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NOTE

SOME NEW TWO-WEIGHT CODES AND STRONGLY REGULAR GRAPHS

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It is well known (and due to Delsarte [3]) that the three concepts (i) two-weight projective code, (ii) strongly regular graph defined by a difference set in a vector space, and (iii) subset X of a projective space such that $|X\cap H|$ takes only two values when H runs over all hyperplanes, are equivalent. Here we construct some new examples (formulated as in (iii)) by taking a quadric defined over a small field and cutting out a quadric defined over a larger field.

Let F be a field with r elements and F_0 a subfield with q elements, so that $r=q^e$ for some $e>1$. Let V be a vector space of dimension d over F and write V_0 for the same vector space but now regarded as a vector space of dimension de over F_0. (We shall use the zero subscript to indicate objects or operations in V_0 corresponding to those indicated without this subscript in V.) Let $\mathrm{Tr}:F\to F_0$ be the trace map (defined by $\mathrm{Tr}(x)=x+x^q+\cdots+x^{q^{e-1}}$). One immediately checks the following observations:

(a) *If $Q:V\to F$ is a quadratic form on V, then $Q_0=\mathrm{Tr}\circ Q$ is a quadratic form on V_0.*

(b) *If $B:V\times V\to F$ is the bilinear form corresponding to Q (defined by $Q(x+y)=Q(x)+Q(y)+B(x,y)$), then $B_0=\mathrm{Tr}\circ B$ is the bilinear form corresponding to Q_0.*

(c) *B_0 is nondegenerate iff B is nondegenerate.*

(d) *Q_0 is nondegenerate iff Q is nondegenerate and either q is odd or d is even.* [If q is odd, then Q is nondegenerate iff B is; if q is even and d is odd and Q is nondegenerate, then $\dim \mathrm{rad}\, V=1$ so that $\dim_0 \mathrm{rad}_0\, V_0=e$ and Q_0 is degenerate.]

(e) *If d is even, then Q_0 has maximal (minimal) Witt index iff Q has.*

Proof. (For details on orthogonal geometry, see e.g. Artin [1, Chapter III].) Let $\varepsilon=+1$ (-1), then $Q(x)=0$ is true for $(r^{d/2}-\varepsilon)(r^{d/2-1}+\varepsilon)$ nonzero vectors in V. Since d is even the number of solutions of $Q(x)=a$ does not depend on the $a\in F\setminus\{0\}$ chosen, so this equation has $r^{d-1}-\varepsilon r^{d/2-1}$ solutions. Since $\mathrm{Tr}\, y=0$ is true for q^{e-1} elements of F among which 0, we see that $\mathrm{Tr}\, Q(x)=0$ is true for

$$(q^{e-1}-1)(r^{d-1}-\varepsilon r^{d/2-1})+(r^{d/2}-\varepsilon)(r^{d/2-1}+\varepsilon)=(q^{de/2}-\varepsilon)(q^{de/2-1}+\varepsilon)$$

nonzero vectors x. Thus Q and Q_0 have simultaneously maximal or minimal Witt index.

Remark. If U is a totally isotropic subspace of V of dimension $\frac{1}{2}d$, then U_0 is totally isotropic of dimension $\frac{1}{2}de$ in V_0 so that Q_0 has maximal index when Q has. But it is not so easy to give a similar proof without counting when Q has minimal index.

If $x^{\perp}$ is a tangent hyperplane to Q in PV, then $x^{\perp 0}$ is a tangent hyperplane to Q_0 in PV_0. [*Note:* the converse does not hold.] [*Note:* PV is the projective space corresponding to V.]

After these preliminaries let us define $X=\{x\in PV_0 \mid Q_0(x)=0 \text{ and } Q(x)\neq 0\}$, where Q_0 is a nondegenerate quadratic form on V_0, and investigate $|X\cap H|$ for hyperplanes H in PV_0. Write $H=a^{\perp 0}$. First assume that d is even. Distinguish three cases.

(i) *$a^{\perp}$ is a tangent hyperplane.*

Now H is a tangent hyperplane, and $H\cap Q_0$ is a cone over a nondegenerate quadric in $de-2$ dimensions and hence contains $1+q(q^{de/2-1}-\varepsilon)(q^{de/2-2}+\varepsilon)/(q-1)$ projective points, i.e., $q-1+q(q^{de/2-1}-\varepsilon)(q^{de/2-2}-\varepsilon)=q^{de-2}-1+\varepsilon q^{de/2-1}(q-1)$ nonzero vectors.

Similarly $a^{\perp}\cap Q$ contains $r^{d-2}-1+\varepsilon r^{d/2-1}(r-1)=q^{de-2e}-1+\varepsilon q^{de/2-e}(q^e-1)$ nonzero vectors.

Since Q contains $q^{de-e}-1+\varepsilon q^{de/2-e}(q^e-1)$ nonzero vectors and each nonzero value of the inner product $B(a,\cdot)$ occurs equally often on $Q\setminus a^{\perp}$ we find that each nonzero value of $B(a,\cdot)$ is taken for q^{de-2e} vectors in $Q\setminus a^{\perp}$.

Now the number of nonzero vectors x with $Q(x)=0$ and $B_0(a,x)=0$ is

$$\begin{aligned} &q^{de-2e}-1+\varepsilon q^{de/2-e}(q^e-1)+(q^{e-1}-1)q^{de-2e}\\ &\quad =q^{de-e-1}-1+\varepsilon q^{de/2-e}(q^e-1). \end{aligned}$$

Finally

$$|X\cap H|=\frac{1}{q-1}(q^{e-1}-1)(q^{de-e-1}-\varepsilon q^{de/2-e}).$$

(ii) *$a^{\perp}$ is a secant hyperplane but H is tangent.*

We find the same value for $|H\cap Q_0|$ as before; this time $a^{\perp}\cap Q$ is a nondegenerate quadric in $d-1$ dimensions and hence contains $r^{d-2}-1$ nonzero vectors.

Each nonzero value of $B(a,\cdot)$ is taken for $q^{de-2e}+\varepsilon q^{de/2-e}$ vectors in $Q\setminus a^{\perp}$ so that $H\cap Q$ contains $q^{de-e-1}-1+\varepsilon q^{de/2-e}(q^{e-1}-1)$ nonzero vectors. Finally

$$|X\cap H|=\frac{1}{q-1}[q^{de-e-1}(q^{e-1}-1)+\varepsilon q^{de/2-e}(q^e-2q^{e-1}+1)].$$

(iii) *Both $a^{\perp}$ and H are secant.*

This time $H \cap Q_0$ contains $q^{de-2}-1$ nonzero vectors, $H \cap Q$ has the same size as under (ii), and

$$|X \cap H| = \frac{1}{q-1}(q^{e-1}-1)(q^{de-e-1}-\varepsilon q^{de/2-e}),$$

the same value as we found under (i).

Theorem. *Let d be even. X is a subset of size $(q^{e-1}-1)(q^{de-e}-\varepsilon q^{de/2-e})/(q-1)$ of PV_0 such that $|X \cap H|$ is either $(q^{e-1}-1)(q^{de-e-1}-\varepsilon q^{de/2-e})/(q-1)$ or*

$$[q^{de-e-1}(q^{e-1}-1)+\varepsilon q^{de-e}(q^{e}-2q^{e-1}+1)]/(q-1)$$

where the latter possibility occurs for precisely $|X|$ hyperplanes H.

The corresponding two-weight code over F_0 has word length $|X|$ and weights $w_0=0$, $w_1=(q^{e-1}-1)q^{de-e-1}$ and $w_2=(q^{e-1}-1)q^{de-e-1}-\varepsilon q^{de/2-1}$.

The corresponding strongly regular graph has $v=|V_0|=q^{de}$ vertices, valency $k=(q-1)|X|=(q^{e-1}-1)(q^{de-e}-\varepsilon q^{de/2-e})$ and eigenvalues $k-qw_i$ $(i=0,1,2)$.

Proof. We already saw the first part. For the connections with two-weight codes and strongly regular graphs see Calderbank & Kantor [2]. □

Comparison with known constructions

For $\varepsilon=+1$ the graphs constructed above have the parameters of Latin square graphs derived from OA(u,g), where

$$u=q^{de/2} \quad \text{and} \quad g=q^{de/2-e}(q^{e-1}-1).$$

Many constructions for graphs with Latin square parameters are known; I do not know whether the graphs constructed above are isomorphic to previously constructed ones.

For $\varepsilon=-1$ these graphs have 'negative Latin square' parameters. When $d=2$ these are known (not surprisingly: Q is empty, so $X=Q_0 \backslash Q=Q_0$) but for $d \geq 4$ they seem to be new. The smallest graph constructed here and not known before has parameters ($q=e=2$, $d=4$):

$$v=256, \quad k=68, \quad \lambda=12, \quad \mu=20, \quad r=4, \quad s=-12.$$

A cyclotomic description of this same graph can be given by taking $V=\mathrm{GF}(256)$, $Q(x)=x^{17}+x^{68}$, $X=\{\alpha^{15i+j} \mid 0 \leq i \leq 16,\ j=1,2,4,8\}$ where α is a primitive element of GF(256).

Case d odd

Similar computations when d is odd show that $|X \cap H|$ takes more than two distinct values here, so that this case is not interesting for our purpose.

References

[1] E. Artin, Geometric Algebra, Interscience Tracts in Pure and Applied Mathematics 3 (Interscience, New York, 1957).

[2] R. Calderbank and W.M. Kantor, The geometry of two-weight codes, Preprint, Bell Labs. (1982).

[3] Ph. Delsarte, Weights of linear codes and strongly regular normed spaces, Discrete Math. 3 (1972) 47–64.