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AN INTRODUCTION TO MULTILINEAR ALGEBRA AND SOME APPLICATIONS

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Abstract

The present paper intends to give a quick and easy introduction to tensors, in particular to the exterior and the symmetric algebra of a vector space and the relations between them. The entries of the k-th exterior, and the k-th symmetric power of a matrix are expressed as determinants, and permanents, respectively. The generating polynomials of these powers are related by their traces. As applications we mention MacDonald's proof of MacMahon's master theorem, Bebiano's permanent expansion [1]) and the permanent version of the solution of Fredholm's integral equation, observed by Kershaw [2]) and by de Bruijn [3]).

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1. Symmetric functions [4])

The *elementary symmetric* polynomial of degree k in d variables, $x_1, \ldots, x_d$, is defined by

$$e_k(x) := \sum_{1 \leqslant j_1 < \ldots < j_k \leqslant d} x_{j_1} \ldots x_{j_k},$$

or, equivalently, by its generating function

$$E(t; x) := \sum_{k \geqslant 0} e_k(x) t^k = \prod_{i=1}^{d} (1 + x_i t).$$

The k-th *complete symmetric* polynomial is defined by

$$h_k(x) := \sum_{1 \leqslant j_1 \leqslant \ldots \leqslant j_k \leqslant d} x_{j_1} \ldots x_{j_k},$$

with generating function

$$H(t; x) := \sum_{k \geqslant 0} h_k(x) t^k = \prod_{i=1}^{d} \frac{1}{1 - x_i t}.$$

Theorem 1.1

$$E(-t; x) H(t; x) = 1.$$

Remark

$\{e_1, \ldots, e_d\}$ forms a basis for the graded ring S_d of symmetric polynomials in d variables. Let j denote the all-one vector of size d, then

$$e_k(j) = \binom{d}{k} \quad \text{and} \quad h_k(j) = \binom{d+k-1}{k}.$$

2. Tensors [6-7])

We start with a quick definition. Let V be a (real) vector space of dimension d, with (positive definite) inner product $(\, , \,)$ and orthonormal basis $e_1, \ldots, e_d$. We define tensor products by their components with respect to this basis:

$$x \otimes y \quad \text{by the } d^2 \text{ components } x_i y_j,$$
$$x \otimes y \otimes z \quad \text{by the } d^3 \text{ components } x_i y_j z_k.$$

The tensor product $V \otimes V$ is the linear space spanned by all tensor products $x \otimes y$; $x, y \in V$. The inner product $(x \otimes y, u \otimes v) := (x, u)(y, v)$ provides $V \otimes V$ with the orthonormal basis $\{e_i \otimes e_j \mid 1 \leqslant i, j \leqslant d\}$.

Similarly tensor products of k vectors, $x_1 \otimes \ldots \otimes x_k$ are defined by their d^k components $(x_1)_i (x_2)_j \ldots (x_k)_m$, with respect to the orthonormal basis

$$\{e_i \otimes e_j \otimes \ldots \otimes e_m \mid \quad 1 \leqslant i \leqslant j \leqslant \ldots \leqslant m \leqslant d\}$$

of $T_k(V) := \otimes^k V$, with inner product

$$(x_1 \otimes \ldots \otimes x_k, y_1 \otimes \ldots \otimes y_k) = \prod_{i=1}^{k} (x_i, y_i).$$

Clearly, $\dim T_k V = d^k$. The *tensor algebra* TV is the direct sum

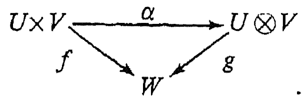
$$TV := T_0 V \oplus T_1 V \oplus \ldots = \sum_{k \geqslant 0}^{\oplus} T_k V,$$

where $T_0 V \simeq R$, $T_1 V \simeq V$. Its Poincaré polynomial (as a graded algebra) is

$$P_T(t) := \sum_{k \geqslant 0} (\dim T_k V)\, t^k = \frac{1}{1 - dt}.$$

Remark

We refer to the literature for details about the following coordinatefree definition. The tensor product of the vector spaces U and V is the pair of a vector space $U \otimes V$ and a bilinear map $\alpha : U \times V \to U \otimes V$ such that for all bilinear f there is a unique linear g such that the following diagram commutes



3. The exterior algebra ΛV [5,6])

$\Lambda_2 V$ is the linear subspace of $T_2 V$ spanned by the *skew* 2-tensors:

$$x \wedge y := \tfrac{1}{2}(x \otimes y - y \otimes x); \quad x, y \in V.$$

Clearly, $x \wedge x = 0$, $x \wedge y = -y \wedge x$ and $\dim \Lambda_2 V = \binom{d}{2}$.

The inner product inherited from $T_2 V$ becomes

$$(x \wedge y, u \wedge v) = \tfrac{1}{2}((x,u)(y,v) - (x,v)(y,u)) = \tfrac{1}{2}\det\begin{pmatrix}(x,u) & (x,v)\\ (y,u) & (y,v)\end{pmatrix}.$$

An orthonormal basis for $\Lambda_2 V$ is provided by $\{\sqrt{2}(e_i \wedge e_j) \mid 1 \leqslant i < j \leqslant d\}$.

Likewise, for $0 \leqslant k \leqslant d$ we define

$$x_1 \wedge \ldots \wedge x_k := \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} (-1)^\sigma x_{\sigma 1} \otimes \ldots \otimes x_{\sigma k},$$

where $(-1)^\sigma$ is 1 (-1) if the permutation σ is even (odd), and γ_k is the symmetric group on k letters, and

$$\Lambda_k V := \langle x_1 \wedge \ldots \wedge x_k \mid x_i \in V\rangle_R,$$

with inner product

$$(x_1 \wedge \ldots \wedge x_k, y_1 \wedge \ldots \wedge y_k) = \frac{1}{k!}\det((x_i, y_j)).$$

Note that $\dim \Lambda_k V = \binom{d}{k} = e_k(j)$; an orthonormal basis for $\Lambda_k V$ is provided by

$$\{\sqrt{k!}\, e_{i_1} \wedge \ldots \wedge e_{i_k} \mid 1 \leqslant i_1 < \ldots < i_k \leqslant d\}.$$

The *exterior algebra* ΛV is the (finite) direct sum

$$\Lambda V = \Lambda_0 V \oplus \Lambda_1 V \oplus \ldots \oplus \Lambda_{d-1} V \oplus \Lambda_d V,$$

where $\Lambda_0 V \simeq \Lambda_d V \simeq R$ and $\Lambda_1 V \simeq \Lambda_{d-1} V \simeq V$. Its Poincaré polynomial is

$$p_\Lambda(t) = (1 + t)^d = E(t, j).$$

Remark

$x_1, \ldots, x_k$ are linearly dependent iff $x_1 \wedge \ldots \wedge x_k = 0$.

4. The symmetric algebra ΣV[5,6])

$\Sigma_2 V$ is the linear space spanned by the *symmetric* 2–tensors,

$$x \vee y := \tfrac{1}{2}(x \otimes y + y \otimes x); \quad x, y \in V.$$

Clearly, $x \vee y = y \vee x$ and $\dim \Sigma_2 V = \binom{d+1}{2}$.

$$(x \vee y, u \vee v) = \tfrac{1}{2}((x,u)(y,v) + (x,v)(y,u)) = \tfrac{1}{2}\,\mathrm{per}\begin{pmatrix}(x,u) & (x,v)\\ (y,u) & (y,v)\end{pmatrix}.$$

Likewise, for $0 \leqslant k < \infty$ we define

$$\Sigma_k V := \langle x_1 \vee \ldots \vee x_k | x_i \in V \rangle_R ,$$

where

$$x_1 \vee \ldots \vee x_k = \frac{1}{k!} \sum_{\sigma \in \mathscr{S}_k} x_{\sigma 1} \otimes \ldots \otimes x_{\sigma k} .$$

Then $\Sigma_k V$ has basis $\{e_{i_1} \vee \ldots \vee e_{i_k} | \ 1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant d\}$, and

$$\dim \Sigma_k V = \binom{d+k-1}{k} = h_k(j).$$

The inner product derives from

$$(x_1 \vee \ldots \vee x_k, y_1 \vee \ldots \vee y_k) = \frac{1}{k!} \operatorname{per}((x_i, y_j)).$$

The *symmetric algebra* $\Sigma V = \sum_{k \geqslant 0}^{\oplus} (\Sigma_k V)$ has Poincaré polynomial

$$p_\Sigma(t) = (1-t)^{-d} = H(t, j).$$

Remark

The basis for $\Sigma_k V$ may be normalized using the factors $\left(\frac{k!}{\mathbf{k}!}\right)^{\frac{1}{2}}$, where $\mathbf{k}! = k_1! \ldots k_d!$ and $|\mathbf{k}| = k_1 + \ldots + k_d = k$, $e_{i_1} \vee \ldots \vee e_{i_k} = e_1^{k_1} \vee \ldots \vee e_d^{k_d}$.

5. Relating Λ to Σ [7,8])

We shall consider power series whose coefficients are vector spaces, with addition defined as direct sum, and multiplication as tensor product.

$$\Lambda(t; V) := \sum_{k \geqslant 0} (\Lambda_k V) t^k; \quad \Sigma(t; V) := \sum_{k \geqslant 0} (\Sigma_k V) t^k.$$

Theorem 5.1

$$\Lambda(t; V \oplus W) = \Lambda(t; V) \otimes \Lambda(t; W); \ \Sigma(t; V \oplus W) = \Sigma(t; V) \otimes \Sigma(t; W).$$

Proof

By considering bases we prove $\Lambda_k(V \otimes W) = \sum_{i+j=k}^{\oplus} \Lambda_i(V) \otimes \Lambda_j(W)$, and the analogous formula for $\Sigma_k(V \oplus W)$.
These imply the relations of the theorem. □

Theorem 5.2

$$\Lambda(-t; V) \otimes \Sigma(t; V) = 1.$$

Proof

First observe that for $\dim V = 1$ we have

$$\Lambda(t; V) = 1 + tV; \quad \Sigma(t; V) = 1 + tV + t^2 V + \ldots = (1 - tV)^{-1}.$$

For an arbitrary V (which is the direct sum of one-dimensional subspaces), the theorem follows by application of theorem 5.1. □

6. Relating Λ and Σ to S [4,8]

The semigroup $(N, +)$ can be extended to the ring $(Z, +, .)$ by a well-known construction. Similarly, the isomorphism classes $[V]$ of vector spaces V, over a fixed field form a semigroup with respect to direct sums, which can be extended to the Grothendieck ring K, with operations direct sum and tensor product. The map $[V] \to \dim V$ identifies the rings K and Z. For V of dimension d we have

$$[\Lambda_k V] \to \binom{d}{k} \quad \text{and} \quad [\Sigma_k V] \to \binom{d+k-1}{k}.$$

More interesting is the isomorphism of the Grothendieck ring K and the ring S of symmetric functions (in infinitely many variables). Here we have the following correspondences: (V of dimension d, $x = (x_1, \ldots, x_d, 0, 0, \ldots)$)

$$\begin{aligned} \Lambda_k V &\to e_k(x); \quad \Lambda(t; V) \to E(t; x), \\ \Sigma_k V &\to h_k(x); \quad \Sigma(t; V) \to H(t; x). \end{aligned}$$

This shows the connection between theorems 5.2 and 1.1.

Remark

The ring S of the symmetric functions is the underlying ring for the representations of the symmetric group $\mathscr{S}$ in the following sense: Let $\pi\colon \mathscr{S}_k \to \mathrm{Aut}(V_\pi)$ denote an irreducible representation of $\mathscr{S}_k$ on the $\mathscr{S}$-module V_π. Let $R(\mathscr{S})$ denote the ring of all such $\mathscr{S}$-modules. The fundamental theorem [5] says that there is an isomorphism (of λ-rings)

$$\Theta : R(\mathscr{S}) \simeq S.$$

We indicate how to obtain the symmetric function $\Theta(V_\pi)$: Let X be any vector space, consider $V_\pi \otimes X^{\otimes k}$ and its $\mathscr{S}_k$-invariant subspace $\pi(X)$. For instance, if $\pi = 1$ then $\pi(X) = \Sigma_k(X)$, if $\pi = +/-$ then $\pi(X) = \Lambda_k(X)$. Any linear $T: X \to X$ induces $T_k(T)$: $T_k X \to T_k X$ (see sec. 9) and $\pi(T)$: $\pi(X) \to \pi(X)$. Now the function $\mathrm{spec}(T) \to \mathrm{trace}\, \pi(T)$ is the desired symmetric function in S which is attached to π.

7. Duality [6]

The dual V^* of the vector space V consists of the linear functionals on V. The action of V^* on V may conveniently be described by a pairing $[V, V^*]$: $[v, w^*] := w^*(v)$. If we identify $(T_k V)^*$ with $T_k(V^*)$ we may write

$$[v_1 \otimes \ldots \otimes v_k, w_1^* \otimes \ldots \otimes w_k^*] = \prod_{i=1}^{k} [v_i, w_i^*] = \prod_{i=1}^{k} w_i^*(v_i).$$

Likewise we have $(\Lambda_k V)^* \simeq \Lambda_k(V^*)$ and $(\Sigma_k V)^* \simeq \Sigma_k(V^*)$ with the induced pairings

$$[v_1 \wedge \ldots \wedge v_k, w_1^* \wedge \ldots \wedge w_k^*] = \frac{1}{k!} \det (w_j^*(v_i)),$$

$$[v_1 \vee \ldots \vee v_k, w_1^* \vee \ldots \vee w_k^*] = \frac{1}{k!} \operatorname{per} (w_j^*(v_i)).$$

Remark

ΣV^* is isomorphic with the ring of polynomial functions on V, with ordinary multiplication. In a certain sense ΛV^* corresponds to the set of square-free polynomials in d variables, cf. sec. 9.

8. Metrics[6])

Let V have basis $e_1, \ldots, e_d$ and let $e_1^*, \ldots, e_d^*$ be the dual basis for V^*, i.e., $[e_i, e_j^*] = \delta_{ij}$.

In V a non-degenerate inner product is defined by a non-singular map $G: V \rightarrow V^*$ and $(v, w) := [v, Gw]$, $v, w \in V$. For the orthogonal geometry $O(r, s)$ the standard matrix for G, with respect to the bases e_i and e_j^* is

$$G = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}, \quad \text{and} \quad G = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

for the symplectic geometry. The map $G: V \rightarrow V^*$ induces $T_k G: T_k V \rightarrow T_k V^*$: $x_1 \otimes \ldots \otimes x_k \rightarrow G x_1 \otimes \ldots G x_k$, which in $T_k V$ leads to the inner product

$$(t, u) := [t, T_k G u]; \; t, u \in T_k V,$$

or

$$(x_1 \otimes \ldots x_k, y_1 \otimes \ldots y_k) = \prod_{i=1}^{k} [x_i, G y_i] = \prod_{i=1}^{k} (x_i, y_i) \,.$$

Likewise the maps $\Lambda_k G: \Lambda_k V \rightarrow \Lambda_k V^*$ and $\Sigma_k G: \Sigma_k V \rightarrow \Sigma_k V^*$ are defined and

$$(x_1 \wedge \ldots \wedge x_k, y_1 \wedge \ldots \wedge y_k) = \frac{1}{k!} \det ((x_i, y_j)),$$

$$(x_1 \vee \ldots \vee x_k, y_1 \vee \ldots \vee y_k) = \frac{1}{k!} \operatorname{per} ((x_i, y_j)).$$

9. Exterior and symmetric powers of a matrix[5,6,9,10])

Let A: $V \rightarrow V$ denote a linear map of V. The k-th tensor power $T_k A$ is defined by $T_k A$: $x_1 \otimes \ldots \otimes x_k \rightarrow A x_1 \otimes \ldots \otimes A x_k$. Likewise the maps $\Lambda_k A$ and $\Sigma_k A$ are defined.

Notation:

For k and l, with $|k| = |l| = k$ the matrix $A(k, l)$ is the $k\times k$ matrix which is obtained from (the $d\times d$ matrix) A by repeating k_i times the i-th row, and l_j times the j-th column, for $i, j = 1, \ldots, d$.

Lemma 9.1

In $(A\,x)^k$ the coefficient of x^l equals $\frac{1}{l!}$ per $A(k, l)$.

Proof

$(A\,x)^k = \prod_{i=1}^{d} \left(\sum_{j=1}^{d} a_{ij} x_j \right)^{k_i}$. Arrange the terms $a_{ij}x_j$ in a $k\times d$ matrix. Change this into a $k\times k$ matrix with columns corresponding to l. Suppress x^l, then we are left with $A(k, l)$ and the result follows. □

In the final remark of sec. 7 we mentioned that $\Lambda_k V$ corresponds in a certain sense to the square-free polynomials, the sense being that

$$x_{i_1} x_{i_2} \ldots x_{i_k} = (-1)^{\sigma} x_{i_1} \ldots x_{i_k}$$

for $\sigma \in \mathscr{S}_k$, i.e., the polynomials are *skew*. In this sense one has the following skew version of lemma 9.1:

Lemma 9.1′

In $(A\,x)^k$ the coefficient of x^l equals $\det A(k, l)$. Note that $\det A(k, l) \neq 0$ implies $l! = 1$. By use of the orthonormal bases for $\Lambda_k V$ and $\Sigma_k V$ introduced in secs 3 and 4, we can now calculate the entries of the power matrices. □

Theorem 9.2

$$\Lambda_k(A)_{k,l} = \det A(k, l); \quad \Sigma_k(A)_{k,l} = \frac{\operatorname{per} A(k, l)}{\sqrt{k!}\sqrt{l!}} \,. \qquad \square$$

From the definition we infer:

Theorem 9.3

Let A have eigenvalues $\alpha_1, \ldots, \alpha_d$. The eigenvalues of $\Lambda_k(A)$ are the $\binom{d}{k}$ squarefree, those of $\Sigma_k(A)$ the $\binom{d+k-1}{k}$ homogeneous monomials of degree k in $\alpha_1, \ldots, \alpha_d$. □

The following expansion of the determinant is well-known:

$$\det(I + tA) = 1 + t \sum_{i=1}^{d} a_{ii} + t^2 \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} + \ldots + t^d \det(A).$$

Its counterpart for permanents is used less frequently:

Theorem 9.4

$$\det(I + tA) = \sum_{k=0}^{d} t^k \cdot \operatorname{trace} \Lambda_k(A) = \sum_{\boldsymbol{k}} t^{|\boldsymbol{k}|} \det A(\boldsymbol{k}, \boldsymbol{k});$$

$$\det^{-1}(I - tA) = \sum_{k \geqslant 0} t^k \cdot \operatorname{trace} \Sigma_k(A) = \sum_{\boldsymbol{k}} t^{|\boldsymbol{k}|} \frac{\operatorname{per} A(\boldsymbol{k}, \boldsymbol{k})}{\boldsymbol{k}!}.$$

Proof

The equalities on the right hand sides follow from theorem 9.2. As for the left hand sides, theorem 9.3 implies that trace $\Sigma_k(A)$ is the k-th complete symmetric polynomial in the eigenvalues α of A, i.e., we have

$$\operatorname{trace} \Sigma_k(A) = h_k(\alpha), \quad \text{similarly trace } \Lambda_k(A) = e_k(\alpha).$$

Hence the sums of the traces are the generating functions $E(t; \boldsymbol{\alpha})$ and $H(t; \boldsymbol{\alpha})$ of sec. 1. The theorem now follows from theorem 1.1. □

Corollary 9.5

$$\det(I + tA) = \operatorname{trace} \Lambda(t; A) = \operatorname{trace}^{-1} \Sigma(t; A),$$

for the generating functions

$$\Lambda(t; A) = \sum_{k \geqslant 0} t^k \Lambda_k(A) \text{ and } \Sigma(t; A) = \sum_{k \geqslant 0} t^k \Sigma_k(A).$$ □

10. Applications

I. G. Macdonald told us the following proof of MacMahon's master theorem.

Theorem 10.1

The coefficient of $\boldsymbol{x}^{\boldsymbol{k}}$ in the symmetric product $(A\,x)^k$ equals the coefficient of $\boldsymbol{x}^{\boldsymbol{k}}$ in $1/\det(I - A\,\Delta(x))$, where $\Delta(x) = \operatorname{diag}(x_1, \ldots, x_d)$.

Proof

By 9.1 and 9.2 the coefficient of $\boldsymbol{x}^{\boldsymbol{k}}$ in $(A\,x)^k$ equals the $(\boldsymbol{k}, \boldsymbol{k})$ entry of $\Sigma_k(A)$ where $k = |\boldsymbol{k}|$. Hence it is the coefficient of $\boldsymbol{x}^{\boldsymbol{k}}$ in trace $\Sigma_k(A\,\Delta(x))$. However, $\sum_{k \geqslant 0} \operatorname{trace} \Sigma_k(A\,\Delta(x)) = 1/\det(I - A\,\Delta(x))$, according to theorem 9.4. □

Theorem 10.2[1])

$$\frac{(x, A\,y)^k}{k!} = \sum_{|\boldsymbol{k}| = |\boldsymbol{l}| = k} \frac{\boldsymbol{x}^{\boldsymbol{k}} \boldsymbol{y}^{\boldsymbol{l}}}{\boldsymbol{k}!\,\boldsymbol{l}!} \operatorname{per} A(\boldsymbol{k}, \boldsymbol{l}).$$

Proof

Write $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + \ldots + x_d \boldsymbol{e}_d$, and take the inner products of the symmetric tensors $\frac{1}{k!} \boldsymbol{x} \vee \ldots \vee \boldsymbol{x}$ and $\frac{1}{k!} A\,y \vee \ldots \vee A\,y$, using

$$\frac{1}{k!}\,x\vee\ldots\vee x=\sum_{|k|=k}\frac{x^k}{k!}\,e_1^{k_1}\vee\ldots\vee e_d^{k_d}.$$ □

Corrolary 10.3 (N. G. de Bruijn)

For any rectangular matrix $A=(a_{ij})$ we have

$$\frac{1}{k!}\,(\sum_{i,j}a_{ij})^k=\sum_{|k|=|l|=k}\frac{\operatorname{per}A(k,l)}{k!\,l!}.$$

Proof

Make A into a square matrix by supplying zeros, and apply theorem 10.2.

□

11. Fredholm's integral equation[2,3,11])

From the theory of integral equations it follows that

$$u(x)=f(x)+\lambda\int_0^1 K(x,t)\,u(t)\,\mathrm{d}t$$

has the solution

$$u(x)=f(x)+\int_0^1\frac{D(x,y;\lambda)}{D(\lambda)}\,f(y)\,\mathrm{d}y,$$

provided $D(\lambda)\neq 0$. To explain the notation, we divide the interval $[0,1]$ into d equal parts by $0<\frac{1}{d}<\ldots<\frac{d-1}{d}<1$. Let f be the d-vector with components $f_i=f\left(\frac{i}{d}\right)$, and let M be the $d\times d$ matrix with entries $M_{ij}=\frac{1}{d}K\left(\frac{i}{d},\frac{j}{\mathrm{d}}\right)$. Then the integral equation is approximated by the following set of matrix equations of increasing size $d=1,2,3,\ldots$

$$(I-\lambda M)u=f.$$

These equations may be solved using Cramer's rule. Fredholm's determinant $D(\lambda)$ is defined by

$$1-\lambda\int_0^1 K(t,t)\,\mathrm{d}t+\frac{\lambda^2}{2!}\int_0^1\int_0^1\begin{vmatrix}K(t_1,t_1)\,K(t_1,t_2)\\K(t_2,t_1)\,K(t_2,t_2)\end{vmatrix}\mathrm{d}t_1\,\mathrm{d}t_2-\frac{\lambda^3}{3!}\int\int\int+\ldots.$$

It is the limit, for $d\to\infty$, of

$$\Delta(\lambda):=1-\lambda\sum_i M_{ii}+\frac{\lambda^2}{2!}\sum_{i,j}\begin{vmatrix}M_{ii}\,M_{ij}\\M_{ji}\,M_{jj}\end{vmatrix}+\ldots+(-1)^d\lambda^d\det M.$$

Fredholm's first minor $D(x,y;\lambda)$ is defined by

$$\lambda K(x,y) - \lambda^2 \int_0^1 \begin{vmatrix} K(x,y) & K(x,t) \\ K(t,y) & K(t,t) \end{vmatrix} \mathrm{d}t + \frac{\lambda^3}{2!} \int_0^1 \int_0^1 \begin{vmatrix} K(x,y) & K(x,s) & K(x,t) \\ K(s,y) & K(s,s) & K(s,t) \\ K(t,y) & K(t,s) & K(t,t) \end{vmatrix} \mathrm{d}s\,\mathrm{d}t + \dots .$$

It is the limit for $d \to \infty$, of

$$\Delta(x,y;\lambda) := \lambda M_{xy} - \lambda^2 \sum_i \begin{vmatrix} M_{xy} & M_{xi} \\ M_{iy} & M_{ii} \end{vmatrix} + \frac{\lambda^3}{2!} \sum_{i,j} \begin{vmatrix} M_{xy} & M_{xi} & M_{xj} \\ M_{iy} & M_{ii} & M_{ij} \\ M_{jy} & M_{ji} & M_{jj} \end{vmatrix} + \dots .$$

By replacing determinants by permanents we get expressions which we denote by $P(-\lambda)$, $\Pi(-\lambda)$, $-P(x,y;-\lambda)$, $-\Pi(x,y;-\lambda)$, respectively. Now in ref. 2 it is proved that

$$D(\lambda)\,P(\lambda) = 1, \quad D(x,y;\lambda)\,P(\lambda) = P(x,y;\lambda)\,D(\lambda). \qquad (*)$$

As a consequence the solution of Fredholm's equation may also be written in terms of permanents:

$$u(x) = f(x) + \int_0^1 \frac{P(x,y;\lambda)}{P(\lambda)} f(y)\,\mathrm{d}y.$$

Now the first equality in (*) is implied by

$$\Delta(\lambda) = \det(I - \lambda M) = (\Pi(\lambda))^{-1}.$$

One can verify that $\Delta(x,y;\lambda)\,\Pi(\lambda) = \Pi(x,y;\lambda)\,\Delta(\lambda)$ also holds, but we lack an elegant proof of this fact.

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