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A VARIATION-OF-CONSTANTS FORMULA FOR NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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A variation-of-constants formula for nonlinear Volterra integral equations of convolution type*)

by

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ABSTRACT

With a Volterra integral equation of convolution type one can associate a semigroup of operators acting on a space of forcing functions. Within this context we derive a variation-of-constants formula for a certain class of nonlinear equations. We indicate how to extend the results to other classes by considering a special equation from mathematical epidemiology.

KEY WORDS & PHRASES: *nonlinear Volterra integral equations of convolution type (renewal equations), variation-of-constants formula, semigroups of operators*

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1. INTRODUCTION

There are (at least) two different ways to associate with Volterra integral equations of convolution type a semigroup of operators:

(i) Write the equation in its translation invariant form and prescribe an initial function on an interval of the right length. The semigroup acts on the space of initial functions and it is defined by translation along the solution.

(ii) Consider a space of forcing functions as the state space and define the semigroup by the formula which shows how the equation transforms under translation.

In the linear case, with an appropriate choice of the spaces, one construction is modulo transposition of the matrix-valued kernel the adjoint of the other [2]. In the process of building a qualitative theory this observation, which applies to other delay equations as well [1,3], can be succesfully exploited in the proof of Fredholm alternatives and in the construction of projection operators.

In this note we shall derive an important tool for a geometric theory within the framework of the *second* construction. It will appear that this somewhat unusual approach has certain advantages. For instance, if

$$\begin{cases} x(t) = \int_0^b B(\tau)g(x(t-\tau))d\tau, & t > 0, \\ x(t) = \phi(t), & -b \le t \le 0, \end{cases}$$

then x is *discontinuous* in t = 0 unless $\phi \in M$ where by definition

$$M = \{\phi \mid \phi(0) = \int_0^b B(\tau)g(\phi(-\tau))d\tau\}.$$

Of course one can restrict one's attention to the manifold M, but, particularly in perturbation problems where B and g, and hence M as well, may depend on parameters, this leads to technical (though not insuperable) difficulties [4, section 12.3;5]. Such difficulties are less prominent in the theory we are going to sketch.

At first we shall deal with equations where the nonlinearity occurs in

the integrand. But in section 5 we shall, by means of an example, indicate how the theory can be extended to equations which contain a nonlinear function of integrals.

2. DEFINITION OF THE SEMIGROUP.

In the following B denotes a given n×n-matrix valued function defined and integrable on $\mathbb{R}_+ = [0,\infty)$. We assume that the support of B is contained in the interval $[0,b]$, where b is some positive number. Let $g:\mathbb{R}^n \to \mathbb{R}^n$ be a given uniformly Lipschitz continuous function. We are interested in the equation

(2.1) $$x = B*g(x) + f$$

where, as usual,

$$(B*g(x))(t) = \int_0^t B(\tau)g(x(t-\tau))d\tau.$$

For reasons which are explained in detail in [2] we put rather severe restrictions on the forcing function f. More precisely, we take $f \in X$ where

$$X = \{f \in C(\mathbb{R}_+) \mid f(t) = 0 \text{ for } t \geq b\}.$$

We provide X with the supremum norm topology.

Let $f \in X$ be arbitrary. Equation (2.1) has a unique continuous solution x defined on $\mathbb{R}_+$. We define, for $s \geq 0$, $S(s)f$ by the relation

(2.2) $$x_s = B*g(x_s) + S(s)f,$$

where $x_s(t) = x(s+t)$. Using the identity

$$(B*g(x_s))(t) = (B*g(x))(t+s) - (B_t*g(x))(s)$$

and (2.1) we obtain

(2.3) $(S(s)f)(t) = f(t+s) + (B_t*g(x))(s).$

From the fact that translation is continuous in the L_1-topology we infer that $(B_t*g(x))(s)$ is continuous as a function of t. Moreover, $(B_t*g(x))(s) = 0$ for $t \geq b$. Hence S(s) is a mapping of X into itself. Since x(t) depends continuously on f, uniformly on compact t-intervals, S(s) is continuous.

THEOREM 2.1. *The mapping* $s \mapsto S(s)$ *defines a strongly continuous semigroup of continuous (nonlinear) operators on* X.

PROOF. From (2.2) we deduce that

$$(x_\sigma)_s = B*g((x_\sigma)_s) + S(s)S(\sigma)f,$$

and

$$x_{s+\sigma} = B*g(x_{s+\sigma}) + S(s+\sigma)f.$$

Since $(x_\sigma)_s = x_{s+\sigma}$ this implies that

$$S(s)S(\sigma) = S(s+\sigma).$$

(Note that we use implicitly the uniqueness of the solution of (2.1).) Clearly $S(0) = I$. Finally,

$$(S(s)f)(t) - (S(\sigma)f)(t) = x_s(t) - x_\sigma(t) + (B*(g(x_s) - g(x_\sigma)))(t) \to 0$$

as $s-\sigma \to 0$ uniformly for $t \in [0,b]$. □

3. THE LINEAR CASE

In the special case that $g(x) = x$ the semigroup constructed above consists of linear operators and will be called T(s). Let R denote the

resolvent of B, i.e. the unique (matrix-valued) solution of the equation (see [6])

$$R = B*R - B, \tag{3.1}$$

(for later use we note that $B*R = R*B$). The solution of

$$x = B*x + f \tag{3.2}$$

is given explicitly as

$$x = f - R*f. \tag{3.3}$$

Substitution of this expression into (2.3) yields an explicit representation of $T(s)$:

$$(T(s)f)(t) = f(t+s) + (B_t - B_t*R)*f(s). \tag{3.4}$$

The formula (3.4) extends the action of $T(s)$ to integrable functions and hence also to the columns of B. The next result will turn out to be useful.

<u>LEMMA 3.1</u>. $(T(s)B)(t) = B_t(s) - B_t*R(s)$.

<u>PROOF</u>. By (3.4) and (3.1) we can write

$$\begin{aligned}(T(s)B)(t) &= B_t(s) + (B_t - B_t*R)*B(s)\\ &= B_t(s) + B_t*B(s) - B_t*(B+R)(s)\\ &= B_t(s) - B_t*R(s).\end{aligned}$$ □

One can show that the infinitesimal generator A of $T(s)$ is given by

$$(Af)(t) = f'(t) + B(t)f(0)$$

with

$$\mathcal{D}(A) = \{f \in X \mid f \text{ absolutely continuous and } f'(.) + B(.)f(0) \text{ continuous}\}.$$

Moreover,

$$\sigma(A) = P\sigma(A) = \{\lambda \mid \det[I - \int_0^b e^{-\lambda\tau}B(\tau)d\tau] = 0\},$$

and one can decompose the space X according to the spectrum of A. We refer to [2] for a detailed account of these matters.

4. THE VARIATION-OF-CONSTANTS FORMULA

Suppose now that $g(x) = x + r(x)$. Let for a given $f \in X$ the functions x and y be the solutions of, respectively,

$$x = B*g(x) + f = B*x + B*r(x) + f, \tag{4.1}$$

$$y = B*y + f. \tag{4.2}$$

LEMMA 4.1 (Miller [6])

$$x - y = -R*r(x).$$

PROOF. Subtracting the equations we obtain

$$x - y = B*(x-y) + B*r(x).$$

Hence, by (3.1),

$$R*(x-y) = R*(x-y) + B*(x-y) + R*r(x) + B*r(x)$$

and so

$$x - y = -R*r(x) - B*r(x) + B*r(x) = -R*r(x). \qquad \square$$

From (2.3) and the corresponding formula for $T(s)$ we deduce, using Lemmas 3.1 and 4.1, that

$$\begin{aligned}(S(s)f)(t) &= (T(s)f)(t) + (B_t*(x+r(x)-y))(s)\\ &= (T(s)f)(t) + ((B_t-B_t*R)*r(x))(s)\\ &= (T(s)f)(t) + \int_0^s (T(s-\tau)B)(t)r(x(\tau))d\tau.\end{aligned}$$

If we define $F:\mathbb{R}_+ \to X$ by $F(s) = S(s)f$ and $\alpha:X \to \mathbb{R}$ by $\alpha(f) = f(0)$ we can rewrite this identity as

$$F(s) = T(s)F(0) + \int_0^s (T(s-\tau)B)r(\alpha(F(\tau)))d\tau \tag{4.3}$$

(indeed, note that, by (2.2), $x(s) = \alpha(S(s)f))$. Our main result formulates the "equivalence" between (4.1) and (4.3).

THEOREM 4.2.

(i) *Let* x *be the solution of* (4.1). *Then* $F:\mathbb{R}_+ \to X$ *defined by* $F(s) = x_s - B*g(x_s)$ *satisfies* (4.3).

(ii) *Conversely, let* F *satisfy* (4.3). *Then* x *defined by* $x(s) = \alpha(F(s))$ *satisfies* (4.1) *with* $f = F(0)$.

PROOF. (i) has been proved above so we concentrate on (ii). Putting $F(0) = f$, $x(s) = \alpha(F(s))$ and applying α to (4.3) we obtain, using Lemma 3.1, (3.4) and (3.1),

$$\begin{aligned}x &= f + (B-B*R)*(f-r(x))\\ &= f - R*f - R*r(x).\end{aligned}$$

Hence

$$\begin{aligned}B*x &= B*f - B*f - R*f - B*r(x) - R*r(x)\\ &= x - f - B*r(x). \qquad \square\end{aligned}$$

REMARKS.

(i) For obvious reasons we call (4.3) the variation-of-constants formula.

(ii) If $r(x) = o(x)$, $x \to 0$, then $T(s)$ is the Fréchet derivative of $S(s)$ in $f = 0$.

(iii) Formal differentiation of (4.3) yields the autonomous ordinary differential equation

$$(4.4)\qquad \frac{dF}{ds} = AF + Br(\alpha F) = F' + Bg(\alpha F)$$

in the Banach space X. So we have demonstrated the correspondence between solutions of (4.1) and mild solutions of (4.4).

5. A SPECIAL EQUATION

The equation

$$(5.1)\qquad x(t) = \gamma(1 - \int_0^1 x(t-\tau)d\tau) \int_0^1 a(\tau)x(t-\tau)d\tau,$$

arises from a model of the spread of a contagious disease, which supplies only temporary immunity, in a closed population. The positive parameter γ is proportional to the population size. The nonnegative kernel $a(\tau)$ describes the infectivity as a function of the time τ elapsed since exposure. This infectivity vanishes for $\tau > 1$. Moreover, an infected individual becomes susceptible again after exactly one unit of time. Finally, $x(t)$ is the frequency of those infected at time t.

If we define

$$(5.2)\qquad b^1(\tau) = \begin{cases} 1 & \text{if } 0 \le \tau \le 1, \\ 0 & \text{otherwise}, \end{cases} \qquad b^2(\tau) = \gamma a(\tau),$$

and if we prescribe x on the interval $-1 \le t \le 0$ we can rewrite (5.1) as

(5.3) $$x = (1 - b^1*x - f^1)(b^2*x + f^2),$$

where f^1 and f^2 incorporate the influence of the past (the prescribed initial function). We observe that the support of f^1 and f^2 is contained in [0,1]. Motivated by this fact we choose

$$X = \{(f^1,f^2) \mid f^i \in C(\mathbb{R}_+) \text{ and } f^i(t) = 0 \text{ for } t \geq 1, \quad i = 1,2\},$$

provided with the topology induced by the norm

$$\|(f^1,f^2)\| = \sup_{0\leq t\leq 1} (|f^1(t)| + |f^2(t)|),$$

as our state space.

Additional properties of f^1 and f^2 will guarantee that (5.3) has a globally defined solution. Here we shall not comment on those properties, but rather we simply assume that they are satisfied.

Let $f = (f^1,f^2)$. The semigroup $S(s)$ is now defined by the formula

(5.4) $$x_s = (1 - b^1*x_s - (S(s)f)^1)(b^2*x_s + (S(s)f)^2),$$

or, in other words,

(5.5) $$(S(s)f)^i(t) = f^i(t+s) + (b_t^i*x)(s), \qquad i = 1,2.$$

Introducing $B = (b^1,b^2)$ we can rewrite (5.5) as

(5.6) $$(S(s)f)(t) = f(t+s) + (B_t*x)(s).$$

The equation (5.1) has two constant solutions. Each of these yields a fixed point of $S(s)$ (for arbitrary s). Here we shall derive the variation-of-constants formula corresponding to the linearization about $f = 0$, but we remark that a similar formula exists for the other case.

The linearized equation is

(5.7) $$y = b^2*y + f^2$$

and the linearized semigroup is

(5.8) $$(T(s)f)(t) = f(t+s) + (B_t*y)(s),$$

(note that, essentially, there is no dependence on f^1 in the linearized problem). Consequently

(5.9) $$S(s)f = T(s)f + \int_0^s B(.+s-\tau)(x(\tau)-y(\tau))d\tau.$$

The following observations are intended to rewrite this identity in a more useful form. We omit the proofs since they are very similar to those of the corresponding results in the foregoing sections.

(i) Let R denote the resolvent corresponding to b^2, i.e. the solution of

$$R = b^2*R - b^2.$$

Define h by

$$x = b^2*x + f^2 + h.$$

Then $x - y = h - R*h$ (see Lemma 4.1).

(ii) The definition of h implies

$$\begin{aligned} h &= -(b^1*x + f^1)(b^2*x + f^2) \\ &= -(S(.)f)^1(0).(S(.)f)^2(0) \\ &= r(\alpha(S(.)f)), \end{aligned}$$

where $\alpha(f) := f(0)$ and $r: \mathbb{R}^2 \to \mathbb{R}$, $r(x_1,x_2) = -x_1x_2$.

(iii)

$$(T(s)B)(t) = B_t(s) - (B_t*R)(s).$$

Using (i) - (iii) and (5.9) we obtain the variation-of-constants formula

(5.10) $$S(s)f = T(s)f + \int_0^s (T(s-\tau)B)r(\alpha(S(\tau)f))d\tau.$$

6. CONCLUDING REMARKS

In work in progress we use the variation-of-constants formula for the construction of (local) invariant manifolds (the stable and unstable manifolds of a saddle point as well as the center manifold in the case of critical stability). We intend to apply these results to concrete problems (special equations). In a prelude to Hopf bifurcation R. Montijn has recently obtained rather detailed information about a characteristic equation associated with (5.1). It appears that lots of roots may cross the imaginary axis with nonzero speed. Detailed results will be given in future publications.

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