A COINDUCTIVE TREATMENT OF INFINITARY REWRITING

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Abstract. We introduce a coinductive definition of infinitary term rewriting. The setup is surprisingly simple, and has in contrast to the usual definitions of infinitary rewriting, neither need for ordinals nor for metric convergence. While the idea of a coinductive treatment of infinitary rewriting is not new, all previous approaches were limited to reductions of length $\leq \omega$. The approach presented in this paper is the first to capture the full infinitary term rewriting with reductions of arbitrary ordinal length. Apart from an elegant reformulation of known concepts, our approach gives rise — in a very natural way — to a novel notion of infinitary equational reasoning.

1. Introduction

Infinitary rewriting is a generalization of the ordinary finitary rewriting to infinite terms and infinite reductions (including reductions of ordinal lengths larger than ω). We present a coinductive treatment of infinitary rewriting free of ordinals, metric convergence and partial orders which have been essential in earlier definitions of the concept [11, 21, 23, 12, 28, 24, 22, 25, 20, 4, 3, 5, 14]. In a slogan one could say: *Infinitary rewriting has never been easier!*

Let us describe the idea. Let R be a term rewriting system (TRS). We write $\to_\varepsilon$ for root steps with respect to R, that is, we define $\to_\varepsilon = \{ (\ell\sigma, r\sigma) \mid \ell \to r \in R,\ \sigma \text{ a substitution} \}$. The crucial ingredient of our definition of infinitary rewriting $\mathrel{\twoheadrightarrow\!\!\!\!\rightarrow}$ are the *coinductive* rules

$$\frac{s\ (\to_\varepsilon \cup \mathrel{\rightarrow\!\!\!\!\!\!/\!/\!/})^*\ t}{s \mathrel{\twoheadrightarrow\!\!\!\!\rightarrow} t} \qquad\qquad \frac{s_1 \mathrel{\twoheadrightarrow\!\!\!\!\rightarrow} t_1 \quad \ldots \quad s_n \mathrel{\twoheadrightarrow\!\!\!\!\rightarrow} t_n}{f(s_1, s_2, \ldots, s_n) \mathrel{\rightarrow\!\!\!\!\!\!/\!/\!/} f(t_1, t_2, \ldots, t_n)} \qquad (1)$$

Here $\mathrel{\twoheadrightarrow\!\!\!\!\rightarrow}$ and $\mathrel{\rightarrow\!\!\!\!\!\!/\!/\!/}$ stand for finite and infinite reductions where $\mathrel{\rightarrow\!\!\!\!\!\!/\!/\!/}$ contains only steps below the root. Note that these relations are defined mutually. The coinductive nature of the rules means that the derivation trees, the proof trees, need not be well-founded. (As we shall see, for the standard notion of infinitary rewriting, we need to restrict the derivation trees.)

To illustrate the use of the rules, let us immediately consider an example.

Example 1.1. Let R be the TRS consisting solely of the following rewrite rule $\mathsf{a} \to \mathsf{C}(\mathsf{a})$. We write C^ω to denote the infinite term $\mathsf{C}(\mathsf{C}(\mathsf{C}(\ldots)))$, the solution of the equation $\mathsf{C}^\omega = \mathsf{C}(\mathsf{C}^\omega)$. We then have $\mathsf{a} \mathrel{\twoheadrightarrow\!\!\!\!\rightarrow} \mathsf{C}^\omega$, that is, an infinite reduction from a to C^ω in the limit:

$$\mathsf{a} \to \mathsf{C}(\mathsf{a}) \to \mathsf{C}(\mathsf{C}(\mathsf{a})) \to \mathsf{C}(\mathsf{C}(\mathsf{C}(\mathsf{a}))) \to \ldots \to^\omega \mathsf{C}^\omega$$

Using the rules above, we can derive $\mathsf{a} \mathrel{\twoheadrightarrow\!\!\!\!\rightarrow} \mathsf{C}^\omega$ as shown in Figure 1. This is an infinite proof tree as indicated by the loop $\cdots\!\!\!>$ in which the rewrite sequence $\mathsf{a} \to_\varepsilon \mathsf{C}(\mathsf{a}) \mathrel{\rightarrow\!\!\!\!\!\!/\!/\!/} \mathsf{C}^\omega$ is written in the form $\mathsf{a} \to_\varepsilon \mathsf{C}(\mathsf{a}) \quad \mathsf{C}(\mathsf{a}) \mathrel{\rightarrow\!\!\!\!\!\!/\!/\!/} \mathsf{C}^\omega$, that is, two separate steps such that the target of the first equals the source of the second step; this is made precise in Notation 1.2, below.

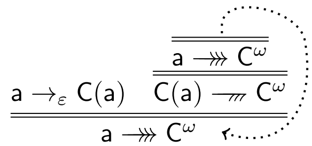

FIGURE 1. A reduction $\mathsf{a} \mathrel{\rightarrow\!\!\!\!\twoheadrightarrow} \mathsf{C}^\omega$ of length ω.

Put in words, the proof tree in Figure 1 can be described as follows. We have an infinitary rewrite sequence $\mathrel{\rightarrow\!\!\!\!\twoheadrightarrow}$ from a to C^ω since we have a root step from a to $\mathsf{C(a)}$, and an infinitary reduction below the root $\mathrel{-\!\!\!\!\!/\!\!/\!\!/}$ from $\mathsf{C(a)}$ to C^ω. The latter reduction $\mathsf{C(a)} \mathrel{-\!\!\!\!\!/\!\!/\!\!/} \mathsf{C}^\omega$ is in turn witnessed by the infinitary rewrite sequence $\mathsf{a} \mathrel{\rightarrow\!\!\!\!\twoheadrightarrow} \mathsf{C}^\omega$ on the direct subterms. □

Notation 1.2. Instead of introducing derivation rules for transitivity, in particular for $(\rightarrow_\varepsilon \cup \mathrel{-\!\!\!\!\!/\!\!/\!\!/})^*$, we will write rewrite sequences $s_0 \rightsquigarrow_0 s_1 \rightsquigarrow_1 \ldots \rightsquigarrow_{n-1} s_n$ where $\rightsquigarrow_i \in \{\rightarrow_\varepsilon, \mathrel{-\!\!\!\!\!/\!\!/\!\!/}\}$ as sequence of single steps $s_0 \rightsquigarrow_0 s_1 \quad s_1 \rightsquigarrow_1 s_2 \quad \ldots \quad s_{n-1} \rightsquigarrow_{n-1} s_n$. That is:

$$\frac{s_0 \rightsquigarrow_0 s_1 \quad s_1 \rightsquigarrow_1 s_2 \quad \ldots \quad s_{n-1} \rightsquigarrow_{n-1} s_n}{s_0 \mathrel{\rightarrow\!\!\!\!\twoheadrightarrow} s_n}$$

This notation is more convenient since it avoids the need for explicitly introducing rules for transitivity, and thereby keeps the proof trees small.

As a second example, let us consider a rewrite sequence of length beyond ω.

Example 1.3. We consider the term rewriting system consisting of the following rules:

$$\mathsf{f}(x, x) \rightarrow \mathsf{D} \qquad\qquad \mathsf{a} \rightarrow \mathsf{C(a)} \qquad\qquad \mathsf{b} \rightarrow \mathsf{C(b)}$$

Then we have the following reduction of length $\omega + 1$:

$$\mathsf{f(a,b)} \rightarrow \mathsf{f(C(a),b)} \rightarrow \mathsf{f(C(a),C(b))} \rightarrow \ldots \rightarrow^\omega \mathsf{f}(\mathsf{C}^\omega, \mathsf{C}^\omega) \rightarrow \mathsf{D}$$

That is, after an infinite rewrite sequence of length ω, we reach the limit term $\mathsf{f}(\mathsf{C}^\omega, \mathsf{C}^\omega)$, and we then continue with a rewrite step from $\mathsf{f}(\mathsf{C}^\omega, \mathsf{C}^\omega)$ to D. Figure 2 shows how this rewrite sequence $\mathsf{f(a,b)} \mathrel{\rightarrow\!\!\!\!\twoheadrightarrow} \mathsf{D}$ can be derived in our setup. The precise meaning of the symbol $\overset{<}{\mathrel{-\!\!\!\!\!/\!\!/\!\!/}}$ in the figure will be explained later; for the moment, we may think of $\overset{<}{\mathrel{-\!\!\!\!\!/\!\!/\!\!/}}$ to be $\mathrel{-\!\!\!\!\!/\!\!/\!\!/}$.

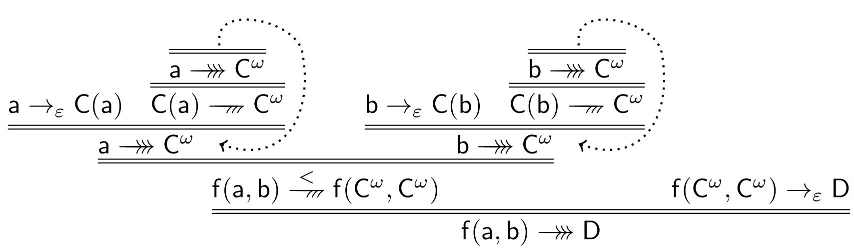

FIGURE 2. A reduction $\mathsf{f(a,b)} \mathrel{\rightarrow\!\!\!\!\twoheadrightarrow} \mathsf{D}$ of length $\omega + 1$.

We note that the rewrite sequence $\mathsf{f(a,b)} \mathrel{\rightarrow\!\!\!\!\twoheadrightarrow} \mathsf{D}$ cannot be 'compressed' to length ω. That is, there exists no reduction $\mathsf{f(a,b)} \rightarrow^{\leq\omega} \mathsf{D}$. □

For the definition of rewrite sequences of ordinal length, there is a design choice concerning the connectedness at limit ordinals: (a) metric convergence, or (b) strong convergence.

The purpose of the connectedness condition is to exclude jumps at limit ordinals, as illustrated in the following non-connected rewrite sequence (where $R = \{\, \mathsf{a} \to \mathsf{a},\ \mathsf{b} \to \mathsf{b} \,\}$):

$$\underbrace{\mathsf{a} \to \mathsf{a} \to \mathsf{a} \to \dots}_{\omega\text{-many steps}}\ \mathsf{b} \to \mathsf{b}$$

The rewrite sequence stays ω steps at a and in the limit step 'jumps' to b.

The connectedness condition with respect to *metric convergence* requires that for every limit ordinal γ, the terms t_α converge with limit t_γ as α approaches γ from below. The *strong convergence* requires additionally that the depth of the rewrite steps $t_\alpha \to t_{\alpha+1}$ tends to infinity as α approaches γ from below. The standard notion of infinitary rewriting [31, 14] is based on strong convergence as it gives rise to a more elegant rewriting theory; for example, allowing to trace symbols and redexes over limit ordinals. This is the notion that we are concerned with in this paper.

The rules (1) give rise to infinitary rewrite sequences in a very natural way, without the need for ordinals, metric convergence, or depth requirements. The depth requirement in the definition of strong convergence arises naturally in the rules (1) by employing coinduction over the term structure. Indeed, it is not difficult to see that the coinductive rules (1) capture all infinitary strongly convergent reductions $s \twoheadrightarrow t$. This is a consequence of a result due to [23] which states that every strongly convergent rewrite sequence contains only a finite number of steps at any depth $d \in \mathbb{N}$. Thus, in particular, only a finite number of root steps $\to_\varepsilon$, and before, in-between and after these root steps, there are strongly convergent rewrite sequences on the arguments. As a consequence, every strongly convergent rewrite sequence is of the shape $(\mathrel{-\!\!\!\!/\!\!\!/\!\!\!/\!\!\!\!\to} \circ \to_\varepsilon)^* \circ \mathrel{-\!\!\!\!/\!\!\!/\!\!\!/\!\!\!\!\to}$. Since strongly convergent rewrite sequences are closed under transitivity, we allow the slightly more general $(\to_\varepsilon \cup \mathrel{-\!\!\!\!/\!\!\!/\!\!\!/\!\!\!\!\to})^*$ in (1).

While this argument shows that every strongly convergent reduction $s \twoheadrightarrow t$ can be derived using the rules (1), it does not guarantee that we can derive precisely the strongly convergent reductions. Actually, the rules do allow to derive more, as the following example shows.

Example 1.4. Let R consist of the rewrite rule $\mathsf{C}(\mathsf{a}) \to \mathsf{a}$. Using the rules (1), we can derive $\mathsf{C}^\omega \twoheadrightarrow \mathsf{a}$ as shown in Figure 3.

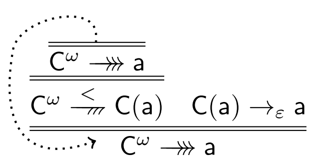

FIGURE 3. A derivation of $\mathsf{C}^\omega \twoheadrightarrow \mathsf{a}$.

We emphasize that with respect to the standard notion of infinitary rewriting $\twoheadrightarrow$ in the literature we do not have $\mathsf{C}^\omega \twoheadrightarrow \mathsf{a}$ since C^ω is a normal form (does not contain an occurrence of the left-hand side $\mathsf{C}(\mathsf{a})$ of the rule). Note that the rule $\mathsf{C}(\mathsf{x}) \to \mathsf{x}$ also gives rise to $\mathsf{C}^\omega \twoheadrightarrow \mathsf{a}$ by the same derivation as in Figure 3. □

This example illustrates that, without further restrictions, the rules (1) give rise to a notion of infinitary rewriting that allows rewrite sequences to extend infinitely forwards, but also infinitely backwards. Here backwards does *not* refer to reversing the arrow $\leftarrow_\varepsilon$. While this is a non-standard notion of infinitary rewriting, it is nevertheless interesting, especially for a theory of infinitary equational reasoning, a field that has remained largely underdeveloped.

From the rules (1), a theory of *infinitary equational reasoning* arises naturally by replacing $\to_\varepsilon$ with $\leftarrow_\varepsilon \cup \to_\varepsilon$ in the first rule. This notion of infinitary equational reasoning has the property of strong convergence built in, and thereby allows to trace redex occurrences forwards as well as backwards over rewriting sequences of arbitrary length. As a consequence, this concept can profit from the well-developed theory of term rewriting and infinitary term rewriting.

The focus of this paper is the standard notion of infinitary rewriting. *How to obtain the strongly convergent rewrite sequences $s \mathrel{\to\!\!\!\!\twoheadrightarrow} t$?* For this purpose it suffices to impose a syntactic restriction on the shape of the proof trees obtained from the rules (1). The idea is that all rewrite sequences $\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}$ in $(\to_\varepsilon \cup \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/})^*$, that are before a root step $\to_\varepsilon$, should be shorter than the rewrite sequence that we are defining. To this end, we change $(\to_\varepsilon \cup \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/})^*$ to $(\to_\varepsilon \cup \overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}})^* \circ \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}$ where $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$ is a marked equivalent of $\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}$, and we employ the marker to exclude infinite nesting of $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$. Then we have an infinitary strongly convergent rewrite sequence from s to t if and only if $s \mathrel{\to\!\!\!\!\twoheadrightarrow} t$ can be derived by the rules

$$\frac{s\ (\to_\varepsilon \cup \overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}})^* \circ \mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}\ t}{s \mathrel{\to\!\!\!\!\twoheadrightarrow} t} \qquad \frac{s_1 \mathrel{\to\!\!\!\!\twoheadrightarrow} t_1 \quad \dots \quad s_n \mathrel{\to\!\!\!\!\twoheadrightarrow} t_n}{f(s_1, s_2, \dots, s_n) \overset{(<)}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}} f(t_1, t_2, \dots, t_n)} \qquad \frac{}{s \overset{(<)}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}} s} \tag{2}$$

in a (not necessarily well-founded) proof tree without infinite nesting of $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$. In other words, we only allow those proof trees in which all paths (ascending through the proof tree) contain only finitely many occurrences of $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$.

We note that the second and third rule are abbreviations for two rules each: the symbol $\overset{(<)}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$ stands for $\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}$ and for $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$. Intuitively, $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$ can be thought of as infinitary rewrite sequence below the root that is 'smaller' than the sequence we are defining. Here 'smaller' refers to the nesting depth of $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$, but can equivalently be thought of the length of the reduction (in some well-founded order).

Example 1.5. Let us revisit Examples 1.1, 1.3 and 1.4. Example 1.1 contains no occurrences of $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$. The proof tree in Example 1.3 has a single occurrence of $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$, but this occurrence is not contained in the indicated loops, and thus not infinitely nested. Only Example 1.4 contains a symbol $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$ on a loop, and hence a path with infinitely many occurrences of $\overset{<}{\mathrel{\rightarrow\!\!\!\!\!\!\!/\!\!/\!\!/}}$, and thus the proof tree is excluded by the syntactic restriction. □

Related Work. The basic idea of a coinductive treatment of infinitary rewriting is not new. Already in 1996, Catarina Coquand and Thierry Coquand [9] have given a coinductive definition of standard reductions in infinitary combinatory logic. Felix Joachimski [18, 19] introduces a coinductive definition of infinite developments, a very restrictive form of infinite reductions. In [15], Jörg Endrullis and Andrew Polonsky present a coinductive definition of infinite rewrite sequences in infinitary λ-calculus. All previous coinductive definitions have in common that they do not capture rewrite sequences of length $> \omega$. The coinductive treatment presented here captures all strongly convergent rewrite sequences of arbitrary ordinal length.

From the topological perspective, various notions of infinitary rewriting and infinitary equational reasoning have been studied in [20]. However, in contrast to the topological notions, our setup captures strong convergence which yields an elegant rewriting theory, and is the basic standard notion of infinitary rewriting in the literature. We note that none of the rewrite notions ($\mathrel{\to\!\!\!\!\twoheadrightarrow}$, $=^\infty$ and ${}^\infty\!\!\to^\infty$) considered in this paper are continuous (forward closed) in general. Here $\to$ is continuous means that $\lim_{i\to\infty} t_i = t$ and $\forall i. s \to t_i$

implies $s \to t$. However, continuity might hold for certain classes of term rewrite systems; see further [14] for continuity for strongly convergent infinitary rewriting $\twoheadrightarrow$.

Outline. In Section 2 we introduce infinitary rewriting in the usual way with ordinal-length rewrite sequences, and convergence at every limit ordinal. We then continue in Section 3 with an introduction to coinduction. In Section 4 we present a novel notion of infinitary equational reasoning, and bi-infinite rewriting, both based on a coinductive treatment. We give two definitions of infinitary rewriting based on mixing induction and coinduction in Section 5. We then discuss these definitions, with particular focus on their formalization in theorem provers, in Section 6. In Section 7, we prove the equivalence of our coinductive definitions of infinitary rewriting with the standard definition.

2. Preliminaries

We give a brief introduction to infinitary rewriting. For further reading on infinitary rewriting we refer to [28, 31, 7, 14], for an introduction to finitary rewriting to [27, 31, 2, 6].

A *signature* Σ is a set of symbols f each having a fixed arity $\#(f) \in \mathbb{N}$. Let $\mathcal{X}$ be an infinite set of variables such that $\mathcal{X} \cap \Sigma = \emptyset$. The set of (finite and) *infinite terms* $\mathit{Ter}^\infty(\Sigma, \mathcal{X})$ over Σ and $\mathcal{X}$ is *coinductively* (see further [8]) defined by the grammar:

$$T ::=^{\mathrm{co}} x \mid f(\underbrace{T, \ldots, T}_{\#(f) \text{ times}}) \qquad\qquad (x \in \mathcal{X},\ f \in \Sigma) \tag{3}$$

Intuitively, coinductively means that the grammar rules may be applied an infinite number of times. The equality on the terms is bisimilarity. For a brief introduction to coinduction, we refer to Section 3.

We define the *identity relation on terms* by $\mathsf{id} = \{\langle s, s\rangle \mid s \in \mathit{Ter}^\infty(\Sigma, \mathcal{X})\}$.

Remark 2.1. Alternatively, the infinite terms arise from the set of finite terms, $\mathit{Ter}(\Sigma, \mathcal{X})$, by metric completion, using the well-known distance function d such that for $t, s \in \mathit{Ter}(\Sigma, \mathcal{X})$, $d(t, s) = 2^{-n}$ if the n-th level of the terms t, s (viewed as labeled trees) is the first level where a difference appears, in case t and s are not identical; furthermore, $d(t, t) = 0$. It is standard that this construction yields $\langle \mathit{Ter}(\Sigma, \mathcal{X}), d\rangle$ as a metric space. Now infinite terms are obtained by taking the completion of this metric space, and they are represented by infinite trees. We will refer to the complete metric space arising in this way as $\langle \mathit{Ter}^\infty(\Sigma, \mathcal{X}), d\rangle$, where $\mathit{Ter}^\infty(\Sigma, \mathcal{X})$ is the set of finite and infinite terms over Σ.

Let $t \in \mathit{Ter}^\infty(\Sigma, \mathcal{X})$ be a finite or infinite term. The set of *positions* $\mathcal{P}\mathit{os}(t) \subseteq \mathbb{N}^*$ *of* t is defined coinductively by:

$$\mathcal{P}\mathit{os}(x) = \{\epsilon\} \qquad\qquad \mathcal{P}\mathit{os}(f(t_1, \ldots, t_n)) = \{\epsilon\} \cup \{ip \mid 1 \le i \le n,\ p \in \mathcal{P}\mathit{os}(t_i)\}$$

For $p \in \mathcal{P}\mathit{os}(t)$, the *subterm* $t|_p$ *of* t *at position* p is defined by:

$$t|_\epsilon = t \qquad\qquad f(t_1, \ldots, t_n)|_{ip} = t_i|_p$$

The set of *variables* $\mathcal{V}\mathit{ar}(t) \subseteq \mathcal{X}$ *of* t is defined by $\mathcal{V}\mathit{ar}(t) = \{x \in \mathcal{X} \mid \exists\, p \in \mathcal{P}\mathit{os}(t).\, t|_p = x\}$.

A *substitution* σ is a map $\sigma : \mathcal{X} \to \mathit{Ter}^\infty(\Sigma, \mathcal{X})$. We extend the domain of substitutions σ to $\mathit{Ter}^\infty(\Sigma, \mathcal{X})$ by coinduction, as follows: $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$. For terms s and substitutions σ, we write $s\sigma$ for $\sigma(s)$. We write $x \mapsto s$ for the substitution defined by $\sigma(x) = s$ and $\sigma(y) = y$ for all $y \neq x$. Let $\Box$ be a fresh variable. A *context* C is a term $\mathit{Ter}^\infty(\Sigma, \mathcal{X} \cup \{\Box\})$ containing precisely one occurrence of the variable $\Box$. For contexts C and terms s we write $C[s]$ for $C(x \mapsto s)$.

A *rewrite rule* $\ell \to r$ over Σ and $\mathcal{X}$ is a pair $(\ell, r) \in Ter^\infty(\Sigma, \mathcal{X}) \times Ter^\infty(\Sigma, \mathcal{X})$ of terms such that the left-hand side ℓ is not a variable ($\ell \notin \mathcal{X}$), and all variables in the right-hand side r occur in ℓ ($\mathcal{V}ar(r) \subseteq \mathcal{V}ar(\ell)$). Note that we do neither require the left-hand side nor the right-hand side of a rule to be finite.

A *term rewrite system (TRS)* $\mathcal{R}$ over Σ and $\mathcal{X}$ is a set of rewrite rules over Σ and $\mathcal{X}$. A TRS induces a rewrite relation on the set of terms as follows. For $p \in \mathbb{N}^*$ we define $\to_{\mathcal{R},p} \subseteq Ter^\infty(\Sigma, \mathcal{X}) \times Ter^\infty(\Sigma, \mathcal{X})$, a *rewrite step at position* p, by

$$C[\ell\sigma] \to_{\mathcal{R},p} C[r\sigma] \quad \text{if} \quad C \text{ a context with } C|_p = \square,\ \ \ell \to r \in \mathcal{R},\ \ \sigma : \mathcal{X} \to Ter^\infty(\Sigma, \mathcal{X})$$

We write $s \to_{\mathcal{R}} t$ if $s \to_{\mathcal{R},p} t$ for some $p \in \mathbb{N}^*$. A *normal form* is a term without a redex occurrence, that is, a term that is not of the form $C[\ell\sigma]$ for some context C, rule $\ell \to r \in \mathcal{R}$ and substitution σ.

A natural consequence of this construction is the emergence of the notion of *metric convergence*: we say that $t_0 \to t_1 \to t_2 \to \ldots$ is an infinite reduction sequence with limit t, if t is the limit of the sequence $t_0, t_1, t_2, \ldots$ in the usual sense of metric convergence. Metric convergence is sometimes also called *weak convergence*. In fact, we will use throughout a stronger notion that has better properties. This is *strong convergence*, which in addition to the stipulation for metric (or weak) convergence, requires that the depth of the redexes contracted in the successive steps tends to infinity when approaching a limit ordinal from below. So this rules out the possibility that the action of redex contraction stays confined at the top, or stagnates at some finite level of depth. See further Figure 4 for an intuitive illustration.

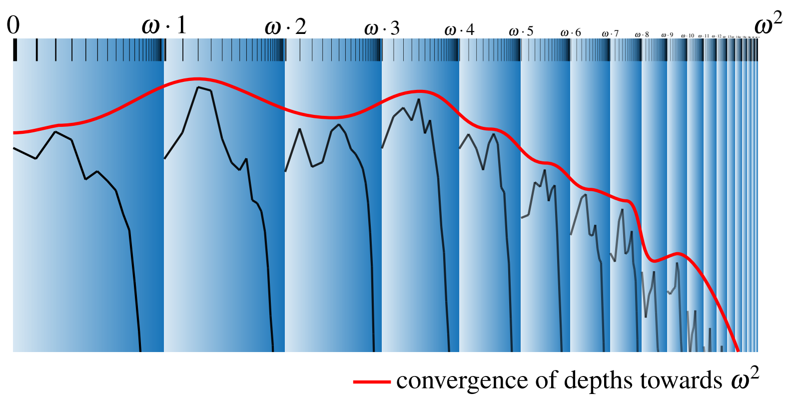

FIGURE 4. Depth of redex contractions tends to infinity at each limit ordinal.

A more precise definition is as follows:

Definition 2.2. A *transfinite rewrite sequence* (of ordinal length α) is a sequence of rewrite steps $(t_\beta \to_{\mathcal{R},p_\beta} t_{\beta+1})_{\beta<\alpha}$ such that for every limit ordinal $\lambda < \alpha$ we have that if β approaches λ from below, then

(i) the distance $d(t_\beta, t_\lambda)$ tends to 0 and, moreover,
(ii) the depth of the rewrite action, i.e., the length of the position p_β, tends to infinity.

The sequence is called *strongly convergent* if α is a successor ordinal, or there exists a term t_α such that the conditions i and ii are fulfilled for every limit ordinal $\lambda \leq \alpha$. In this case we write $t_0 \twoheadrightarrow\!\!\!\!\!\rightarrow_{ord,\mathcal{R}} t_\alpha$, or $t_0 \to^\alpha t_\alpha$ to explicitly indicate the length α of the sequence. The sequence is called *divergent* if it is not strongly convergent.

There are several reasons why strong convergence is beneficial; the foremost being that in this way we can define the notion of *descendant* (also *residual*) over limit ordinals. Also the well-known Parallel Moves Lemma and the Compression Lemma fail for weak convergence, see [30] and [11] respectively. It is further easy to establish that strongly convergent reductions can have any countable length; weakly convergent reductions can have any length, as the one-rule TRS with $\mathtt{c} \to \mathtt{c}$ demonstrates.

3. Introduction to Coinduction

We briefly introduce the relevant concepts from (co)algebra and (co)induction that will be used later throughout this paper. For a more thorough introduction, we refer to [17]. There will be two main points where coinduction will play a role, in the definition of terms and in the definition of the term rewriting.

Terms are usually defined with respect to a signature Σ. For instance, consider the type of lists with elements in a given set A.

```
type List a = Empty | Cons a (List a)
```

The above grammar corresponds to the signature (or type constructor) $\Sigma(X) = 1 + A \times X$ where the 1 is used as a placeholder for the empty list `Empty` and the second component represents the `Cons` constructor. Such a grammar can be interpreted in two ways: The *inductive* interpretation yields as terms the set of finite lists, and corresponds to the *least fixed point* of Σ. The *coinductive* interpretation yields as terms the set of all finite or infinite lists, and corresponds to the *greatest fixed point* of Σ. More generally, the inductive interpretation of a signature yields finite terms (with well-founded syntax trees), and dually, the coinductive interpretation yields possibly infinite terms. For readers familiar with the categorical definitions of algebras and coalgebras, these two interpretations amount to defining finite terms as the *initial* Σ*-algebra*, and possibly infinite terms as the *final* Σ*-coalgebra*.

Equality on finite terms is the expected syntactic/inductive definition. Equality of possibly infinite terms is observational equivalence (or bisimilarity). For instance, in the above example, two infinite lists σ and τ are equal if and only if they are related by a List-bisimulation. A relation $R \subseteq \mathtt{List\ a} \times \mathtt{List\ a}$ is a List-bisimulation if and only if for all pairs $(\mathtt{Cons\ a}\ \sigma, \mathtt{Cons\ b}\ \tau) \in R$, it holds that $a = b$ and $(\sigma, \tau) \in R$.

Formally, term rewriting is a relation on a set T of terms, and hence an element of the complete lattice $L := \mathcal{P}(T \times T)$, i.e., the powerset of $T \times T$. Relations on terms can thus be defined using least and greatest fixed points of monotone operators on L. In this setting, an inductively defined relation is a least fixed point μF of a monotone $F \colon L \to L$; and dually, a coinductively defined relation is a greatest fixed point νF of a monotone $F \colon L \to L$. These notions of induction and coinduction are, in fact, also instances of the more abstract categorical definitions. This can be seen by viewing L as a partial order (ordered by set inclusion). In turn, a partial order $(P, \leq)$ can be seen as a category whose objects are the elements of P and there is a unique arrow $X \to Y$ if $X \leq Y$. A functor on $(P, \leq)$ is then nothing but a monotone map F; an F-coalgebra $X \to F(X)$ is a post-fixed point of F; and a final F-coalgebra is a greatest fixed point of F. The existence of the final F-coalgebra is guaranteed by the Knaster-Tarski fixed point theorem. Coinduction, and similarly induction, can now be formulated as proof rules:

$$\frac{X \leq F(X)}{X \leq \nu F}(\nu\text{-rule}) \qquad\qquad \frac{F(X) \leq X}{\mu F \leq X}(\mu\text{-rule}) \tag{4}$$

that express the fact that νF is the greatest post-fixed point of F, and μF is the least pre-fixed point of F.

4. Infinitary Equational Reasoning

From the basic rules (1) arises in a very natural way a novel notion of infinitary equational reasoning $=^\infty$. This notion is the natural counterpart of strongly convergent infinitary rewriting. Like infinitary strongly convergent reductions, the theory of infinitary equational reasoning has the property that every derivation contains only a finite number of reasoning steps at any depth $d \in \mathbb{N}$. We consider an *equational specification (ES)* as a TRS.

Definition 4.1. Let E be an equational specification over Σ. We define *infinitary equational reasoning* $=^\infty \subseteq T \times T$ on terms $T = Ter^\infty(\Sigma, \mathcal{X})$ by the following coinductive rules

$$\frac{s\ (\leftarrow_\varepsilon \cup \rightarrow_\varepsilon \cup \cong^\infty)^*\ t}{s =^\infty t} \qquad \frac{t_1 =^\infty t_1' \quad \dots \quad t_n =^\infty t_n'}{f(t_1, t_2, \dots, t_n) \cong^\infty f(t_1', t_2', \dots, t_n')}$$

where $\cong^\infty \subseteq T \times T$ stands for infinitary equational reasoning below the root. □

Example 4.2. Let E be an equational specification consisting of the equations (rules):

$$\mathsf{a} = \mathsf{f}(\mathsf{a}) \qquad \mathsf{b} = \mathsf{f}(\mathsf{b}) \qquad \mathsf{C}(\mathsf{b}) = \mathsf{C}(\mathsf{C}(\mathsf{a}))$$

Then $\mathsf{a} =^\infty \mathsf{b}$ as derived in Figure 5 (top), and $\mathsf{C}(\mathsf{a}) =^\infty \mathsf{C}^\omega$ as in Figure 5 (bottom).

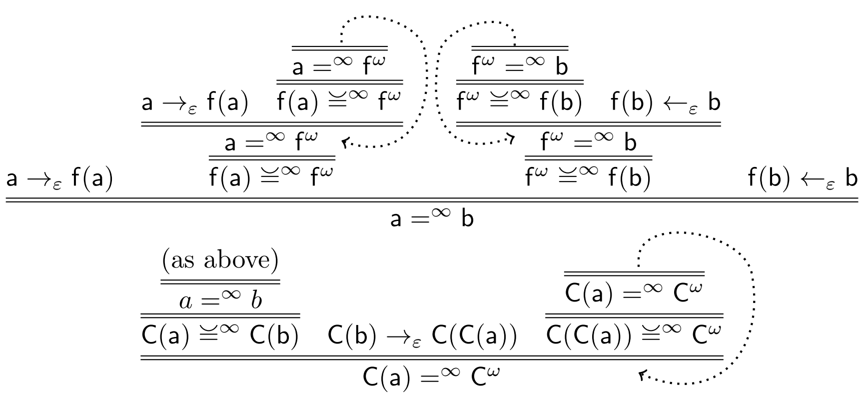

Figure 5. Infinitary equational reasoning.

It is easy to see that $(\twoheadleftarrow\!\!\!\leftarrow \circ \twoheadrightarrow\!\!\!\rightarrow)^* \subseteq =^\infty$, and $\mathsf{C}(\mathsf{a}) =^\infty \mathsf{C}^\omega$ shows that this inclusion is strict.

Definition 4.1 of $=^\infty$ can be equivalently be defined using a greatest fixed point as follows.

Definition 4.3. Let E be an equational specification over Σ, and $T = Ter^\infty(\Sigma, \mathcal{X})$. For $R \in \mathcal{P}(T \times T)$, we define its *lifting* as

$$\overline{R} = \{\, \langle f(s_1, \dots, s_n),\ f(t_1, \dots, t_n) \rangle \mid s_1\ R\ t_1, \dots, s_n\ R\ t_n \,\} \cup \mathsf{id}$$

We define the relation $=^\infty$ as $\nu\, x.\ (\leftarrow_\varepsilon \cup \rightarrow_\varepsilon \cup \overline{x})^*$. □

It is easy to verify that the function $x \mapsto (\leftarrow_\varepsilon \cup \rightarrow_\varepsilon \cup \overline{x})^*$ is monotone, and consequently the greatest fixed point in Definition 4.3 exists.

Another notion that arises naturally in our setup is that of bi-infinite rewriting, allowing rewrite sequences to extend infinitely forwards and backwards. We emphasize that each of the steps $\rightarrow_\varepsilon$ in such sequences is a forward step.

Definition 4.4. Let R be a term rewriting system over Σ, and let $T = \mathit{Ter}^\infty(\Sigma, \mathcal{X})$. We define *bi-infinite rewrite relation* ${}^\infty\!\!\rightarrow^\infty \subseteq T \times T$ by the following coinductive rules

$$\frac{s \; (\rightarrow_\varepsilon \cup {}^\infty\!\!\rightarrowtriangle^\infty)^* \; t}{s \; {}^\infty\!\!\rightarrow^\infty \; t} \qquad\qquad \frac{t_1 \; {}^\infty\!\!\rightarrow^\infty \; t_1' \quad \ldots \quad t_n \; {}^\infty\!\!\rightarrow^\infty \; t_n'}{f(t_1, t_2, \ldots, t_n) \; {}^\infty\!\!\rightarrowtriangle^\infty \; f(t_1', t_2', \ldots, t_n')}$$

where ${}^\infty\!\!\rightarrowtriangle^\infty \subseteq T \times T$ stands for bi-infinite rewriting below the root. □

Examples 1.1, 1.3 and 1.4 are illustrations of this rewrite relation. Note that these examples employ the symbols $\twoheadrightarrow\!\!\!\rightarrow$ and $\rightarrowtriangle$ instead of ${}^\infty\!\!\rightarrow^\infty$ and ${}^\infty\!\!\rightarrowtriangle^\infty$, respectively. In the infinitary conversion in Example 4.2 we need to reverse the rule $\mathsf{b} = \mathsf{f}(\mathsf{b})$ in order to obtain a bi-infinite rewrite sequence $\mathsf{a} \; {}^\infty\!\!\rightarrow^\infty \; \mathsf{b}$.

5. Infinitary Term Rewriting

We present two – ultimately equivalent – definitions of infinitary rewriting $s \twoheadrightarrow\!\!\!\rightarrow t$, based on mixing induction and coinduction. We summarize the definitions:

A. *Derivation Rules.* First, we define $s \twoheadrightarrow\!\!\!\rightarrow t$ via a syntactic restriction on the proof trees that arise from the coinductive rules (2). The restriction excludes all proof trees that contain ascending paths with an infinite number of marked symbols. This can be viewed as a two phase process: first generating all finite and infinite proof trees with respect to (2), and in a post-processing step we filter all well-formed proof terms.

B. *Mixed Induction and Coinduction.* Second, we define $s \twoheadrightarrow\!\!\!\rightarrow t$ based on mutually mixing induction and coinduction, that is, least fixed points μ and greatest fixed points ν. This rendering allows for the surprisingly succinct definition:

$$\twoheadrightarrow\!\!\!\rightarrow \quad = \quad \mu x.\, \nu y.\, (\rightarrow_\varepsilon \cup \overline{x})^* \circ \overline{y}$$

where $\overline{R} = \{\, \langle f(s_1, \ldots, s_n),\, f(t_1, \ldots, t_n) \rangle \mid s_1 \; R \; t_1, \ldots, s_n \; R \; t_n \,\} \cup \mathsf{id}$.

In contrast to previous coinductive definitions [9, 18, 19, 15], the setup proposed here captures all strongly convergent rewrite sequences (of arbitrary ordinal length).

Throughout this section, we fix a signature Σ and a term rewriting system $\mathcal{R}$ over Σ. The notation $\rightarrow_\varepsilon$ denotes a root step with respect to $\mathcal{R}$.

5.1. **Derivation Rules.** The first definition has already been discussed in the introduction. The strongly convergent rewrite sequences are obtained by a syntactic restriction on the formation of the proof trees that arise from rules (1).

Definition 5.1. We define the relation $\twoheadrightarrow\!\!\!\rightarrow$ on terms $T = \mathit{Ter}^\infty(\Sigma, \mathcal{X})$ as follows. We have $s \twoheadrightarrow\!\!\!\rightarrow t$ if there exists a (finite or infinite) proof tree δ deriving $s \twoheadrightarrow\!\!\!\rightarrow t$ using the rules

$$\frac{s \; (\rightarrow_\varepsilon \cup \overset{<}{\rightarrowtriangle})^* \circ \rightarrowtriangle \; t}{s \twoheadrightarrow\!\!\!\rightarrow t}\;\mathsf{split} \qquad \frac{s_1 \twoheadrightarrow\!\!\!\rightarrow t_1 \quad \ldots \quad s_n \twoheadrightarrow\!\!\!\rightarrow t_n}{f(s_1, s_2, \ldots, s_n) \overset{(<)}{\rightarrowtriangle} f(t_1, t_2, \ldots, t_n)}\;\mathsf{lift} \qquad \frac{}{s \overset{(<)}{\rightarrowtriangle} s}\;\mathsf{id}$$

such that δ does not contain infinite nesting[1] of $\stackrel{\leq}{\to\!\!\!\!\!/\!/\!/}$. The symbol $\stackrel{(<)}{\to\!\!\!\!\!/\!/\!/}$ stands for $\to\!\!\!\!\!/\!/\!/$ or $\stackrel{\leq}{\to\!\!\!\!\!/\!/\!/}$; so the second rule is an abbreviation for two rules; similarly for the third rule. □

Let us give some intuition for the rules in Definition 5.1. The relation $\stackrel{\leq}{\to\!\!\!\!\!/\!/\!/}$ can be thought of as an infinitary reduction below the root, that is 'shorter' than the reduction that we are deriving. The three rules (split, lift and id) can be interpreted as follows:

(i) The split-rule: the term s rewrites infinitarily to t, $s \twoheadrightarrow\!\!\!\!\!/\!/\!/\, t$, if s rewrites to t using a finite sequence of (a) root steps, and (b) infinitary reductions $\to\!\!\!\!\!/\!/\!/$ below the root (where infinitary reductions preceding root steps must be shorter than the derived reduction).

(ii) The lift-rule: the term s rewrites infinitarily to t below the root, $s \stackrel{(<)}{\to\!\!\!\!\!/\!/\!/} t$, if the terms are of the shape $s = f(t_1, t_2, \ldots, t_n)$ and $t = f(t'_1, t'_2, \ldots, t'_n)$ and there exist infinitary reductions $\twoheadrightarrow\!\!\!\!\!/\!/\!/$ on the arguments: $t_1 \twoheadrightarrow\!\!\!\!\!/\!/\!/\, t'_1, \ldots, t_n \twoheadrightarrow\!\!\!\!\!/\!/\!/\, t'_n$.

(iii) The id-rule allows the rewrite relation $\stackrel{(<)}{\to\!\!\!\!\!/\!/\!/}$ to be reflexive, and this in turn yields reflexivity of $\twoheadrightarrow\!\!\!\!\!/\!/\!/$. For variable-free terms, reflexivity can already be derived using the first two rules. However, for terms with variables, this third rule is needed (unless we treat variables as function symbols of arity 0).

For example proof trees using the rules from Definition 5.1, we refer to Examples 1.1 and 1.3 in the introduction.

5.2. **Mixed Induction and Coinduction.** The next definition is based on mixing induction and coinduction. The inductive part is used to model the restriction to finite nesting of $\stackrel{\leq}{\to\!\!\!\!\!/\!/\!/}$ in the proofs in Definition 5.1. The induction corresponds to a least fixed point μ, while a coinductive rule to a greatest fixed point ν.

Definition 5.2. Let $T = \mathit{Ter}^{\infty}(\Sigma, \mathcal{X})$ be the set of terms, and let L be the set of all relations on terms $L = \mathcal{P}(T \times T)$. For $R \in \mathcal{P}(T \times T)$, we define its *lifting* as

$$\overline{R} = \{\, \langle f(s_1, \ldots, s_n),\ f(t_1, \ldots, t_n) \rangle \mid s_1\ R\ t_1, \ldots, s_n\ R\ t_n \,\} \cup \mathsf{id}$$

We define the relation $\twoheadrightarrow\!\!\!\!\!/\!/\!/$ by

$$\twoheadrightarrow\!\!\!\!\!/\!/\!/ \quad = \quad \mu x.\, \nu y.\, (\to_\varepsilon \cup\, \overline{x})^* \circ \overline{y}$$

We define functions $G \colon L \times L \to L$ and $F \colon L \to L$ by

$$G(x, y) = (\to_\varepsilon \cup\, \overline{x})^* \circ (\overline{y}) \qquad \text{and} \qquad F(x) = \nu y.\, G(x, y) = \nu y.\, (\to_\varepsilon \cup\, \overline{x})^* \circ (\overline{y}) \qquad (5)$$

We then have $\twoheadrightarrow\!\!\!\!\!/\!/\!/ \;=\; \mu x.\, F(x) \;=\; \mu x.\, \nu y.\, G(x, y) \;=\; \mu x.\, \nu y.\, (\to_\varepsilon \cup\, \overline{x})^* \circ \overline{y}$ □

It can easily be verified that F and G are monotone (in all their arguments). Recall that a function f over sets is monotone if $X \subseteq Y \implies f(\ldots, X, \ldots) \subseteq f(\ldots, Y, \ldots)$. Hence F and G have unique least and greatest fixed points.

The reflexive, transitive closure $(\cdot)^*$ in Definition 5.2 can, of course, also be defined using a least fixed point, for example, as follows:

$$R^* = \mu z.\, (\mathsf{id} \cup R \circ z) \qquad \text{or equivalently} \qquad R^* = \mu z.\, (\mathsf{id} \cup R \cup z \circ z)$$

Unfolding this definition of the reflexive, transitive closure in Definition 5.2 we obtain: $\twoheadrightarrow\!\!\!\!\!/\!/\!/ = \mu x.\, \nu y.\, (\mu z.\, \mathsf{id} \cup (\to_\varepsilon \cup\, \overline{x}) \circ z) \circ \overline{y}$

[1] No infinite nesting of $\stackrel{\leq}{\to}$ means that there exists no path ascending through the proof tree that meets an infinite number of symbols $\stackrel{\leq}{\to}$.

5.3. **Comparing Definitions 5.1 and 5.2.** We emphasize the close connection between Definitions 5.1 and 5.2. Observe that the clause $(\to_\varepsilon \cup \overline{x})^* \circ (\overline{y})$ in Definition 5.2 models $(\to_\varepsilon \cup \overset{\leq}{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/})^* \circ \rightarrow\!\!\!\!\!\!/\!\!/\!\!/$ in the first rule of Definition 5.1. Here $\overline{x}$ corresponds to $\overset{\leq}{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}$, and $\overline{y}$ to $\rightarrow\!\!\!\!\!\!/\!\!/\!\!/$. The least fixed point μx caters for the restriction of the proof tree formation to finite nesting of $\overset{\leq}{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}$.

Definitions 5.1 and 5.2 of the rewrite relation $\twoheadrightarrow\!\!\!\!\!\!\twoheadrightarrow$ both have their merits. Definition 5.2, which is based on mixing induction and coinduction, is a succinct, mathematically precise formulation of $\twoheadrightarrow\!\!\!\!\!\!\twoheadrightarrow$. The derivation rules, Definition 5.1, on the other hand, are easy to understand, and easy to use for humans.

6. Towards a Formalization

One of the advantages of the definitions we propose, over the standard definition, is that they are suitable for formalization in theorem provers. The standard definition of infinitary rewriting, using ordinal length rewrite sequences and strong convergence at limit ordinals, appears to be difficult to formalize. Martijn Vermaat has formalized infinitary rewriting using metric convergence in the Coq proof assistant [32], and proved that weakly orthogonal infinitary rewriting does not have the property UN of unique normal forms, see [13]. While his formalization could be extended to strong convergence, it remains to be investigated to what extent it can be used for the further development of the theory of infinitary rewriting. Another route is the formalization of restricted variants of infinitary rewriting. One such variant is that of 'computable infinite reductions' [26], where terms as well as reductions are computable; there is no formalization yet, but they have an implementation of the Compression Lemma in Haskell.

Agda has support for mixed inductive and coinductive definitions. Unfortunately, it still has a restriction that the coinduction has to be on the outside. If this restriction is lifted, then Definition 5.2 would be a perfect candidate for a formalization. The theorem prover Coq has regrettably no support for mutual inductive and coinductive definitions.

There are several papers on studying mixing induction and conduction [1, 10, 16, 29], most of them in the context of extending proof assistants with the capability of doing proofs by (co)induction. Our setting is very specific to infinitary term rewriting and the form of coinduction we use here is very concrete, not requiring general formulations, parametric on the functor type.

In principle, Definition 5.1 can be formalized as follows:

(i) First, we define all coinductive proof trees derivable using the rules (a purely coinductive definition poses no problems).
(ii) Afterwards, define an accessibility relation on the proof trees in order to filter out those proof trees where $\overset{\leq}{\rightarrow\!\!\!\!\!\!/\!\!/\!\!/}$ has a well-founded nesting.

Then a proof of $s \twoheadrightarrow\!\!\!\!\!\!\twoheadrightarrow t$ is a pair of a proof tree δ deriving $s \twoheadrightarrow\!\!\!\!\!\!\twoheadrightarrow t$ and proof of accessibility of δ. While this works in principle, and the definition is accepted in Coq, problems arise when trying to work with this definition. The transitivity in the split-rule harms the guardedness of the proof terms, and makes it basically impossible to define actual reductions.

This problem can be solved by avoiding the explicit corecursion in the formalization. The idea is to employ the largest relation semantics of coinductive definitions. We can render Definition 4.1 of infinitary equational reasoning as follows:

Definition 6.1. Let E be an equational specification over Σ. We define $=^\infty \subseteq T \times T$ by $s =^\infty t$ if there exists a relation $X \subseteq T \times T$ such that $s \mathrel{X} t$ and

– for all $s', t' \in T$ we have that $s' \; X \; t'$ implies $s' \; (\leftarrow_{\varepsilon} \cup \rightarrow_{\varepsilon} \cup \overline{X})^* \; t'$

where $\overline{X}$ is the lifting as in Definition 4.3. $\square$

This definition can be straightforwardly formalized in Coq, Agda or other theorem provers. For infinitary rewriting, we additionally need to ensure well-foundedness of certain paths. This can be achieved by taking a family of relations $\{X_i\}_{i \in I}$ indexed by a well-founded order $(I, >)$. The well-foundedness along certain paths is then no longer imposed on top of the definition of the proof trees, but built into the proof tree right from the start. Definition 5.1 of infinitary rewriting can now be rendered as follows:

Definition 6.2. We define the relation $\twoheadrightarrow\!\!\!\!\twoheadrightarrow$ on terms $T = \mathit{Ter}^{\infty}(\Sigma, \mathcal{X})$ as follows. We have $s \twoheadrightarrow\!\!\!\!\twoheadrightarrow t$ if there exists a well-founded order $(I, >)$ together with a family of relations $\{X_i\}_{i \in I}$ such that $s \; X_i \; t$ for some $i \in I$ and

– for all $i \in I$ and all $s', t' \in T$ we have that $s' \; X_i \; t'$ implies $s' \; (\rightarrow_{\varepsilon} \cup \overline{X_{<i}})^* \circ \overline{X_i} \; t'$

where $X_{<i} = \bigcup_{j<i} X_j$. $\square$

This definition can be formalized directly in Coq or other theorem provers, and this formalization is infinitely easier than a formalization of the standard definition employing ordinal length rewrite sequences with convergence at every limit ordinal.

Although the formalizations above are succinct and easy to work with, we think that the coinductive rules are more elegant. We therefore plan, for future work, to investigate whether there are ways to overcome the problems (e.g. with transitivity and guardedness) in order to get the coinductive definitions to work in theorem provers. As mentioned before, there is work [1, 10, 29] addressing this problem, and we are interested in knowing whether these solutions are applicable in our setting.

7. Equivalence with the Standard Definition

In this section we prove the equivalence of the coinductively defined infinitary rewrite relations $\twoheadrightarrow\!\!\!\!\twoheadrightarrow$ from Definitions 5.1, and 5.2 with the standard definition based on ordinal length rewrite sequences with metric and strong convergence at every limit ordinal (Definition 2.2).

7.1. **Derivation Rules.** Let $\twoheadrightarrow\!\!\!\!\twoheadrightarrow$ be the relation defined in Definition 5.1. The definition requires that the nesting structure of $\stackrel{\le}{\to\!\!\!\!\!/\!\!\!/\!\!\!/}$ in proof trees is well-founded. As a consequence, we can associate to every proof tree a (countable) ordinal that allows to embed the nesting structure in an order-preserving way. We use ω_1 to denote the first uncountable ordinal, and we view ordinals as the set of all smaller ordinals (then the elements of ω_1 are all countable ordinals).

Definition 7.1. Let δ be a proof tree as in Definition 5.1, and let α be an ordinal. An *α-labeling of δ* is a labeling of all symbols $\stackrel{\le}{\to\!\!\!\!\!/\!\!\!/\!\!\!/}$ in δ with elements from α such that each label is strictly greater than all labels occurring in the subtrees (all labels above). $\square$

Lemma 7.2. *Every proof tree as in Definition 5.1 has an α-labeling for some $\alpha \in \omega_1$.* $\square$

Definition 7.3. Let δ be a proof tree as in Definition 5.1. We define the *nesting depth* of δ as the least ordinal $\alpha \in \omega_1$ such that δ admits an α-labeling. For every $\alpha \le \omega_1$, we define a relation $\twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\alpha} \subseteq \twoheadrightarrow\!\!\!\!\twoheadrightarrow$ as follows: $s \twoheadrightarrow\!\!\!\!\twoheadrightarrow_{\alpha} t$ whenever $s \twoheadrightarrow\!\!\!\!\twoheadrightarrow t$ can be derived using a proof with nesting depth $< \alpha$. Likewise we define relations $\to\!\!\!\!\!/\!\!\!/\!\!\!/_{\alpha}$ and $\stackrel{\le}{\to\!\!\!\!\!/\!\!\!/\!\!\!/}_{\alpha}$. $\square$

As a direct consequence of Lemma 7.2 we have:

Corollary 7.4. *We have* $\twoheadrightarrow_{\omega_1} = \twoheadrightarrow$. □

We will now show that the coinductively defined infinitary rewrite relation $\twoheadrightarrow$ (Definition 5.1) coincides with the standard definition of $\twoheadrightarrow_{ord}$ (Definition 2.2) based on ordinal length rewrite sequences with metric and strong convergence at every limit ordinal. The crucial observation is the following theorem from [28]:

Theorem 7.5 (Theorem 2 of [28])**.** *A transfinite reduction is divergent if and only if for some N there are infinitely many steps at depth N.*

We are now ready to prove the equivalence of both notions:

Theorem 7.6. *We have* $\twoheadrightarrow = \twoheadrightarrow_{ord}$.

Proof. We write $\rightharpoondown$ for steps that are not at the root, and $\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}_{ord}$ to denote a reduction $\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}$ without root steps.

We begin with the direction $\twoheadrightarrow_{ord} \subseteq \twoheadrightarrow$. We show by induction on the ordinal length α that we have both $\to_{ord}^{\alpha} \subseteq \twoheadrightarrow$ and $\rightharpoondown^{\alpha} \subseteq \overset{(<)}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}}$. Let α be an ordinal and s, t terms. We proceed by coinduction on the structure of the proof tree to derive $\twoheadrightarrow$:

(i) Assume that $s \to_{ord}^{\alpha} t$, that is, we have a strongly convergent reduction σ from s to t of length α. By Theorem 7.5 the rewrite sequence σ contains only a finite number of root steps. As a consequence, σ is of the form: $s\ (\to_{\varepsilon} \cup \rightharpoondown^{<\alpha})^{*} \circ \rightharpoondown^{\leq\alpha}\ t$. Note that the reductions $\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}_{ord}$ preceding root steps must be shorter than α since the last root step is contracted at an index $< \alpha$ in the reduction σ. By induction hypothesis we have $\rightharpoondown^{<\alpha} \subseteq \overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}}$. Then $s\ (\to_{\varepsilon} \cup \overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}})^{*} \circ \rightharpoondown^{\leq\alpha}\ t$. Hence, $s \twoheadrightarrow t$ can be derived using the split-rule since by coinduction hypothesis we have $\rightharpoondown^{\leq\alpha} \subseteq \twoheadrightarrow$. Observe that the thereby constructed proof tree for $s \twoheadrightarrow t$ contains no infinite nesting of $\overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}}$ because every marker $\overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}}$ occurs in a node where the induction hypothesis has been applied. An infinite nesting of markers would thus give rise to an infinite descending chain of ordinals, which is impossible by well-foundedness of α.

(ii) Assume that $s \rightharpoondown^{\alpha} t$, that is, we have a strongly convergent reduction σ without root steps from s to t of length α. Then the terms s, t must be of the shape $s = \mathsf{f}(s_1, \ldots, s_n)$ and $t = \mathsf{f}(t_1, \ldots, t_n)$, and σ can be split in reductions $s_1 \to_{ord}^{\leq\alpha} t_1, \ldots, s_n \to_{ord}^{\leq\alpha} t_n$ on the arguments. By (i) we have $s_1 \twoheadrightarrow t_1, \ldots, s_n \twoheadrightarrow t_n$. Hence by the lift-rule we obtain $s \mathrel{\rightharpoondown\!\!\!\!\rightharpoondown} t$ and $s \overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}} t$ (the nesting of $\overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}}$ stays well-founded).

We now show $\twoheadrightarrow \subseteq \twoheadrightarrow_{ord}$. We prove by well-founded induction on $\alpha \leq \omega_1$ that $\twoheadrightarrow_{\alpha} \subseteq \twoheadrightarrow_{ord}$. This suffices since $\twoheadrightarrow = \twoheadrightarrow_{\omega_1}$. Let $\alpha \leq \omega_1$ and assume that $s \twoheadrightarrow_{\alpha} t$. Let δ be a proof tree of nesting depth $\leq \alpha$ deriving $s \twoheadrightarrow_{\alpha} t$. The only possibility to derive $s \twoheadrightarrow t$ is an application of the split-rule with the premise $s\ (\to_{\varepsilon} \cup \overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}})^{*} \circ \mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}\ t$. Since $s \twoheadrightarrow_{\alpha} t$, we have $s\ (\to_{\varepsilon} \cup \overset{<}{\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}}_{\alpha})^{*} \circ \mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}_{\alpha}\ t$. By induction hypothesis we have $s\ (\to_{\varepsilon} \cup \twoheadrightarrow_{ord})^{*} \circ \mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}_{\alpha}\ t$, and thus $s \twoheadrightarrow_{ord} \circ \mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}_{\alpha}\ t$. We have $\mathrel{\rightharpoondown\!\!\!\!\rightharpoondown}_{\alpha} = \overline{\twoheadrightarrow_{\alpha}}$, and consequently $s \twoheadrightarrow_{ord} s_1 \overline{\twoheadrightarrow_{\alpha}}\ t$ for some term s_1. Repeating this argument on $s_1 \overline{\twoheadrightarrow_{\alpha}}\ t$, we get $s \twoheadrightarrow_{ord} s_1 \overline{\twoheadrightarrow_{ord}}\ s_2 \overline{\overline{\twoheadrightarrow_{\alpha}}}\ t$. After n iterations, we obtain

$$s \twoheadrightarrow_{ord} s_1 \overline{\twoheadrightarrow_{ord}}\ s_2 \overline{\overline{\twoheadrightarrow_{ord}}}\ s_3 \overline{\overline{\overline{\twoheadrightarrow_{ord}}}}\ s_4 \cdots (\twoheadrightarrow_{\alpha})^{-(n-1)}\ s_n\ (\twoheadrightarrow_{\alpha})^{-n}\ t$$

where $(\twoheadrightarrow_{\alpha})^{-n}$ denotes the n'th iteration of $x \mapsto \overline{x}$ on $\twoheadrightarrow_{\alpha}$.

Clearly, the limit of $\{s_n\}$ is t. Furthermore, each of the reductions $s_n \twoheadrightarrow_{ord} s_{n+1}$ are strongly convergent and take place at depth greater than or equal to n. Thus, the infinite concatenation of these reductions yields a strongly convergent reduction from s to t (there is only a finite number of rewrite steps at any depth n). □

7.2. Mixed Induction and Coinduction.

Theorem 7.7. *The Definitions 5.1 and 5.2 give rise to the same relation* $\twoheadrightarrow$.

Proof. To avoid confusion we write $\twoheadrightarrow_{nest}$ for the relation $\twoheadrightarrow$ defined in Definition 5.1, and $\twoheadrightarrow_{fp}$ for the relation $\twoheadrightarrow$ defined in Definition 5.2. We show $\twoheadrightarrow_{nest} = \twoheadrightarrow_{fp}$.

We begin with $\twoheadrightarrow_{fp} \subseteq \twoheadrightarrow_{nest}$. Employing the μ-rule from (4), it suffices to show that $F(\twoheadrightarrow_{nest}) \subseteq \twoheadrightarrow_{nest}$. We prove this fact by coinduction on the structure of coinductively defined proof trees (Definition 5.1). We have $\rightharpoonup_{nest} = \overset{<}{\rightharpoonup}_{nest} = \overline{\twoheadrightarrow_{nest}}$, and thus

$$\begin{aligned} F(\twoheadrightarrow_{nest}) &= (\to_\varepsilon \cup \overline{\twoheadrightarrow_{nest}})^* \circ \overline{F(\twoheadrightarrow_{nest})} = (\to_\varepsilon \cup \overset{<}{\rightharpoonup}_{nest})^* \circ \overline{F(\twoheadrightarrow_{nest})} \\ \overline{F(\twoheadrightarrow_{nest})} &= \mathsf{id} \cup \{\, \langle f(\vec{s}),\, f(\vec{t})\rangle \mid \vec{s}\ F(\twoheadrightarrow_{nest})\ \vec{t}\,\} \end{aligned}$$

where $\vec{s}$, $\vec{t}$ abbreviate $s_1, \dots, s_n$ and $t_1, \dots, t_n$, respectively, and we write $\vec{s}\ R\ \vec{t}$ if we have $s_1\ R\ t_1, \dots, s_n\ R\ t_n$. Now we apply the split-rule to derive $(\to_\varepsilon \cup \overset{<}{\rightharpoonup}_{nest})^* \circ \overline{F(\twoheadrightarrow_{nest})}$ and $\overline{F(\twoheadrightarrow_{nest})}$ can be derived via the id-rule, or the lift-rule; for the arguments $\vec{s}$, $\vec{t}$ of the lift-rule we have by coinduction that $\vec{s} \twoheadrightarrow_{nest} \vec{t}$ since $\vec{s}\ F(\twoheadrightarrow_{nest})\ \vec{t}$.

We now show that $\twoheadrightarrow_{nest} \subseteq \twoheadrightarrow_{fp}$. We prove by well-founded induction on $\alpha \le \omega_1$ that $\twoheadrightarrow_{\alpha,nest} \subseteq \twoheadrightarrow_{fp}$. This yields the claim $\twoheadrightarrow_{\omega_1,nest} = \twoheadrightarrow_{nest}$ by Corollary 7.4. Since $\twoheadrightarrow_{fp}$ is a fixed point of F, we obtain $\twoheadrightarrow_{fp} = F(\twoheadrightarrow_{fp})$, and since $F(\twoheadrightarrow_{fp})$ is a greatest fixed point, using the ν-rule from (4), it suffices to show that $(*)$ $\twoheadrightarrow_{\alpha,nest} \subseteq G(\twoheadrightarrow_{fp}, \twoheadrightarrow_{\alpha,nest})$. Thus assume that $s \twoheadrightarrow_{\alpha,nest} t$, and let δ be a proof tree of nesting height $\le \alpha$ deriving $s \twoheadrightarrow_{\alpha,nest} t$. The only possibility to derive $s \twoheadrightarrow_{nest} t$ is an application of the split-rule with the premise $s\ (\to_\varepsilon \cup \overset{<}{\rightharpoonup}_{nest})^* \circ \rightharpoonup_{nest} t$. Since $s \twoheadrightarrow_{\alpha,nest} t$, we have $s\ (\to_\varepsilon \cup \overset{<}{\rightharpoonup}_{\alpha,nest})^* \circ \rightharpoonup_{\alpha,nest} t$. Let τ be one of the steps $\overset{<}{\rightharpoonup}_{\alpha,nest}$ displayed in the premise. Let u be the source of τ and v the target, so $\tau : u \overset{<}{\rightharpoonup}_{\alpha,nest} v$. The step τ is derived either via the id-rule or the lift-rule. The case of the id-rule is not interesting since we then can drop τ from the premise. Thus let the step τ be derived using the lift-rule. Then the terms u, v are of form $u = f(u_1, \dots, u_n)$ and $v = f(v_1, \dots, v_n)$ and for every $1 \le i \le n$ we have $u_i \twoheadrightarrow_{\beta,nest} v_i$ for some $\beta < \alpha$. Thus by induction hypothesis we obtain $u_i \twoheadrightarrow_{fp} v_i$ for every $1 \le i \le n$, and consequently $u\ \overline{\twoheadrightarrow_{fp}}\ v$. We then have $s\ (\to_\varepsilon \cup \overline{\twoheadrightarrow_{fp}})^* \circ \rightharpoonup_{\alpha,nest} t$, and hence $s\ G(\twoheadrightarrow_{fp}, \twoheadrightarrow_{\alpha,nest})\ t$. This concludes the proof. ☐

8. Conclusion

We have proposed a coinductive framework of infinitary rewriting. From the framework arise three natural variants of infinitary rewriting:

(a) infinitary equational reasoning,
(b) bi-infinite rewriting, and
(c) infinitary rewriting,

of which (c) is the standard definition of infinitary rewriting with respect to strong convergence. The variants (a) and (b) are novel and have to the best knowledge of the authors not yet been studied. For example, we are interested in a comparison of the Church-Rosser properties $=^\infty \subseteq \twoheadrightarrow \circ \twoheadleftarrow$ and $(\twoheadleftarrow \circ \twoheadrightarrow)^* \subseteq \twoheadrightarrow \circ \twoheadleftarrow$. As a consequence of the coinduction over the term structure, all of the infinitary rewriting notions (a), (b) and (c) have the strong convergence built in, and thus can profit from the well-developed techniques (such as tracing) in infinitary rewriting.

We emphasize that our framework captures the full infinitary rewriting with rewrite sequences of arbitrary ordinal length. Previously, coinductive definitions of infinitary rewriting have been limited to rewrite sequences of length at most ω.

Last but not least, our work contributes towards a formalization of infinitary rewriting in theorem provers. As discussed in Section 6, variants of the definitions we propose are very suitable for a formalization.

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