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D. QUADE

AVERAGE INTERNAL RANK CORRELATION

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat 49, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

SUMMARY

Statistics proposed in the literature for the analysis of n independent rankings are in most instances equivalent to an average internal rank correlation, obtained as follows: Choose some index of rank correlation, calculate its value for each pair of rankings in the data, and average these values over all pairs. This paper is concerned with the asymptotic sampling theory (as n tends to infinity) of such average correlations, and their use in testing hypotheses, particularly the hypothesis of random ranking. Spearman's rho and Kendall's tau are discussed in detail as special cases, and new tables of average correlation based on these indices are provided.

CONTENTS

1. INTRODUCTION

How to analyze independent rankings of the same objects, or more general sorts of data sets which it seems desirable to reduce thereto, has long been a question of considerable interest. It is hoped that this paper will provide some new insights into methods previously advanced for this purpose.

It is assumed throughout that the data consist of a random sample of independent rankings $\underline{Y}_i = (Y_{i1},\ldots,Y_{im})'$, $i = 1,\ldots,n$. A vector $\underline{s} = (s_1,\ldots,s_m)'$ is called a ranking if

$$s_k = \frac{m+1}{2} + \frac{1}{2}\sum_{j=1}^{m} \mathrm{sgn}(s_j - s_k) \qquad \text{for } k = 1,\ldots,m,$$

where $\mathrm{sgn}(x) = +1$, 0 or -1 according as x is positive, zero, or negative. In particular, a ranking is untied if its components are all distinct; they must then constitute some permutation of the integers $1,\ldots,m$. In all cases $\sum s_i = m(m+1)/2$.

A real-valued function $c(\underline{s}_u,\underline{s}_v)$, defined for all rankings $\underline{s}_u$ and $\underline{s}_v$, will be called an index of rank correlation if it satisfies the following three Fundamental Properties:

I For all rankings $\underline{s}_u$ and $\underline{s}_v$,

$$c(\underline{s}_u,\underline{s}_v) = c(\underline{s}_v,\underline{s}_u).$$

II For all rankings $\underline{s}_u$ and $\underline{s}_v$, if $\underline{s}_u^*$ is any permutation of $\underline{s}_u$, and $\underline{s}_v^*$ is the same permutation of $\underline{s}_v$, then

$$c(\underline{s}_u^*,\underline{s}_v^*) = c(\underline{s}_u,\underline{s}_v).$$

III For all rankings $\underline{s}_u$ and $\underline{s}_v$,

$$|c(\underline{s}_u,\underline{s}_v)| \le 1,$$

with equality attained in at least one instance.

It may be noted that the third property involves no real restriction; if for some function c it were not satisfied then a simple change of scale would establish it.

The statistics which have been proposed for the summary of n independent rankings, though they may seem quite diverse at first glance, can in most instances be obtained by the following procedure: Choose some index of rank correlation, calculate its value for each pair of rankings in the data, and average these values over all n(n-1)/2 pairs. The resulting statistic is called an <u>average internal rank correlation</u>. (Having introduced the word "internal", which distinguishes this statistic from the result of averaging the correlations of the n rankings with a fixed or "external" ranking, I shall omit it in what follows.)

Consider, for example, the familiar <u>Spearman rank correlation</u>, or <u>Spearman's rho</u>. For simplicity, suppose the data to be without ties, so that the Spearman correlation between any two Y's, say $\underline{Y}_i$ and $\underline{Y}_j$, is

$$r_{ij} = \frac{12 \sum_k Y_{ik}^2 - 3m(m+1)^2}{m^3 - m} .$$

(When ties are present this formula produces the "type-a" form of the index.) Then <u>average rho</u> is

$$R = \frac{2}{n(n-1)} \sum_{i<j} r_{ij} = \frac{1}{n-1} \{\frac{12K}{n(m^3-m)} - 1\},$$

where

$$K = \sum_k \{\sum_i Y_{ik} - \frac{n(m+1)}{2}\}^2 = \sum_k (\sum_i Y_{ik})^2 - \frac{nm(m+1)^2}{4} .$$

It is seen that two better-known statistics are linearly related to R: the <u>coefficient of concordance</u>

$$W = \frac{1}{n} \{1 + (n-1)R\} = \frac{12K}{n^2(m^3-m)}$$

of Kendall (1948, chapter 6), and the approximate chi-square

$$X_F = (m-1) \{1 + (n-1)R\} = \frac{12K}{n(m^2+m)}$$

of Friedman (1937).

Another average correlation may be based on the <u>Kendall rank correlation</u>, or <u>Kendall's tau</u>, which may be defined for two rankings $\underline{Y}_i$ and $\underline{Y}_j$ as

$$t_{ij} = \frac{2}{m(m-1)} \sum_{k<l} \operatorname{sgn}(Y_{ik}-Y_{il}) \operatorname{sgn}(Y_{jk}-Y_{jl}).$$

(When ties are present this formula produces the "type-a" form of the index.) Then <u>average tau</u> is

$$T = \frac{2}{n(n-1)} \sum_{i<j} t_{ij}.$$

This statistic has the same form as the coefficient of agreement proposed for paired comparisons by Kendall & Smith (1940), but was first seriously proposed for rankings by Ehrenberg (1952). An alternative approach is as follows. Consider any two rankings, say the i-th and j-th, and any two positions within them, say the k-th and l-th. Then the two positions are said to form a <u>concordant pair</u> with respect to the two rankings if the component in one position is larger in both rankings: i.e.

if $\quad Y_{ik} > Y_{il}\, ,\ Y_{jk} > Y_{jl} \quad$ or $\quad Y_{ik} < Y_{il}\, ,\ Y_{jk} < Y_{jl}.$

On the other hand, the pair is <u>discordant</u> if the position which has the larger component in one ranking has the smaller component in the other: i.e.

if $\quad Y_{ik} > Y_{il}\, ,\ Y_{jk} < Y_{jl} \quad$ or $\quad Y_{ik} < Y_{il}\, ,\ Y_{jk} > Y_{jl}.$

Define the integer L as the number of concordant pairs in the sample, less the number of discordant pairs. Then T is the ratio of L to the total number of possible pairs:

$$T = \frac{L}{\binom{n}{2}\binom{m}{2}} = \frac{4L}{(m^2-m)(n^2-n)}\ .$$

The remainder of this paper is concerned with the asymptotic sampling theory (as n tends to infinity) of average rank correlations, and their use in testing hypotheses, particularly the hypothesis of random ranking to be defined in Section 9. The Spearman and Kendall indices are discussed in detail as special cases.

2. AVERAGE CORRELATION AS A U-STATISTIC

Let there be given n independent observations $Y_1,\ldots,Y_n$ on a random variable Y. We shall consider statistics of the form

$$C = \sum_{i<j} c(Y_i,Y_j)/\binom{n}{2},$$

where the function c is symmetric: that is, $c(u,v) \equiv c(v,u)$. We see that C is the average of the function c taken over all pairs of observations. There exists a considerable body of theory about such averages, which are called U-statistics. A recent expository summary may be found in Puri & Sen (1971, chapter 3). However, we shall need only a few of the most elementary results.

Define

$$\gamma = E[c(Y_1,Y_2)],$$

where Y_1 and Y_2 are independent observations on Y; then clearly

$$E[C] = \gamma.$$

The parameter γ may be interpreted in the context of this paper as a measure of agreement; more specifically, it is the expected correlation between 2 independent rankings. Write also the variance

$$\eta = V[c(Y_1,Y_2)] = E[c^2(Y_1,Y_2)] - \gamma^2$$

and

$$\zeta = E[c(Y_1,Y_2)\ c(Y_1,Y_3)] - \gamma^2,$$

assuming these to exist. Now let us find an expression for the variance of C. For convenience, write

$$c_{ij} = c(Y_i,Y_j), \qquad i,j = 1,\ldots,n.$$

Then

$$n(n-1)C = 2\sum_{i<j} c_{ij} = \sum_i \sum_j c_{ij} - \sum_i c_{ii},$$

whence

$$\{n(n-1)C\}^2 = \sum_i \sum_j \sum_k \sum_l c_{ij}\ c_{kl} - 2\sum_i \sum_j \sum_k c_{ij}\ c_{kk} + \sum_i \sum_j c_{ii}\ c_{jj}$$

and

$$
\begin{aligned}
E[\{(n(n-1)C\}^2] = \{&nE[c_{11}^2] + 4n(n-1)E[c_{11}c_{12}] + n(n-1)E[c_{11}c_{22}] \\
&+ 2n(n-1)E[c_{12}^2] + 2n(n-1)(n-2)E[c_{11}c_{23}] \\
&+ 4n(n-1)(n-2)E[c_{12}c_{13}] \\
&+ n(n-1)(n-2)(n-3)E[c_{12}c_{34}]\} \\
- 2\{&nE[c_{11}^2] + n(n-1)E[c_{11}c_{22}] + 2E[c_{11}c_{12}] \\
&+ n(n-1)(n-2)E[c_{11}c_{23}]\} \\
+ \{&nE[c_{11}^2] + n(n-1)E[c_{11}c_{23}]\} \\
= 2&n(n-1)E[c_{12}^2] + 4n(n-1)(n-2)E[c_{12}c_{13}] \\
&+ n(n-1)(n-2)(n-3)E[c_{12}c_{34}].
\end{aligned}
$$

But $E[c_{12}^2] = \eta + \gamma^2$, $E[c_{12}c_{13}] = \zeta + \gamma^2$, and $E[c_{12}c_{34}] = \gamma^2$; hence

$$
\begin{aligned}
V[n(n-1)C] &= E[\{n(n-1)C\}^2] - \{n(n-1)\gamma\}^2 \\
&= 4n(n-1)(n-2)\zeta + 2n(n-1)\eta
\end{aligned}
$$

Thus

$$V[C] = \frac{4(n-2)\zeta + 2\eta}{n(n-1)} ,$$

and for large n $V[C] \sim 4\zeta/n$.

The quantity ζ may be estimated by the following simple method due to Sen (1960). For each $i = 1,\ldots,n$ define the <u>component</u>

$$C_i = \frac{1}{n-1} \sum_{j \neq i} c_{ij} ;$$

note that

$$C = \frac{1}{n} \sum_i C_i .$$

Now consider

$$Z = \frac{1}{n-1} \sum_i (C_i - C)^2.$$

An expansion similar to the one just given for V[C] shows that

$$E[Z] = \frac{(n-2)\{(n-4)\zeta + \eta\}}{(n-1)^2}\ .$$

An exact expression for V[Z] can also be obtained, but it is somewhat complicated. In any event, Sen shows that V[Z] tends to zero with increasing n, so that Z is a consistent estimator of ζ. Thus we may consider

$$S = \sqrt{\frac{4Z}{n}}$$

to be the asymptotic standard error of C, and write more briefly $C \pm S$. Combining this result with the central limit theorem for U-statistics due to Hoeffding (1948) yields

Theorem 1. (Hoeffding-Sen). If $0 < \zeta < \infty$ then as n increases without limit the quantity $(C-\gamma)/S$ is asymptotically distributed as a standard normal deviate.

This fundamental theorem provides a basis for statistical inference concerning the parameter γ, at least in large samples. For example, if $Q(\alpha)$ is defined by the relation

$$\frac{1}{\sqrt{2\pi}} \int_{-Q(\alpha)}^{Q(\alpha)} e^{-\frac{x^2}{2}}\, dx = \alpha$$

then a confidence interval on γ, with approximate confidence coefficient $100(1-\alpha)\%$, is

$$(C-SQ(\alpha),\ C+SQ(\alpha)).$$

Also, a test of size α for the hypothesis H: $\gamma = \gamma_0$ is obtained by rejecting if and only if the confidence interval fails to include γ_0. It is clear from Theorem 1 that this test is consistent against the general alternative H': $\gamma \neq \gamma_0$.

These results can easily be extended to the comparison of agreement within different groups of rankings. Thus suppose that $C_1,\ldots,C_k$ are the observed average correlations in independent samples of sizes $n_1,\ldots,n_k$ respectively, where the expected correlations are $\gamma_1,\ldots,\gamma_k$; and let $Z_1,\ldots,Z_k$ be the corresponding estimates of $\zeta_1,\ldots,\zeta_k$. Then the hypothesis

$$H: \gamma_1 = \ldots = \gamma_k$$

can be tested by referring the statistic

$$X^2 = \sum \frac{(C_i-\bar{C})^2}{4Z_i/n_i}, \quad \text{where} \quad \bar{C} = \sum \frac{n_i C_i}{Z_i} \Big/ \sum \frac{n_i}{Z_i}$$

to the χ^2-distribution with k-1 degrees of freedom. For the case k = 2 this is equivalent to rejecting at level α when

$$|C_1 - C_2| > S_{12}Q(\alpha), \quad \text{where} \quad S_{12} = \sqrt{\frac{4Z_1}{n_1} + \frac{4Z_2}{n_2}}$$

is the standard error of the difference (C_1-C_2).

The comparison of agreement has previously been considered by Linhart (1960) and Hays (1960). Linhart proposed, on heuristic grounds, a rather complicated test for comparing two coefficients of concordance (equivalent to average Spearman correlations). Hays proposed a narrower definition of agreement: his hypothesis is that the probability distribution is the same in each group of rankings, which may be false even though γ is the same as required by H above. He conjectured that "a relatively simple chi-square statistic might serve to test this hypothesis of equal agreement", where agreement is measured by Kendall correlation. The tests presented here provide a simple method for comparison of values of γ, though only a partial solution to Hays' problem.

It must be stressed, however, that all the asymptotic procedures presented so far depend on the assumption that $\zeta > 0$. Asymptotic results for the case $\zeta = 0$ are derived in the next section, and inference in the most common situations where this occurs is considered later. Theorems 3 through 6 give some simple criteria which often indicate whether $\zeta = 0$ or not.

3. AVERAGE CORRELATION AS A QUADRATIC FORM

To this point we have made no use of the fact that the random variable $\underline{Y}$ in our context is a ranking. One important consequence of this is that the sample space of $\underline{Y}$ is finite. Given any ranking $\underline{s}_u = (s_{u1},\ldots,s_{um})'$ of m components, define its inverse as $\underline{s}_u^- = (m+1-s_{u1},\ldots,m+1-s_{um})'$. Then one ranking is its own inverse, namely the completely tied ranking $((m+1)/2,\ldots,(m+1)/2)$, which we label $\underline{s}_0$. The other rankings form finitely many pairs, say f(m) of them; let them be labeled $\underline{s}_1,\ldots,\underline{s}_{2f}$, where $\underline{s}_{u+f}=\underline{s}_u^-$ for $u = 1,\ldots,f$. We also specify that $\underline{s}_1 = (1,\ldots,m)'$. Then in particular if m = 2 we have f = 1 and the 3 possible rankings are

$$\underline{s}_0 = (1.5,1.5)', \quad \underline{s}_1 = (1,2)', \quad \underline{s}_2 = (2,1)'.$$

Again, if m = 3 then f = 6 and one suitable enumeration of the 13 possible rankings is

$$\begin{array}{lll}
\underline{s}_0 = (2,2.2)' & \underline{s}_1 = (1,2,3)' & \underline{s}_7 = (3,2,1)' \\
 & \underline{s}_2 = (3,1,2)' & \underline{s}_8 = (1,3,2)' \\
 & \underline{s}_3 = (2,3,1)' & \underline{s}_9 = (2,1,3)' \\
 & \underline{s}_4 = (1,2.5,2.5)' & \underline{s}_{10} = (3,1.5,1.5)' \\
 & \underline{s}_5 = (2.5,1,2.5)' & \underline{s}_{11} = (1.5,3,1.5)' \\
 & \underline{s}_6 = (2.5,2.5,1)' & \underline{s}_{12} = (1.5,1.5,3)'
\end{array}$$

Define the symmetric $(2f+1) \times (2f+1)$ matrix

$$\Gamma = ((\gamma_{uv})) \quad \text{where} \quad \gamma_{uv} = c(\underline{s}_u,\underline{s}_v),$$

and suppose the probability assignment is the vector

$$\underline{p} = (p_0,p_1,\ldots,p_{2f})', \quad \text{where} \quad p_u = \Pr[\underline{Y}=\underline{s}_u].$$

Of course, we must also have $p_u \geq 0$ for all u, and $\sum p_u = 1$. Then the expected average correlation can be written

$$\gamma = E[c(Y_1,Y_2)] = \sum_u \sum_v p_u\, p_v\, \gamma_{uv} = \underline{p}'\Gamma\underline{p}.$$

Write also

$$\underline{\theta} = \Gamma\underline{p} = (\theta_0,\theta_1,\ldots,\theta_{2f})'$$

where

$$\theta_u = \sum_v p_u \gamma_{uv} \quad \text{for} \quad u = 0,1,\ldots,2f.$$

Then we see that

$$\gamma = \sum_u p_u \theta_u$$

and

$$\zeta = \sum_u p_u(\theta_u-\gamma)^2,$$

so clearly $\zeta > 0$ unless for every u either $p_u = 0$ or $\theta_u = \gamma$. In matrix notation

$$\zeta = \underline{p}'\Gamma\Omega\Gamma\underline{p}$$

where $\Omega = \Delta - \underline{pp}'$ and $\Delta = \text{diag}(p_0,p_1,\ldots,p_{2f})$.

Now, in the sample of n observations on $\underline{Y}$, let n_u be the number of times that $\underline{Y} = \underline{s}_u$ for $u = 0,1,\ldots,2f$, so that $\sum n_u = n$, and write $\underline{n} = (n_0,n_1,\ldots,n_{2f})'$. The average correlation is then

$$C = \frac{2}{n(n-1)} \sum_{i<j} c(\underline{Y}_i,\underline{Y}_j)$$

$$= \frac{2}{n(n-1)} \{ \sum_{u<v} n_u n_v \; c(\underline{s}_u,\underline{s}_v) + \sum_u \frac{n_u(n_u-1)}{2} c(\underline{s}_u,\underline{s}_u)\}$$

$$= \frac{1}{n(n-1)} \{\sum_u \sum_v n_u n_v \; c(\underline{s}_u,\underline{s}_v) - \sum_u n_u \; c(\underline{s}_u,\underline{s}_u)\},$$

or in matrix notation

$$C = \frac{\underline{n}'\Gamma\underline{n} - \underline{\phi}'\underline{n}}{n^2 - n} \quad \text{where} \quad \underline{\phi} = (\gamma_{00},\gamma_{11},\ldots,\gamma_{2f,2f})'.$$

Thus C is essentially a quadratic form.

Consider now the asymptotic distribution of C. To begin with, the random vector $\underline{n}$ has the multinomial distribution with parameters n and $\underline{p}$: its mean vector is $n\underline{p}$, and its variance matrix is $n\Omega$ where Ω is as already defined. Then

$$E[\underline{n}'\Gamma\underline{n}] = (n\underline{p})'\Gamma(n\underline{p}) + \mathrm{tr}[n\Omega\Gamma]$$

$$= n^2\underline{p}'\Gamma\underline{p} + n(\underline{\phi}'\underline{p}-\underline{p}'\Gamma\underline{p})$$

and

$$E[\underline{\phi}'\underline{n}] = n\underline{\phi}'\underline{p},$$

from which we verify that (for all n)

$$E[C] = \underline{p}'\Gamma\underline{p} = \gamma.$$

Define the random vector

$$\underline{w} = \frac{1}{\sqrt{n}}(\underline{n}-n\underline{p})$$

which has mean vector $\underline{0}$ and variance matrix Ω; then on substituting $\underline{n} = n\underline{p} + \sqrt{n}\,\underline{w}$ into the matrix expression for C a little matrix algebra yields

$$C = \frac{n^2\underline{p}'\Gamma\underline{p} + 2n\sqrt{n}\,\underline{p}'\Gamma\underline{w} + n\underline{w}'\Gamma\underline{w} - n\underline{\phi}'\underline{p} - \sqrt{n}\,\underline{\phi}'\underline{w}}{n^2 - n}$$

or

$$\frac{(n-1)(C-\gamma)}{\sqrt{n}} = 2\underline{p}'\Gamma\underline{w} + \frac{1}{\sqrt{n}}(\underline{w}'\Gamma\underline{w} - \underline{\phi}'\underline{p} + \gamma) - \frac{1}{n}\underline{\phi}'\underline{w}.$$

As n tends to infinity the last two terms on the right of this equation can be neglected. The term $2\underline{p}'\Gamma\underline{w}$ has (for all n) mean 0 and variance $4\underline{p}'\Gamma\Omega\Gamma p$ $= 4\zeta$ and is asymptotically normal. Thus we conclude that $\sqrt{n}(C-\gamma)$ is asymptotically normal with mean 0 and variance 4ζ: that is, we have reproved Hoeffding's result for our special case.

However, if $\zeta = 0$ the asymptotic normal distribution is degenerate. But then we have

Theorem 2. If $\zeta = 0$ then as n increases without limit the quantity $(n-1)(C-\gamma)$ has asymptotically the same distribution as $\Sigma\lambda_i(Q_i^2-1)$, where the Q's are independent standard normal variables and the λ's are the nonzero characteristic roots of $\Gamma\Omega$.

Proof. If $\zeta = 0$ then $\underline{p}'\ \underline{w} = 0$ with certainty. Hence

$$(n-1)(C-\gamma) = (\underline{w}'\Gamma\underline{w} - \underline{\phi}'\underline{p} + \gamma) - \frac{1}{\sqrt{n}}\underline{\phi}'\underline{w}.$$

For large n the last term can be neglected. Note that the trace of $\Gamma\Omega$ is $\underline{\phi}'\underline{p} - \gamma = \Sigma\lambda_i$. The theorem then follows from well-known results on the distribution of quadratic forms in normal variables: see for example Puri & Sen (1971, chapter 2).

The first of several theorems which help to decide whether $\zeta = 0$ or not is

Theorem 3. If the probability assignment $\underline{p}$ produces a maximum or minimum of γ among those assignments for which all components in some specified set (possibly null) vanish, then $\zeta = 0$.

Proof. Write $\gamma(\varepsilon) = \underline{q}'\Gamma\underline{q}$ where $\underline{q} = (q_0, q_1, \ldots, q_{2f})'$ and

$$q_u = p_u\{1 + \varepsilon(\theta_u - \underline{p}'\Gamma\underline{p}) \qquad \text{for } u = 0,1,\ldots,2f.$$

Note that $q_u = 0$ if $p_u = 0$; also, $\Sigma q_u = 1$, and hence $\underline{q}$ is a probability assignment if $|\varepsilon|$ is small enough that $1 + \varepsilon(\theta_u - \underline{p}'\Gamma p) \geq 0$ for all u such that $p_u > 0$. But a simple calculation shows that

$$\frac{d}{d\varepsilon}\gamma(\varepsilon)\Big|_{\varepsilon=0} = 2\zeta.$$

Hence $\zeta = 0$, since otherwise $\gamma(0) = \underline{p}'\Gamma\underline{p} = \gamma$ would not be a maximum or minimum as hypothesized.

(The fact that $\zeta = 0$ if any $p_u = 1$ may be regarded as a trivial verification of this theorem.)

To this point our use of the term "correlation" has been purely gratuitous, for we have not really began to exploit the properties of correlation as it is ordinarily understood; all the results obtained so far hold for any function γ of two rankings which satisfies only Fundamental Property I. One consequence of Fundamental Property II is of some interest, however. Define a permutation set as a set of rankings consisting of all the permutations of a single ranking: two examples are the set of all untied rankings and the set containing only the one ranking $\underline{s}_0$. Then consider the sum $\Sigma c(\underline{s}_u,\underline{s}_v)$ where $\underline{s}_v$ ranges over a permutation set V. By Fundamental Property II this sum is the same for all $\underline{s}_u$ in the same permutation set U; indeed, for different rankings $\underline{s}_u$ in U the sum involves only different permutations of the same quantities. Thus it is legitimate to define the function

$$\alpha(U,V) = \sum_{\underline{s}_v \in V} c(\underline{s}_u,\underline{s}_v) \qquad \text{where } \underline{S}_u \in U,$$

U and V being any permutation sets. And then it is trivial to prove

Theorem 4. Let the h(V) members of some permutation set V all have equal probability 1/h(V). Then, for every permutation set U, $\theta_u = \alpha(U,V)/h(V)$ for every ranking $\underline{s}_u$ in U; also $\gamma = \alpha(V,V)/h(V)$ and $\zeta = 0$.

4. SYMMETRIC CORRELATION INDICES

An index of correlation will be called symmetric if, for all rankings $\underline{s}_u$ and $\underline{s}_v$,

$$c(\underline{s}_u,\underline{s}_v^-) = -c(\underline{s}_u,\underline{s}_v).$$

An immediate consequence of symmetry is that

$$c(\underline{s}_u,\underline{s}_0) = c(\underline{s}_0,\underline{s}_u) = 0$$

for every ranking $\underline{s}_u$. The indices of rank correlation in common use are all symmetric, but others do appear in the literature. One such is "Spearman's footrule". Also, those measures of association which do not distinguish positive from negative correlation cannot be symmetric: for example, the indices of "squared correlation" discussed briefly in Section 7.

The first nontrivial consequence of symmetry is

Theorem 5. For a symmetric index, if each ranking has the same probability as its inverse then the vector $\underline{\theta}$ is null, and hence $\gamma = \zeta = 0$.

Proof. For a symmetric index of correlation, Γ has the form

$$\Gamma = \left(\begin{array}{c|c|c} 0 & \underline{0}' & \underline{0}' \\ \hline \underline{0} & \Gamma_1 & -\Gamma_1 \\ \hline \underline{0} & -\Gamma_1 & \Gamma_1 \end{array}\right)$$

where $\underline{0}$ is the null vector of order f and Γ_1 is a symmetric f×f matrix. Write

$$\underline{p} = \left(\begin{array}{c} \underline{p}_0 \\ \hline \underline{p}_1 \\ \hline \underline{p}_2 \end{array}\right)$$

where $\underline{p}_1$ and $\underline{p}_2$ are vectors of f components each. By hypothesis $\underline{p}_1 = \underline{p}_2$. Hence $\underline{\theta} = \Gamma\underline{p} = \underline{0}$.

A second consequence of symmetry, a simple condition sufficient to ensure that $\zeta > 0$ (and hence that C is asymptotically normal), is

Theorem 6. For a symmetric index, if $\gamma \neq 0$, and if there exists any ranking (possibly $\underline{s}_0$) such that it and its inverse both have positive probabilities, then $\zeta > 0$.

Proof. Suppose first that $\underline{s}_0$ satisfies the hypothesis, so $p_0 > 0$. By symmetry $\gamma_{0w} = 0$ for all w, and hence $\theta_0 = \Sigma p_w \gamma_{0w} = 0$. Then

$$\zeta = \sum_w p_w(\theta_w - \gamma)^2 > p_0(\theta_0 - \gamma)^2 = p_0\gamma^2 > 0.$$

Now suppose some other ranking $\underline{s}_u$ satisfies the hypothesis and write $\underline{s}_v = \bar{\underline{s}}_u$; we have $p_u > 0$ and $p_v > 0$. By symmetry $\gamma_{vw} = -\gamma_{uw}$ for all w, and hence

$$\theta_v = \sum_w p_w \gamma_{vw} = -\sum_w p_w \gamma_{uw} = -\theta_u.$$

Then

$$\zeta > p_u(\theta_u - \gamma)^2 + p_v(\theta_v - \gamma)^2 = p_u(\theta_u - \gamma)^2 + p_v(-\theta_u - \gamma)^2 > 0.$$

5. MULTIPLICATIVE CORRELATION INDICES

We shall say that an index of correlation is multiplicative if there exists a function $g(x,y)$ such that for all rankings $\underline{s}_u$ and $\underline{s}_v$

$$c(\underline{s}_u,\underline{s}_v) = \alpha_u\alpha_v \sum_{k=1}^{m} \sum_{l=1}^{m} g(s_{uk},s_{ul})\, g(s_{vk},s_{vl}),$$

where $\alpha_0,\alpha_1,\ldots,\alpha_{2f}$ are constants chosen to standardize the index. There are two main multiplicative types, defined by the way in which the α's are determined. In type a

$$\alpha_u = \frac{1}{\sqrt{\beta}} \qquad \text{for all } u,$$

and in type b

$$\alpha_u = \frac{1}{\sqrt{\beta_u}} \qquad \text{if } \beta_u > 0, \text{ and otherwise } \alpha_u = 0,$$

where

$$\beta_u = \sum_k \sum_l g^2(s_{uk},s_{ul}), \qquad \beta = \max_u \beta_u.$$

To see that the type a and b indices are indeed standardized, let $\underline{s}_w$ be any ranking such that

$$\sum_k \sum_l g^2(s_{wk},s_{wl}) = \beta;$$

clearly $c(\underline{s}_w,\underline{s}_w) = 1$, and then by Cauchy's inequality $c^2(\underline{s}_u,\underline{s}_v) \leq 1$ for all rankings $\underline{s}_u$ and $\underline{s}_v$.

Well-known multiplicative indices include Spearman's rho and Kendall's tau. Some indices which are not multiplicative, though symmetric, are the coefficient "gamma" of Goodman & Kruskal (1954) and the variously-defined indices of quadrant or median correlation. The terminology "a" and "b" corresponds to the usage in Kendall (1948, chapter 3) where he defines the indices ρ_a, ρ_b, τ_a, and τ_b.

Theorem 7. For a multiplicative index, the matrix Γ is positive semidefinite with rank at most $m(m-1)/2$.

Proof. The expression given above for γ_{uv} can be rewritten as

$$\gamma_{uv} = 2\alpha_u\alpha_v \sum_{k<l} g(s_{uk},s_{ul})\, g(s_{vk},s_{vl})$$

$$= 2\alpha_u\alpha_v \sum_{j=1}^{m(m-1)/2} g(s_{uk_j},s_{ul_j})\, g(s_{vk_j},s_{vl_j})$$

where the integers k_j and l_j are uniquely defined so that $1 \le k_j \le l_j \le m$ and $j = k_j + (l_j-1)(l_j-2)/2$. Thus the matrix $\Gamma = \Xi'\Xi$ where $\Xi = ((\xi_{ju}))$ has $m(m-1)/2$ rows and $(2f+1)$ columns with

$$\xi_{ju} = \sqrt{2}\ \alpha_u\ g(s_{uk_j},s_{ul_j}).$$

An immediate consequence is the following

Corollary. For a multiplicative index, let

$$\psi_{kl} = \sum_u p_u\ \alpha_u\ g(s_{uk},s_{ul}) \qquad \text{for } k,l = 1,\dots,m.$$

Then

$$\theta_u = \alpha_u \sum_k \sum_l \psi_{kl}\ g(s_{uk},s_{ul})$$

and hence

(i) $\gamma = \sum_k \sum_l \psi_{kl}^2 \ge 0,$

(ii) $\gamma = 0$ if and only if $\psi_{kl} = 0$ for all k and l,

(iii) if $\gamma = 0$ then $\zeta = 0$ also.

Two further results concerning multiplicative indices are as follows:

Theorem 8. If a multiplicative index of type a or b has

$$g(x,y) \equiv -g(m+1-x,m+1-y)$$

then it is symmetric.

Proof. Under the hypothesized condition on g the β's corresponding to inverse rankings are equal, and hence also the α's. Thus

$$c(\underline{s}_u,\underline{s}_v^-) = \alpha_u\alpha_v \sum_k \sum_l g(s_{uk},s_{ul})\ g(m+1-s_{vk},m+1-s_{vl})$$

$$= -\alpha_u\alpha_v \sum_k \sum_l g(s_{uk},s_{ul})\ g(s_{vk},s_{vl})$$

$$= -c(\underline{s}_u,\underline{s}_v).$$

Theorem 9. If a multiplicative index of type a or b has

$$g(x,y) \equiv -g(y,x)$$

then $\alpha(U,V) = 0$ for all permutation sets U and V.

Proof. Let $\underline{s}_u$ be any ranking in U, and write

$$\alpha(U,V) = \sum_{\underline{s}_v \in V} c(\underline{s}_u,\underline{s}_v) = \alpha_u \sum_{k=1}^{m} \sum_{l=1}^{m} g(\underline{s}_{uk},\underline{s}_{ul}) \sum_{\underline{s}_v \in V} \alpha_v\ g(\underline{s}_{vk},\underline{s}_{vl}).$$

Now β_v does not vary for $\underline{s}_v$ in V, so neither does α_v. Thus α_v may be taken out of the inner summation, which then vanishes because of the condition on g.

The reader may notice that a multiplicative index of type b which satisfies the hypothesis of Theorem 9 is the same as the generalized correlation coefficient of Daniels (1944).

6. SPEARMAN AND KENDALL CORRELATION

Consider a multiplicative index of correlation with

$$g(x,y) = x - y.$$

With this definition of g - which, it may be noted, satisfies the hypotheses of both Theorems 8 and 9 - we have

$$\gamma_{uv} = \alpha_u \alpha_v \sum_k \sum_l (\underline{s}_{uk} - \underline{s}_{ul})(s_{vk} - s_{vl})$$

$$= 2m\, \alpha_u \alpha_v \sum_k (s_{uk} - \frac{m+1}{2})(s_{vk} - \frac{m+1}{2}).$$

For any untied ranking $\underline{s}_u$,

$$\beta_u = \sum_k \sum_l (s_{uk} - s_{ul})^2 = \sum_k \sum_l (k-l)^2 = \frac{m^2(m^2-1)}{6} ,$$

and this is also the value of β, since β_u is smaller for all tied rankings. A little algebra now shows that the type a index is identical with the Spearman correlation as defined in Section 1. For the type b index, so long as neither of the rankings $\underline{s}_u$ and $\underline{s}_v$ is completely tied, we have

$$\gamma_{uv} = \frac{\sum_k (s_{uk} - \frac{m+1}{2})\ (s_{vk} - \frac{m+1}{2})}{\sqrt{\sum_k (s_{uk} - \frac{m+1}{2})^2 \sum_k (s_{vk} - \frac{m+1}{2})^2}} ,$$

which is the same as the ordinary Pearsonian product-moment correlation calculated from the ranks. We can also prove

<u>Theorem 10</u>. For Spearman correlation, the rank of Γ does not exceed m-1.

<u>Proof</u>. Define the matrix $\Xi = ((\xi_{ju}))$, of m rows and (2f+1) columns, where

$$\xi_{ju} = \sqrt{2m\alpha_u}\ (s_{uj} - \frac{m+1}{2}).$$

Then clearly $\Gamma = \Xi'\Xi$. But the row sums of Ξ are all zero; so the rank of Ξ, and hence of Γ, is at most (m-1).

Finally, for k = 1,...,m define the <u>expected rank</u>

$$\varepsilon_k = E[Y_{ik}] = \sum_u p_u\, s_{uk};$$

then we have

Theorem 11. For the Spearman index of type a,

$$\gamma = \frac{12}{m^3-m} \sum_k (\varepsilon_k - \frac{m+1}{2})^2,$$

and $\gamma = 0$ if and only if the expected ranks are all equal.

Proof. Write

$$\psi_{kl} = \frac{1}{\sqrt{\beta}} \sum_k p_u(s_{uk}-s_{ul}) = \sqrt{\frac{6}{m^2(m-1)}}\,(\varepsilon_k-\varepsilon_l)$$

in the Corollary to Theorem 7.

This result is well-known for the case where no ties are allowed, in which the Spearman indices of types a and b are equal. That it is not true in general for the type b index is shown by Example 5.

Now consider a multiplicative index of correlation with

$$g(x,y) = \operatorname{sgn}(x-y).$$

With this definition of g - which also satisfies the hypotheses of both Theorems 8 and 9 - we have for any untied ranking $\underline{s}_u$ that

$$\beta_u = \sum_k \sum_l \operatorname{sgn}^2(s_{uk}-s_{ul}) = \sum_k \sum_l \operatorname{sgn}^2(k-l) = m(m-1),$$

and this is also the value of β, since β_u is smaller for all tied rankings. Thus clearly the type a index is identical with the Kendall correlation as defined in Section 1. For the type b index, so long as neither of the rankings $\underline{s}_u$ and $\underline{s}_v$ is completely tied, we have

$$\gamma_{uv} = \frac{\sum\limits_{k<l} \operatorname{sgn}(Y_{ik}-Y_{il})\, \operatorname{sgn}(Y_{jk}-Y_{jl})}{\sqrt{\beta_u\beta_v}},$$

where β_w is the number of untied pairs of components of the ranking $\underline{s}_w$ for $\underline{w} = 0,1,\ldots,2f$. We have now

Theorem 12. For the Kendall index of type a,

$$\gamma = \frac{1}{m(m-1)} \sum_k \sum_l (\Pr[Y_{ik}>Y_{il}] - \Pr[Y_{ik}<Y_{il}])^2,$$

and $\gamma = 0$ if and only if $\Pr[Y_{ik}>Y_{il}] = \Pr[Y_{ik}<Y_{il}]$ for $k,l = 1,\ldots,m$.

Proof. Write

$$\psi_{kl} = \frac{1}{\sqrt{\beta}} \sum_u p_u \operatorname{sgn}(s_{uk}-s_{ul}) = \frac{\Pr[Y_{ik}>Y_{il}] - \Pr[Y_{ik}<Y_{il}]}{\sqrt{m(m-1)}}$$

in the Corollary to Theorem 7.

This result had previously been discovered by Hays (1960), for the case where no ties are allowed, in which the Kendall indices of types a and b are equal. That it is not true in general for the type b index is shown by Example 5.

A final result of some interest is

Theorem 13. If for some probability assignment the type a Kendall index has expectation zero, then so does the type a Spearman index.

Proof. For each $k = 1,\ldots,m$ write

$$\varepsilon_k = \sum_u p_u s_{uk}$$

$$= \sum_u p_u \left\{\frac{m+1}{2} - \frac{1}{2} \sum_l \operatorname{sgn}(s_{uk}-s_{ul})\right\}$$

$$= \frac{m+1}{2} - \frac{1}{2} \sum_l \sum_u p_u \operatorname{sgn}(s_{uk}-s_{ul}).$$

From the proof of Theorem 12, if the type a Kendall index has expectation zero then

$$\sum_k p_u \operatorname{sgn}(s_{uk}-s_{ul}) = 0$$

for all k and l. Hence $\varepsilon_k = (m+1)/2$ for all k, and the type a Spearman index has expectation zero by Theorem 11.

That the theorem does not hold for the type b indices is shown by Example 5.

7. SQUARED CORRELATION

It was suggested by Ehrenberg (1952), though apparently not seriously, that an average rank correlation might be based on squared Kendall correlations. More generally, starting with any index of correlation defined by a function $c(\underline{Y}_i, \underline{Y}_j)$, consider the statistic

$$S = \sum_{i<j} c^2(\underline{Y}_i, \underline{Y}_j)/\binom{n}{2}.$$

Let the matrix $\Psi = ((\psi_{uv}))$ where $\psi_{uv} = c^2(\underline{s}_u, \underline{s}_v)$, and $E[S] = \psi = \underline{p}'\Psi\underline{p}$; it may be of interest to ask what values ψ can assume. An answer to this question is provided by

Theorem 14. For a squared correlation based on a symmetric multiplicative index, with ties disallowed, the expected value is minimized if for each ranking the sum of its probability and that of its inverse is the same.

Proof. Having eliminated those rows and columns of Ψ which correspond to tied rankings, note from the symmetry of the original index that we can write

$$\Psi = \left(\begin{array}{c|c} \Psi_1 & \Psi_1 \\ \hline \Psi_1 & \Psi_1 \end{array}\right)$$

where Ψ_1 is of order $m!/2$. By Theorem 7, since the original index is multiplicative, Γ is positive semidefinite; thence also Ψ - see Theorem 12.2.8 of Graybill (1969); and thence also Ψ_1, which is a principal minor of Ψ. Now eliminating also the unnecessary elements of the vector $\underline{p}$, write

$$p = \left(\begin{array}{c} \underline{p}_1 \\ \hline \underline{p}_2 \end{array}\right)$$

where $\underline{p}_1$ and $\underline{p}_2$ each have $m!/2$ elements. Then

$$\psi = (\underline{p}_1+\underline{p}_2)'\ \Psi_1(\underline{p}_1+\underline{p}_2).$$

Now write

$$\underline{p}_1 + \underline{p}_2 = \frac{2}{m!}\,\underline{j} + (\underline{p}_1+\underline{p}_2-\frac{2}{m!}\,\underline{j})$$

where $\underline{j}$ is the vector of $m!/2$ components each equal to unity; then

$$\psi = (\frac{2}{m!})^2\, \underline{j}'\Psi_1\underline{j} + \frac{4}{m!}\, j'\Psi_1(\underline{p}_1+\underline{p}_2-\frac{2}{m!}\,\underline{j}) + (\underline{p}_1+\underline{p}_2-\frac{2}{m!}\,\underline{j})'\Psi_1(\underline{p}_1+\underline{p}_2-\frac{2}{m!}\,\underline{j}).$$

From Fundamental Property II it can be seen that each column of Ψ_1 contains a different permutation of the same elements, so that the column sums are all equal to some constant, say ξ: that is, $\underline{j}'\Psi_1 = \xi\underline{j}'$. Thus the first term of this expression for ψ equals $2\xi/m!$, and the middle term always equals zero. The last term is zero if $\underline{p}_1 + \underline{p}_2 = \frac{2}{m!}\,\underline{j}$, and it cannot be negative since Ψ_1 is positive semidefinite.

Incidentally, we also have immediately

Theorem 15. For a symmetric multiplicative index, with ties disallowed, if $\gamma = 0$ then η (and hence also the variance of C) is minimized if for each ranking the sum of its probability and that of its inverse is the same.

The actual value of the minimum in Theorems 14 and 15 is of course $2\xi/m!$, where ξ is the common column sum of Ψ_1.

With the definition of squared correlation as given the case where ties may occur is of little interest: for example, if $p_0 = 1$ then $\psi = 0$. Kendall (1948, chapter 3) shows that the type a Spearman and Kendall indices are equivalent to the result of defining the correlation in the presence of ties as the average of the values which would be obtained if the ties were broken in all possible ways. The same idea might be used in defining squared correlations. However, this topic will not be pursued further here.

8. THE HYPOTHESIS OF ZERO CORRELATION

Consider testing the hypothesis of zero correlation, that is

$$H_0\colon \gamma = 0,$$

using an index for which $\gamma = 0$ implies $\zeta = 0$, so that Theorem 2 applies: for example, any multiplicative index.

By Theorem 2 the asymptotic distribution of the quantity $\Sigma\lambda_i+(n-1)C$

under H_0 is the same as that of $\Sigma\lambda_i Q_i^2 = S$, say, where the λ's are the non-zero characteristic roots of $\Gamma\Omega$ and the Q's are independent standard normal variables. Thus for any c

$$\Pr[C \geq c] \doteq \Pr[S \geq \Sigma\lambda_i + (n-1)C].$$

Now, assuming $\Sigma\lambda_i \geq 0$, we have

$$S \leq (\Sigma\lambda_i) \max_i Q_i^2,$$

so

$$\Pr[C \geq c] \leq \Pr[\max_i Q_i^2 \geq 1 + (n-1)C/\Sigma\lambda_i].$$

But by standardization $\Sigma\lambda_i \leq 1$, so

$$\Pr[C \geq c] \leq 1 - \{\Pr[Q^2 \leq 1 + (n-1)C]\}^k$$

where Q is a standard normal variable and k is the rank of $\Gamma\Omega$. To put this another way, if C_α is the critical value for the test of H_0 based on C, and if $Q(\alpha)$ is the normal critical value as defined in Section 2, then asymptotically

$$C_\alpha \leq \frac{Q^2(1-\sqrt[k]{1-\alpha}) - 1}{n-1} \leq \frac{Q^2(\alpha/k) - 1}{n-1} .$$

The procedure thus derived is clearly quite conservative, but nevertheless it is consistent against the general alternative H_0': $\gamma > 0$. This is because C always converges to γ in probability, and so under H_0' it must eventually exceed the stated bound for C_α. Note that for a multiplicative index $k \leq m(m-1)/2$ by Theorem 7, and for Spearman correlation in particular $k \leq m-1$ by Theorem 10.

Particularly for large m, it may be preferable to use simple Chebyshev-type bounds, based on the fact that if a random variable G cannot be negative then $\Pr[G \geq g]$ is less than $E[G^x]/g^x$ for any $x > 0$. Thus let $G = 1 + (n-1)C$, so that $\Pr[C \geq c] = P[G \geq 1 + (n-1)C]$. Substituting $\gamma = \zeta = 0$ into the formulas of Section 2 yields $E[C] = 0$, so $E[G] = 1$ and hence

$$\Pr[C \geq c] \leq \frac{1}{1 + (n-1)C} ;$$

also, $E[C^2] = 2\eta/n(n-1)$, so $E[G^2] = 1 + 2\eta(n-1)/n \leq 3$ and

$$\Pr[C \geq c] \leq \frac{3}{\{1 + (n-1)C\}^2} .$$

Further such bounds could be obtained using other values of x. These bounds, though crude, have at least the advantage of being exact for all n.

An alternative approach involves approximating the exact distribution under H_0. Given the asymptotic form, it seems reasonable to fit a chi-square: so let say $X = A + BC$ be approximated by a chi-square with D degrees of freedom, determining A, B, and D to give the first three moments correctly. The first two moments are as stated in the previous paragraph. For the third moment we may make an expansion similar to that used in Section 2 for the variance, starting from

$$C^3 = \sum_{i_1<i_2} \sum_{j_1<j_2} \sum_{k_1<k_2} c_{i_1i_2} c_{j_1j_2} c_{k_1k_2} / \binom{n}{2}^3 .$$

Since $\zeta = 0$, it follows that for every u either $\theta_u = \gamma = 0$ or else $p_u = 0$, and hence that

$$E[c_{i_1i_2} c_{j_1j_2} c_{k_1k_2}] = 0$$

if any one of the 6 subscripts is different from all the others; this considerably reduces the number of terms involved. A bit of algebra now yields

$$E[C^3] = \{\frac{2}{n(n-1)}\}^2 \{2(n-2)\omega+\mu\},$$

where

$$\omega = E[c(\underline{Y}_1,\underline{Y}_2)\ c(\underline{Y}_2,\underline{Y}_3)\ c(\underline{Y}_3,\underline{Y}_1)],$$

$$\mu = E[c^3(\underline{Y}_1,\underline{Y}_2)].$$

The three parameters of the chi-square approximation are then

$$A = D = \frac{4n(n-1)\eta^3}{\{2(n-1)\omega+\mu\}^2}, \quad B = \frac{2n(n-1)\eta}{2(n-2)\omega+\mu} .$$

The unknown parameters η, ω, and μ can be estimated by simple U-statistics:

$$\hat{\eta} = \sum_{i<j} c^2(\underline{Y}_i,\underline{Y}_j)/\binom{n}{2},$$

$$\hat{\omega} = \sum_{i<j<k} c(\underline{Y}_i,\underline{Y}_j)\ c(\underline{Y}_j,\underline{Y}_k)\ c(\underline{Y}_k,\underline{Y}_i)/\binom{n}{3},$$

$$\hat{\mu} = \sum_{i<j} c^3(\underline{Y}_i,\underline{Y}_j)/\binom{n}{2}.$$

Thus the suggested procedure for testing H_0: $\gamma = 0$ is to take the quantity

$$X = D\{1 + \frac{2(n-2)\hat{\omega} + \hat{\mu}}{2\hat{\eta}^2} C\}$$

as a chi-square with

$$D = \frac{4n(n-1)\hat{\eta}^3}{\{2(n-2)\hat{\omega} + \hat{\mu}\}^2}$$

degrees of freedom. As n tends to infinity this becomes asymptotically equivalent to using as critical value for nC the quantity $(\omega/\eta)(\chi^2_\alpha - \eta^2/\omega^2)$, where χ^2 is the critical value for a χ^2_α with η^3/ω^2 degrees of freedom; and certainly such a test is consistent against the general alternative H_0': $\gamma > 0$. This might be called an "asymptotically distribution-free approximate test", where "asymptotic" refers to the fact that parameters have been estimated, and "approximate" to the fact that the true distribution is not in general a single chi-square but a mixture even for infinite n. It may be noted that Stuart (1951) used a somewhat similar procedure, but with less justification, for Spearman correlation only.

For future reference let us calculate the fourth moment of the average correlation, still assuming $\gamma = \zeta = 0$, but applying the same method. This turns out to be

$$E[C^4] = \{\frac{2}{n(n-1)}\}^3 \{(n-2)(n-3)\ (\frac{3\eta^2}{2} + 6\varepsilon) + 6(n-2)(\phi+12\nu) + \psi\},$$

where

$$\varepsilon = E[c(\underline{Y}_1,\underline{Y}_2)\ c(\underline{Y}_2,\underline{Y}_3)\ c(\underline{Y}_3,\underline{Y}_4)\ c(\underline{Y}_4,\underline{Y}_1)],$$

$$\phi = E[c^2(\underline{Y}_1,\underline{Y}_2)\ c^2(\underline{Y}_1,\underline{Y}_3)],$$

$$\nu = E[c^2(\underline{Y}_1,\underline{Y}_2)\ c(\underline{Y}_1,\underline{Y}_3)\ c(\underline{Y}_2,\underline{Y}_3)],$$

$$\psi = E[c^4(\underline{Y}_1,\underline{Y}_2)].$$

Let us also set down the standard measures of skewness

$$\beta_1 = \frac{(E[C^3])^2}{(E[C^2])^3} = \frac{2\{2(n-2)\omega + \mu\}^2}{n(n-1)\eta^3}$$

$$= \frac{8\omega^2}{\eta^3} - \frac{24\omega^2 - 8\omega\mu}{n\eta^3} + \frac{8\omega^2 - 8\omega\mu + 2\mu}{n^2\eta^3} + O(\frac{1}{n^3})$$

and kurtosis

$$\beta_2 = \frac{E[C^4]}{(E[C^2])^2} = \frac{3(n-2)(n-3)(\eta^2+4\varepsilon) + 12(n-2)(\phi+2\nu) + 2\psi}{n(n-1)\eta^2}$$

$$= 3 + \frac{12\varepsilon}{\eta^2} - \frac{12(\eta^2+4\varepsilon-\phi-2\nu)}{n\eta^2} + \frac{6(\eta^2+4\varepsilon-2\phi-4\nu+\psi/3)}{n^2\eta^2} + O(\frac{1}{n^3}) .$$

9. RANDOM RANKING

In this section we consider using an average correlation to test the narrow hypothesis

H_1: ranking is at random,

where by definition random ranking occurs if $p_v = p_u$ whenever $\underline{s}_v$ is a permutation of $\underline{s}_u$: that is, if within any permutation set V all rankings have the same value of p, say q(v). This hypothesis commonly arises when the observed rankings $\underline{Y}_1,\ldots,\underline{Y}_n$ have been obtained by converting underlying more general observations $\underline{X}_1,\ldots,\underline{X}_n$ into ranks. Then the ranking is at random if (though not only if) the components of each $\underline{X}$ are independent and identically distributed, and the rejection of H_1 implies the rejection of one or both of these conditions on $\underline{X}$.

An important special case is that where q(v) = 0 for all permutations except one. Then the asymptotic result obtained by combining Theorems 2 and 4 is given by

Theorem 16. Let the h members of some permutation set V all have equal probability $1/h$, and let $\alpha(V,V) = 0$. Then as n increases without limit the quantity $h(n-1)C$ has asymptotically the same distribution as $\Sigma\xi_i(Q_i^2-1)$, where the Q's are independent standard normal variables, and the ξ's are the nonzero characteristic roots of Γ_0, that portion of Γ which pertains to the rankings in V.

Proof. By Theorem 4 $\zeta = 0$, so Theorem 2 may be applied, specializing to the case where $\gamma = 0$. Let the matrix Γ be rearranged to put Γ_0 in its upper left corner, so that

$$\Gamma\Delta = \left(\begin{array}{c|c} \frac{1}{h}\Gamma_0 & 0 \\ \hline A & 0 \end{array}\right) \qquad \text{and} \qquad \Gamma\underline{pp}' = \left(\begin{array}{c|c} 0 & 0 \\ \hline B & 0 \end{array}\right)$$

Then the characteristic roots of $\Gamma\Omega = \Gamma\Delta - \Gamma\underline{pp}'$ are seen to be $1/h$ times those of Γ_0, independently of the matrices A and B.

Suppose in particular that $V = V_1$, the set of untied rankings, with $h(V) = m!$. Then in principle the exact sampling distribution of the average correlation coefficient can be worked out - by complete enumeration if no more convenient method can be found, although in practice this is feasible only for very small m and n. But if $\alpha(V_1,V_1) = 0$ then it is necessary only to set down the corresponding matrix Γ_0 and find its characteristic roots in order to obtain an approximate test of H_1. The parameters which determine the first four moments of C are

$$\eta = \Sigma(\xi_i/m!)^2, \quad \omega = \Sigma(\xi_i/m!)^3, \quad \varepsilon = \Sigma(\xi_i/m!)^4$$

$$\psi = \Sigma\Sigma(\gamma_{uv}^2/m!)^2, \quad \phi = \Sigma\Sigma\Sigma(\gamma_{uv}\gamma_{uw}/m!)^2$$

$$\mu = \Sigma\Sigma\gamma_{uv}^3/(m!)^2, \quad \nu = \Sigma\Sigma\Sigma\gamma_{uv}^2\gamma_{uw}\gamma_{vw}/(m!)^3.$$

Assume the index of correlation being used is symmetric: this is a simple condition sufficient to ensure that $\alpha(V_1,V_1) = 0$, although the example of Spearman's footrule shows it not to be necessary. Then $\phi = \eta^2$, $\mu = \nu = 0$, and we may take the test statistic

$$X = \frac{n(n-1)}{(n-2)^2} \cdot \frac{\eta^3}{\omega^2} [1 + (n-2) \frac{\omega}{\eta^2} C]$$

as a chi-square with $\{n(n-1)/(n-2)^2\}(\eta^3/\omega^2)$ degrees of freedom. The first three moments of X have been arranged to agree exactly with those of the approximating chi-square. The kurtosis of X, or C, is

$$\beta_2 = \frac{3(n-2)(n-3)(\eta^2+4\varepsilon) + 12(n-2)\eta^2 + 2\psi}{n(n-1)\eta^2}$$

$$= 3 + \frac{12\varepsilon}{\eta^2} - \frac{48\varepsilon}{n\eta^2} + \frac{24\varepsilon - 6\eta^2 + 2\psi}{n^2\eta^2} + 0(\frac{1}{n^3}),$$

while that of the approximating chi-square is

$$3 + \frac{12(n-2)^2\omega^2}{n(n-1)\eta^3} = \beta_2 - \frac{12(\varepsilon\eta-\omega^2)}{\eta^3} + \frac{12(4\varepsilon\eta-3\omega\eta^2)}{n\eta^3}$$

$$+ \frac{12\omega^2 - 24\varepsilon\eta + 6\eta^3 - 2\psi\eta}{n^2\eta^3} + 0(\frac{1}{n^3}).$$

By the Cauchy inequality $\omega^2 \leq \varepsilon\eta$, with equality only if the ξ's are all equal: in that case the approximation is asymptotically correct, whereas otherwise its kurtosis is asymptotically too small.

By the argument of Section 8, this test of H_1 will be consistent against the alternative H_0': $\gamma > 0$. However, if $\gamma = 0$ the limiting power of the test will be less than unity: that is, it will not be consistent. To obtain a test which is consistent against the general alternative H_1' that ranking is not at random one may proceed as follows. Let $c(\underline{s}_u,\underline{s}_v) = 1$ or 0 according as $\underline{s}_u = \underline{s}_v$ or not, so that Γ is the identity matrix. Note that for this index $\alpha(V_1,V_1) = 1$, so Theorem 16 does not apply. However, $\gamma = \Sigma p_u^2$; and under random ranking $\gamma = 1/m!$, while otherwise γ is strictly greater then this, so $\zeta = 0$ by Theorem 4. Then Theorem 2 shows that $(n-1)(C-1/m!)$ has asymptotically the same distribution as $\Sigma\lambda_i(Q_i^2-1)$, where the λ's are the nonzero characteristic roots of $\Gamma\Omega = \Omega$, namely $1/m!$ repeated $(m!-1)$ times. Hence, if $X = m!(n-1)C + (m!-n)$ then X is asymptotically a chi-square with $(m!-1)$ degrees of freedom. But

$$C = \Sigma\binom{n_u}{2}/\binom{n}{2} = \frac{\Sigma n_u^2 - n}{n^2 - n},$$

where n_u is the number of times the ranking $\underline{s}_u$ is observed in the sample, so $X = m!\ \Sigma n_u^2/n - n$. This is, of course, exactly the test one would have arrived at without the average correlation concept.

With ties allowed, a stronger condition must be imposed on the index before progress is possible. We have

Theorem 17. If $\alpha(U,V) = 0$ for all permutation sets U and V, then under random ranking the vector $\underline{\theta}$ is null, and hence $\gamma = \zeta = 0$.

Proof. For any ranking $\underline{s}_u$, in permutation set U, say,

$$\theta_u = \sum p_v \gamma_{uv} = \sum_V \sum_{\underline{s}_v \in V} p_v\ c(\underline{s}_u, \underline{s}_v).$$

By the randomness of the ranking p_v may be replaced by $q(V)$ and taken out of the inner summation, which is then $\alpha(U,V) = 0$.

Under the condition stated, for which simple symmetry is not sufficient, one may use the asymptotically distribution-free test of Section 8. The necessary parameters η, ω, and μ now depend only on the quantities $q(V)$, but these must still be estimated (or the test performed conditionally). The parameter μ may no longer vanish, even for a symmetric index, but one might well ignore it, however: it enters only into the skewness, and its coefficient there is of lower order in n than that of ω. At any rate, simple bounds on μ can be established. Let U be the set consisting of every ranking whose inverse is a permutation of it - note that U includes V_1 - and let $P = \Pr[\underline{Y} \epsilon U]$. Then

$$\begin{aligned}\mu &= E[c^3(\underline{Y}_1, \underline{Y}_2)] \\ &= E[c^3(\underline{Y}_1, \underline{Y}_2) \mid \underline{Y}_1, \underline{Y}_2 \epsilon U]\ \Pr[\underline{Y}_1, \underline{Y}_2 \epsilon U] \\ &\qquad + E[c^3(\underline{Y}_1, \underline{Y}_2) \mid \underline{Y}_1 \text{ or } \underline{Y}_2 \notin U]\ \Pr[\underline{Y}_1 \text{ or } \underline{Y}_2 \notin U]\end{aligned}$$

and

$$|\mu| \le 0\times P^2 + 1\times(1-P^2) = 1 - P^2.$$

If the index is multiplicative, then Γ is positive semidefinite, and hence also the matrix $((\gamma^3_{uv}))$, so that the further bound $\mu \ge 0$ holds.

10. AVERAGE RHO UNDER RANDOM RANKING

The first four moments of average rho (R) under the hypothesis of random ranking can be obtained using the results given in the preceding section. For the present we shall consider only the case where there are no ties. Then the matrix Γ_0, for Spearman correlation, turns out to have (m-1) nonzero characteristic roots, all equal to m(m-2)! Thence $\eta = 1/(m-1)$, $\omega = \eta^2$, and $\varepsilon = \eta^3$; and since this is a symmetric index of correlation, $\phi = \eta^2$ and $\mu = \nu = 0$. Thus we obtain

$$E[R] = 0, \quad V[R] = \frac{1}{m-1} \cdot \frac{2}{n(n-1)}, \quad \beta_1 = \frac{8}{m-1} \cdot \frac{(n-2)^2}{n(n-1)}$$

The kurtosis is then

$$\beta_2 = \frac{3(m+3)(n-2)(n-3) + 12(m-1)(n-2) + 2(m-1)^3\psi}{(m-1)n(n-1)}$$

where ψ is the fourth moment of Spearman's rho, given by Kendall (1948, chapter 5) as

$$\psi = \frac{3(25m^3-38m^2-35m+72)}{25m(m+1)(m-1)^3}.$$

More simply,

$$\beta_2 = 3 + \frac{12}{m-1} - \frac{48}{(m-1)n} + \frac{12(31m^2+45m+36)}{25(m^3-m)n^2} + O(\frac{1}{n^3}).$$

This result for β_2 is not consistent with the formula for the fourth moment of W as given by Kendall (1948, chapter 7), which appears to be incorrect.

Various tabulations of the exact distribution are available. With n = 2, R is just the ordinary Spearman rank correlation coefficient; and for m = 2 it can be shown equivalent to the sign test statistic. Kendall & Smith (1939) tabulated the statistic $K = n(m^3-m)\{1+(n-1)R\}/12$, which is integral-valued unless n is odd and m-2 is a multiple of 4, for m=3 with

$n = 2(1)10$, for $m = 4$ with $n = 2(1)6$, and for $m = 5$ with $n = 3$. Owen (1962) tabulated Friedman's statistic $X_F = (m-1)\{1+(n-1)R\}$ for $m = 3$ with $n = 2(1)15$, and for $m = 4$ with $n = 2(1)8$. Finally, Michaelis (1971) added two further cases: $(m,n) = (5,4)$ and $(6,3)$. A comparison of these published tables reveals numerous discrepancies, however, so I have performed an independent computation. My results indicate that Owen's tabulation is quite unreliable, except for $m = 3$ with $n = 3(1)8$, although interestingly enough it yields the first three moments correctly in at least two other instances: (4,3) and (4,4). The tables of Kendall & Smith appear to be entirely correct; so also is Michaelis' extension to (5,4), but I did not check him at (6,3). However, these latter tables are neither as extensive nor as detailed as one might wish. I therefore include as Appendix I my own more complete version, identical in extent and similar in format to that of Owen. These tables were obtained by the method described in Kendall (1948, chapter 7). Moments calculated from them agree in every instance with the formulas given in the preceding paragraph.

For values of m or n beyond the scope of the tables one may resort to various approximations to the distribution. In an appendix to the original paper of Friedman (1937), Wilks showed (by a totally different method from Theorem 16) that as n increases without limit the distribution of X_F tends to that of a chi-square with m-1 degrees of freedom. This asymptotic result provides quite a simple approximation to the distribution of R, but unfortunately it is extremely conservative for moderate values of n. It fits only one moment exactly, since the true variance of X_F, namely $2(m-1)(n-1)/n$, depends on n. A two-moment fit could be achieved by a simple linear transformation, of course, but the approximation would still have skewness

$$\frac{8}{m-1} = \beta_1 + \frac{24}{(m-1)n} + O(\frac{1}{n^2})$$

and kurtosis

$$3 + \frac{12}{m-1} = \beta_2 + \frac{48}{(m-1)n} + O(\frac{1}{n^2}).$$

Kendall & Smith (1939) proposed instead to fit a $\beta_1(u,v)$ distribution to

$$W = \frac{1}{n}\{1+(n-1)R\} = \frac{K}{(m^3-m)n/12} = \frac{X_F}{(m-1)n} ,$$

setting the left end of the range at 0 (which is correct unless m is even and n odd) and the right end at 1 (which is correct). Then, determining u and v so as to fit the first two moments exactly, they obtained

$$u = \frac{m-1}{2} - \frac{1}{n}, \qquad v = (n-1)u.$$

Their approximation is asymptotically correct for increasing n. For finite n, its skewness is

$$\frac{8(n-2)^2}{n-1} \cdot \frac{(m-1)n}{\{(m-1)n+2\}^2} = \beta_1 - \frac{32}{(m-1)^2 n} + O(\frac{1}{n^2})$$

and its kurtosis is

$$3 + \frac{12\{(m-1)(n^3-5n^2+5n)-2(n-1)\}}{(n-1)\{(m-1)n+2\}\{(m-1)n+4\}} = \beta_2 - \frac{72}{(m-1)^2 n} + O(\frac{1}{n^2}).$$

The approximation is much closer than Friedman's, but in the tail it tends to be anticonservative: that is, to indicate levels of significance smaller than the true values. A refinement is afforded by applying a continuity correction, which consists of subtracting 1 from the numerator of the formula given for W in terms of K, and adding 2 to its denominator. Michaelis (1971) has given 5% and 1% critical values based on this for m = 3(1)15 with n = 3(1)20.

Kendall & Smith point out that their approximation is equivalent to taking

$$V = \frac{1+(n-1)R}{1-R} = \frac{(n-1)K}{(m^3-m)n^2/12-K} = \frac{(n-1)W}{1-W} = \frac{(n-1)X_F}{(m-1)n-X_F}$$

as an F with (m-1-2/n) and (n-1)(m-1-2/n) degrees of freedom, where V is the same as the variance ratio which would be obtained from an analysis of variance of the ranks. Since the nonintegral degrees of freedom are somewhat awkward to work with, consider replacing them by the nearest integers, namely m-1 and (m-1)(n-1)-2 if $n \geq 4$. This suggests a conceptually simpler approximate test, as follows: Perform an ordinary two-way analysis of variance using the ranked data, but subtract from the denominator 2 degrees of freedom "for ranking" before looking up the result in the F-table. Incidentally, this analysis of variance technique would be particularly advantageous with ties in the data, since it automatically incorporates the correction for ties given by Kendall (1948, chapter 6). And it appears that ties, even when numerous, generally will be found to have had little effect on the degrees of freedom when the more complicated calculations which they entail have been performed. It must be admitted, however, that to incorporate any correction for continuity would destroy the intuitive simplicity of the procedure, and the lack of such works together with the slight error in degrees of freedom to accentuate the anticonservative nature of the approximation.

But suppose we take the approach of the preceding two sections. This involves abandoning all requirements on the range, but the extreme tails of the distribution, particularly the left one, are of little interest in practice anyway. We then obtain

$$X = \frac{n(n-1)}{(n-2)^2}(m-1)\{1+(n-2)R\} = \frac{n}{n-2}\{\frac{12K}{(m^2+m)n} + \frac{m-1}{n-2}\} = \frac{n}{n-2}\{X_F + \frac{m-1}{n-2}\}$$

as approximately a chi-square with $(m-1)n(n-1)/(n-2)^2$ degrees of freedom. A correction for continuity can be supplied, if desired, by subtracting 1 from K. This approximation may be particularly appealing at $n = 3$ and $n = 4$, where it gives integral degrees of freedom $6(m-1)$ and $3(m-1)$, respectively. It is asymptotically correct for increasing n. Its skewness is of course exactly correct for all n, and its kurtosis is

$$3 + \frac{12(n-2)^2}{(m-1)n(n-1)} = \beta_2 + \frac{12}{(m-1)n} + O(\frac{1}{n^2}),$$

with error smaller than that of the Kendall & Smith approximation if $n(7-m) \geq 24$. Thus the present approximation, although no more complicated than that of Kendall & Smith, will generally be more accurate except perhaps for very small values of n.

In Table 10.1 the four approximations described in this section are compared for the case where $m = 3$ with $n = 10$. The statistic

$$K = \sum_{k=1}^{3} (\sum_{i=1}^{10} Y_{ij} - 20)^2,$$

whose values are integral, is taken as the index. Then

$$R = \frac{1}{9}(\frac{K}{20} - 1).$$

The true significance level P, to 5 decimal places, is taken from Appendix I. The Friedman approximation is

$$P_1 = \Pr[\chi^2(2) \geq X_F] \quad \text{where } X_F = K/10.$$

The Kendall & Smith approximation, with continuity correction (indicated in the notation by the prime), is

$$P_2 = \Pr[F(1.8,16.2) \geq V'] \quad \text{where } V' = 9(k-1)/(203-K).$$

The simplified analysis of variance version of this gives

$$P_3 = \Pr[F(2,16) \geq V] \quad \text{where } V = 9K/(200-K).$$

Finally, the new chi-square approximation, corrected for continuity, is

$$P_4 = \Pr[\chi^2(2.8125) \geq X'] \quad \text{where } X' = (3+2K)/16.$$

All instances where $.0001 < P < .1000$ are shown in the table. In the upper part of this range the approximations are all fairly good, that of Kendall & Smith being best. Farther out in the tail the two approximations based on the χ^2-distribution become conservative while the two based on F become anticonservative; these tendencies are more and more accentuated for values of P still smaller than those shown in the table. The new chi-square approximation is clearly best for all instances where the true $P < .005$. The example just presented seems typical of various cases examined.

In the more general case where ties are permitted it appears simplest, for both computation and interpretation, to use the type a index. For each $i = 1,\ldots,n$ define

$$Q_i = \frac{12 \sum_k (Y_{ik} - \frac{m+1}{2})^2}{m^3 - m}$$

Note that $Q_i = 0$ if $\underline{Y}_i$ is completely tied, $Q_i = 1$ if $\underline{Y}_i$ is untied, and otherwise $0 < Q_i < 1$. Then the conditional expected value of r_{ij}^2 under random ranking, given that $\underline{Y}_i$ and $\underline{Y}_j$ have tie patterns such as to produce quantities Q_i and Q_j, is $\eta_{ij} = Q_iQ_j/(m-1)$; and similarly the conditional

Table 10.1

Approximations to the significance level of average rho for testing randomness of ranking when m = 3 with n = 10

	Average rho	Significance Levels				
			Approximate			
		Exact	Friedman	K-S	ANOVA	New χ^2
K	R	P	P_1	P_2	P_3	P_4
50	.167	.09236	.08209	.08921	.07827	.08048
54	.189	.07810	.06721	.07158	.06184	.06420
56	.200	.06647	.06081	.06398	.05485	.05731
62	.233	.04556	.04505	.04531	.03791	.04069
72	.289	.03033	.02732	.02468	.01979	.02287
74	.300	.02587	.02472	.02174	.01728	.02037
78	.322	.01793	.02024	.01678	.01310	.01615
86	.367	.01153	.01357	.00974	.00733	.01012
96	.422	.00747	.00823	.00468	.00335	.00563
98	.433	.00634	.00745	.00401	.00284	.00500
104	.467	.00336	.00552	.00248	.00170	.00351
114	.522	.00198	.00335	.00104	.00067	.00194
122	.567	.00125	.00224	.00048	.00030	.00121
126	.589	.00083	.00184	.00032	.00019	.00095
128	.600	.00051	.00166	.00026	.00015	.00084
134	.633	.00037	.00123	.00013	.00007	.00059
146	.700	.00018	.00068	.00003	.00001	.00029

expected value of $r_{ij}r_{jk}r_{ki}$ is $\omega_{ijk} = Q_iQ_jQ_k/(m-1)^2$. Hence the U-statistics for estimating η and ω under the hypothesis of random ranking, when ties may be present, are

$$\hat{\eta} = \frac{(\Sigma Q_i)^2 - \Sigma Q_i^2}{(m-1)(n^2-n)}$$

and

$$\hat{\omega} = \frac{(\Sigma Q_i)^3 - 3\Sigma Q_i \Sigma Q_i^2 + 2\Sigma Q_i^3}{(m-1)^2(n^3-3n+2n)}$$

These estimates may be substituted into the new approximate chi-square test, giving as chi-square

$$X = D + \frac{(\Sigma Q_i)^2 - \Sigma Q_i^2}{(\Sigma Q_i)^3 - 3\Sigma Q_i \Sigma Q_i^2 + 2\Sigma Q_i^3}\{\frac{12K}{m^2 + m} - n(m-1)\}$$

where the degrees of freedom are

$$D = \frac{(m-1)\ \{(\Sigma Q_i)^2 - \Sigma Q_i^2\}^3}{\{(\Sigma Q_i)^3 - 3\Sigma Q_i \Sigma Q_i^2 + 2\Sigma Q_i^3\}^3}$$

The parameter μ has here been ignored, as explained at the end of Section 9, although with ties allowed it may not equal zero even under random ranking: Example 6 below, in which $\mu = 27(m^2-2m-2)/m(m-1)^2(m+1)^3$, is a case in point.

On the other hand, if one prefers to work with the type b index, the new chi-square approximation can be used as presented for the untied case, provided only that it is agreed to discard any completely tied rankings. This is because (in obvious notation)

$$\rho_b(\underline{Y}_i,\underline{Y}_j) = \begin{cases} \rho_a(\underline{Y}_i,\underline{Y}_j)/\sqrt{Q_iQ_j} & \text{if } Q_iQ_j > 0 \\ 0 & \text{if } Q_iQ_j = 0 \end{cases}$$

and thus for ρ_b

$$\eta = \frac{(1-p_0)^2}{m-1} \quad \text{and} \quad \omega = \frac{(1-p_0)^2}{(m-1)^2},$$

where p_0 is the probability of a completely tied ranking. Discarding such rankings yields $p_0 = 0$, whereupon η and ω are the same as in the untied case. However, it must be noted that the shortcut formulas based on K do not apply to ρ_b: one must proceed from the definition of average correlation, calculating ρ_b for each pair of rankings and then averaging. Furthermore, the alternatives against which this test is consistent are those for which $\rho_b > 0$: in particular, the simple interpretation in terms of expected ranks is not valid. Of course, to obtain a test for equal expected ranks, one should use instead the procedure of Section 8 with ρ_a as the index of correlation.

11. AVERAGE TAU UNDER RANDOM RANKING

The first four moments of average tau (T) under the hypothesis of random ranking can be obtained from the results given in Section 9. For the present we shall consider only the case where there are no ties. Then the matrix Γ_0, for Kendall correlation, turns out to have $m(m-1)/2$ nonzero characteristic roots, of which $m-1$ are equal to $2(m+1)(m-2)!/3$ and $(m-1)(m-2)/2$ are equal to $2(m-2)!/3$. Thence

$$\eta = \frac{2(2m+5)}{9m(m-1)}\,, \qquad \omega = \frac{4(2m^2+6m+7)}{27m^2(m-1)^2}\,, \qquad \varepsilon = \frac{8(2m^3+8m^2+12m+9)}{81m^3(m-1)^3}\,;$$

and, since this is a symmetric index of correlation, $\phi = \eta^2$ and $\mu = \nu = 0$. Thus we obtain

$$E[T] = 0, \qquad V[T] = \frac{2(2m+5)}{9m(m-1)} \cdot \frac{2}{n(n-1)}\,,$$

$$\beta_1 = \frac{16(2m^2+6m+7)^2}{m(m-1)(2m+5)^3} \cdot \frac{(n-2)^2}{n(n-1)}\,.$$

The kurtosis is then

$$\beta_2 = \frac{3(n+1)(n-2)}{n(n-1)} + \frac{24(2m^3+8m^2+12m+9)}{m(m-1)(2m+5)^2} \cdot \frac{(n-2)(n-3)}{n(n-1)} + \frac{81m^2(m-1)^2\psi}{2(2m+5)^2n(n-1)}$$

where ψ is the fourth moment of Kendall's tau, which can be obtained from the results of Silverstone (1950) as

$$\psi = \frac{100m^4 + 328m^3 - 127m^2 - 997m - 372}{1350\{m(m-1)/2\}^3} .$$

More simply

$$\beta_2 = 3 + \frac{24(2m^3+8m^2+12m+9)}{m(m-1)(2m+5)^2} - \frac{96(2m^3+8m^2+12m+9)}{m(m-1)(2m+5)^2} \cdot \frac{1}{n} + 0(\frac{1}{n^2}).$$

This result for β_2 is not consistent with the formula for the fourth moment of T given by Ehrenberg (1952), which appears to be incorrect.

No satisfactory tabulation of the exact distribution has been published. With n = 2, T is just the ordinary Kendall rank correlation coefficient; and for m = 2 it can be shown equivalent to the sign test statistic. Ehrenberg (1952) gave the distribution for three further cases, namely (m,n) = (3,4),(3,5)(4,3). Van Elteren (1957) gave four cases, namely m = 3 with n = 3(1)6. A more extensive tabulation, covering m = 3 with n = 3(1)10, m = 4 with n = 3(1)6, and (m,n) = (5,3),(5,4), and (6,3), is given as Appendix II. These tables were obtained by complete enumeration of all possibilities (before it was realized that Kendall's method for average rho could be extended to average tau also). Moments calculated from them agree in every instance with the formulas given in the preceding paragraph.

For values of m and n beyond the scope of the tables one may resort to various approximations to the distribution. Van Elteren (1957) showed (by a totally different method from Theorem 16) that as n increases without limit the distribution of

$$Z = \frac{3m(m-1)}{2}\{1 + (n-1)T\} = \frac{3m(m-1)}{2} + \frac{6}{n} L$$

tends to that of $\{(m+1)X_1 + X_2\}$, where X_1 and X_2 are independently distributed as χ^2 with m-1 and (m-1)(m-2)/2 degrees of freedom respectively. He suggested that this asymptotic result would provide a relatively good approximation to the exact distribution even for quite small values of n. However, Van Elteren's approximation fits only one moment exactly, since the variance of Z, namely m(m-1)(2m+5)(n-1)/n, depends on n; but a two-moment fit is easily achieved, by changing n to $\sqrt{n(n-1)}$ in the formula for Z in terms of L. Also, Van Elteren made no provision for a continuity correction; but one is easily supplied, by subtracting C from L where C = 1 or 2 according as n is even or odd. Thus an improved approximation is obtained on replacing Z by

$$Z' = \frac{3m(m-1)}{2} + \frac{6(L-C)}{\sqrt{n(n-1)}} .$$

One remaining disadvantage of this proposal is that it requires tabulating a new nonstandard distribution, although Van Elteren did give explicit formulas for the case where m is odd.

Ehrenberg (1952) had previously proposed for this situation the same approach as has been developed more generally in this paper: that is, to approximate the distribution of a linear function of T by a chi-square, determining the coefficients and degrees of freedom so as to make the first three moments agree. Starting from the general expression in Section 9, one obtains

$$X = D\{1 + (n-2)f_1(m)T\} = D + \frac{f_2(m)L}{n-2}$$

as a chi-square with

$$D = \frac{n(n-1)}{(n-2)^2} f_3(m)$$

degrees of freedom, where

$$f_1(m) = \frac{\omega}{\eta^2} = \frac{3(2m^2+6m+7)}{(2m+5)^2} ,$$

$$f_2(m) = \frac{\eta}{\omega m(m-1)} = \frac{3(2m+5)}{2(2m^2+6m+7)} ,$$

$$f_3(m) = \frac{\eta^3}{\omega^2} = \frac{m(m-1)(2m+5)^3}{2(2m^2+6m+7)^2} .$$

These coefficients have been calculated for the first few values of m, and for convenience are given below:

m	3	4	5	6	7	8
f_1	1.0661	1.1183	1.1600	1.1938	1.2216	1.2449
f_2	.3837	.3095	.2586	.2217	.1939	.1721
f_3	2.1595	3.3212	4.4590	5.5724	6.6657	7.7431

The Ehrenberg approximation can also be corrected for continuity, by subtracting C from L.

Since the fractional degrees of freedom of Ehrenberg's chi-square are inconvenient, Hays (1960) suggested the following simplification: approximate the distribution of a linear function of T by a chi-square with D' degrees of freedom, where the coefficients are determined so as to make the mean and variance agree, but D' is a given integer. A little algebra yields the linear function

$$H = D' + 3T\sqrt{\frac{(m^3-m)(n^2-n)D'}{2(2m+5)}} = D' + \frac{6L\sqrt{2D'}}{\sqrt{(m^2-m)(2m+5)(n^2-n)}} .$$

Hays noted that if m and n are both large the nearest integer to Ehrenberg's D is m, and he therefore proposed taking D' = m in all cases. It appears worthwhile, however, actually to calculate D and then let D' be the nearest integer to it.

(Remark. The approach of the previous paragraph can of course be taken in all the chi-square approximations of this paper. Suppose we have the approximation

$$X \doteqdot \chi^2(F),$$

where F is an inconvenient fraction which we desire to replace by the convenient integer I - ordinarily, the nearest integer to F. Then take

$$X' = I - \sqrt{IF} + X\sqrt{I/F} \doteqdot \chi^2(I),$$

where the modified approximation has two correct moments if the original one did.)

Since Ehrenberg's chi-square approximation for average tau is not asymptotically correct, it must be inferior to Van Elteren's approximation for sufficiently large n. For any given m, the true values of the skewness β_1 and the kurtosis β_2 both increase with n. The skewness and kurtosis of Van Elteren's approximation are for all n equal to the corresponding asymptotic values. The skewness of Ehrenberg's approximation is exact, but its kurtosis is

$$3 + 1.5\beta_1 = \beta_2 - \frac{48(m^3-4m+2)}{m(m-1)(2m+5)^3} + O(\frac{1}{n}).$$

This is clearly smaller than β_2 for large n, but it happens to be larger for smaller n. Thus there is a value of n, say $n_1(m)$, where the true kurtosis is equal to that of Ehrenberg's approximation (if for convenience we treat n in the moment formulas as though it were continuous); and there is a larger value of n, say $n_2(m)$, where the true kurtosis is halway between those of Ehrenberg's and Van Elteren's approximations. Then for $n < n_2$ Ehrenberg fits each of the first four moments at least as closely as Van Elteren - indeed he fits the first three exactly whereas Van Elteren does not - and hence at least for these values of n Ehrenberg's approximation should be the better of the two. A little algebra shows that for all m we have $n_1 > 2m$ and $n_2 > 10m$; for small m the exact values of n_1 and n_2 are:

m	3	4	5	6	7	8
n_1	61.1	43.0	38.4	37.1	37.1	37.8
n_2	296.4	206.0	182.7	176.2	176.4	179.8

The obvious conclusion is that in practice the Ehrenberg approximation will almost always be preferable to that of Van Elteren.

A logical next approximation would be to use four moments, fitting perhaps a Pearson curve: this will be of Type I (beta) if $n < n_1$, and of Type IV if $n > n_1$. The procedure is complicated but well described in various texts and will not be discussed here.

In Table 11.1 the approximations described in this section are compared for the case where m = 3 with n = 10. The statistic L is taken as the index; then T = L/135. The true significance level P, to 5 decimal places, is taken from Appendix II. The Van Elteren approximation, as originally given, is

$$P_1 = \Pr[4\chi^2(2) + \chi^2(1) \geq Z] \quad \text{where } Z = 9 + .6L.$$

The improved version of this is

$$P_2 = \Pr[4\chi^2(2) + \chi^2(1) \geq Z'] \quad \text{where } Z' = 9 + \sqrt{.4}(L-1).$$

The Ehrenberg approximation is

$$P_3 = \Pr[\chi^2(3.0369) \geq X'] \quad \text{where } X' = 3.0369 + .1918(L-1).$$

Table 11.1

Approximations to the significance level of average tau for testing randomness of ranking when m = 3 with n = 10

	Average tau	Significance Levels				
		Exact	Approximate			
			Van Elteren	(Improved)	Ehrenberg	Pearson
L	T	P	P_1	P_2	P_3	P_5
21	.1556	.08457	.07760	.07713	.07807	.08105
23	.1704	.07036	.06679	.06585	.06590	.06872
25	.1852	.05661	.05749	.05622	.05557	.05815
27	.2000	.05010	.04948	.04800	.04683	.04913
29	.2148	.03860	.04259	.04098	.03942	.04142
33	.2444	.03146	.03155	.02987	.02788	.02929
35	.2593	.02747	.02716	.02550	.02343	.02456
37	.2741	.02269	.02337	.02177	.01967	.02056
39	.2889	.01469	.02012	.01859	.01651	.01717
43	.3185	.01344	.01490	.01355	.01161	.01191
45	.3333	.00872	.01283	.01157	.00973	.00989
49	.3630	.00772	.00950	.00843	.00682	.00677
51	.3778	.00685	.00818	.00720	.00571	.00559
53	.3926	.00499	.00704	.00614	.00478	.00460
55	.4074	.00369	.00606	.00525	.00400	.00378
57	.4222	.00281	.00522	.00448	.00334	.00310
61	.4519	.00207	.00386	.00326	.00233	.00206
67	.4963	.00150	.00246	.00203	.00136	.00110
69	.5111	.00101	.00212	.00173	.00113	.00089
71	.5259	.00059	.00183	.00148	.00095	.00072
75	.5556	.00046	.00135	.00108	.00066	.00046
81	.6000	.00022	.00086	.00067	.00038	.00023
85	.6296	.00020	.00064	.00049	.00027	.00015

The Hays simplification of this would be

$$P_4 = \Pr[\chi^2(3) \geq H'] \quad \text{where } H' = 3 + .1907(L-1),$$

but this is not shown, since it is barely distinguishable from P_3. Finally, the Pearson approximation turns out to be

$$P_5 = \Pr[B(1.0860, 17.7839) > .0576 + .004067(L-1)].$$

All instances where $.0001 < P < .1000$ are shown in the table. In the upper part of this range the approximations are all fairly good. Farther out in the tail the Ehrenberg approximation becomes conservative, the improved Van Elteren approximation more so, and the original version even more so; these tendencies are more and more accentuated for values of P still smaller than those shown in the table. The Pearson approximation remains reasonably accurate to the extreme tail of the distribution; indeed, a graph of the results suggests that no smooth curve could yield any substantial improvement. Whether the additional accuracy provided by the Pearson fit justifies the additional effort it requires is left for the reader to decide; the author, who programmed these computations himself, votes "no". Comparisons similar to this were made for all values of m and n covered in Appendix II, except that the Van Elteren approximations were not computed for $m > 3$. The example just presented seem typical.

In the more general case where ties are permitted it appears simplest, for both computation and interpretation, to use the type-a index. Suppose the observed ranking $\underline{Y}_i = (Y_{i1}, \ldots, Y_{im})$ contains m_i distinct tied groups, their sizes constituting the set $G_i = \{G_{i1}, \ldots, G_{im_i}\}$, where of course $G_{i1} + \ldots + G_{im_i} = m$. Define

$$A_i = 1 - \frac{\Sigma G_{iu}(G_{iu}-1)}{m(m-1)}, \quad B_i = 1 - \frac{\Sigma G_{iu}(G_{iu}-1)(G_{iu}-2)}{m(m-1)(m-2)}.$$

It may be seen that $0 \leq A_i \leq B_i \leq 1$ for all $i = 1, \ldots, n$; $A_i = B_i = 0$ if $\underline{Y}_i$ is completely tied, and $A_i = B_i = 1$ if $\underline{Y}_i$ is untied. Then the conditional expected value of t_{ij}^2 under random ranking, given that $\underline{Y}_i$ and $\underline{Y}_j$ have tie patterns G_i and G_j respectively, can be found from Kendall (1948, chapter 4) as

$$\eta_{ij} = \frac{2}{9(m^2-m)} \{2(m-2)B_iB_j + 9A_iA_j\},$$

and the conditional expected value of $t_{ij}t_{jk}t_{kj}$ can be worked out similarly as

$$\omega_{ijk} = \frac{4}{27(m^2-m)^2} \{2(m-2)(m-4)B_iB_jB_k +$$

$$+ 6(m-2)(A_iB_jB_k+B_iA_jB_k+B_iB_jA_k) + 27A_iA_jA_k\}.$$

Hence the U-statistics for estimating η and ω under the hypothesis of random ranking, when ties may be present, are

$$\hat{\eta} = \frac{2}{9(m^2-m)(n^2-n)} [2(m-2)\{(\Sigma B_i)^2-\Sigma B_i^2\} + 9\{(\Sigma A_i)^2-\Sigma A_i^2\}]$$

and

$$\hat{\omega} = \frac{4}{27(m^2-m)^2(n^3-3n^2+2n)} [2(m-2)(m-4) \{(\Sigma B_i)^3-3\Sigma B_i\Sigma B_i^2+2\Sigma B_i^3\}$$

$$+ 18(m-2) \{\Sigma A_i(\Sigma B_i)^2-2\Sigma B_i\Sigma A_iB_i-\Sigma A_i\Sigma B_i^2+2\Sigma B_i^2\}$$

$$+ 27\{(\Sigma A_i)^3-3\Sigma A_i\Sigma A_i^2+2\Sigma A_i^3\}].$$

These estimates may be substituted for η and ω in the formulas for f_1, f_2, and f_3, thus yielding the approximate chi-square test; the parameter μ is here being ignored.

12. RANKINGS WITH AT MOST TWO DISTINCT COMPONENTS

Suppose that $\underline{Y}_1,\ldots,\underline{Y}_n$ are obtained by converting underlying observations $\underline{X}_1,\ldots,\underline{X}_n$ into ranks, where $\underline{X}_i = (X_{i1},\ldots,X_{im})'$ for $i = 1,\ldots,n$, and each X_{ik} equals 0 or 1 only. Write $W_i = X_{i1} + \ldots + X_{ik}$. Then the possible values of $\underline{Y}_i$ are only the completely tied ranking, corresponding to the case where $W_i = 0$ or m, and those rankings which contain exactly two distinct components: W_i components each equal to $(1+W_i)/2$, corresponding to those components of $\underline{X}_i$ which equal 0, and $(m-W_i)$ components each equal to $(W_i+1+m)/2$, corresponding to those components of $\underline{X}_i$ which equal 1. And suppose we are interested in the hypothesis H_1 that such rankings are at random. Cochran (1950) proposed basing a conditional test, given the W_i's, on the statistic

$$Q = \frac{m(m-1)\ \Sigma(G_k-G)^2}{\Sigma W_i(m-W_i)}, \quad \text{where } G_k = \sum_i X_{ik}, \quad G = \frac{1}{m}\sum_k G_k;$$

it is assumed that the rankings are not all completely tied, so that $\Sigma W_i(m-W_i) > 0$. For small values of n the exact permutation distribution of Q may be calculated. For large n Cochran suggested using as an approximation the chi-square with m-1 degrees of freedom; as he showed, this is asymptotically correct provided only that $\Sigma W_i(m-W_i)$ tends to infinity as n increases. The same test was later proposed independently by Van Elteren (1963).

Now, the tau-a correlation between $\underline{Y}_i$ and $\underline{Y}_j$ (or $\underline{X}_i$ and $\underline{X}_j$) turns out to be

$$t_a(\underline{Y}_i,\underline{Y}_j) = \frac{2(m \sum_k X_{ik}X_{jk} - W_iW_j)}{m^2 - m} \qquad i,j = 1,\dots,m$$

and the corresponding average correlation is then (as Van Elteren shows)

$$T_a = \frac{\sum_{i<j} t_a(\underline{Y}_i,\underline{Y}_j)}{\binom{n}{2}} = \frac{2(\frac{Q}{m-1} - 1)\ \Sigma W_i(m-W_i)}{(m^2-m)(n^2-n)}.$$

Furthermore, the rho-a correlation differs only by a change of scale: in fact,

$$\frac{r_a(\underline{Y}_i,\underline{Y}_j)}{t_a(\underline{Y}_i,\underline{Y}_j)} = \frac{R_a}{T_a} = \frac{3m}{2(m+1)}.$$

Thus, when conditioned on the W's, or indeed on the quantity $\Sigma W_i(m-W_i)$, Cochran's statistic is equivalent to average type-a Kendall or Spearman correlation.

A particularly simple situation is that where $W_i = w$ for all i. This might arise, for example, if each of n judges independently were required to select w objects as the best from a group of m. Then Cochran's statistic simplifies to

$$Q = \frac{m(m-1)\ \Sigma(G_k-nw/m)^2}{nw(m-w)},$$

where G_k is the number of judges who select the k-th object. For this situation Van Elteren (1963) has tabulated the exact distribution under H_1 of the integral-valued statistic $S = nw(m-w)Q/(m-1)$ for 34 cases with very small m and n.

It may also be of interest to consider the broader hypothesis

$$H_2\colon\ P_r[X_{ik}=1] \text{ is independent of } k.$$

Cochran's test has been proposed for this hypothesis also, but here it is no longer valid: For example, suppose $m = 4$, $W_i = 2$ for all i, and the conditional distribution of $\underline{X}$ given $W = 2$ assigns probability $\frac{1}{2}$ each to the two points $(0,0,1,1)'$ and $(1,1,0,0)'$. Then clearly H_2 holds, since $Pr[X_{ik}=1] = \frac{1}{2}$ for $k = 1,2,3,4$. But $Q = 3(2A-n)^2/n$, where A is the number of times $\underline{X} = (0,0,1,1)$ in the sample, so that A is binomial with parameters $(n,\frac{1}{2})$. Hence we conclude that Q/3 is asymptotically $\chi^2(1)$ rather than that Q is $\chi^2(3)$. But for $k,l = 1,\ldots,n$ let us write

$$\phi_{kl} = Pr[X_{ik}=X_{il}=1] = E[X_{ik}X_{il}];$$

note that

$$\phi_{kk} = Pr[X_{ik}=1] = E[X_{ik}]$$

and that H_2 is equivalent to stating that $\phi_{kk} = E[W_i]/m$ for all k. Then it was shown by Bhapkar (1970) that the unconditional distribution of Q under H_2 is asymptotically chi-square with m-1 degrees of freedom if and only if ϕ_{kl} is equal to some constant, say ϕ, for all $k \neq l$; since then

$$E[W_i^2] = \sum_k \sum_l \phi_{kl} = \sum_k \phi_{kk} + m(m-1)\ ,$$

it follows that $\phi = (E[W_i^2]-E[W_i])/(m^2-m)$. Bhapkar's condition is clearly more restrictive than H_2; yet it is also less restrictive than random ranking, unless m = 3.

On the other hand, a little algebra quickly establishes that the (unconditional) expected value of $t_a(\underline{Y}_i,\underline{Y}_j)$ is

$$\tau_a = \frac{2}{m-1}\,\Sigma(\phi_{kk} - E[W_k]/m)^2,$$

and this vanishes if and only if H_2 is true. That is, H_2 is entirely equivalent to the hypothesis H_0 of zero correlation treated in Section 8, and the methods proposed there can be applied. Alternatively, see Bhapkar (1970) for a different approach to testing this hypothesis.

13. THEORETICAL EXAMPLES

In each of the following examples, only a few of the conceivable rankings have positive probability. The condense the presentation, therefore, these rankings are relabeled $\underline{s}_1, \underline{s}_2, \ldots$; then matrices and vectors such as Γ and $\underline{p}$ are reordered to correspond, and unnecessary elements are dropped without further ado. Also, $\rho(\tau)$ is written for γ whenever Spearman (Kendall) correlation is used as the index.

Example 1. Suppose m = 3, ties are disallowed, the index of correlation being used is symmetric, and the correlation between identical rankings is unity. Taking the 6 possible rankings in the order (1,2,3)', (3,1,2)', (2,3,1)', (3,2,1)', (1,3,2)', (2,1,3)', we find on applying Fundamental Property II that the matrix Γ must be of the form

$$\Gamma = \begin{pmatrix} 1 & -c & -c & -1 & c & c \\ -c & 1 & -c & c & -1 & c \\ -c & -c & 1 & c & c & -1 \\ -1 & c & c & 1 & -c & -c \\ c & -1 & c & -c & 1 & -c \\ c & c & -1 & -c & -c & 1 \end{pmatrix} = (1+c)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} - c\begin{pmatrix} J & -J \\ -J & J \end{pmatrix}.$$

where I is the 3×3 identity matrix and J the 3×3 matrix in which every element is unity. The value $c = \frac{1}{2}$ corresponds to Spearman correlation, and $c = \frac{1}{3}$ to Kendall correlation; c = 1 gives the median correlation as defined by Blomqvist (1950). Define $a_i = p_i + p_{i+3}$ and $d_i = p_i - p_{i+3}$ for i = 1,2,3; then a little calculation yields

$$\gamma = (1+c)\ \Sigma d_i^2 - c(\Sigma d_i)^2,$$

$$\begin{aligned} \zeta = (1+c)^2\ &\{\Sigma a_i d_i^2 - (\Sigma d_i^2)^2\} \\ &- 2c(1+c)\ \Sigma d_i \{\Sigma a_i d_i - \Sigma d_i \Sigma d_i^2\} \\ &+ c^2 (\Sigma d_i)^2\ \{1 - (\Sigma d_i)^2\}. \end{aligned}$$

If $d_1 = d_2 = d_3 = d$, then $\gamma = 3d^2(1-2c)$ and $\zeta = d^2(1-9d^2)(1-2c)^2$. The value $d = 0$ produces a verification of Theorem 5, and $d = \frac{1}{3}$ is a special case of Example 3.

Example 2. Let $m = 4$, and let the rankings $(1,2,3,4)'$, $(1,4,3,2)'$ and $(3.2.1,4)'$ have probabilities p_1, p_2, p_3 respectively, where $p_1 + p_2 + p_3 = 1$. For Spearman correlation we have

$$\Gamma = \begin{pmatrix} 1 & .2 & .2 \\ .2 & 1 & -.6 \\ .2 & -.6 & 1 \end{pmatrix}, \quad \rho = 1 - 1.6(p_2+p_3) + .8(p_2+p_3) + .8(p_2-p_3)^2,$$

and for Kendall correlation

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}, \quad \tau = p_1^2 + (p_2+p_3)^2 - \frac{8}{3} p_2 p_3.$$

The minimum value of ρ is achieved at $p_1 = 0$, $p_2 = p_3 = .5$, with $\rho = .2$; the minimum τ is at $p_1 = .25$, $p_2 = p_3 = .375$, with $\tau = .25$. If one of the p's is required to vanish, the minimum ρ or τ is achieved by setting the other two p's equal to .5 each, as follows: if $p_1 = p_2 = .5$, $p_3 = 0$ or $p_1 = p_3 = .5$, $p_2 = 0$ then $\rho = .6$, $\tau = .5$, and if $p_2 = p_3 = .5$, $p_1 = 0$ then $\rho = .2$ (as before), $\tau = \frac{1}{3}$. If two p's vanish the third must be set equal to 1, producing trivially the value 1 as the minimum ρ or τ; these three points also provide the only maxima. It is not difficult to verify that $\zeta = 0$ in all the cases cited, and indeed in no others. This example illustrates Theorem 3.

Example 3. Suppose the possible rankings are $(1,2,3,\ldots,m-1,m)'$, $(m,1,2,\ldots,m-2,m-1)',\ldots,(2,3,4,\ldots,m,1)'$. Then by Fundamental Property II we see that for any index of correlation the matrix Γ corresponding to this cycle of untied rankings is a symmetric circulant, with elements $\gamma_{ij} = f(|i-j|)$ where $f(x) \equiv f(m-x)$. Suppose also that the m rankings have equal probability $1/m$ each, then

$$\gamma = \frac{1}{m} \sum_{x=0}^{m-1} f(x), \qquad \zeta = 0, \qquad \eta = \frac{1}{m} \sum_{x=0}^{m-1} f^2(x) - \gamma^2.$$

If the index used is Spearman's rho then $f(x) = 1 - 6x(m-x)/(m^2-1)$, and we have $\rho = 0$, $\eta = (m^2+11)/5(m^2-1)$. The case $m = 3$ is particularly interesting because then the nonzero characteristic roots of $\Gamma\Omega$ are .5 and .5, the same as under random ranking. This means that the asymptotic power against this alternative of the test of H_1 based on R is equal to its significance level; indeed, the asymptotic distributions of R under hypothesis and alternative are the same. For larger m the variance of R under this "cyclical" alternative is greater than under random ranking, but the mean is still zero and the test of H_1 (or of H_0) is not consistent.

On the other hand, if the index used is Kendall's tau, for which $f(x) = 1 - 4x(m-x)/(m^2-m)$, then $\tau = (m-2)/3m > 0$, so that the tests of H_0 and H_1 based on T are both consistent. Here $\eta = 4(m+1)(m^2+11)/45m^2(m-1)$, and note that

$$\Pr[Y_{ik}>Y_{il}] - \Pr[Y_{ik}<Y_{il}] = 2(k-l)/m - \operatorname{sgn}(k-l).$$

This example also shows that the converse to Theorem 13 is false.

Example 4. According to Theorem 5, for the expected correlation based on any symmetric correlation to vanish it is sufficient that each ranking have the same probability as its inverse. Examples 1 and 3 show that this is not a necessary condition, if Spearman's rho is used as the index. With Kendall's tau, Example 1 shows that the condition is necessary for $m = 3$ if ties are disallowed, but this is not true for larger m: suppose the possible rankings and their probabilities are as follows:

$\underline{s}'$	2134	2143	2314	2341	2413	2431	4123	4132	4213	4231	4312
p	.10	.15	.10	.10	.15	.15	.05	.05	.05	.05	.05

Then a simple computation shows that $\tau = 0$, even though there is no ranking such that it and its inverse have equal probabilities (other than zero). Of course, by Theorem 13 $\rho = 0$ for this example, and by the Corollary to Theorem 7 $\zeta = 0$ for both indices; these results are easily verified for the example.

Example 5. Suppose $m = 3$, and the rankings $(1,2,3)'$, $(1,3,2)'$, and $(3,1.5,1.5)'$ have probabilities p, p, and 1-2p respectively, where

$0 \le p \le .5$.

For the Goodman-Kruskal index we then have

$$\Gamma = \begin{pmatrix} 1 & 1/3 & -1 \\ 1/3 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \gamma = 1 - 8p + \frac{44}{3} p^2.$$

This shows that symmetry is insufficient for Theorem 7, since Γ is not semi-definite, and if say $p = 1/4$ then $\gamma = -1/12 < 0$, contradicting part (i) of the Corollary. Furthermore, if $p = (6 \pm \sqrt{3})/22$ then $\gamma = 0$ but a simple calculation yields $\zeta = (27 \mp \sqrt{3})/363 > 0$ (read the upper signs in both expressions, or the lower in both), so part (iii) fails also.

For Spearman correlation:

$$\text{(type a)} \qquad \Gamma = \begin{pmatrix} 1 & .5 & -.75 \\ .5 & 1 & -.75 \\ -.75 & -.75 & .75 \end{pmatrix}, \quad \rho_a = .75(1-4p)^2;$$

$$\text{(type b)} \qquad \Gamma = \begin{pmatrix} 1 & .5 & -\sqrt{.75} \\ .5 & 1 & -\sqrt{.75} \\ -\sqrt{.75} & -\sqrt{.75} & 1 \end{pmatrix}, \quad \rho_b = 1 - 4p + 7p^2 - 2p(1-2p)\sqrt{3}.$$

The expected ranks are $\varepsilon_1 = 3 - 4p$, $\varepsilon_2 = \varepsilon_3 = -1.5 + 2p$, and these are equal if and only if $p = .25$, in which case $\rho_a = 0$, thus verifying Theorem 11. But $\rho_b = (7- 48)/16 > 0$ if $p = .25$, while $\rho_b = 0$ if $p = 1/(2+\sqrt{3})$.

For Kendall correlation:

$$\text{(type a)} \qquad \Gamma = \begin{pmatrix} 1 & 1/3 & -2/3 \\ 1/3 & 1 & -2/3 \\ -2/3 & -2/3 & 2/3 \end{pmatrix}, \quad \tau_a = \frac{2}{3}(1-4p)^2;$$

$$\text{(type b)} \qquad \Gamma = \begin{pmatrix} 1 & 1/3 & -\sqrt{2/3} \\ 1/3 & 1 & -\sqrt{2/3} \\ -\sqrt{2/3} & -\sqrt{2/3} & 1 \end{pmatrix}, \quad \tau_b = 1 - 4p + \frac{20}{3} p^2 - 4p(1-2p)\sqrt{\frac{2}{3}}\,.$$

Now $\Pr[Y_{i2}>Y_{i3}] - \Pr[Y_{i2}<Y_{i3}] = 0$, but $\Pr[Y_{i1}>Y_{i2}] - \Pr[Y_{i1}<Y_{i2}] =$ $= \Pr[Y_{i1}>Y_{i3}] - \Pr[Y_{i1}<Y_{i3}] = 1 - 4p$, and this also vanishes if and only if $p = .25$, in which case $\tau_a = 0$, thus verifying Theorem 12. But $\tau_b = (5-\sqrt{24})/12 > 0$ if $p = .25$, while $\tau_b = 0$ if $p = (3\sqrt{3}+3\sqrt{2})/(10\sqrt{3}+12\sqrt{2})$. This also shows that the result stated for $m = 3$ in Example 4 fails if ties are allowed.

Example 6. Suppose the possible rankings are $\underline{s}_1,\ldots,\underline{s}_m$ where $\underline{s}_u = (s_{u1},\ldots,s_{um})$ and $s_{uk} = m$ or $m/2$ according as $k = u$ or $k \neq u$; these rankings constitute a single permutation set, say V. This is of course a special case of the situation discussed in Section 12. Let I be the m×m identity matrix, and J the m×m matrix in which every element is unity; then by Fundamental Property II we have $\Gamma = (\sigma-\delta)I + \delta J$, where σ (or δ) is the common correlation between any member of V and itself (or a different member of V); note that $\alpha(V,V) = m\delta + (\sigma-\delta)$. If $p_u = \Pr[\underline{Y}=\underline{S}_u]$ then

$$\gamma = (\sigma-\delta)\Sigma p_u^2 + \delta, \quad \eta = (\sigma-\delta)^2\{\Sigma p_u^2 - (\Sigma p_u^2)^2\},$$

$$\zeta = (\sigma-\delta)^2\{\Sigma p_u^3 - (\Sigma p_u^2)^2\}.$$

If ranking is at random, so that $p_u = 1/m$ for all u, then $\gamma = \delta + (\sigma-\delta)/m$ and $\zeta = 0$, in agreement with Theorem 4. For rho-b and tau-b we have $\sigma = 1$, $\delta = -1/(m-1)$, and $\alpha(V,V) = 0$, so that $\gamma = (m\Sigma p_u^2-1)/(m-1)$, and $\gamma = 0$ under random ranking; for rho-a multiply σ and δ by $3/(m+1)$, and for tau-a multiply by $2/m$. For the Goodman-Kruskal coefficient, however, $\sigma = 1$, $\delta = -1$, $\alpha(V,V) = 2-m$, and $\gamma = 2\Sigma p_u^2 - 1$. Then under random ranking $\gamma = 2/m-1 < 0$, while γ may vanish when ranking is not random: for instance, if $m = 3$, $p_1 = 2/3$, $p_2 = p_3 = 1/6$. This again shows the failure of Theorem 7 and its corollary for a nonmultiplicative index.

Nevertheless, all indices of correlation for which $\sigma > \delta$ are really equivalent for testing H_1 in this example. We can see that γ is minimized only if H_1 holds, so the test of H_1 based on the average correlation will be consistent against all alternatives. Writing G_u for the number of times the ranking $\underline{s}_u$ occurs in the sample, we find the average correlation to be

$$C = \delta + (\sigma-\delta)\ \Sigma\binom{G_u}{2}/\binom{n}{2} = \delta + \frac{\sigma-\delta}{m(n-1)}\ \{Q+(n-m)\},$$

where Q is as defined in Section 12. Now, under H_1, $\Gamma\Omega = (\sigma-\delta)(I-J/m)/m$, with m-1 nonzero characteristic roots each equal to $(\sigma-\delta)/m$. Then from Theorem 2 we find, after some calculation, that Q is asymptotically a χ^2 with m-1 degrees of freedom no matter what the values of σ and δ.

14. A NUMERICAL EXAMPLE

Hays (1960) presents the orderings of m = 6 objects by two groups of 16 judges each. Translated into rankings, the data are as given in Table 14.1.

According to Hays' (p. 340), "The problem is to measure the agreement among judges within each group, and to compare agreement within and between the two groups". This will be attacked by the methods of Section 2, using Spearman's rho and Kendall's tau as alternative indices of correlation. A summary of the calculations is given in Table 14.2. By "coefficient of concordance" is meant the quantity $\{1 + (n-1)C\}/n$, which ranges from 0 to 1 if a multiplicative index is used; ordinarily the term refers only to Kendall's W calculated in this way from average rho, but Hays proposed the same rescaling for average tau also. The value of average tau for the combined tau for the combined group differs from that given by Hays since he used n = 32 instead of n-1 = 31 in the last term of the next-to-last displayed expression on page 340. Hays was unable to state any conclusion with respect to the comparison of agreement; in this analysis the difference between the within-group agreements is found not significant at usual levels.

The hypothesis H_0 of zero correlation, for each group separately and for the two combined, may be tested by the methods of Section 8. A significant result is anticipated in each case, of course, since the approximate lower 99% confidence limit on C is positive. The results are summarized in Table 14.3. The upper bounds of the first two methods are too weak to achieve significance, but the chi-square approximation establishes it firmly. It may be noted that assuming $\mu = 0$ in this example, instead of calculating an estimate of it, would reduce the approximate P-value in each instance; this suggests that the estimation of μ may be useful to guard against an anti-conservative procedure.

We come finally to the hypothesis H_1 of random ranking. Substituting m = 6 into the new chi-square approximation for average rho, we have that

Table 14.1

Two groups of 16 rankings of 6 objects

Group I		Group II	
124365	215346	432165	425316
142563	315246	643152	124365
213564	314256	524163	534216
316425	314265	624153	643215
216534	412356	624135	652413
521463	513264	645231	641325
431265	416235	654132	624153
341265	314256	614352	462315

Table 14.2

Analysis of data in Table 14.1 by methods of Section 2

	Using Rho			Using Tau		
	I	II	Combined	I	II	Combined
(Measurement of agreement among judges)						
n	16	16	32	16	16	32
K (if rho) or L (if tau)	2042	1360	3780	610	372	1062
Average rank correlation, C	.4195	.2571	.1855	.3389	.2067	.1427
Coefficient of concordance	.4558	.3036	.2109	.3802	.2562	.1695
Estimate of ζ, Z	.0308	.0320	.0291	.0225	.0225	.0174
Standard error of C, $s=\sqrt{4Z/n}$	.0877	.0895	.0603	.0750	.0749	.0466
Lower 99% confidence limit on C	.2160	.0490	.0452	.1644	.0324	.0342
(Comparison of two groups of judges)						
Difference, $C_I - C_{II}$		.1624			.1322	
Standard error		.1253			.1060	
Corresponding normal deviate		1.2958			1.2473	
P-value (2-sided)		.1951			.2123	

Table 14.3

Testing the hypothesis of zero correlation in the data of Table 14.1

	Using Rho			Using Tau		
	I	II	Combined	I	II	Combined
(First method)						
$1 + (n-1)C$	7.2929	4.8571	6.7500	6.0833	4.1000	5.4250
$P = \Pr[Q^2 \leq 1+(n-1)C]$	.9931	.9725	.9906	.9864	.9571	.9802
k	5	5	5	15	15	15
$U_0 = 1 - P^k$	.0341	.1303	.0460	.1863	.4818	.2597
(Second method: Chebyshev bounds)						
$U_1 = 1/\{1+(n-1)C\}$	.1371	.2059	.1481	.1644	.2439	.1843
$U_2 = 3/\{1+(n-1)C\}^2$	.0564	.1272	.0658	.0811	.1785	.1019
(Third method: χ^2 approximation)						
$\hat{\eta}$	.3224	.2457	.2359	.2187	.1689	.1520
$\hat{\omega}$	.1239	.0644	.0600	.0683	.0362	.0295
$\hat{\mu}$	.2130	.1284	.0947	.1287	.0727	.0537
D	2.373	3.817	3.812	2.407	3.915	4.201
X'	19.994	19.505	27.287	19.795	19.289	27.816
P-value	$.0^4 77$	$.0^3 52$	$.0^4 14$	$.0^4 89$	$.0^3 63$	$.0^4 17$

Table 14.4

Testing the hypothesis of random ranking in the data of Table 14.1

	Using Rho			Using Tau		
	I	II	Combined	I	II	Combined
n	16	16	32	16	16	32
K (if rho) or L (if tau)	2042	1360	3780	610	372	1062
D	6.122	6.122	5.511	6.823	6.823	6.142
X'	42.061	28.143	36.168	45.406	30.328	37.511
P-value	$.0^{6}2$	$.0^{4}98$	$.0^{5}16$	$.0^{6}1$	$.0^{4}71$	$.0^{5}16$

$$X' = D + \frac{1}{n-2}\left(\frac{k-1}{17.5} - n\right)$$

is approximately a chi-square with $D = 5(n^2-n)/(n-2)^2$ degrees of freedom. Using average tau instead yields

$$X' = D + .2217(L-1)/(n-2)$$

as a chi-square with $D = 5.5724\ (n^2-n)/(n-2)^2$ degrees of freedom. The results are summarized in Table 14.4.

This work was performed while I held a Research Career Development Award (No. GM-38906) from the National Institute of General Medical Sciences. I wish also to acknowledge the invaluable assistance of Katie Yelverton and Anna Colosi with the computer.

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APPENDIX I

EXACT DISTRIBUTION OF AVERAGE RHO

m = 3, n = 3

K	R	W	X_F	f	Σf	P
0	-.5000	0.0000	0.000	2	36	1.00000
2	-.3333	.1111	.667	15	34	.94444
6	0.0000	.3333	2.000	6	19	.52778
8	.1167	.4444	2.667	6	13	.36111
14	.6667	.7778	4.667	6	7	.19444
18	1.0000	1.0000	6.000	1	1	.02778

m = 3, n = 4

K	R	W	X_F	f	Σf	P
0	-.3333	0.0000	0.000	15	216	1.00000
2	-.2500	.0625	.500	60	201	.93056
6	-.0833	.1875	1.500	48	141	.65278
8	0.0000	.2500	2.000	34	93	.43056
14	.2500	.4375	3.500	32	59	.27315
18	.4167	.5625	4.500	12	27	.12500
24	.6667	.7500	6.000	6	15	.06944
26	.7500	.8125	6.500	8	9	.04167
32	1.0000	1.0000	8.000	1	1	.00463

m = 3, n = 5

K	R	W	X_F	f	Σf	P
0	-.2500	.0000	0.000	60	1296	1.00000
2	-.2000	.0400	.400	340	1236	.95370
6	-.1000	.1200	1.200	220	896	.69136
8	-.0500	.1600	1.600	200	676	.52160
14	.1000	.2800	2.800	240	476	.36728
18	.2000	.3600	3.600	75	236	.18210
24	.3500	.4800	4.800	40	161	.12423
26	.4000	.5200	5.200	70	121	.09336
32	.5500	.6400	6.400	20	51	.03935
38	.7000	.7600	7.600	20	31	.02392
42	.8000	.8400	8.400	10	11	.00849
50	1.0000	1.0000	10.000	1	1	$.0^3772$

m = 3, n = 6

K	R	W	X_F	f	Σf	P
0	-.2000	0.0000	0.000	340	7776	1.00000
2	-.1667	.0278	.333	1680	7436	.95628
6	-.1000	.0833	1.000	1320	5756	.74023
8	-.0667	.1111	1.333	1095	4436	.57047
14	.0333	.1944	2.333	1380	3341	.42966
18	.1000	.2500	3.000	530	1961	.25219
24	.2000	.3333	4.000	330	1431	.18403
26	.2333	.3611	4.333	540	1101	.14159
32	.3333	.4444	5.333	156	561	.07215
38	.4333	.5278	6.333	180	405	.05208
42	.5000	.5833	7.000	132	225	.02894
50	.6333	.6944	8.333	30	93	.01196
54	.7000	.7500	9.000	20	63	.00810
56	.7333	.7778	9.333	30	43	.00553
62	.8333	.8611	10.333	12	13	.00167
72	1.0000	1.0000	12.000	1	1	$.0^3129$

m = 3, n = 7

K	R	W	X_F	f	Σf	P
0	-.1667	0.0000	0.000	1680	46656	1.00000
2	-.1429	.0204	.286	9135	44976	.96399
6	-.0952	.0612	.857	6930	35841	.76820
8	-.0714	.0816	1.143	6230	28911	.61966
14	0.0000	.1429	2.000	8470	22681	.48613
18	.0476	.1837	2.571	3171	14211	.30459
24	.1190	.2449	3.429	2100	11040	.23663
26	.1429	.2653	3.714	3724	8940	.19162
32	.2143	.3265	4.571	1232	5216	.11180
38	.2857	.3878	5.429	1582	3984	.08539
42	.3333	.4286	6.000	1134	2402	.05148
50	.4286	.5102	7.143	301	1263	.02718
54	.4762	.5510	7.714	210	967	.02073
56	.5000	.5714	8.000	364	757	.01623
62	.5714	.6327	8.857	224	393	.00842
72	.6905	.7347	10.286	42	169	.00362
74	.7143	.7551	10.571	70	127	.00272
78	.7619	.7959	11.143	42	57	.00122
86	.8571	.8776	12.286	14	15	$.0^3322$
98	1.0000	1.0000	14.000	1	1	$.0^4214$

m = 3, n = 8

K	R	W	X_F	f	Σf	P
0	-.1429	0.0000	0.000	9135	279936	1.00000
2	-0.1250	.0156	.250	48440	270801	.96737
6	-.0893	.0469	.750	39200	222361	.79433
8	-.0714	.0625	1.000	34636	183161	.65430
14	-.0179	.1094	1.750	49056	148525	.53057
18	.0179	.1406	2.250	19656	99469	.35533
24	.0714	.1875	3.000	13776	79813	.28511
26	.0893	.2031	3.250	24192	66037	.23590
32	.1429	.2500	4.000	8330	41845	.14948
38	.1964	.2969	4.750	11424	33515	.11972
42	.2321	.3281	5.250	8960	22091	.07891
50	.3036	.3906	6.250	2632	13131	.04691
54	.3393	.4219	6.750	2016	10499	.03751
56	.3571	.4375	7.000	3472	8483	.03030
62	.4107	.4844	7.750	2240	5011	.01790
72	.5000	.5625	9.000	540	2771	.00990
74	.5179	.5781	9.250	896	2231	.00797
78	.5536	.6094	9.750	672	1335	.00477
86	.6250	.6719	10.750	352	663	.00237
96	.7143	.7500	12.000	70	311	.00111
98	.7321	.7656	12.250	168	241	$.0^3861$
104	.7857	.8125	13.000	56	73	$.0^3261$
114	.8750	.8906	14.250	16	17	$.0^4607$
128	1.0000	1.0000	16.000	1	1	$.0^5357$

$m = 3,\ n = 9$

K	R	W	X_F	f	Σf	P
0	-0.1250	0.0000	0.000	48440	1679616	1.00000
2	-.1111	.0123	.222	264726	1631176	.97116
6	-.0833	.0370	.667	215208	1366450	.81355
8	-.0694	.0494	.889	195552	1151242	.68542
14	-.0278	.0864	1.556	287784	955690	.56899
18	0.0000	.1111	2.000	116214	667906	.39765
24	.0417	.1481	2.667	84672	551692	.32846
26	.0556	.1605	2.889	152964	467020	.27805
32	.0972	.1975	3.556	55440	314056	.18698
38	.1389	.2346	4.222	79632	258616	.15397
42	.1667	.2593	4.667	63252	178984	.10656
50	.2222	.3086	5.556	20070	115732	.06890
54	.2500	.3333	6.000	15792	95662	.05695
56	.2639	.3457	6.222	28224	79870	.04755
62	.3056	.3827	6.889	19800	51646	.03075
72	.3750	.4444	8.000	5280	31846	.01896
74	.3889	.4568	8.222	9324	26566	.01582
78	.4167	.4815	8.667	7128	17242	.01027
86	.4722	.5309	9.556	4176	10114	.00602
96	.5417	.5926	10.667	1008	5938	.00354
98	.5556	.6049	10.889	2673	4930	.00294
104	.5972	.6420	11.556	1152	2257	.00134
114	.6667	.7037	12.667	522	1105	$.0^3658$
122	.7222	.7531	13.556	252	583	$.0^3347$
126	.7500	.7778	14.000	168	331	$.0^3197$
128	.7639	.7901	14.222	72	163	$.0^4970$
134	.8056	.8272	14.889	72	91	$.0^4542$
146	.8889	.9012	16.222	18	19	$.0^4113$
162	1.0000	1.0000	18.000	1	1	$.0^6595$

m = 3, n = 10

K	R	W	X_F	f	Σf	P
0	-.1111	0.0000	0.000	264726	10077696	1.00000
2	-.1000	.0100	.200	1446060	9812970	.97373
6	-.0778	.0300	.600	1208340	8366910	.83024
8	-.0667	.0400	.800	1099140	7158570	.71034
14	-.0333	.0700	1.400	1664040	6059430	.60127
18	-.0111	.0900	1.800	691740	4395390	.43615
24	.0222	.1200	2.400	520380	3703650	.36751
26	.0333	.1300	2.600	943320	3183270	.31587
32	.0667	.1600	3.200	352500	2239950	.22227
38	.1000	.1900	3.800	525000	1887450	.18729
42	.1222	.2100	4.200	431640	1362450	.13519
50	.1667	.2500	5.000	143772	930810	.09236
54	.1889	.2700	5.400	117180	787038	.07810
56	.2000	.2800	5.600	210720	669858	.06647
62	.2333	.3100	6.200	153480	459138	.04556
72	.2889	.3600	7.200	44955	305658	.03033
74	.3000	.3700	7.400	80040	260703	.02587
78	.3222	.3900	7.800	64440	180663	.01793
86	.3667	.4300	8.600	40980	116223	.01153
96	.4222	.4800	9.600	11340	75243	.00747
98	.4333	.4900	9.800	30090	63903	.00634
104	.4667	.5200	10.400	13830	33813	.00336
114	.5222	.5700	11.400	7380	19983	.00198
122	.5667	.6100	12.200	4200	12603	.00125
126	.5889	.6300	12.600	3240	8403	$.0^{3}834$
128	.6000	.6400	12.800	1450	5163	$.0^{3}512$
134	.6333	.6700	13.400	1860	3713	$.0^{3}368$
146	.7000	.7300	14.600	740	1853	$.0^{3}184$
150	.7222	.7500	15.000	252	1113	$.0^{3}110$
152	.7333	.7600	15.200	420	861	$.0^{4}854$
158	.7667	.7900	15.800	240	441	$.0^{4}438$
162	.7889	.8100	16.200	90	201	$.0^{4}199$
168	.8222	.8400	16.800	90	111	$.0^{4}110$
182	.9000	.9100	18.200	20	21	$.0^{5}208$
200	1.0000	1.0000	20.000	1	1	$.0^{7}992$

m = 3, n = 11

K	R	W	X_F	f	Σf	P
0	-.1000	0.0000	0.000	1446060	60466176	1.00000
2	-.0909	.0083	.182	7996296	59020116	.97608
6	-.0727	.0248	.546	6754440	51023820	.84384
8	-.0636	.0331	.727	6218520	44269380	.73213
14	-.0364	.0579	1.273	9646560	38050860	.62929
18	-.0182	.0744	1.636	4059000	28404300	.46976
24	.0091	.0992	2.182	3132360	24345300	.40263
26	.0182	.1074	2.364	5749920	21212940	.35082
32	.0455	.1322	2.909	2210472	15463020	.25573
38	.0727	.1570	3.455	3385800	13252548	.21917
42	.0909	.1736	3.818	2825064	9866748	.16318
50	.1273	.2066	4.546	982575	7041684	.11646
54	.1455	.2231	4.909	815760	6059109	.10021
56	.1545	.2314	5.091	1488960	5243349	.08672
62	.1818	.2562	5.636	1125234	3754389	.06209
72	.2273	.2975	6.546	348282	2629155	.04348
74	.2364	.3058	6.727	632280	2280873	.03772
78	.2545	.3223	7.091	519090	1648593	.02726
86	.2909	.3554	7.818	349140	1129503	.01868
96	.3364	.3967	8.727	104280	780363	.01291
98	.3455	.4050	8.909	283195	676083	.01118
104	.3727	.4298	9.456	137940	392888	.00650
114	.4182	.4711	10.364	80410	254948	.00422
122	.4545	.5041	11.091	51084	174538	.00289
126	.4727	.5207	11.455	40590	123454	.00204
128	.4818	.5289	11.636	18260	82864	.00137
134	.5091	.5537	12.182	25520	64604	.00107
146	.5636	.6033	13.273	12430	39084	$.0^3646$
150	.5818	.6198	13.636	4620	26654	$.0^3441$
152	.5909	.6281	13.818	8184	22034	$.0^3364$
158	.6182	.6529	14.364	5610	13850	$.0^3229$
162	.6364	.6694	14.727	2211	8240	$.0^3136$
168	.6636	.6942	15.273	2860	6029	$.0^4997$
182	.7273	.7521	16.546	1936	3169	$.0^4524$

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table continued

K	R	W	X_F	f	Σf	P
186	.7455	.7686	16.909	660	1233	$.0^4204$
194	.7818	.8017	17.636	330	573	$.0^5948$
200	.8091	.8264	18.182	110	243	$.0^5402$
206	.8364	.8512	18.727	110	133	$.0^5220$
222	.9091	.9174	20.182	22	23	$.0^6380$
242	1.0000	1.0000	22.000	1	1	$.0^7165$

m = 3, n = 12

K	R	W	X_F	f	Σf	P
0	-.0909	0.0000	0.000	7996296	362797056	1.00000
2	-.0833	.0069	.167	44396352	354800760	.97796
6	-.0682	.0208	.500	38076192	310404408	.85559
8	-.0606	.0278	.667	35210736	272328216	.75064
14	-.0379	.0486	1.167	55725120	237117480	.65358
18	-.0227	.0625	1.500	23825472	181392360	.49998
24	0.0000	.0833	2.000	18782280	157566888	.43431
26	.0076	.0903	2.167	34661088	138784608	.38254
32	.0303	.1111	2.667	13616559	104123520	.28700
38	.0530	.1319	3.167	21345984	90506961	.24947
42	.0682	.1458	3.500	18136008	69160977	.19063
50	.0985	.1736	4.167	6509052	51024969	.14064
54	.1136	.1875	4.500	5507040	44515917	.12270
56	.1212	.1944	4.667	10118988	39008877	.10752
62	.1439	.2153	5.167	7843968	28889889	.07963
72	.1818	.2500	6.000	2551714	21045921	.05801
74	.1894	.2569	6.167	4668840	18494207	.05098
78	.2045	.2708	6.500	3921984	13825367	.03811
86	.2348	.2986	7.167	2748768	9903383	.02730
96	.2727	.3333	8.000	871794	7154615	.01972
98	.2803	.3403	8.167	2385636	6282821	.01732
104	.3030	.3611	8.667	1203180	3897185	.01074
114	.3409	.3958	9.500	751080	2694005	.00743
122	.3712	.4236	10.167	506088	1942925	.00536
126	.3864	.4375	10.500	416768	1436837	.00396
128	.3939	.4444	10.667	189816	1020069	.00281
134	.4167	.4653	11.167	279840	830253	.00229
146	.4621	.5069	12.167	150744	550413	.00152
150	.4773	.5208	12.500	60192	399669	.00110
152	.4848	.5278	12.667	108108	339477	$.0^{3}936$
158	.5076	.5486	13.167	78144	231369	$.0^{3}638$
162	.5227	.5625	13.500	31812	153225	$.0^{3}422$
168	.5455	.5833	14.000	45012	121413	$.0^{3}335$
182	.5985	.6319	15.167	39072	76401	$.0^{3}211$

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table continued

K	R	W	X_F	f	Σf	P
186	.6136	.6458	15.500	15048	37329	$.0^{3}103$
194	.6439	.6736	16.167	9240	22281	$.0^{4}614$
200	.6667	.6944	16.667	3246	13041	$.0^{4}359$
206	.6894	.7153	17.167	4224	9795	$.0^{4}270$
216	.7273	.7500	18.000	924	5571	$.0^{4}154$
218	.7348	.7569	18.167	1584	4647	$.0^{4}128$
222	.7500	.7708	18.500	1344	3063	$.0^{5}844$
224	.7576	.7778	18.667	990	1719	$.0^{5}474$
234	.7955	.8125	19.500	440	729	$.0^{5}201$
242	.8258	.8403	20.167	132	289	$.0^{6}797$
248	.8485	.8611	20.667	132	157	$.0^{6}433$
266	.9167	.9236	22.167	24	25	$.0^{7}689$
288	1.0000	1.0000	24.000	1	1	$.0^{8}276$

m = 3, n = 13

K	R	W	X_F	f	Σf	P
0	-.0833	0.0000	0.000	44396352	2176782336	1.00000
2	-.0769	.0059	.154	248133600	2132386984	.97960
6	-.0641	.0178	.462	214939296	1884252384	.86561
8	-.0577	.0237	.615	200099328	1669313088	.76687
14	-.0385	.0414	1.077	322175568	1469213760	.67495
18	-.0256	.0533	1.385	139213503	1147038192	.52694
24	-.0064	.0710	1.846	111732192	1007824689	.46299
26	0.0000	.0769	2.000	207655734	896092497	.41166
32	.0192	.0947	2.462	83131620	688436763	.31626
38	.0385	.1124	2.923	132840708	605305143	.27807
42	.0513	.1243	3.231	114221250	472464435	.21705
50	.0769	.1479	3.846	42148249	358243185	.16457
54	.0897	.1598	4.154	36133812	316094936	.14521
56	.0962	.1657	4.308	66930864	279961124	.12861
62	.1154	.1834	4.769	53047280	213030260	.09786
72	.1474	.2130	5.538	17897880	159982980	.07350
74	.1538	.2189	5.692	33057024	142085100	.06527
78	.1667	.2308	6.000	28173288	109028076	.05009
86	.1923	.2544	6.615	20415824	80854788	.03714
96	.2244	.2840	7.385	6764472	60438964	.02777
98	.2308	.2899	7.538	18701826	53674492	.02466
104	.2500	.3077	8.000	9707984	34972666	.01607
114	.2821	.3373	8.769	6354348	25264682	.01161
122	.3077	.3609	9.385	4484480	18910334	.00869
126	.3205	.3728	9.692	3764904	14425854	.00663
128	.3269	.3787	9.846	1729728	10660950	.00490
134	.3462	.3964	10.308	2644928	8931222	.00410
146	.3846	.4320	11.231	1539252	6286294	.00289
150	.3974	.4438	11.538	633204	4747042	.00218
152	.4038	.4497	11.692	1155440	4113838	.00189
158	.4231	.4675	12.154	874016	2958398	.00136
162	.4359	.4793	12.462	363870	2084382	$.0^3958$
168	.4551	.4970	12.923	542256	1720512	$.0^3790$
182	.5000	.5385	14.000	537420	1178256	$.0^3541$

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table continued

K	R	W	X_F	f	Σf	P
186	.5128	.5503	14.308	217074	640836	$.0^{3}294$
194	.5385	.5740	14.923	143858	423762	$.0^{3}195$
200	.5577	.5917	15.385	53352	279904	$.0^{3}129$
206	.5769	.6095	15.846	76232	226552	$.0^{3}104$
216	.6090	.6391	16.615	20592	150320	$.0^{4}691$
218	.6154	.6450	16.769	36894	129728	$.0^{4}596$
222	.6282	.6568	17.077	31200	92834	$.0^{4}426$
224	.6346	.6627	17.231	26312	61634	$.0^{4}283$
234	.6667	.6923	18.000	14586	35322	$.0^{4}162$
242	.6923	.7160	18.615	4615	20736	$.0^{5}953$
248	.7115	.7337	19.077	6032	16121	$.0^{5}741$
254	.7308	.7515	19.538	3432	10089	$.0^{5}463$
258	.7436	.7633	19.846	2574	6657	$.0^{5}306$
266	.7692	.7870	20.462	3172	4083	$.0^{5}188$
278	.8077	.8225	21.385	572	911	$.0^{6}419$
288	.8397	.8521	22.154	156	339	$.0^{6}156$
294	.8590	.8698	22.615	156	183	$.0^{7}841$
314	.9231	.9290	24.154	26	27	$.0^{7}124$
338	1.0000	1.0000	26.000	1	1	$.0^{9}459$

m = 3, n = 14

K	R	W	X_F	f	Σf	P
0	-.0769	0.0000	0.000	248133600	13060694016	1.00000
2	-.0714	.0051	.143	1392623232	12812560416	.98100
6	-.0604	.0153	.429	1218641424	11419937184	.87437
8	-.0549	.0204	.571	1139401263	10201295760	.78107
14	-.0385	.0357	1.000	1861799940	9061894497	.69383
18	-.0275	.0459	1.286	813062250	7200094557	.55128
24	-.0110	.0612	1.714	662672010	6387032307	.48903
26	-.0055	.0663	1.857	1237392156	5724360297	.43829
32	.0110	.0816	2.286	503238736	4486968141	.34355
38	.0275	.0969	2.714	817380564	3983729405	.30502
42	.0385	.1071	3.000	711034324	3166348841	.24243
50	.0604	.1276	3.571	268298030	2455314517	.18799
54	.0714	.1378	3.857	232828596	2187016487	.16745
56	.0769	.1429	4.000	433607174	1954187891	.14962
62	.0934	.1582	4.429	349833484	1520580717	.11642
72	.1209	.1837	5.143	121852731	1170747233	.08964
74	.1264	.1888	5.286	226418192	1048894502	.08031
78	.1374	.1990	5.571	195619424	822476310	.06297
86	.1593	.2194	6.143	145561416	626856886	.04800
96	.1868	.2449	6.857	50001952	481295470	.03685
98	.1923	.2500	7.000	139114404	431293518	.03302
104	.2088	.2653	7.429	73813740	292179114	.02237
114	.2363	.2908	8.143	50250200	218365374	.01672
122	.2582	.3112	8.714	36660624	168115174	.01287
126	.2692	.3214	9.000	31327296	131454550	.01006
128	.2747	.3265	9.143	14498848	100127254	.00767
134	.2912	.3418	9.571	22782760	85628406	.00656
146	.3242	.3724	10.429	14000168	62845646	.00481
150	.3352	.3827	10.714	5905900	48845478	.00374
152	.3407	.3878	10.857	10872862	42939578	.00329
158	.3571	.4031	11.286	8468460	32066716	.00246
162	.3681	.4133	11.571	3593772	23598256	.00181
168	.3846	.4286	12.000	5549726	20004484	.00153
182	.4231	.4643	13.000	6006364	14454758	.00111

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table continued

K	R	W	X_F	f	Σf	P
186	.4341	.4745	13.286	2502500	8448394	$.0^3647$
194	.4560	.4949	13.857	1750476	5945894	$.0^3455$
200	.4725	.5102	14.286	671125	4195418	$.0^3321$
206	.4890	.5255	14.714	1009372	3524293	$.0^3270$
216	.5165	.5510	15.429	306306	2514921	$.0^3193$
218	.5220	.5561	15.571	556556	2208615	$.0^3169$
222	.5330	.5663	15.857	471380	1652059	$.0^3126$
224	.5385	.5714	16.000	416416	1180679	$.0^4904$
234	.5659	.5969	16.714	254436	764263	$.0^4585$
242	.5879	.6173	17.286	86450	509827	$.0^4390$
248	.6044	.6327	17.714	124488	423377	$.0^4324$
254	.6209	.6480	18.143	84084	298889	$.0^4229$
258	.6319	.6582	18.429	68068	214805	$.0^4164$
266	.6538	.6786	19.000	91000	146737	$.0^4112$
278	.6868	.7092	19.857	22204	55737	$.0^5427$
288	.7143	.7347	20.571	6384	33533	$.0^5257$
294	.7308	.7500	21.000	11804	27149	$.0^5208$
296	.7363	.7551	21.143	6006	15345	$.0^5118$
302	.7527	.7704	21.571	4004	9339	$.0^6715$
312	.7802	.7959	22.286	2002	5335	$.0^6408$
314	.7857	.8010	22.429	2212	3333	$.0^6255$
326	.8187	.8316	23.286	728	1121	$.0^7858$
338	.8516	.8622	24.143	182	393	$.0^7301$
344	.8681	.8776	24.571	182	211	$.0^7162$
366	.9286	.9337	26.143	28	29	$.0^8222$
392	1.0000	1.0000	28.000	1	1	$.0^{10}77$

m = 3, n = 15

K	R	W	X_F	f	Σf	P
0	-.0714	0.0000	0.000	1392623232	78364164096	1.00000
2	-.0667	.0044	.133	7850732175	76971540864	.98223
6	-.0571	.0133	.400	6925848930	69120808689	.88205
8	-.0524	.0178	.533	6504768270	62194959759	.79367
14	-.0381	.0311	.933	10766745990	55690191489	.71066
18	-.0286	.0400	1.200	4741832095	44923445499	.57327
24	-.0143	.0533	1.600	3916572660	40181613404	.51275
26	-.0095	.0578	1.733	7348160820	36265040744	.46278
32	.0048	.0711	2.133	3029786760	28916879924	.36901
38	.0190	.0844	2.533	4990415430	25887093164	.33034
42	.0286	.0933	2.800	4381286910	20896677734	.26666
50	.0476	.1111	3.333	1685959275	16515390824	.21075
54	.0571	.1200	3.600	1477405930	14829431549	.18924
56	.0691	.1244	3.733	2765943180	13352025619	.17038
62	.0762	.1378	4.133	2266123860	10586082439	.13509
72	.1000	.1600	4.800	810090710	8319958579	.10617
74	.1048	.1644	4.933	1514022510	7509867869	.09583
78	.1143	.1733	5.200	1321950630	5995845359	.07651
86	.1333	.1911	5.733	1006017870	4673894729	.05964
96	.1571	.2133	6.400	355795440	3667876859	.04681
98	.1619	.2178	6.533	996006375	3312081419	.04227
104	.1762	.2311	6.933	538257720	2316075044	.02956
114	.2000	.2533	7.600	377951340	1777817324	.02269
122	.2190	.2711	8.133	283393110	1399866984	.01786
126	.2286	.2800	8.400	245315070	1116472874	.01425
128	.2333	.2844	8.533	114241920	871157804	.01112
134	.2476	.2978	8.933	183409590	756915884	.00966
146	.2762	.3244	9.733	117791310	573506294	.00732
150	.2857	.3333	10.000	50516466	455714984	.00582
152	.2905	.3378	10.133	93753660	405198518	.00517
158	.3048	.3511	10.533	74785620	311444858	.00397
162	.3143	.3600	10.800	32193525	236659238	.00302
168	.3286	.3733	11.200	51034620	204465713	.00261
182	.3619	.4044	12.133	58899750	153431043	.00196

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table continued

K	R	W	X_F	f	Σf	P
186	.3714	.4133	12.400	25047750	94531343	.00121
194	.3905	.4311	12.933	18200910	69483593	$.0^3887$
200	.4048	.4444	13.333	7174986	51282683	$.0^3654$
206	.4190	.4578	13.733	11173890	44107697	$.0^3563$
216	.4429	.4800	14.400	3623620	32933807	$.0^3420$
218	.4476	.4844	14.533	6666660	29310187	$.0^3374$
222	.4571	.4933	14.800	5719350	22643527	$.0^3289$
224	.4619	.4978	14.933	5176080	16924177	$.0^3216$
234	.4857	.5200	15.600	3367000	11748097	$.0^3150$
242	.5048	.5378	16.133	1195845	8381097	$.0^3107$
248	.5190	.5511	16.533	1812720	7185252	$.0^4917$
254	.5333	.5644	16.933	1334190	5372532	$.0^4686$
258	.5429	.5733	17.200	1111110	4038342	$.0^4515$
266	.5619	.5911	17.733	1559250	2927232	$.0^4374$
278	.5905	.6178	18.533	434070	1367982	$.0^4175$
288	.6143	.6400	19.200	135800	933912	$.0^4119$
294	.6286	.6533	19.600	286860	798112	$.0^4102$
296	.6333	.6578	19.733	163020	511252	$.0^5652$
302	.6476	.6711	20.133	120120	348232	$.0^5444$
312	.6714	.6933	20.800	70980	228112	$.0^5291$
314	.6762	.6978	20.933	68670	157132	$.0^5201$
326	.7048	.7244	21.733	32760	88462	$.0^5113$
338	.7333	.7511	22.533	21495	55702	$.0^6711$
342	.7429	.7600	22.800	10010	34207	$.0^6437$
344	.7476	.7644	22.933	11340	24197	$.0^6309$
350	.7619	.7778	23.333	6006	12857	$.0^6164$
362	.7905	.8044	24.133	2730	6851	$.0^7874$
366	.8000	.8133	24.400	2760	4121	$.0^7526$
378	.8286	.8400	25.200	910	1361	$.0^7174$
392	.8619	.8711	26.133	210	451	$.0^8576$
398	.8762	.8844	26.533	210	241	$.0^8308$
422	.9333	.9378	28.133	30	31	$.0^9396$
450	1.0000	1.0000	30.000	1	1	$.0^{10}13$

m = 4, n = 3

K	R	W	X_F	f	Σf	P
1	-.4667	.0222	.200	24	576	1.00000
3	-.4000	.0667	.600	28	552	.95833
5	-.3333	.1111	1.000	105	524	.90972
9	-.2000	.2000	1.800	69	419	.72743
11	-.1333	.2444	2.200	48	350	.60764
13	-.0667	.2889	2.600	45	302	.52431
17	.0667	.3778	3.400	60	257	.44618
19	.1333	.4222	3.800	24	197	.34201
21	.2000	.4667	4.200	54	173	.30035
25	.3333	.5556	5.000	18	119	.20660
27	.4000	.6000	5.400	16	101	.17535
29	.4667	.6444	5.800	42	85	.14757
33	.6000	.7333	6.600	18	43	.07465
35	.6667	.7778	7.000	12	31	.05382
37	.7333	.8222	7.400	9	19	.03299
41	.8667	.9111	8.200	9	10	.01736
45	1.0000	1.0000	9.000	1	1	.00174

$m = 4, n = 4$

K	R	W	X_F	f	Σf	P
0	-.3333	0.0000	0.000	105	13824	1.00000
2	-.3000	.0250	.300	888	13719	.99240
4	-.2667	.0500	.600	384	12831	.92817
6	-.2333	.0750	.900	1392	12447	.90039
8	-.2000	.1000	1.200	633	11055	.79970
10	-.1667	0.1250	1.500	1068	10422	.75391
12	-.1333	.1500	1.800	384	9354	.67665
14	-.1000	.1750	2.100	1728	8970	.64887
16	-.0667	.2000	2.400	225	7242	.52387
18	-.0333	.2250	2.700	1044	7017	.50760
20	.0000	0.2500	3.000	592	5973	.43207
22	.0333	.2750	3.300	480	5381	.38925
24	.0667	.3000	3.600	420	4901	.35453
26	.1000	.3250	3.900	1140	4481	.32415
30	.1667	.3750	4.500	576	3341	.24168
32	.2000	.4000	4.800	142	2765	.20001
34	.2333	.4250	5.100	432	2623	.18974
36	.2667	.4500	5.400	240	2191	.15849
38	.3000	.4750	5.700	496	1951	.14113
40	.3333	.5000	6.000	150	1455	.10525
42	.3667	.5250	6.300	240	1305	.09440
44	.4000	.5500	6.600	128	1065	.07704
46	.4333	.5750	6.900	192	937	.06778
48	.4667	.6000	7.200	30	745	.05389
50	.5000	.6250	7.500	212	715	.05172
52	.5333	.6500	7.800	48	503	.03639
54	.5667	.6750	8.100	192	455	.03291
56	.6000	.7000	8.400	68	263	.01902
58	.6333	.7250	8.700	36	195	.01411
62	.7000	.7750	9.300	64	159	.01150
64	.7333	.8000	9.600	9	95	.00687
66	.7667	.8250	9.900	48	86	.00622
68	.8000	.8500	10.200	16	38	.00275
72	.8667	.9000	10.800	9	22	.00159
74	.9000	.9250	11.100	12	13	$.0^3940$
80	1.0000	1.0000	12.000	1	1	$.0^4723$

$m = 4,\ n = 5$

K	R	W	X_F	f	Σf	P
1	−.2400	.0080	.120	8430	331776	1.00000
3	−.2200	.0240	.360	10200	323346	.97459
5	−.2000	.0400	.600	28960	323146	.94385
9	−.1600	.0720	1.080	28410	284186	.85656
11	−.1400	.0880	1.320	20560	255776	.77093
13	−.1200	.1040	1.560	18840	235216	.70896
17	−.0800	.1360	2.040	30180	216376	.65217
19	−.0600	.1520	2.280	13480	186196	.56121
21	−.0400	.1680	2.520	25200	172716	.52058
25	.0000	.2000	3.000	12290	147516	.44463
27	.0200	.2160	3.240	11800	135226	.40758
29	.0400	.2320	3.480	24480	123426	.37202
33	.0800	.2640	3.960	12580	98946	.29823
35	.1000	.2800	4.200	11360	86366	.26031
37	.1200	.2960	4.440	5460	75006	.22607
41	.1600	.3280	4.920	15920	69546	.20962
43	.1800	.3440	5.160	3400	53626	.16163
45	.2000	.3600	5.400	9345	50226	.15139
49	.2400	.3920	5.880	5510	40881	.12322
51	.2600	.4080	6.120	4400	35371	.10661
53	.2800	.4240	6.360	5935	30971	.09335
57	.3200	.4560	6.840	2940	25036	.07546
59	.3400	.4720	7.080	3920	22096	.06660
61	.3600	.4880	7.320	3465	18176	.05478
65	.4000	.5200	7.800	3550	14711	.04434
67	.4200	.5360	8.040	760	11161	.03364
69	.4400	.5520	8.280	2900	10401	.03135
73	.4800	.5840	8.760	1010	7501	.02261
75	.5000	.6000	9.000	960	6491	.01956
77	.5200	.6160	9.240	1550	5531	.01667
81	.5600	.6480	9.720	1080	3981	.01200
83	.5800	.6640	9.960	680	2901	.00874
85	.6000	.6800	10.200	410	2221	.00669
89	.6400	.7120	10.680	770	1811	.00546

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table continued

K	R	W	X_F	f	Σf	P
91	.6600	.7280	10.920	280	1041	.00314
93	.6800	.7440	11.160	150	761	.00229
97	.7200	.7760	11.640	80	611	.00184
99	.7400	.7920	11.880	80	531	.00160
101	.7600	.8080	12.120	240	451	.00136
105	.8000	.8400	12.600	100	211	$.0^{3}636$
107	.8200	.8560	12.840	40	111	$.0^{3}335$
109	.8400	.8720	13.080	25	71	$.0^{3}214$
113	.8800	.9040	13.560	30	46	$.0^{3}139$
117	.9200	.9360	14.040	15	16	$.0^{4}482$
125	1.0000	1.0000	15.000	1	1	$.0^{5}301$

m = 4, n = 6

K	R	W	X_F	f	Σf	P
0	-.2000	0.0000	0.000	28960	7962624	1.00000
2	-.1867	.0111	.200	310080	7933664	.99636
4	-.1733	.0222	.400	141960	7623584	.95742
6	-.1600	.0333	.600	522240	7481624	.93959
8	-.1467	.0444	.800	240240	6959384	.87401
10	-.1333	.0556	1.000	438720	6719144	.84384
12	-.1200	.0667	1.200	137160	6280424	.78874
14	-.1067	.0778	1.400	738720	6143264	.77151
16	-.0933	.0889	1.600	86580	5404544	.67874
18	-.0800	.1000	1.800	466140	5317964	.66787
20	-.0667	.1111	2.000	283665	4851824	.60932
22	-.0533	.1222	2.200	256800	4568159	.57370
24	-.0400	.1333	2.400	234840	4311359	.54145
26	-.0267	.1444	2.600	647130	4076519	.51196
30	.0000	.1667	3.000	359580	3429389	.43069
32	.0133	.1778	3.200	82980	3069809	.38553
34	.0267	.1889	3.400	296850	2986829	.37511
36	.0400	.2000	3.600	169525	2689979	.33783
38	.0533	.2111	3.800	371400	2520454	.31654
40	.0667	.2222	4.000	114060	2149054	.26989
42	.0800	.2333	4.200	203280	2034994	.25559
44	.0933	.2444	4.400	95280	1831714	.23004
46	.1067	.2556	4.600	169920	1736434	.21807
48	.1200	.2667	4.800	25440	1566514	.19673
50	.1333	.2778	5.000	240900	1541074	.19354
52	.1467	.2889	5.200	62145	1300174	.16328
54	.1600	.3000	5.400	228700	1238029	.15548
56	.1733	.3111	5.600	101160	1009329	.12676
58	.1867	.3222	5.800	47370	908169	.11405
62	.2133	.3444	6.200	149280	860799	.10810
64	.2267	.3556	6.400	9360	711519	.08936
66	.2400	.3667	6.600	122400	702159	.08818
68	.2533	.3778	6.800	55020	579759	.07281
70	.2667	.3889	7.000	46620	524739	.06590

table continued on next page

table continued

K	R	W	X_F	f	Σf	P
72	.2800	.4000	7.200	33860	478119	.06005
74	.2933	.4111	7.400	99240	444259	.05579
76	.3067	.4222	7.600	17280	345019	.04333
78	.3200	.4333	7.800	30720	327739	.04116
80	.3333	.4444	8.000	14856	297019	.03730
82	.3467	.4556	8.200	26910	282163	.03544
84	.3600	.4667	8.400	22830	255253	.03206
86	.3733	.4778	8.600	50880	232423	.02919
88	.3867	.4889	8.800	8160	181543	.02280
90	.4000	.5000	9.000	39338	173383	.02177
94	.4267	.5222	9.400	24840	134045	.01683
96	.4400	.5333	9.600	5400	109205	.01371
98	.4533	.5444	9.800	22080	103805	.01304
100	.4667	.5556	10.000	5526	81725	.01026
102	.4800	.5667	10.200	8160	76199	.00957
104	.4933	.5778	10.400	10260	68039	.00854
106	.5067	.5889	10.600	8850	57779	.00726
108	.5200	.6000	10.800	3920	48929	.00614
110	.5333	.6111	11.000	13344	45009	.00565
114	.5600	.6333	11.400	5640	31665	.00398
116	.5733	.6444	11.600	3870	26025	.00327
118	.5867	.6556	11.800	3900	22155	.00278
120	.6000	.6667	12.000	2472	18255	.00229
122	.6133	.6778	12.200	4110	15783	.00198
126	.6400	.7000	12.600	4480	11673	.00147
128	.6533	.7111	12.800	240	7193	$.0^{3}903$
130	.6667	.7222	13.000	1152	6953	$.0^{3}873$
132	.6800	.7333	13.200	660	5801	$.0^{3}729$
134	.6933	.7444	13.400	1980	5141	$.0^{3}646$
136	.7067	.7556	13.600	300	3161	$.0^{3}397$
138	.7200	.7667	13.800	660	2861	$.0^{3}359$
140	.7333	.7778	14.000	312	2201	$.0^{3}276$
144	.7600	.8000	14.400	100	1889	$.0^{3}237$
146	.7733	.8111	14.600	810	1789	$.0^{3}225$

table continued on next page

table continued

K	R	W	X_F	f	Σf	P
148	.7867	.8222	14.800	225	979	$.0^3123$
150	.8000	.8333	15.000	264	754	$.0^4947$
152	.8133	.8444	15.200	120	490	$.0^4615$
154	.8267	.8556	15.400	180	370	$.0^4465$
158	.8533	.8778	15.800	60	190	$.0^4239$
160	.8667	.8889	16.000	36	130	$.0^4163$
162	.8800	.9000	16.200	30	94	$.0^4118$
164	.8933	.9111	16.400	45	64	$.0^5804$
170	.9333	.9444	17.000	18	19	$.0^5239$
180	1.0000	1.0000	18.000	1	1	$.0^6126$

m = 4, n = 7

K	R	W	X_F	f	Σf	P
1	-.1619	.0041	.086	3151680	191102976	1.00000
3	-.1524	.0122	.257	3900960	187951396	.98351
5	-.1429	.0204	.429	10913385	184050336	.96310
9	-.1238	.0367	.771	11679045	173136951	.90599
11	-.1143	.0449	.943	8664180	161457906	.84487
13	-.1048	.0531	1.114	8045205	152793726	.79954
17	-.0857	.0694	1.457	13778800	144748521	.75744
19	-.0762	.0776	1.629	6367200	130969721	.68534
21	-.0667	.0857	1.800	11849670	124602521	.65202
25	-.0476	.1020	2.143	6313545	112752851	.59001
27	-.0381	.1102	2.314	6235320	106439306	.55697
29	-.0286	.1184	2.486	12976215	100203986	.52435
33	-.0095	.1347	2.829	7351470	87227771	.45644
35	.0000	.1429	3.000	6796160	79876301	.41798
37	.0095	.1510	3.171	3184755	73080141	.38241
41	.0286	.1673	3.514	10669750	69895386	.36575
43	.0381	.1755	3.686	2437260	59225636	.30991
45	.0476	.1837	3.857	6778821	56788376	.29716
49	.0667	.2000	4.200	4324530	50009555	.26169
51	.0762	.2082	4.371	3554880	45685025	.23906
53	.0857	.2163	4.543	4906951	42130145	.22046
57	.1048	.2327	4.886	2733990	37223194	.19478
59	.1143	.2408	5.057	3783164	34489204	.18047
61	.1238	.2490	5.229	3444714	30706040	.16068
65	.1429	.2653	5.571	3911005	27261326	.14265
67	.1524	.2735	5.743	890400	23350321	.12219
69	.1619	.2816	5.914	3332448	22459921	.11753
73	.1810	.2980	6.257	1379553	19127473	.10009
75	.1905	.3061	6.429	1446144	17747920	.09287
77	.2000	.3143	6.600	2283932	16301776	.08530
81	.2190	.3306	6.943	2021208	14017844	.07335
83	.2286	.3388	7.114	1314208	11996636	.06278
85	.2381	.3469	7.286	816060	10682428	.05590
89	.2571	.3633	7.629	1976401	9866368	.05163

table continued on next page

table continued

K	R	W	X_F	f	Σf	P
91	.2667	.3714	7.800	633108	7889967	.04129
93	.2762	.3796	7.971	540708	7256859	.03797
97	.2952	.3959	8.314	440811	6716151	.03514
99	.3048	.4041	8.486	597408	6275340	.03284
101	.3143	.4122	8.657	1303155	5677932	.02971
105	.3333	.4286	9.000	628278	4374777	.02289
107	.3429	.4367	9.171	409752	3746499	.01960
109	.3524	.4449	9.343	375333	3336747	.01746
113	.3714	.4612	9.686	420462	2961414	.01550
115	.3810	.4694	9.857	179424	2540952	.01330
117	.3905	.4776	10.029	421575	2361528	.01236
121	.4095	.4939	10.371	208383	1939953	.01015
123	.4190	.5020	10.543	124824	1731570	.00906
125	.4286	.5102	10.714	330582	1606746	.00841
129	.4476	.5265	11.057	258426	1276164	.00668
131	.4571	.5347	11.229	202496	1017738	.00533
133	.4667	.5429	11.400	62244	815242	.00427
137	.4857	.5592	11.743	112588	752998	.00394
139	.4952	.5673	11.914	71988	640410	.00335
141	.5048	.5755	12.086	92106	568422	.00297
145	.5238	.5918	12.429	73038	476316	.00249
147	.5333	.6000	12.600	37632	403278	.00211
149	.5429	.6082	12.771	93212	365646	.00191
153	.5619	.6245	13.114	58506	272434	.00143
155	.5714	.6327	13.286	31584	213928	.00112
157	.5810	.6408	13.457	31143	182344	$.0^3954$
161	.6000	.6571	13.800	46256	151201	$.0^3791$
163	.6095	.6653	13.971	4704	104945	$.0^3549$
165	.6190	.6735	14.143	14826	100241	$.0^3525$
169	.6381	.6898	14.486	14763	85415	$.0^3447$
171	.6476	.6980	14.657	15708	70652	$.0^3370$
173	.6571	.7061	14.829	18438	54944	$.0^3288$
177	.6762	.7224	15.171	3360	36506	$.0^3191$
179	.6857	.7306	15.343	8288	33146	$.0^3173$

table continued on next page

table continued

K	R	W	X_F	f	Σf	P
181	.6952	.7388	15.514	4557	24858	$.0^{3}130$
185	.7143	.7551	15.857	5943	20301	$.0^{3}106$
187	.7238	.7633	16.029	1260	14358	$.0^{4}751$
189	.7333	.7714	16.200	4452	13098	$.0^{4}685$
193	.7524	.7878	16.543	609	8646	$.0^{4}452$
195	.7619	.7959	16.714	1344	8037	$.0^{4}421$
197	.7714	.8041	16.886	1827	6693	$.0^{4}350$
201	.7905	.8204	17.229	1890	4866	$.0^{4}255$
203	.8000	.8286	17.400	728	2976	$.0^{4}156$
205	.8095	.8367	17.571	693	2248	$.0^{4}118$
209	.8286	.8531	17.914	938	1555	$.0^{5}814$
213	.8476	.8694	18.257	294	617	$.0^{5}323$
219	.8762	.8939	18.771	84	323	$.0^{5}169$
221	.8857	.9020	18.943	154	239	$.0^{5}125$
225	.9048	.9184	19.286	63	85	$.0^{6}445$
233	.9429	.9510	19.971	21	22	$.0^{6}115$
245	1.0000	1.0000	21.000	1	1	$.0^{8}523$

m = 4, n = 8

K	R	W	X_F	f	Σf	P
0	-.1429	0.0000	0.000	10913385	4586471424	1.00000
2	-.1357	.0062	.150	122164560	4575558039	.99762
4	-.1286	.0125	.300	57141280	4453393479	.97098
6	-.1214	.0187	.450	213839360	4396252199	.95853
8	-.1143	.0250	.600	100027480	4182412839	.91190
10	-.1071	.0312	.750	187046720	4082385359	.89009
12	-.1000	.0375	.900	58405760	3895338639	.84931
14	-.0929	.0437	1.050	326824960	3836932879	.83658
16	-.0857	.0500	1.200	38330320	3510108919	.76532
18	-.0786	.0562	1.350	214187120	3471777599	.75696
20	-.0714	.0625	1.500	133357056	3257590479	.71026
22	-.0643	.0687	1.650	124221440	3124233423	.68118
24	-.0571	.0750	1.800	116049920	3000012983	.65410
26	-.0500	.0812	1.950	325462144	2883962063	.62880
30	-.0357	.0937	2.250	189066752	2558499919	.55784
32	-.0286	.1000	2.400	44058889	2369433167	.51661
34	-.0214	.1062	2.550	163902312	2325374278	.50701
36	-.0143	.1125	2.700	95567584	2161472966	.47127
38	-.0071	.1187	2.850	214183424	2065904382	.45043
40	.0000	.1250	3.000	66847060	1851721958	.40374
42	.0071	.1312	3.150	123546752	1784873898	.38916
44	.0143	.1375	3.300	57948800	1661327146	.36222
46	.0214	.1437	3.450	107614304	1603378346	.34959
48	.0286	.1500	3.600	16535680	1495764042	.32613
50	.0357	.1562	3.750	162509424	1479228362	.32252
52	.0429	.1625	3.900	43198624	1316718938	.28709
54	.0500	.1687	4.050	161319872	1273520314	.27767
56	.0571	.1750	4.200	74435760	1112200442	.24250
58	.0643	.1812	4.350	35039200	1037764682	.22627
62	.0786	.1937	4.650	119820288	1002725482	.21863
64	.0857	.2000	4.800	7173537	882905194	.19250
66	.0929	.2062	4.950	103648048	875731657	.19094
68	.1000	.2125	5.100	48135584	772083709	.16834
70	.1071	.2187	5.250	43799392	723948025	.15784

table continued on next page

table continued

K	R	W	X_F	f	Σf	P
72	.1143	.2250	5.400	31249820	680148633	.14829
74	.1214	.2312	5.550	95900448	648898813	.14148
76	.1286	.2375	5.700	17524864	552998365	.12057
78	.1357	.2437	5.850	32544512	535473601	.11675
80	.1429	.2500	6.000	15414336	502928989	.10965
82	.1500	.2562	6.150	28890008	487514653	.10629
84	.1571	.2625	6.300	26188288	458624645	.10000
86	.1643	.2687	6.450	60618432	432436357	.09429
88	.1714	.2750	6.600	10799152	371817925	.08107
90	.1786	.2812	6.750	51364096	361018773	.07871
94	.1929	.2937	7.050	35221984	309654677	.06751
96	.2000	.3000	7.200	8019284	274432693	.05984
98	.2071	.3062	7.350	33837832	266413409	.05809
100	.2143	.3125	7.500	8746752	232575577	.05071
102	.2214	.3187	7.650	1311048	223828825	.04880
104	.2286	.3250	7.800	17883852	210718777	.04594
106	.2357	.3312	7.950	16409344	192834925	.04204
108	.2429	.3375	8.100	6903680	176425581	.03847
110	.2500	.3437	8.250	27664448	169521901	.03696
114	.2643	.3562	8.550	15013040	141857453	.03093
116	.2714	.3625	8.700	10560032	126844413	.02766
118	.2786	.3687	8.850	9999360	116284381	.02535
120	.2857	.3750	9.000	6292160	106285021	.02317
122	.2929	.3812	9.150	13789216	99992861	.02180
126	.3071	.3937	9.450	14579936	86203645	.01880
128	.3143	.4000	9.600	1046990	71623709	.01562
130	.3214	.4062	9.750	4117512	70576719	.01539
132	.3286	.4125	9.900	3525760	66459207	.01449
134	.3357	.4187	10.050	11538688	62933447	.01372
136	.3429	.4250	10.200	2866640	51394759	.01121
138	.3500	.4312	10.350	5451712	48528119	.01058
140	.3571	.4375	10.500	2515072	43076407	.00939
142	.3643	.4437	10.650	2080512	40561335	.00884
144	.3714	.4500	10.800	1268512	38480823	.00839

table continued on next page

table continued

K	R	W	X_F	f	Σf	P
146	.3786	.4562	10.950	7803600	37212311	.00811
148	.3857	.4625	11.100	1054592	29408711	.00641
150	.3929	.4687	11.250	3993248	28354119	.00618
152	.4000	.4750	11.400	2219168	24360871	.00531
154	.4071	.4812	11.550	2769536	22141703	.00483
158	.4214	.4937	11.850	2157568	19372167	.00422
160	.4286	.5000	12.000	579110	17214599	.00375
162	.4357	.5062	12.150	1975064	16635489	.00368
164	.4429	.5125	12.300	1652000	14660425	.00320
166	.4500	.5187	12.450	1662752	13008425	.00284
168	.4571	.5250	12.600	625072	11345673	.00247
170	.4643	.5312	12.750	1888096	10720601	.00234
172	.4714	.5375	12.900	226688	8832505	.00193
174	.4786	.5437	13.050	1546720	8605817	.00188
176	.4857	.5500	13.200	241024	7059097	.00154
178	.4929	.5562	13.350	859264	6818073	.00149
180	.5000	.5625	13.500	497120	5958809	.00130
182	.5071	.5687	13.650	1004640	5461689	.00119
184	.5143	.5750	13.800	310128	4457049	$.0^{3}972$
186	.5214	.5812	13.950	817152	4146921	$.0^{3}904$
190	.5357	.5937	14.250	174272	3329769	$.0^{3}726$
192	.5429	.6000	14.400	18270	3155497	$.0^{3}688$
194	.5500	.6062	14.550	808328	3137227	$.0^{3}684$
196	.5571	.6125	14.700	171808	2328899	$.0^{3}508$
198	.5643	.6187	14.850	351232	2157091	$.0^{3}470$
200	.5714	.6250	15.000	193564	1805859	$.0^{3}394$
202	.5786	.6312	15.150	148064	1612295	$.0^{3}352$
204	.5857	.6375	15.300	115584	1464231	$.0^{3}319$
206	.5929	.6437	15.450	402752	1348647	$.0^{3}294$
208	.6000	.6500	15.600	36800	945895	$.0^{3}206$
210	.6071	.6562	15.750	108304	909095	$.0^{3}198$
212	.6143	.6625	15.900	98560	800791	$.0^{3}175$
214	.6214	.6687	16.050	93408	702231	$.0^{3}153$
216	.6286	.6750	16.200	95536	608823	$.0^{3}133$

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table continued

K	R	W	X_F	f	Σf	P
218	.6357	.6812	16.350	70784	513287	$.0^3112$
222	.6500	.6937	16.650	103072	442503	$.0^4965$
224	.6571	.7000	16.800	16660	339431	$.0^4740$
226	.6643	.7062	16.950	46592	322771	$.0^4704$
228	.6714	.7125	17.100	12992	276179	$.0^4602$
230	.6786	.7187	17.250	82880	263187	$.0^4574$
232	.6857	.7250	17.400	10780	180307	$.0^4393$
234	.6929	.7312	17.550	45920	169527	$.0^4370$
236	.7000	.7375	17.700	14464	123607	$.0^4270$
238	.7071	.7437	17.850	19712	109143	$.0^4238$
242	.7214	.7562	18.150	21672	89431	$.0^4195$
244	.7286	.7625	18.300	2912	67759	$.0^4148$
246	.7357	.7687	18.450	10976	64847	$.0^4141$
248	.7429	.7750	18.600	6944	53871	$.0^4117$
250	.7500	.7812	18.750	12224	46927	$.0^4102$
254	.7643	.7937	19.050	9184	34703	$.0^5757$
256	.7714	.8000	19.200	1225	25519	$.0^5556$
258	.7786	.8062	19.350	3920	24294	$.0^5530$
260	.7857	.8125	19.500	4256	20374	$.0^5444$
262	.7929	.8187	19.650	2464	16118	$.0^5351$
264	.8000	.8250	19.800	3040	13654	$.0^5298$
266	.8071	.8312	19.950	4928	10614	$.0^5231$
270	.8214	.8437	20.250	1344	5686	$.0^5124$
272	.8286	.8500	20.400	784	4342	$.0^6947$
274	.8357	.8562	20.550	952	3558	$.0^6776$
276	.8429	.8625	20.700	896	2606	$.0^6568$
278	.8500	.8687	20.850	704	1710	$.0^6373$
282	.8643	.8812	21.150	448	1006	$.0^6219$
288	.8857	.9000	21.600	105	558	$.0^6122$
290	.8929	.9062	21.750	280	453	$.0^7988$
292	.9000	.9125	21.900	64	173	$.0^7377$
296	.9143	.9250	22.200	84	109	$.0^7238$
306	.9500	.9562	22.950	24	25	$.0^8545$
320	1.0000	1.0000	24.000	1	1	$.0^9218$

APPENDIX II

EXACT DISTRIBUTION OF AVERAGE TAU

m = 3, n = 3

L	T	f	Σf	P
-3	-.3333	17	36	1.00000
1	.1111	12	19	.52778
5	.5556	6	7	.19444
9	1.0000	1	1	.02778

m = 3, n = 4

L	T	f	Σf	P
-6	-.3333	15	216	1.00000
-4	-.2222	48	201	.93056
-2	-.1111	60	153	.70833
0	0.0000	28	93	.43056
2	.1111	6	65	.30093
4	.2222	24	59	.27315
6	.3333	20	35	.16204
10	.5556	6	15	.06944
12	.6667	8	9	.04167
18	1.0000	1	1	.00463

m = 3, n = 5

L	T	f	Σf	P
-6	-.2000	370	1296	1.00000
-2	-.0667	430	926	.71451
2	.0667	240	496	.38272
6	.2000	95	256	.19753
10	.3333	100	161	.12423
14	.4667	30	61	.04707
18	.6000	20	31	.02392
22	.7333	10	11	.00849
30	1.0000	1	1	$.0^{3}772$

m = 3, n = 6

L	T	f	Σf	P
-9	-.2000	310	7776	1.00000
-7	-.1556	1200	7466	.96013
-5	-.1111	1680	6266	.80581
-3	-.0667	825	4586	.58976
-1	-.0222	300	3761	.48367
1	.0222	1080	3461	.44509
3	.0667	900	2381	.30620
7	.1556	300	1481	.19046
9	.2000	470	1181	.15188
11	.2444	120	711	.09144
13	.2889	120	591	.07600
15	.3333	66	471	.06057
17	.3778	120	405	.05208
19	.4222	180	285	.03665
25	.5556	42	105	.01350
27	.6000	20	63	.00810
29	.6444	30	43	.00553
35	.7778	12	13	.00167
45	1.0000	1	1	$.0^3129$

m = 3, n = 7

L	T	f	Σf	P
-9	-.1429	9135	46656	1.00000
-5	-.0794	13440	37521	.80421
-1	-.0159	8190	24081	.51614
3	.0476	4641	15891	.34060
7	.1111	5250	11250	.24113
11	.1746	1764	6000	.12860
15	.2381	1540	4236	.09079
19	.3016	1386	2696	.05778
23	.3651	252	1310	.02808
27	.4286	581	1058	.02268
31	.4921	294	477	.01022
39	.6190	126	183	.00392
43	.6825	42	57	.00122
51	.8095	14	15	$.0^{3}322$
63	1.0000	1	1	$.0^{4}214$

m = 3, n = 8

L	T	f	Σf	P
-12	-.1429	7455	279936	1.00000
-10	-.1190	31920	272481	.97337
-8	-.0952	47880	240561	.85934
-6	-.0714	24696	192681	.68830
-4	-.0476	11200	167985	.60008
-2	-.0238	39200	156785	.56007
0	0.0000	33096	117585	.42004
4	.0476	11620	84489	.30182
6	.0714	20272	72869	.26031
8	.0952	7840	52597	.18789
10	.1190	7616	44757	.15988
12	.1429	3220	37141	.13268
14	.1667	7280	33921	.12117
16	.1905	11648	26641	.09517
20	.2381	70	14993	.05356
22	.2619	3696	14923	.05331
24	.2857	2352	11227	.04011
26	.3095	2576	8875	.03170
28	.3333	840	6299	.02250
30	.3571	1456	5459	.01950
32	.3810	1176	4003	.01430
36	.4286	476	2827	.01010
38	.4524	560	2351	.00840
40	.4762	896	1791	.00640
42	.5000	120	895	.00320
46	.5476	448	775	.00277
52	.6190	70	327	.00117
54	.6429	112	257	$.0^{3}918$
56	.6667	72	145	$.0^{3}518$
60	.7143	56	73	$.0^{3}261$
70	.8333	16	17	$.0^{4}607$
84	1.0000	1	1	$.0^{5}357$

m = 3, n = 9

L	T	f	Σf	P
-12	-.1111	243306	1679616	1.00000
-8	-.0741	411768	1436310	.85514
-4	-.0370	268380	1024542	.60999
0	0.0000	184940	756162	.45020
4	.0370	212688	571222	.34009
8	.0741	76608	358534	.21346
12	.1111	78246	281926	.16785
16	.1481	82656	203680	.12127
20	.1852	18396	121024	.07205
24	.2222	41364	102628	.06110
28	.2593	24066	61264	.03648
32	.2963	3528	37198	.02215
36	.3333	14340	33670	.02005
40	.3704	8568	19330	.01151
44	.4074	3024	10762	.00641
48	.4444	1728	7738	.00461
52	.4815	3096	6010	.00358
56	.5185	1512	2914	.00173
60	.5556	153	1402	$.0^{3}835$
64	.5926	648	1249	$.0^{3}744$
68	.6296	252	601	$.0^{3}358$
72	.6667	168	349	$.0^{3}208$
76	.7037	90	181	$.0^{3}108$
80	.7407	72	91	$.0^{4}542$
92	.8519	18	19	$.0^{4}113$
108	1.0000	1	1	$.0^{6}595$

m = 3, n = 10

L	T	f	Σf	P
-15	-.1111	195426	10077696	1.00000
-13	-.0963	890820	9882270	.98061
-11	-.0815	1399860	8991450	.89221
-9	-.0667	749490	7591590	.75331
-7	-.0519	383040	6842100	.67893
-5	-.0370	1335600	6459060	.64093
-3	-.0222	1144080	5123460	.50840
1	.0074	415800	3979380	.39487
3	.0222	779940	3563580	.35361
5	.0370	365400	2783640	.27622
7	.0519	348390	2418240	.23996
9	.0667	134520	2069850	.20539
11	.0815	322560	1935330	.19204
13	.0963	539280	1612770	.16003
17	.1259	8820	1073490	.10652
19	.1407	212400	1064670	.10565
21	.1556	143220	852270	.08457
23	.1704	138600	709050	.07036
25	.1852	65520	570450	.05661
27	.2000	115920	504930	.05010
29	.2148	72000	389010	.03860
33	.2444	40200	317010	.03146
35	.2593	48132	276810	.02747
37	.2741	80640	228678	.02269
39	.2889	12615	148038	.01469
43	.3185	47520	135423	.01344
45	.3333	10080	87903	.00872
49	.3630	8820	77823	.00772
51	.3778	18720	69003	.00685
53	.3926	13140	50283	.00499
55	.4074	8820	37143	.00369
57	.4222	7440	28323	.00281
61	.4519	5760	20883	.00207
67	.4963	4950	15123	.00150

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table continued

L	T	f	Σf	P
69	.5111	4200	10173	.00101
71	.5259	1350	5973	$.0^{3}593$
75	.5556	2400	4623	$.0^{3}459$
81	.6000	190	2223	$.0^{3}221$
85	.6296	1152	2033	$.0^{3}202$
87	.6444	420	881	$.0^{4}874$
93	.6889	240	461	$.0^{4}457$
99	.7333	110	221	$.0^{4}219$
103	.7630	90	111	$.0^{4}110$
117	.8667	20	21	$.0^{5}208$
135	1.0000	1	1	$.0^{7}993$

m = 4, n = 3

L	T	f	Σf	P
-6	-.3333	151	576	1.00000
-2	-.1111	165	425	.73785
2	.1111	135	260	.45139
6	.3333	82	125	.21701
10	.5556	33	43	.07465
14	.7778	9	10	.01736
18	1.0000	1	1	.00174

m = 4, n = 4

L	T	f	Σf	P
-12	-.3333	99	13824	1.00000
-10	-.2778	540	13725	.99284
-8	-.2222	1368	13185	.95378
-6	-.1667	2052	11817	.85482
-4	-.1111	1929	9765	.70638
-2	-.0556	1296	7836	.56684
0	.0000	1120	6540	.47309
2	.0556	1332	5420	.39207
4	.1111	1071	4088	.29572
6	.1667	588	3017	.21824
8	.2222	624	2429	.17571
10	.2778	684	1805	.13057
12	.3333	326	1121	.08109
14	.3889	180	795	.05751
16	.4444	288	615	.04449
18	.5000	156	327	.02365
20	.5556	21	171	.01237
22	.6111	72	150	.01085
24	.6667	56	78	.00564
28	.7778	9	22	.00159
30	.8333	12	13	$.0^{3}940$
36	1.0000	1	1	$.0^{4}723$

m = 4, n = 5

L	T	f	Σf	P
-12	-.2000	33820	331776	1.00000
-8	-.1333	65640	297956	.89806
-4	-.0667	68230	232316	.70022
0	0.0000	51270	164086	.49457
4	.0667	36380	112816	.34004
8	.1333	28660	76436	.23038
12	.2000	18455	47776	.14400
16	.2667	12170	29321	.08838
20	.3333	8075	17151	.05169
24	.4000	4440	9076	.02736
28	.4667	2405	4636	.01397
32	.5333	1330	2231	.00672
36	.6000	470	901	.00272
40	.6667	300	431	.00130
44	.7333	85	131	$.0^3395$
48	.8000	30	46	$.0^3139$
52	.8667	15	16	$.0^4482$
60	1.0000	1	1	$.0^5301$

m = 4, n = 6

L	T	f	Σf	P
-18	-.2000	18400	7962624	1.00000
-16	-.1778	128970	7944224	.99769
-14	-.1556	395670	7815254	.98149
-12	-.1333	679740	7419584	.93180
-10	-.1111	710400	6739844	.84644
-8	-.0889	554580	6029444	.75722
-6	-.0667	599175	5474864	.68757
-4	-.0444	773100	4875689	.61232
-2	-.0222	628920	4102589	.51523
0	0.0000	375420	3473669	.43625
2	.0222	463260	3098249	.38910
4	.0444	532080	2634989	.33092
6	.0667	302280	2102909	.26410
8	.0889	213750	1800629	.22614
10	.1111	305565	1586879	.19929
12	.1333	254430	1281314	.16092
14	.1556	164280	1026884	.12896
16	.1778	141030	862604	.10833
18	.2000	132680	721574	.09062
20	.2222	141300	588894	.07396
22	.2444	95640	447594	.05621
24	.2667	43920	351954	.04420
26	.2889	66495	308034	.03868
28	.3111	65610	241539	.03033
30	.3333	39216	175929	.02209
32	.3556	29880	136713	.01717
34	.3778	18660	106833	.01342
36	.4000	24600	88173	.01107
38	.4222	22800	63573	.00798
40	.4444	5868	40773	.00512
42	.4667	7710	34905	.00438
44	.4889	9090	27195	.00342
46	.5111	5400	18105	.00227
48	.5333	4590	12705	.00160

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L	T	f	Σf	P
50	.5556	930	8115	.00102
52	.5778	1800	7185	$.0^{3}902$
54	.6000	2840	5385	$.0^{3}676$
56	.6222	540	2545	$.0^{3}320$
58	.6444	465	2005	$.0^{3}252$
60	.6667	426	1540	$.0^{3}193$
62	.6889	360	1114	$.0^{3}140$
64	.7111	540	754	$.0^{4}947$
70	.7778	120	214	$.0^{4}269$
72	.8000	30	94	$.0^{4}118$
74	.8222	45	64	$.0^{5}804$
80	.8889	18	19	$.0^{5}239$
90	1.0000	1	1	$.0^{6}126$

m = 5, n = 3

L	T	f	Σf	P
-10	-.3333	1899	14400	1.00000
-6	-.2000	2826	12501	.86812
-2	-.0667	3051	9675	.67188
2	.0667	2667	6624	.46000
6	.2000	1958	3957	.27479
10	.3333	1140	1999	.13882
14	.4667	564	859	.05965
18	.6000	219	295	.02049
22	.7333	63	76	.00528
26	.8667	12	13	$.0^3903$
30	1.0000	1	1	$.0^4694$

m = 5, n = 4

L	T	f	Σf	P
-20	-.3333	771	1728000	1.00000
-18	-.3000	6120	1727229	.99955
-16	-.2667	23940	1721109	.99601
-14	-.2333	60228	1697169	.98216
-12	-.2000	107448	1636941	.94730
-10	-.1667	143076	1529493	.88512
-8	-.1333	150300	1386417	.80232
-6	-.1000	139824	1236117	.71535
-4	-.0667	135339	1096293	.63443
-2	-.0333	137592	960954	.55611
0	0.0000	128140	823362	.47648
2	.0333	108420	695222	.40233
4	.0667	98157	586802	.33958
6	.1000	94636	488645	.28278
8	.1333	79956	394009	.22801
10	.1667	61680	314053	.18174
12	.2000	55332	252373	.14605
14	.2333	49764	197041	.11403
16	.2667	35544	147277	.08523
18	.3000	26544	111733	.06466
20	.3333	25170	85189	.04930
22	.3667	18456	60019	.03473
24	.4000	10388	41563	.02405
26	.4333	9492	31175	.01804
28	.4667	8328	21683	.01255
30	.5000	3964	13355	.00773
32	.5333	2796	9391	.00543
34	.5667	3096	6595	.00382
36	.6000	1397	3499	.00202
38	.6333	504	2102	.00122
40	.6667	828	1598	$.0^3925$
42	.7000	444	770	$.0^3446$
44	.7333	45	326	$.0^3189$
46	.7667	144	281	$.0^3163$

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L	T	f	Σf	P
48	.8000	108	137	$.0^{4}793$
52	.8667	12	29	$.0^{4}168$
54	.9000	16	17	$.0^{5}984$
60	1.0000	1	1	$.0^{6}579$

m = 6, n = 3

L	T	f	Σf	P
-15	-.3333	31711	518400	1.00000
-11	-.2444	59931	486689	.93883
-7	-.1556	79014	426758	.82322
-3	-.0667	85365	347744	.67080
1	.0222	80811	262379	.50613
5	.1111	66957	181568	.35025
9	.2000	48892	114611	.22109
13	.2889	31761	65719	.12677
17	.3778	18393	33958	.06551
21	.4667	9364	15565	.03003
25	.5556	4101	6201	.01196
29	.6444	1527	2100	.00405
33	.7333	455	573	.00111
37	.8222	102	118	$.0^3228$
41	.9111	15	16	$.0^4309$
45	1.0000	1	1	$.0^5193$