^n - k_n^{-\frac{1}{2}} \mathbb{S}^n) \to 0 \tag{4.29}$$

*in* $P^n$ *probability as* $n \to \infty$*. Then*

$$z^n = exp\{2k_n^{-\frac{1}{2}} \mathfrak{m}(\mathbb{S}^{nT}) - \frac{1}{2} < W > + r'^n\} \tag{4.30}$$

*where a reminder term* $r'^n$ *and the first term* $2k_n^{-\frac{1}{2}} \mathfrak{m}(\mathbb{S}^n)$ *satisfy (4.26) and, respectively (4.27) and (4.28) with* $k_n^{-\frac{1}{2}}(\mathbb{S}^n)$ *in place of* $\mathbb{U}^n$*.*

*Finally, if* $k_n^{-\frac{1}{2}}(\mathbb{S}^n)$ *satisfies the Conditions* $\mathbb{H}'$ *then the first term in (4.30) is simplified to* $2k_n^{-\frac{1}{2}} \int_0^t \mathbb{S}^{nT} d\mathbb{N}^n$*.*

**Proof.** Obviously, for the proof of Assertion (i) it suffices to check (4.26) for

$$r^n = R^n - 2(\mathcal{K}^n - \frac{1}{4} < W >)$$

(see (3.6) and (3.9) ), in view of Lemmas 4.1 - 4.3 and the footnote [1] on p. 4.

As for Assertion (ii), we apply Theorem 4.1 and its Corollary 4.1, and then the fact that (4.29) implies

$$< \mathfrak{m}(\mathbb{U}^n - k_n^{-\frac{1}{2}} \mathbb{S}^n) > \to 0$$

in $P^n$ probability as $n \to \infty$, each $t$, $0 \leqslant t \leqslant 1$, which is established by using the arguments leading to (3.13). The latter fact ensures the property (4.26) for

$$r'^n = r^n + 2\mathfrak{m}(\mathbb{U}^n - k_n^{-\frac{1}{2}} \mathbb{S}^n) .$$

□

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