COMBINATORICA
Bolyai Society – Springer-Verlag

Combinatorica 27pp.
DOI: 10.1007/s00493-016-3469-8

COUNT MATROIDS OF GROUP-LABELED GRAPHS*

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Received July 5, 2015
Revised June 29, 2016

A graph $G=(V,E)$ is called (k,ℓ)-sparse if $|F|\leq k|V(F)|-\ell$ for any nonempty $F\subseteq E$, where $V(F)$ denotes the set of vertices incident to F. It is known that the family of the edge sets of (k,ℓ)-sparse subgraphs forms the family of independent sets of a matroid, called the (k,ℓ)-count matroid of G. In this paper we shall investigate lifts of the (k,ℓ)-count matroids by using group labelings on the edge set. By introducing a new notion called near-balancedness, we shall identify a new class of matroids whose independence condition is described as a count condition of the form $|F|\leq k|V(F)|-\ell+\alpha_\psi(F)$ for some function α_ψ determined by a given group labeling ψ on E.

1. Count matroids

A *Γ-labeled graph* (G,ψ) is a pair of a directed graph $G=(V,E)$ and an assignment ψ of an element of a group Γ with each oriented edge. Although G is directed, its orientation is used only for the reference of the gains, and we are free to change the orientation of each edge by requiring that if an edge has a label g in one direction, then it has g^{-1} in the other direction. Therefore we often do not distinguish between G and the underlying undirected graph. By using the group-labeling one can define variants of graphic matroids. Among such variants, *Dowling geometries* [2], or their restrictions, *frame matroids* [18,19], are of most importance in the theory of matroid representations. In the frame matroid of (G,ψ), an edge set I is independent if and only if each connected component of I contains no cycle or just one

Mathematics Subject Classification (2000): 05B35, 52C25

* Part of this research was conducted when the first author was a graduate student at the University of Tokyo.

cycle which is *unbalanced*, i.e., the total gain through the cycle is not equal to the identity. By extending the notion of balancedness to any edge subsets such that $F \subseteq E$ is *unbalanced* (resp. *balanced*) if it contains (resp. does not contain) an unbalanced cycle, the independence condition in the frame matroid can be equivalently written as

$$|F| \leq |V(F)| - 1 + \begin{cases} 0 & \text{if } F \text{ is balanced} \\ 1 & \text{otherwise} \end{cases} \qquad (\emptyset \neq F \subseteq I), \tag{1}$$

where $V(F)$ denotes the set of vertices incident to F. Notice that, if we ignore the last term, this condition is nothing but the independence condition in the graphic matroid of G, and hence the count condition exhibits how the graphic matroid is lifted (see [17] for a discussion based on submodular functions).

There is a natural generalization of the count condition for cycle-freeness, known as (k, ℓ)-*sparsity*. We say that an edge set I is (k, ℓ)-*sparse* if $|F| \leq k|V(F)| - \ell$ holds for any nonempty $F \subseteq I$. It is known that the set of (k, ℓ)-sparse edge sets in G forms a matroid on E, called the (k, ℓ)-*count matroid* of G. For $k \geq \ell$, the (k, ℓ)-count matroids appear in several contexts in graph theory and combinatorial optimization as they are the unions of copies of the graphic matroid and the bicircular matroid (see, e.g., [4]), and in particular the (k, k)-sparsity condition is Nash-Williams' condition for a graph to be decomposed into k edge-disjoint forests. The (k, ℓ)-count matroids appear in rigidity theory and scene analysis for various kinds of pairs of k and ℓ (see, e.g., [16]).

Since the $(1, 1)$-count matroid coincides with the graphic matroid, it is natural to ask when a count condition of the form

$$|F| \leq k|V(F)| - \ell + \alpha_\psi(F) \qquad (\emptyset \neq F \subseteq I), \tag{2}$$

for some function α_ψ determined by the group labeling induces a matroid of (G, ψ). In this paper we shall establish a general construction of α_ψ for which the count condition induces a matroid. Our work is in fact motivated from characterizations of the rigidity of graphs with symmetry. Recent works on this subject reveal connections of the infinitesimal rigidity of symmetric bar-joint frameworks with count conditions of the form (2) on the quotient group-labeled graphs [9,10,12,15,7,11], where each symmetry and each rigidity model gives a distinct α_ψ. In Section 2 we give examples, several of which were not known to form matroids before. In this context it is crucial to know whether a necessary count condition forms a matroid or not (see, e.g., [9,10,15,7,11]).

Our construction uses more refined properties of group-labelings than balancedness. To explain this we need to introduce some notation. Let (G,ψ) be a Γ-labeled graph. The set of nonempty connected edge sets in G is denoted by $\mathcal{C}(G)$. A *walk* in G is a sequence $W=v_0,e_1,v_1,e_2,\dots,e_k,v_k$ of vertices and edges such that v_{i-1} and v_i are the endvertices of e_i for every $1\le i\le k$. The *gain* $\psi(W)$ of the walk W is defined to be $\psi(e_1)^{\sigma(e_1)}\cdot\psi(e_2)^{\sigma(e_2)}\cdot\dots\cdot\psi(e_k)^{\sigma(e_k)}$, where $\sigma(e)=1$ if W traces e in the forward direction and otherwise $\sigma(e)=-1$. For $F\in\mathcal{C}(G)$ and $v\in V(F)$ let $\langle F\rangle_{v,\psi}$ be the subgroup of Γ generated by $\psi(W)$ for all closed walks W starting at v and using only edges in F. It is known that $\langle F\rangle_{v,\psi}$ is conjugate to $\langle F\rangle_{u,\psi}$ for any $u,v\in V(F)$ (see, e.g., [7]). Hence the conjugate class is uniquely determined for each $F\in\mathcal{C}(G)$, which is denoted by $[F]$.

For a group Γ and $S\subseteq\Gamma$, let $\langle S\rangle$ be the subgroup generated by elements in S and let $[S]$ be the conjugate class of $\langle S\rangle$ in Γ. Also the identity of Γ is denoted by 1_Γ.

We say that a function $\alpha\colon 2^\Gamma\to\mathbb{Z}$ is *polymatroidal* if

(c1) $\alpha(\emptyset)=0$,
(c2) $\alpha(X)+\alpha(Y)\ge\alpha(X\cup Y)+\alpha(X\cap Y)$ for any $X,Y\subseteq\Gamma$,
(c3) $\alpha(X)\le\alpha(Y)$ for any $X\subseteq Y\subseteq\Gamma$,
(c4) $\alpha(\gamma X\gamma^{-1})=\alpha(X)$ for any $X\subseteq\Gamma$ and $\gamma\in\Gamma$,
(c5) $\alpha(\langle X\rangle)=\alpha(X)$ for any $X\subseteq\Gamma$.

Since α is closed under taking the closure and the conjugate, α induces a class function (i.e., a function on the conjugate classes), which is denoted by $\tilde\alpha$. For $F\in\mathcal{C}(G)$ we often abbreviate $\tilde\alpha([F])$ by $\tilde\alpha(F)$.

The following was proved in [15].

Theorem 1.1 (Tanigawa [15]). *Let (G,ψ) be a Γ-labeled graph, $\alpha\colon 2^\Gamma\to\{0,1,\dots,k\}$ be a polymatroidal function. Define $f_\alpha\colon\mathcal{C}(G)\to\mathbb{Z}$ by*

$$f_\alpha(F)=k|V(F)|-k+\tilde\alpha(F)\qquad(F\in\mathcal{C}(G)).$$

Then the set $\mathcal{I}_\alpha(G)=\{I\subseteq E(G)\mid |F|\le f_\alpha(F)\ \forall F\in\mathcal{C}(G)\cap 2^I\}$ forms the family of independent sets in a matroid.

In this paper we shall extend Theorem 1.1 for general ℓ. Interestingly, replacing just "$k|V(F)|-k$" with "$k|V(F)|-\ell$" in the definition of f_α may not produce a matroid in general as shown in Example 3 in the next section, and our extension is achieved by introducing a new notion, called near-balancedness. Let v be a vertex of (G,ψ) and $\{E_1,E_2\}$ be a bipartition of the set of non-loop edges incident to v. If v is not incident to a loop, then a *split* of (G,ψ) (at a vertex v with respect $\{E_1,E_2\}$) is defined to be a Γ-labeled graph (G',ψ') obtained from (G,ψ) by splitting v into two vertices

v_1 and v_2 such that v_i is incident to all the edges in E_i for $i=1,2$. If v is incident to a loop, then the split is defined to be a Γ-labeled graph (G',ψ') obtained from (G,ψ) by splitting v into two vertices v_1 and v_2 such that v_i is incident to the edges in E_i for $i=1,2$, each balanced loop at v is connected to v_1, and each unbalanced loop at v is regarded as an arc from v_1 to v_2, keeping the group-labeling[1], where a loop is called *balanced* (resp., *unbalanced*) if its label is identity (resp., non-indentity).

We say that a connected set F is *near-balanced* if it is not balanced and there is a split of (G,ψ) in which F results in a balanced set.

Example 1. We give an example of near-balanced sets using Figure 1. Let e_1 denote the edge from v_2 to v_3, and let e_2 and e_3 denote the edges from v_1 to v_2 with $\psi(e_2)=1_\Gamma$ and $\psi(e_3)=g\neq 1_\Gamma$, respectively. Consider $I_1=E(G)\setminus\{e_1\}$ and $I_2=E(G)\setminus\{e_2,e_3\}$ for example. Then I_1 is not near-balanced since it contains two vertex-disjoint unbalanced cycles, and I_2 is near-balanced since it is balanced in a split of (G,ψ) at v_3. See Figure 1(d). By the same reason $I_2\cup\{e_2\}$ is near-balanced. On the other hand the property of $I_2\cup\{e_3\}$ differs according to the order of g. In fact $I_2\cup\{e_3\}$ is near-balanced if and only if $g^2=1_\Gamma$.

We also remark that, for a polymatroidal function $\alpha\colon 2^\Gamma\to\{0,1,\dots,\ell\}$, there is a unique maximum set $S\subseteq\Gamma$ with $\alpha(S)=0$ and S actually forms a normal subgroup of Γ due to the submodularity and the invariance under conjugation. Hence, taking the quotient of Γ by S, throughout the paper we may assume that

(c6) $\alpha(\{g\})\neq 0$ for any non-identity $g\in\Gamma$ and $\alpha(\{1_\Gamma\})=0$.

A polymatroidal function α is said to be *normalized* if it satisfies (c6). Note that by (c6) we implicitly assume $\ell\geq 1$ when Γ is nontrivial.

Now we are ready to state our main theorem for $\ell\leq k+1$. The statement for k and ℓ with $\ell\leq 2k-1$ is given in Section 4.

Theorem 1.2. *Let k,ℓ be integers with $k\geq 1$ and $0\leq\ell\leq k+1$, (G,ψ) be a Γ-labeled graph, $\alpha\colon 2^\Gamma\to\{0,1,\dots,\ell\}$ be a normalized polymatroidal function such that $\alpha(\Gamma')\leq k$ for any $\Gamma'\subseteq\Gamma$ with $\Gamma'\simeq\mathbb{Z}_2$. Define $f_\alpha\colon\mathcal{C}(G)\to\mathbb{Z}$ by*

$$f_\alpha(F)=k|V(F)|-\ell+\begin{cases}\min\{\tilde\alpha(F),k\} & (\textit{if } F \textit{ is near-balanced})\\ \tilde\alpha(F) & (\textit{otherwise}).\end{cases}$$

Then the set $\mathcal{I}_\alpha(G)=\{I\subseteq E(G)\mid |F|\leq f_\alpha(F)\ \forall F\in\mathcal{C}(G)\cap 2^I\}$ forms the family of independent sets in a matroid.

[1] By definition of group-labeled graphs, the label of a loop is freely invertible. So, for an unbalanced loop e at v in (G,ψ), the label of the new edge corresponding to e in the split can be either $\psi(e)$ or $\psi(e)^{-1}$.

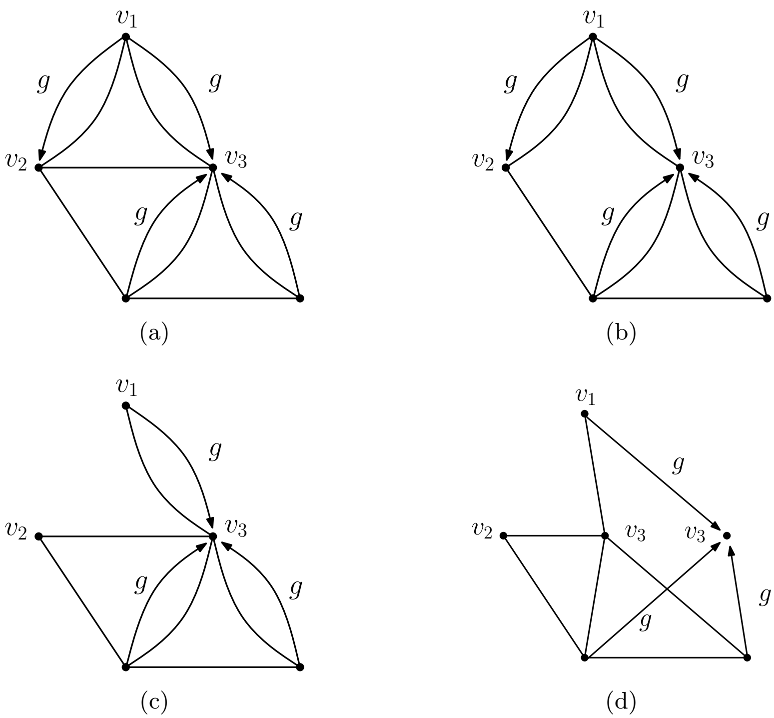

Figure 1. (a) An example of a Γ-labeled graph (G, ψ), where $g \in \Gamma$ is not the identity and every non-labeled edge has the identity label 1_Γ. (b) A non near-balanced edge set I_1, (c) a near-balanced edge set I_2, and (d) I_2 in a split of (G, ψ) at v_3.

Examples given in the next section show the necessity of the lifting value condition for near-balanced sets and the value condition for $\alpha(\mathbb{Z}_2)$ in Theorem 1.2.

2. Examples of matroids

Here we give examples of matroids given in Theorem 1.2.

Example 2. The union of two copies of the frame matroid followed by Dilworth truncation results in a matroid whose independence condition is written by the following count:

$$|F| \leq 2|V(F)| - 3 + \begin{cases} 0 & \text{if } F \text{ is balanced} \\ 2 & \text{otherwise} \end{cases} \qquad (F \in \mathcal{C}(G)).$$

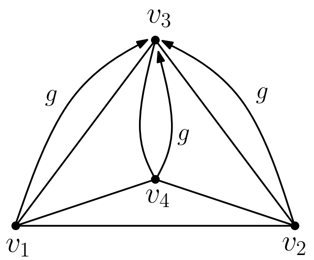

Figure 2. An example of a Γ-labeled graph (G, ψ) not being a matroid in the count condition in Example 3, where $g \in \Gamma$ is not the identity and every non-labeled edge has label 1_Γ. Let e_1 denote the edge from v_1 to v_2 and e_2 and e_3 denote the edges from v_1 to v_3 with $\psi(e_2)=1_\Gamma$ and $\psi(e_3)=g$, respectively. Then $E_1=E(G)\setminus\{e_1\}$ and $E_2=E(G)\setminus\{e_2,e_3\}$ are maximal edge sets satisfying the count condition with distinct cardinalities. Indeed, they are maximal because $E_1\cup\{e_1\}$ violates the $(2,0)$-sparsity while each of $E_2\cup\{e_2\}$ and $E_2\cup\{e_3\}$ contains a balanced K_4, which indicates the violation of the $(2,3)$-sparsity for balanced sets.

This is the case when $k=2, \ell=3$, and

$$\alpha(X) = \begin{cases} 0 & \langle X\rangle \text{ is trivial} \\ 2 & \text{otherwise} \end{cases} \qquad (X \subseteq \Gamma).$$

Example 3. In the context of graph rigidity, the following count condition appears as a necessary condition for the infinitesimal rigidity of symmetric bar-joint frameworks in the plane:

$$|F| \leq 2|V(F)| - 3 + \begin{cases} 0 & \text{if } F \text{ is balanced} \\ 3 & \text{otherwise} \end{cases} \qquad (F \in \mathcal{C}(G)).$$

The corresponding α is given by

$$\alpha(X) = \begin{cases} 0 & \langle X\rangle \text{ is trivial} \\ 3 & \text{otherwise} \end{cases} \qquad (X \subseteq \Gamma).$$

Csaba Király pointed out that this condition does not induce a matroid in general. In Figure 2 we give a smaller example for general groups.

Suppose that Γ does not contain an element of order two. Then Theorem 1.2 implies that adding one additional condition for near-balanced sets gives rise to a matroid. Its independence condition is written as

$$|F| \leq 2|V(F)| - 3 + \begin{cases} 0 & \text{if } F \text{ is balanced} \\ 2 & \text{if } F \text{ is near-balanced} \\ 3 & \text{otherwise} \end{cases} \qquad (F \in \mathcal{C}(G)).$$

This count condition still may not induce a matroid if Γ contains an element of order two. Consider the Γ-labeled graph in Figure 1, and define I_1 and I_2 as in Example 1. Suppose that $g^2 = 1_\Gamma$. Then I_1 and I_2 are maximal sets in $\mathcal{I}_\alpha(G)$. Indeed, by counting, it can easily be checked that $I_1, I_2 \in \mathcal{I}_\alpha(G)$. As for the maximality of I_2, observe that, for each $i = 2, 3$, $I_2 \cup \{e_i\}$ is a near-balanced edge set with $|I_2 \cup \{e_i\}| = 2|V(I_2 \cup \{e_i\})|$, which violates the $(2,1)$-sparsity condition for near-balanced sets. Since I_1 and I_2 have distinct cardinalities, $\mathcal{I}_\alpha(G)$ does not form the family of independent sets of a matroid. This example indicates the necessity of the assumption on the value of $\alpha(\mathbb{Z}_2)$ in Theorem 1.2.

Theorem 1.2 implies that, even if Γ contains an element of order two, the following condition induces a matroid:

$$|F| \leq 2|V(F)| - 3 + \begin{cases} 0 & \text{if } F \text{ is balanced} \\ 2 & \text{if } F \text{ is near-balanced, or } \langle F\rangle_{v,\psi} \simeq \mathbb{Z}_2 \text{ for some } v \in V(F) \\ 3 & \text{otherwise.} \end{cases}$$

Interestingly these additional conditions turn out to be necessary for the infinitesimal rigidity of symmetric bar-joint frameworks [14,6].

Example 4. The following count condition appears when analyzing the infinitesimal rigidity of frameworks with dihedral symmetry on the plane [7]:

$$|F| \leq 2|V(F)| - 3 + \begin{cases} 0 & \text{if } F \text{ is balanced} \\ 2 & \text{if } \langle F\rangle_{v,\psi} \text{ is nontrivial and cyclic for some } v \in V(F) \\ 3 & \text{otherwise} \end{cases}$$

($F \in \mathcal{C}(G)$). In [7] it was shown that the count induces a matroid when Γ is dihedral. The following lemma gives a condition for the corresponding α to be polymatroidal.

Lemma 2.1. *The function $\alpha \colon 2^\Gamma \to \mathbb{Z}$ defined by*

$$\alpha(X) = \begin{cases} 0 & \langle X\rangle \text{ is trivial} \\ 2 & \langle X\rangle \text{ is nontrivial and cyclic} \\ 3 & \text{otherwise} \end{cases} \qquad (X \subseteq \Gamma)$$

is polymatroidal if and only if for each element $g \in \Gamma \setminus \{1_\Gamma\}$ a maximal cyclic subgroup containing g is unique.

Proof. Note that α satisfies the monotonicity, the invariance under conjugation, and the invariance under taking the closure. We prove that α is submodular if and only if for each element $g \in \Gamma \setminus \{1_\Gamma\}$ a maximal cyclic subgroup containing g is unique.

Suppose a maximal cyclic group containing each element is unique. The submodularity can be checked as follows. Take any $X, Y \subseteq \Gamma$. If $\langle X \rangle$ or $\langle Y \rangle$ is not cyclic, the submodular inequality is trivial. If $\langle X \rangle$ and $\langle Y \rangle$ are nontrivial and cyclic, there are unique maximal cyclic subgroups Γ_X and Γ_Y containing X and Y, respectively. If $\Gamma_X \cap \Gamma_Y = \{1_\Gamma\}$, then $\alpha(X)+\alpha(Y)=4>3 \geq \alpha(X\cap Y)+\alpha(X\cup Y)$. If $\Gamma_X \cap \Gamma_Y \neq \{1_\Gamma\}$, then it is cyclic and there is a unique maximal cyclic subgroup containing $\Gamma_X \cap \Gamma_Y$. However, since Γ_X and Γ_Y are maximal, we have $\Gamma_X = \Gamma_Y$, implying $\alpha(X)+\alpha(Y)=\alpha(\Gamma_X)+\alpha(\Gamma_Y) \geq \alpha(X\cap Y)+\alpha(X\cup Y)$.

Conversely, if there is an element $g \in \Gamma$ that is contained in two distinct maximal cyclic subgroups Γ_1 and Γ_2. Then $\alpha(\Gamma_1 \cap \Gamma_2) \geq \alpha(\{g\}) \geq 2$ and $\alpha(\Gamma_1 \cup \Gamma_2) = 3$. Hence the submodularity does not hold. ∎

A dihedral group is an example satisfying this property while $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ is an example not having the property.

It was shown in [7] that the so-called symmetry-forced rigidity of 2-dimensional bar-joint frameworks with dihedral symmetry with order $2n$ for some odd n can be characterized in terms of this count condition (under a certain generic assumption).

Example 5. Let n, i be positive integers with $i < n$, and let

$$\begin{aligned}
S_0(n,i) &= \{n' \in \mathbb{Z} \colon 2 \leq n' \leq n, n' \text{ divides } n \text{ and } i\} \\
S_{-1}(n,i) &= \{n' \in \mathbb{Z} \colon 2 \leq n' \leq n, n' \text{ divides } n \text{ and } i-1\} \\
S_1(n,i) &= \{n' \in \mathbb{Z} \colon 2 \leq n' \leq n, n' \text{ divides } n \text{ and } i+1\} \\
S(n,i) &= \begin{cases} S_0(n,i) \cup S_{-1}(n,i) \cup S_1(n,i) & \text{if } i \text{ is even} \\ S_0(n,i) \cup S_{-1}(n,i) \cup S_1(n,i) \setminus \{2\} & \text{if } i \text{ is odd.} \end{cases}
\end{aligned}$$

Suppose that we have a $\mathbb{Z}_n$-labeled graph (G, ψ). The following count condition appears when analyzing the infinitesimal rigidity of frameworks with cyclic symmetry:

$$\begin{aligned}
&|F| \leq 2|V(F)| - 3 \\
&\qquad + \begin{cases} 0 & \text{if } F \text{ is balanced} \\ 1 & \text{if } i \text{ is odd and } \langle F \rangle_{v,\psi} \simeq \mathbb{Z}_2 \text{ for some } v \in V(F) \\ 2 & \text{if } \langle F \rangle_{v,\psi} \simeq \mathbb{Z}_k \text{ for some } k \in S(n,i), \text{ or } F \text{ is near-balanced} \\ 3 & \text{otherwise.} \end{cases}
\end{aligned}$$

This count indeed determines a matroid since the corresponding α is polymatroidal as shown below.

Lemma 2.2. *The function $\alpha\colon 2^{\mathbb{Z}_n} \to \mathbb{Z}$ defined by*

$$\alpha(X) = \begin{cases} 0 & \text{if } \langle X\rangle \text{ is trivial} \\ 1 & \text{if } i \text{ is odd and } \langle X\rangle \simeq \mathbb{Z}_2 \\ 2 & \text{if } \langle X\rangle \simeq \mathbb{Z}_k \text{ for some } k\in S(n,i) \\ 3 & \text{otherwise} \end{cases} \qquad (X\subseteq \mathbb{Z}_n)$$

is polymatroidal.

Proof. Only the submodularity of α is nontrivial. Take any $X,Y\subseteq\Gamma$. Since $\alpha(\langle X\rangle\cap\langle Y\rangle)+\alpha(\langle X\rangle\cup\langle Y\rangle)\geq\alpha(X\cap Y)+\alpha(X\cup Y)$, it suffices to consider the case when X and Y are subgroups of $\mathbb{Z}_n$. Let n_X and n_Y be positive integers dividing n such that $X\simeq\mathbb{Z}_{n_X}$ and $Y\simeq\mathbb{Z}_{n_Y}$, and let $g=\gcd(n_X,n_Y)$ and $l=\mathrm{lcm}(n_X,n_Y)$. Then we have $X\cap Y=\{0,\frac{n}{g},\dots,\frac{(g-1)n}{g}\}\simeq\mathbb{Z}_g$ and $\langle X\cup Y\rangle=\gcd(\frac{n}{n_X},\frac{n}{n_Y})\mathbb{Z}/n\mathbb{Z}=\frac{n}{l}\mathbb{Z}/n\mathbb{Z}\simeq\mathbb{Z}_l$, implying $\alpha(X\cap Y)+\alpha(X\cup Y)\leq\alpha(X\cap Y)+\alpha(\langle X\cup Y\rangle)=\alpha(\mathbb{Z}_g)+\alpha(\mathbb{Z}_l)$. Hence it suffices to show that

$$\alpha(\mathbb{Z}_{n_X})+\alpha(\mathbb{Z}_{n_Y})\geq\alpha(\mathbb{Z}_g)+\alpha(\mathbb{Z}_l). \tag{3}$$

Suppose that i is odd. If $n_X=1$, then $g=1$ and $l=n_Y$, implying (3). Also, if $n_X\notin S(n,i)\cup\{1,2\}$, then $l\notin S(n,i)\cup\{1,2\}$ and hence $\alpha(\mathbb{Z}_{n_X})=\alpha(\mathbb{Z}_l)=3$. Since $\alpha(\mathbb{Z}_{n_Y})\geq\alpha(\mathbb{Z}_g)$ always holds, we get (3). Therefore, we may suppose that $n_X,n_Y\in S(n,i)\cup\{2\}$.

If $n_X=n_Y=2$, then $g=l=2$, and hence (3) follows.

If $n_X\in S(n,i)$ and $n_Y=2$, then $g\leq 2$. When $g=1$, $\alpha(\mathbb{Z}_{n_X})+\alpha(\mathbb{Z}_{n_Y})=3\geq\alpha(\mathbb{Z}_l)=\alpha(\mathbb{Z}_g)+\alpha(\mathbb{Z}_l)$. When $g=2$, $l=n_X$ and $g=n_Y$ hold, and thus (3) holds.

Suppose finally that $n_X\in S(n,i)$ and $n_Y\in S(n,i)$. If $g\notin S(n,i)$, then $\alpha(\mathbb{Z}_{n_X})+\alpha(\mathbb{Z}_{n_Y})-\alpha(\mathbb{Z}_g)\geq 3\geq\alpha(\mathbb{Z}_l)$. On the other hand, if $g\in S(n,i)$, then $l\in S(n,i)$ holds, which implies (3). Indeed, if $n_X\in S_{j_X}(n,i)$ and $n_Y\in S_{j_Y}(n,i)$ for some $j_X,j_Y\in\{-1,0,1\}$, then j_X-j_Y is an integer multiple of g. Since $g>2$ by $g\in S(n,i)$, this implies $j_X=j_Y$, and hence $l\in S(n,i)$ holds as we claimed.

Suppose that i is even. We can do the same case analysis, and the only nontrivial case will be when $n_X,n_Y,g\in S(n,i)$. We again show $l\in S(n,i)$. Let j_X and j_Y be as above. Then j_X-j_Y is an integer multiple of g. Since $j_X=j_Y$ implies $l\in S(n,i)$, assume $j_X\neq j_Y$. Since $g>1$, we have $g=2$ and $j_Xj_Y=-1$. However, since i is even, $i+j_X$ and $i+j_Y$ are both odd. Since n_X

and n_Y divide $i+j_X$ and $i+j_Y$, respectively, g must be odd, contradicting $g=2$. Therefore, $j_X=j_Y$ always holds, and $l\in S(n,i)$ implies (3). ∎

It was shown in [6] that the infinitesimal rigidity of 2-dimensional bar-joint frameworks with cyclic symmetry of odd order n can be characterized in terms of these count conditions (under a certain generic assumption).

3. Near-balancedness

In this section we shall prepare notation and present several properties of near-balancedness.

Let $G=(V,E)$ be a connected graph. For $F\subseteq E(G)$ and $v\in V(F)$ let F_v be the set of edges in F incident to v, and let $G_F=(V(F),F)$. For $v\in V$, we denote by L_v the set of loops in G incident to v, and by L_v° the set of balanced loops incident to v. For a vertex v, the subgraph of $G-L_v$ induced by v and the vertex set of a connected component of $G-v$ is called a *fraction* of v. Note that if v is not a cut vertex, then $G-L_v$ is a fraction of v.

Let (G,ψ) be a Γ-labeled graph. For $v\in V(G)$ and $g\in\Gamma$, a *switching* at v with g is an operation that creates a new gain function ψ' from ψ as follows:

$$\psi'(e)=\begin{cases} g\cdot\psi(e)\cdot g^{-1} & \text{if } e \text{ is a loop incident with } v\\ g\cdot\psi(e) & \text{if } e \text{ is a non-loop edge and is directed from } v\\ \psi(e)\cdot g^{-1} & \text{if } e \text{ is a non-loop edge and is directed to } v\\ \psi(e) & \text{otherwise}\end{cases}$$

($e\in E(G)$). A gain function ψ is said to be *equivalent* to ψ' if ψ can be obtained from ψ' by a sequence of switchings. It is easy to see that $\langle F\rangle_{v,\psi}$ is conjugate to $\langle F\rangle_{v,\psi'}$ for any equivalent ψ and ψ'. (See, e.g., [5, Section 2.5.2].)

For a forest $F\subseteq E(G)$, a gain function ψ' is said to be *F-respecting* if $\psi'(e)=1_\Gamma$ for every $e\in F$. For any forest $F\subseteq E(G)$, there always exists an F-respecting gain function equivalent to ψ.

A frequently used fact in the subsequence discussion is that, for any $F\subseteq E(G)$ and $v\in V(F)$, $\langle F\rangle_{v,\psi'}=\langle\psi'(F)\rangle$ holds if ψ' is T-respecting for a spanning tree T of G_F, where $\psi'(F)=\{\psi'(e)\colon e\in F\}$ (see, e.g., [7, Section 2.2]). Hence $\tilde{\alpha}(F)=\alpha(\psi'(F))$.

We say that a Γ-labeled graph (G,ψ) is *near-balanced* if $E(G)$ is near-balanced. The following proposition gives an alternative definition for near-balancedness.

Proposition 3.1. *Let (G,ψ) be a connected and unbalanced Γ-labeled graph with $G=(V,E)$. Then (G,ψ) is near-balanced if and only if there are $v\in V$, $g\in\Gamma\setminus\{1_\Gamma\}$, $E'_v\subseteq E_v$, and an equivalent gain function ψ' such that, assuming that all edges incident to v are directed to v,*

- *$\psi'(e)=1_\Gamma$ for $e\in E\setminus E'_v$, and*
- *$\psi'(e)=g$ for $e\in E'_v$.*

Proof. Suppose that the split (H,ψ) of (G,ψ) at $v\in V$ with a partition $\{E_1,E_2\}$ of $E_v\setminus L_v$ results in a balanced graph. Let v_1 and v_2 be the new vertices after the split. If H is disconnected, then G can be obtained from H by identifying v_1 and v_2, and hence (G,ψ) turns out to be balanced, which is a contradiction. Hence H is connected.

Take a spanning tree T of G such that $T\setminus E_2$ is a maximal forest of $G-E_2$, and consider a T-respecting equivalent gain function ψ'. Note that (H,ψ') is still balanced. Let $\mathcal{G}_1$ be the family of fractions G' of v in (G,ψ') with $E_1\cap E(G')\neq\emptyset$, and let $E'_2=\{e\in E_2\cap E(G')\colon G'\in\mathcal{G}_1\}$. We show that ψ' satisfies the property of the statement for $E'_v:=E'_2\cup(L_v\setminus L^\circ_v)$.

The first condition of the statement can be checked as follows. Since T spans $V(H)-v_2$ in H and (H,ψ') is balanced, $\psi'(e)=1_\Gamma$ holds for every $e\in E\setminus(E_2\cup(L_v\setminus L^\circ_v))$. Also, for every $e\in E_2\setminus E'_2$, the fraction G' of v in (G,ψ') containing e satisfies $E_1\cap E(G')=\emptyset$ by $e\notin E'_2$. Hence $(E_2\cap E(G'))\cap T\neq\emptyset$ should hold as T is spanning. Since ψ' is T-respecting and (H,ψ') is balanced, we have $\psi'(e)=1_\Gamma$ for $e\in E_2\setminus E'_2$. Thus $\psi'(e)=1_\Gamma$ holds for every $e\in E\setminus E'_v$.

To see the second condition, we pick any $e\in E'_v$ and let $g=\psi'(e)$. Now, observe that for each $f\in E'_v\setminus\{e\}(=(E'_2\cup(L_v\setminus L^\circ_v))\setminus\{e\})$, H contains a closed walk starting at v_2 and consisting of e,f and edges in T. See Figure 3. This implies $\psi'(e)^{-1}\psi'(f)=1_\Gamma$, meaning $\psi'(f)=\psi'(e)=g$. Thus ψ' also satisfies the second condition.

Conversely, if there are $v\in V$, $g\in\Gamma\setminus\{1_\Gamma\}$, $E'_v\subseteq E_v$, and an equivalent gain function ψ' satisfying the statement, then we let $E_1=E_v\setminus(E'_v\cup L_v)$ and $E_2=E'_v\setminus L_v$. We consider the split of (G,ψ') at v with the partition $\{E_1,E_2\}$ of $E_v\setminus L_v$. Then the resulting graph is balanced. ∎

Suppose that (G,ψ) is near-balanced. Then there is a balanced split of (G,ψ) at $v\in V(G)$ with a partition $\{E_1,E_2\}$ of $E_v\setminus L_v$. This v is called a *base* for the near-balancedness and $E_2\cup(L_v\setminus L^\circ_v)$ (or $E_1\cup(L_v\setminus L^\circ_v)$) is called an *extra edge set*.

The proof of Proposition 3.1 also implies the following useful fact.

Proposition 3.2. *Let (G,ψ) be a connected near-balanced graph and let E' be an extra edge set for the near-balancedness. Suppose that ψ is T-*

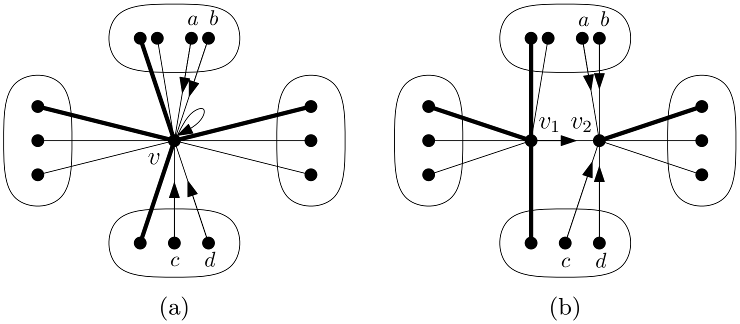

Figure 3. The proof of Proposition 3.1. (a) (G,ψ') and (b) its split (H,ψ') at v. Every unoriented edge has the identity label and $E_2'=\{av,bv,cv,dv\}$. The bold edges represent edges in T. Note that the fraction of v on the right side of v does not belong to $\mathcal{G}_1$.

respecting for some spanning tree $T\subseteq E$ with $T\cap E'=\emptyset$. Then ψ satisfies the following.

- *There is a nonidentity element $g\in\Gamma$ such that $\psi(e)=g$ for every $e\in E'$.*
- *$\psi(e)=1_\Gamma$ for $e\in E\setminus E'$.*

4. Main Theorem

Let k and ℓ be integers with $k\geq 1$ and $0\leq\ell\leq 2k-1$. Our main theorem given below is described under the following smoothness condition on a normalized polymatroidal function $\alpha\colon 2^\Gamma\to\{0,1,\dots,\ell\}$: for any $\emptyset\neq S\subseteq\Gamma$ and $g\in\Gamma$,

$$\alpha(S\cup\{g\})-\alpha(S)>k \quad\Rightarrow\quad S=\{1_\Gamma\} \text{ and } g^2\neq 1_\Gamma. \tag{4}$$

Since α is normalized, we have $\alpha(\{g\})>0$ for any non-identity $g\in\Gamma$. Hence, if $\ell\leq k+1$, then (4) is equivalent to

$$\alpha(\Gamma')\leq k \quad \text{for any subgroup } \Gamma'\subseteq\Gamma \text{ isomorphic to } \mathbb{Z}_2. \tag{5}$$

Now we are ready to state our main theorem.

Theorem 4.1. *Let k,ℓ be integers with $k\geq 1$ and $0\leq\ell\leq 2k-1$, (G,ψ) be a Γ-labeled graph, and $\alpha\colon 2^\Gamma\to\{0,1,\dots,\ell\}$ be a normalized polymatroidal function satisfying the smoothness condition* (4)*, and define $f_\alpha\colon\mathcal{C}(G)\to\mathbb{Z}$ by*

$$f_\alpha(F)=k|V(F)|-\ell+\begin{cases}\min\{\tilde\alpha(F),k\} & (\textit{if } F \textit{ is near-balanced})\\ \tilde\alpha(F) & (\textit{otherwise}).\end{cases}$$

Then the set $\mathcal{I}_\alpha(G)=\{I\subseteq E(G)\mid |F|\leq f_\alpha(F)\ \forall F\in\mathcal{C}(G)\cap 2^I\}$ forms the family of independent sets in a matroid.

The case when $\ell \leq k+1$ implies Theorem 1.2 due to the equivalence between (4) and (5).

Before moving to the proof, we give a remark on the technical difference between Theorem 4.1 and the previous work. In [15] the second author proved Theorem 1.1 (corresponding to the case for $\ell=k$) by showing that a set function $\hat{f}_\alpha\colon 2^E \to \mathbb{R}$ defined by

$$\hat{f}_\alpha(F) = \sum_{C:\ \text{connected component of } F} f_\alpha(C) \qquad (F \subseteq E)$$

is monotone submodular. Then the theorem immediately follows from Edmonds' theorem [3] on intersecting submodular functions. However, for $\ell > k$, $\hat{f}_\alpha$ may not be submodular in general and we do not know whether our main theorem (Theorem 4.1) is a consequence of a general theory of intersecting submodular functions. In [7] a special case (given in Example 4) was proved by directly checking the independence axiom, and here we will follow the same approach.

The main observation in the proof is Lemma 4.6, which asserts the submodular relation among sets that intersect "nicely". To prove this, we further investigate properties of near-balanced graphs in Subsection 4.1, and then we move to a proof of Theorem 4.1 in Subsection 4.2.

For simplicity of description, denote $\beta\colon \mathcal{C}(G) \to \mathbb{Z}$ by

$$\beta(F) = \begin{cases} \min\{\tilde{\alpha}(F), k\} & \text{(if } F \text{ is near-balaced)} \\ \tilde{\alpha}(F) & \text{(otherwise)} \end{cases} \qquad (F \in \mathcal{C}(G)).$$

We say that (G,ψ) is *f_α-sparse* if $|F| \leq f_\alpha(F)$ holds for every $F \in \mathcal{C}(G)$. A Γ-labeled graph (G,ψ) is called *f_α-tight* if it is connected f_α-sparse with $|E(G)| = f_\alpha(E(G))$. Also (G,ψ) is called *f_α-full* if it contains a connected f_α-sparse subgraph G' such that

- G' is spanning, i.e., $V(G') = V(G)$,
- $\beta(E(G')) = \beta(E(G))$, and
- $|E(G')| \geq k|V(G')| - \ell + \min\{\beta(E(G')), 2k - \ell + 1\}$.

Note that any f_α-tight graph is f_α-full. An edge set F is called f_α-sparse, f_α-tight, and f_α-full, respectively, if so is the induced subgraph G_F.

4.1. Further properties of near-balancedness

Assuming f_α-fullness, near-balanced graphs have further nice properties. In the subsequent discussion, α always denotes a normalized polymatroidal function.

Lemma 4.2. *Suppose that* (G,ψ) *is near-balanced and* f_α*-full with* $\beta(E(G))\geq 2k-\ell+1$*. Then a base for the near-balancedness is unique.*

Proof. By definition, (G,ψ) contains a spanning connected f_α-sparse subgraph (G',ψ) with

$$|E(G')| \geq k|V(G)| - 2\ell + 2k + 1 \tag{6}$$

and $\beta(E(G'))=\beta(E(G))$. Note that (G',ψ) is also near-balanced, since otherwise (G',ψ) would be balanced and $0=\beta(E(G'))=\beta(E(G))=\tilde{\alpha}(E(G))$, contradicting that (G,ψ) is unbalanced. Thus, it suffices to show the uniqueness of the base for (G',ψ). Let $E'=E(G')$.

Suppose that there are two distinct base vertices u and v for the near-balancedness of (G',ψ). Clearly, G' cannot contain an unbalanced loop since otherwise, say if u is incident to an unbalanced loop, then any split at v cannot be balanced. Without loss of generality, assume that all edges incident to v are directed to v. By Proposition 3.1 there are $g\in\Gamma\setminus\{1_\Gamma\}$, $F_v\subseteq E'_v$, and an equivalent gain function ψ' such that

$$\psi'(e) = g \text{ for } e \in F_v \text{ and } \psi'(e) = 1_\Gamma \text{ for } e \in E' \setminus F_v. \tag{7}$$

Note also that

$$F_v \neq \emptyset \text{ and } E'_v \setminus F_v \neq \emptyset, \tag{8}$$

since otherwise G would be balanced.

Let K be the union of the edge sets of all simple walks W in G' starting at v with the following property:

$$\begin{gathered}\psi'(W) = g^{-1} \text{ and } W \text{ does not contain } u \text{ as an internal node} \\ \text{(but may be the last).}\end{gathered} \tag{9}$$

By (7), $F_v\subseteq K$ and $E'_v\setminus F_v\subseteq E'\setminus K$. Hence, (7) again implies that K and $E'\setminus K$ are balanced. Since they are also nonempty by (8), we get

$$|K| \leq k|V(K)| - \ell \text{ and } |E' \setminus K| \leq k|V(E' \setminus K)| - \ell \tag{10}$$

by f_α-sparsity. We also claim that

$$V(K) \cap V(E' \setminus K) \subseteq \{u, v\}. \tag{11}$$

To see this, suppose that there is a vertex $w\in V(K)\cap V(E'\setminus K)$ other than u and v, and let e' be an edge of $E'\setminus K$ incident to w. Then the other endvertex of e' should be v since otherwise there would be a simple walk passing e' and satisfying (9). However, by $w\in V(K)$, the concatenation of e'

and a simple path from v to w with gain g^{-1} is an unbalanced cycle which does not pass through u, contradicting that a split of (G,ψ) at u results in a balanced graph. Hence (11) holds. Combining (10) and (11), we get $|E'|=|K|+|E'\setminus K|\leq k|V(K)|+k|V(E'\setminus K)|-2\ell\leq k|V(E')|-2\ell+2k=k|V(G)|-2\ell+2k$, which contradicts (6). ∎

Lemma 4.3. *Suppose that (G,ψ) is near-balanced and f_α-full with $\beta(E(G))\geq 2k-\ell+1$. Then each fraction of a base v is near-balanced. In particular, for each extra edge set K of the near-balancedness, $G-K$ is connected.*

Proof. It suffices to show that each fraction S of v is unbalanced. Suppose that S is balanced. By definition, (G,ψ) contains a spanning f_α-sparse subgraph (G',ψ) with $|E(G')|\geq k|V(G)|-2\ell+2k+1$. Then

$$\begin{aligned}|E(G')| &= |E(G')\setminus E(S)|+|E(G')\cap E(S)|\\ &\leq k|(V(G')\setminus V(S))\cup\{v\}|-\ell+k+k|V(G')\cap V(S)|-\ell\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(by } f_\alpha\text{-sparsity)}\\ &= k|V(G)|-2\ell+2k, \qquad\qquad \text{(since } S \text{ is a fraction)},\end{aligned}$$

which is a contradiction. Hence S is unbalanced. ∎

A Γ-labeled graph (G,ψ) (resp. an edge set E) is called *α-critical* if it is connected and near-balanced with $\tilde{\alpha}(E(G))>k$. If (G,ψ) is α-critical, then $\ell\geq\tilde{\alpha}(E(G))>k$ and hence $\beta(E(G))=k>2k-\ell$ follows. This in turn implies that an α-critical graph always satisfies the assumption for $\beta(E(G))$ in Lemmas 4.2 and 4.3.

The following lemma (Lemma 4.4) says that, for an α-critical graph, even an extra edge set for the near-balancedness is uniquely determined (up to complementation of non-loop edges).

Lemma 4.4. *Suppose that (G,ψ) is α-critical and f_α-full, and let v be the base. If there are two distinct extra edge sets E_1 and E_2 for the near-balancedness, then $\{E_1\setminus L_v, E_2\setminus L_v\}$ is a partition of $E_v\setminus L_v$.*

Proof. By Lemma 4.3, $G-E_1$ is connected and hence G contains a spanning tree T with $T\cap E_1=\emptyset$. We may assume that ψ is T-respecting. Then there is an element $g\in\Gamma\setminus\{1_\Gamma\}$ such that $\psi(e)=g$ for $e\in E_1$ and $\psi(e)=1_\Gamma$ for $e\in E\setminus E_1$ by Proposition 3.2.

Let S be a fraction of v. By Lemma 4.3, $\emptyset\neq E_i\cap E(S)\neq E_v\cap E(S)$ for $i=1,2$. Since $S-v$ is connected, if $E_2\cap E(S)$ contains an edge with label g and an edge with label 1_Γ, then the split of (S,ψ) at v with the partition of $\{E_2\cap E(S),(E_v\cap E(S))\setminus E_2\}$ contains an unbalanced cycle, which contradicts

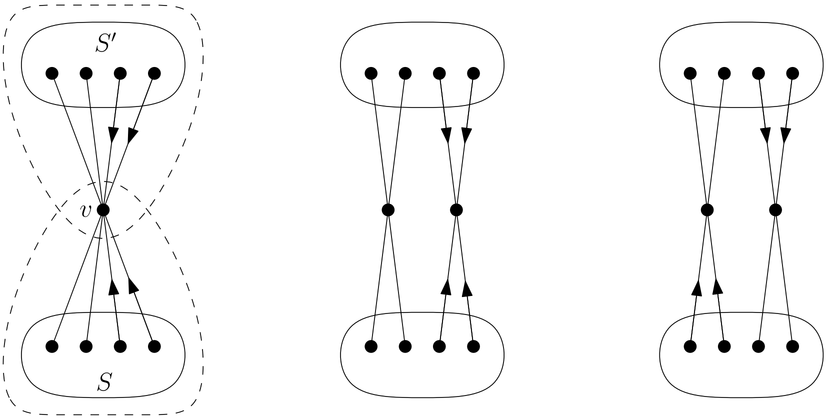

Figure 4. Proof of Lemma 4.4. (a) $(S\cup S',\psi)$, (b) the splitting at v with the partition $\{E_1\cap E(S\cup S'),(E_v\cap E(S\cup S'))\setminus E_1\}$, and (c) the splitting at v with the partition $\{E_2\cap E(S\cup S'),(E_v\cap E(S\cup S'))\setminus E_2\}$, where the oriented edges have the label g and other edges have the identity label.

that the split is balanced. Similarly, $(E_v\cap E(S))\setminus E_2$ cannot contain an edge with label g and an edge with label 1_Γ simultaneously. These imply that

(i) $E_1\cap E(S)=E_2\cap E(S)$, or
(ii) $\{E_1\cap E(S),E_2\cap E(S)\}$ is a partition of $E_v\cap E(S)$

for each fraction S of v.

Since $E_1\neq E_2$, there is a fraction S of v satisfying (ii). If there is another fraction S' of v satisfying (i), then the split of $(S\cup S',\psi)$ at v with the partition $\{E_2\cap E(S\cup S'),(E_v\cap E(S\cup S'))\setminus E_2\}$ contains a closed walk with gain g^2. See Figure 4. Thus $g^2=1_\Gamma$. However, since G is α-critical, $\tilde{\alpha}(E)=\alpha(\{g\})>k$. This contradicts the smoothness assumption (4). Therefore each fraction satisfies (ii), and $\{E_1\setminus L_v,E_2\setminus L_v\}$ is a partition of $E_v\setminus L_v$. ∎

We also remark the following easy lemma.

Lemma 4.5. *Suppose that (G,ψ) is α-critical. Then any connected subgraph of (G,ψ) is either α-critical or balanced.*

Proof. An α-critical graph (G,ψ) is near-balanced, and hence by Proposition 3.1 there are $v\in V$, $g\in\Gamma\setminus\{1_\Gamma\}$, $E'_v\subseteq E_v$, and an equivalent gain function ψ' such that, assuming that all edges incident to v are directed to v, $\psi'(e)=g$ for $e\in E'_v$ and $\psi'(e)=1_\Gamma$ for $e\in E\setminus E'_v$. Note that $\alpha(\{g\})=\tilde{\alpha}(E(G))>k$.

If a connected subgraph G' is not balanced, then it contains a closed walk of gain g. Thus $\tilde{\alpha}(E(G'))>k$. Clearly G' is near-balanced, and hence it is α-critical. ∎

4.2. Proof of Theorem 4.1

The proof of Theorem 4.1 follows from Lemma 4.7 and Lemma 4.9, which are analogs of well-known properties of (k,ℓ)-sparse graphs. The core of the proofs of those two lemmas is the following hidden submodularity of β.

Lemma 4.6. *Suppose that $X,Y \in \mathcal{C}(G)$ are f_α-full sets such that*

- *$(V(X)\cap V(Y), X\cap Y)$ is connected,*
- *$X\cap Y$ is f_α-sparse, and*
- *$|X\cap Y| > k|V(X\cap Y)| - 2\ell + \min\{2k, \beta(X)+\beta(Y)\}$.*

Then $\beta(X)+\beta(Y) \geq \beta(X\cap Y)+\beta(X\cup Y)$.

Proof. Since $(V(X)\cap V(Y), X\cap Y)$ is connected, $G_{X\cap Y} = (V(X)\cap V(Y), X\cap Y)$ holds, and there is a spanning tree $T\subseteq X\cup Y$ of $G_{X\cup Y}$ such that $T\cap X, T\cap Y$, and $T\cap X\cap Y$ are spanning trees of G_X, G_Y, and $G_{X\cap Y}$, respectively. We may assume that ψ is T-respecting. Then we have

$$\begin{aligned}
\tilde{\alpha}(X)+\tilde{\alpha}(Y) &= \alpha(\psi(X))+\alpha(\psi(Y)) && \text{(by (c5))}\\
&\geq \alpha(\psi(X)\cap\psi(Y))+\alpha(\psi(X)\cup\psi(Y)) && \text{(by (c2))}\\
&\geq \alpha(\psi(X\cap Y))+\alpha(\psi(X\cup Y)) && \text{(by (c3))}\\
&= \tilde{\alpha}(X\cap Y)+\tilde{\alpha}(X\cup Y) && \text{(by (c5))}. \qquad (12)
\end{aligned}$$

We split the proof into three cases.

(Case 1.) Suppose that neither X nor Y are α-critical. Then by (12) we have $\beta(X)+\beta(Y) = \tilde{\alpha}(X)+\tilde{\alpha}(Y) \geq \tilde{\alpha}(X\cap Y)+\tilde{\alpha}(X\cup Y) = \beta(X\cap Y)+\beta(X\cup Y)$.

(Case 2.) Suppose that X is α-critical but Y is not α-critical. Let v be the base and X'_v be an extra edge set for the near-balancedness of X. Also let Z be the set of all non-loop edges of $X\cap Y$ incident to v. Since $((X_v\setminus X'_v)\cup L_v)\setminus L^\circ_v$ is an extra edge set of X, we can always take X'_v such that

$$Z\setminus X'_v \neq \emptyset \text{ if } Z\neq\emptyset. \qquad (13)$$

We first show

$$G_{X\cap Y} - X'_v \text{ is connected.} \qquad (14)$$

Suppose not. Then v is in $G_{X\cap Y}$ and there is a fraction of v in $G_{X\cap Y}$ which is balanced. Let C be the edge set of such a fraction. Since v is in $G_{X\cap Y}$, Z

is nonempty. Therefore by (13) $\emptyset \neq Z \setminus X'_v \subseteq (X \cap Y) \setminus C$. Hence, both C and $(X \cap Y) \setminus C$ are nonempty and connected, and we get

$$
\begin{aligned}
|X \cap Y| &= |C| + |(X \cap Y) \setminus C| \\
&\leq f_\alpha(C) + f_\alpha((X \cap Y) \setminus C) && \text{(by the } f_\alpha\text{-sparsity)} \\
&\leq k|V(C)| + k|V((X \cap Y) \setminus C)| - 2\ell + \beta((X \cap Y) \setminus C) \\
& && \text{(since } C \text{ is balanced)} \\
&\leq k|V(X \cap Y)| - 2\ell + k + \beta((X \cap Y) \setminus C) && \text{(since } C \text{ is a fraction)} \\
&\leq k|V(X \cap Y)| - 2\ell + \min\{2k, \beta(X) + \beta(Y)\},
\end{aligned}
$$

where the last inequality follows from $\beta((X \cap Y) \setminus C) \leq \min\{\beta(X), \beta(Y)\} = \min\{k, \beta(Y)\}$. This upper bound of $|X \cap Y|$ contradicts the lemma assumption, and (14) follows.

By (14), we can take the above spanning tree T such that $T \cap X'_v = \emptyset$. Then by Proposition 3.2 there is an element $g \in \Gamma \setminus \{1_\Gamma\}$ such that

$$\psi(e) = g \text{ for every } e \in X'_v \text{ and } \psi(e) = 1_\Gamma \text{ for every } e \in X \setminus X'_v. \tag{15}$$

If $X'_v \cap Y \neq \emptyset$, then $g \in \psi(Y)$ by (15), and hence $\alpha(\psi(Y)) = \alpha(\psi(Y) \cup \{g\}) = \alpha(\psi(X \cup Y))$. Therefore, we have

$$
\begin{aligned}
\beta(X) + \beta(Y) &= \beta(X) + \alpha(\psi(Y)) \\
&= \beta(X) + \alpha(\psi(X \cup Y)) \geq \beta(X \cap Y) + \beta(X \cup Y),
\end{aligned}
$$

where the first equation follows since Y is not α-critical and the third inequality follows due to the definition of β.

On the other hand, if $X'_v \cap Y = \emptyset$, then $X \cap Y$ is balanced since $\psi(e) = 1_\Gamma$ for every $e \in X \cap Y$ by (15). If Y is also balanced, then $\psi(e) = 1_\Gamma$ for every $e \in Y$, which means that $X \cup Y$ is α-critical by Proposition 3.1. Thus

$$\beta(X) + \beta(Y) = \beta(X) = k = \beta(X \cup Y) = \beta(X \cup Y) + \beta(X \cap Y).$$

If Y is unbalanced, then

$$\beta(X \cup Y) \leq \tilde{\alpha}(X \cup Y) = \alpha(\psi(X \cup Y)) = \alpha(\psi(Y) \cup \{g\}),$$

and we get

$$\beta(X \cup Y) - \beta(Y) \leq \alpha(\psi(Y) \cup \{g\}) - \alpha(\psi(Y)) \leq k,$$

where the first inequality follows since Y is not α-critical and the last inequality follows from (4). Therefore,

$$\beta(X) + \beta(Y) = k + \beta(Y) \geq \beta(X \cup Y) = \beta(X \cap Y) + \beta(X \cup Y),$$

where the last equality follows since $X \cap Y$ is balanced.

(Case 3.) Suppose that both X and Y are α-critical. If $X \cap Y$ is not α-critical, then $X \cap Y$ is balanced by Lemma 4.5. Since $\beta(X \cup Y) \leq \ell$ and $\beta(Y) = k$, we get

$$\beta(X) - \beta(X \cap Y) = k > \ell - k \geq \beta(X \cup Y) - \beta(Y),$$

as required. Hence we may assume that $X \cap Y$ is α-critical. Also, by the cardinality assumption for $X \cap Y$ with $\beta(X) + \beta(Y) = 2k$, we have that $X \cap Y$ is an f_α-sparse set with $|X \cap Y| \geq k|V(X \cap Y)| - 2\ell + 2k + 1$. Hence $X \cap Y$ is f_α-full. Therefore, by Lemma 4.2, there is a unique base v for the near-balancedness of $X \cap Y$. Now let $F_X \subseteq X$ and $F_Y \subseteq Y$ be extra edge sets for the near-balancedness of X and the near-balancedness of Y, respectively. Then $F_X \cap X \cap Y$ and $F_Y \cap X \cap Y$ are extra edge sets for the near-balancedness for $X \cap Y$. However, the extra edge set is uniquely determined (up to complementation of non-loop edges) by Lemma 4.4, and hence we may assume that F_Y is taken so that $F_X \cap X \cap Y = F_Y \cap X \cap Y$. Moreover, since $X \cap Y$ has a unique base, the bases of X, Y and $X \cap Y$ coincide.

By Lemma 4.3, $G_X - F_X$, $G_Y - F_Y$, and $G_{X \cap Y} - F_X - F_Y$ are connected, and by $F_X \cap X \cap Y = F_Y \cap X \cap Y$ we can take the above spanning tree T of $G_{X \cup Y}$ such that $T \cap F_X = \emptyset$ and $T \cap F_Y = \emptyset$. By Proposition 3.2, we get $\psi(e) = g$ for $e \in F_X \cup F_Y$ and $\psi(e) = 1_\Gamma$ for $e \notin F_X \cup F_Y$. Therefore by Proposition 3.1 $X \cup Y$ is near-balanced, and moreover it is α-critical by $\tilde{\alpha}(X \cup Y) = \alpha(\psi(X \cup Y)) = \alpha(\{g\}) > k$. Therefore, we get $\beta(X) + \beta(Y) = 2k = \beta(X \cup Y) + \beta(X \cap Y)$. This completes the proof. ∎

For $F \subseteq E(G)$, let $d_F = k|V(F)| - |F|$. Note that, if G is f_α-sparse, then $d_F \geq \ell - \beta(F) \geq 0$ for every $F \in \mathcal{C}(G)$.

Lemma 4.7. *Suppose that (G, ψ) is f_α-sparse. Then, for any f_α-tight sets $X, Y \in \mathcal{C}(G)$ with $X \cap Y \neq \emptyset$, $X \cup Y$ is f_α-tight.*

Proof. Since (G, ψ) is f_α-sparse, we have $d_{X \cup Y} \geq \ell - \beta(X \cup Y)$, and what we have to prove is $d_{X \cup Y} \leq \ell - \beta(X \cup Y)$. In particular, if $d_{X \cup Y} \leq 0$ holds, then we can conclude that $X \cup Y$ is f_α-tight.

Let $G_1 = (V(X) \cap V(Y), X \cap Y)$. Let c_0 and c_1 be the numbers of trivial and non-trivial connected components in G_1, where a connected component is said to be trivial if it consists of a single vertex without a loop. Without loss of generality we assume $\beta(X) \geq \beta(Y)$. Due to the monotonicity of β, we have $\beta(Y) \geq \beta(F)$ for each edge set F of the connected component of G_1.

Hence

$$
\begin{aligned}
d_{X\cup Y} &= k|V(X\cup Y)| - |X\cup Y| \\
&= k(|V(X)| + |V(Y)| - |V(X)\cap V(Y)|) - (|X| + |Y| - |X\cap Y|) \\
&= d_X + d_Y - kc_0 - d_{X\cap Y} && (16)\\
&= 2\ell - \beta(X) - \beta(Y) - kc_0 - d_{X\cap Y} && (17)\\
&\le 2\ell - \beta(X) - \beta(Y) - kc_0 - (\ell - \beta(Y))c_1 && (18)\\
&= \ell - \beta(X) - kc_0 - (\ell - \beta(Y))(c_1 - 1). && (19)
\end{aligned}
$$

We first remark the following.

Claim 4.8. *If $d_{X\cup Y} > 0$, then $|X\cap Y| > k|V(X\cap Y)| - 2\ell + \beta(X) + \beta(Y)$, $c_0 \le 1$, and $c_1 = 1$ hold.*

Proof. If $|X\cap Y| \le k|V(X\cap Y)| - 2\ell + \beta(X) + \beta(Y)$, then $d_{X\cap Y} = k|V(X\cap Y)| - |X\cap Y| \ge 2\ell - \beta(X) - \beta(Y)$. Combining this with (17), we get $d_{X\cup Y} \le -kc_0 \le 0$.

If $c_1 \ge 2$, then we have $d_{X\cup Y} \le 0$ by (19).

If $c_1 \le 1$, then $c_1 = 1$ holds by $X\cap Y \ne \emptyset$. Now (19) implies $0 \le d_{X\cup Y} \le \ell - \beta(X) - kc_0$, and hence $kc_0 \le \ell \le 2k-1$. Therefore $c_0 \le 1$. ∎

As remarked at the beginning of the proof, $d_{X\cup Y} \le 0$ immediately implies the f_α-tightness of $X\cup Y$. Therefore, we may assume $d_{X\cup Y} > 0$, and by Claim 4.8 we have $c_0 \le 1$, $c_1 = 1$, and

$$
|X\cap Y| > k|V(X\cap Y)| - 2\ell + \min\{2k, \beta(X) + \beta(Y)\}. \tag{20}
$$

By $c_1 = 1$, $X\cap Y$ is connected. We split the proof into two cases depending on the value of (c_0, c_1).

(Case 1.) Suppose that $(c_0, c_1) = (0, 1)$. By (16), we have

$$
\begin{aligned}
\ell - \beta(X\cup Y) \le d_{X\cup Y} &= d_X + d_Y - d_{X\cap Y} \\
&\le \ell - \beta(X) - \beta(Y) + \beta(X\cap Y).
\end{aligned} \tag{21}
$$

By $(c_0, c_1) = (0, 1)$ and (20), we can apply Lemma 4.6 to get $\beta(X) + \beta(Y) \ge \beta(X\cap Y) + \beta(X\cup Y)$. This means that each inequality holds with equality in (21), and in particular we get $d_{X\cup Y} = \ell - \beta(X\cup Y)$. In other words, $X\cup Y$ is f_α-tight.

(Case 2.) Suppose that $(c_0, c_1) = (1, 1)$. By (16), we have

$$
\begin{aligned}
\ell - \beta(X\cup Y) \le d_{X\cup Y} &\le d_X + d_Y - d_{X\cap Y} - k \\
&\le \ell - \beta(X) - \beta(Y) + \beta(X\cap Y) - k.
\end{aligned} \tag{22}
$$

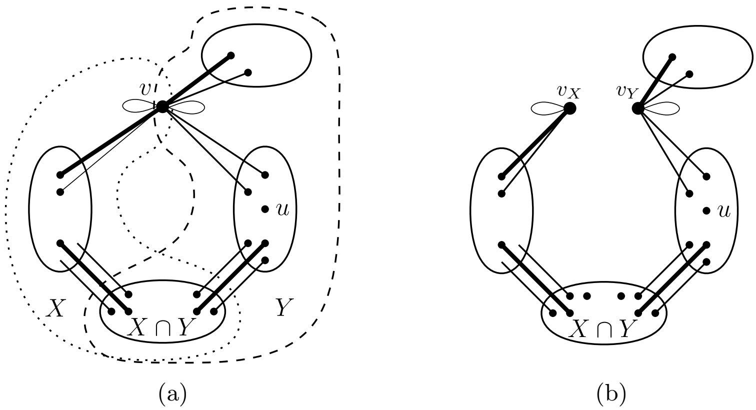

Figure 5. Proof of Lemma 4.7. (a) $G_{X\cup Y}$, where G_X is the dotted region and G_Y is the dashed region. The bold edges represent edges in T. (b) (H, ψ').

Hence, to prove $d_{X\cup Y} = \ell - \beta(X \cup Y)$, it suffices to show that

$$\beta(X) + \beta(Y) \geq \beta(X \cup Y) + \beta(X \cap Y) - k. \tag{23}$$

Let v be the vertex isolated in G_1, and assume that all edges in G incident to v are directed to v. Since $(c_0, c_1) = (1,1)$, there is a unique fraction of v in G_Y whose edge set intersects X. See Figure 5(a), and denote the edge set of the fraction by Y'.

We take a spanning tree T of $G_{X\cup Y}$ such that $T \cap X \cap Y$ is a spanning tree of $G_{X\cap Y}$, $T \cap X$ is a spanning tree of G_X, and $T \cap Y'_v = \emptyset$. Let ψ' be a T-respecting equivalent gain function and let $\mathit{\Gamma}_Y = \langle Y \rangle_{u,\psi'}$ for some $u \in V(Y') \setminus \{v\}$. Take an edge $e \in Y'_v$ and let $g = \psi'(e)$. For each $f' \in Y'_v$, there is a closed walk in $(T \cap Y) \cup \{e, f'\}$ starting at u and passing through e and f' consecutively. The gain of this walk is $\psi'(e)\psi'(f')^{-1}$, and hence $\psi'(e)\psi'(f')^{-1} \in \mathit{\Gamma}_Y$. This implies

$$\psi'(f') \in \mathit{\Gamma}_Y g \text{ for each } f' \in Y'_v. \tag{24}$$

On the other hand, for $f \in Y \setminus (Y' \cup T)$, there is a closed walk in $(T \cap Y) \cup \{e, f\}$ starting at u and passing through e, f, and then e (in the reversed direction for the last e). Its gain is $g\psi'(f)g^{-1}$, and we get

$$\psi'(f) \in g^{-1}\mathit{\Gamma}_Y g \text{ for each } f \in Y \setminus Y'_v. \tag{25}$$

Also, since $X \cup Y$ contains a cycle with gain g, we have

$$\langle X \cup Y \rangle_{u,\psi'} = \langle \psi'(X) \cup \mathit{\Gamma}_Y \cup \{g\} \rangle \text{ in } (G, \psi'). \tag{26}$$

Now to see (23), we consider (H,ψ') obtained from $(G_{X\cup Y},\psi')$ by splitting v into two vertices v_X and v_Y such that all edges in X_v are incident to v_X and those in Y_v are incident to v_Y. Then $V(X\cap Y)=V(X)\cap V(Y)$ in the resulting graph (Figure 5(b)), and by Lemma 4.6 with (20) we have $\beta(X)+\beta(Y)\geq\beta(X\cap Y)+\beta(X\cup Y)$ in (H,ψ'). We now identify the two split vertices of H to get back $G_{X\cup Y}$. Then $\beta(X\cup Y)$ may increase, but we claim that the amount of the increase is bounded by k. To see this, observe that $\langle X\cup Y\rangle_{u,\psi'}=\langle\psi'(X)\cup\Gamma_Y\rangle$ in (H,ψ') by (24) and (25). On the other hand, by (26), $\langle X\cup Y\rangle_{u,\psi'}=\langle\psi'(X)\cup\Gamma_Y\cup\{g\}\rangle$ in $(G_{X\cup Y},\psi')$. Therefore, if $\alpha(X\cup Y)$ changes by more than k (i.e., $\alpha(\psi'(X)\cup\Gamma_Y\cup\{g\})-\alpha(\psi'(X)\cup\Gamma_Y)>k$), then $\psi'(X)\cup\Gamma_Y=\{1_\Gamma\}$ by (4). This means that $X\cup Y$ is near-balanced in (G,ψ'), and $\beta(X\cup Y)$ is bounded by k after the identification. Hence the increase of the β-value is bounded by k when identifying the split vertices, and we obtain (23). ∎

Lemma 4.9. *Let $X\in\mathcal{C}(G)$ be an f_α-tight set, $Y\in\mathcal{C}(G)$ be an f_α-full set, and $e\in E(G)\setminus Y$. Suppose that $X\subseteq Y$, $X+e\in\mathcal{C}(G)$, and $f_\alpha(X+e)=f_\alpha(X)$. Then $f_\alpha(Y+e)=f_\alpha(Y)$. Moreover $Y+e$ is f_α-full.*

Proof. Since $f_\alpha(X+e)=f_\alpha(X)$, it can be easily checked that both endvertices of e are contained in $V(X)$ and $\beta(X)=\beta(X+e)$. Thus $|V(Y+e)|=|V(Y)|$, and for $f_\alpha(Y+e)=f_\alpha(Y)$ it suffices to show that $\beta(Y+e)=\beta(Y)$. This is trivial if $\beta(Y)=\ell$. So we assume $\beta(Y)<\ell$.

Since the endvertices of e are contained in $V(X)$ and $\beta(X+e)=\beta(X)$, $X+e$ is f_α-full. Moreover, since X is f_α-tight, $|X|=k|V(X)|-\ell+\beta(X)>k|V(X)|-2\ell+\beta(X+e)+\beta(Y)$ by $\beta(X+e)=\beta(X)$ and $\beta(Y)<\ell$. Therefore, we can apply Lemma 4.6 to get $0=\beta(X+e)-\beta(X)\geq\beta(Y+e)-\beta(Y)$, implying $\beta(Y+e)=\beta(Y)$ due to the monotonicity of β. This also implies that $Y+e$ is f_α-full. ∎

We are now ready to prove Theorem 4.1. Our proof also gives an explicit formula for the rank and hence we shall restate it in a different form.

Theorem 4.10. *Let (G,ψ) be a Γ-labeled graph with $G=(V,E)$ and $\mathcal{I}_\alpha$ be the family of all f_α-sparse edge subsets in E. Then $(E,\mathcal{I}_\alpha)$ is a matroid on the ground-set E. The rank of the matroid is equal to*

$$\min\left\{|E_0|+\sum_{i=1}^{t}f_\alpha(E_i)\;\middle|\;E_0\subseteq E, E_i\in\mathcal{C}(G):\ \{E_0,E_1,\ldots,E_t\}\text{ is a partition of }E\right\}.$$

Proof. We say that a partition $\mathcal{P}=\{E_0,E_1,\dots,E_t\}$ of E is *valid* if $E_i\in\mathcal{C}(G)$ for $1\le i\le t$. For a valid partition $\mathcal{P}$, we denote $\mathrm{val}(\mathcal{P})=|E_0|+\sum_{i=1}^t f_\alpha(E_i)$. We shall check the following independence axiom of matroids: (I1) $\emptyset\in\mathcal{I}_\alpha$; (I2) for any $X,Y\subseteq E$ with $X\subseteq Y$, $Y\in\mathcal{I}_\alpha$ implies $X\in\mathcal{I}_\alpha$; (I3) for any $E'\subseteq E$, maximal subsets of E' belonging to $\mathcal{I}_\alpha$ have the same cardinality.

It is obvious that $\mathcal{I}_\alpha$ satisfies (I1). Also (I2) follows from the definition of the f_α-sparsity. To see (I3), take a maximal f_α-sparse subset F of E. For any valid partition $\mathcal{P}$, we have $|F|\le\mathrm{val}(\mathcal{P})$ by $|F|=\sum_{i=0}^t|F\cap E_i|\le |F\cap E_0|+\sum_{i=1}^t f_\alpha(E_i)\le\mathrm{val}(\mathcal{P})$. We shall prove that there is a valid partition $\mathcal{P}$ of E with $|F|=\mathrm{val}(\mathcal{P})$, from which (I3) follows.

Let E_0 be the set of edges which are not contained in any f_α-tight set in F, and consider the family $\{F_1,F_2,\dots,F_t\}$ of all inclusion-wise maximal f_α-tight sets in F. Then $E_0\cup\bigcup_{i=1}^t F_i=F$ holds. Since $F_i\cap F_j=\emptyset$ for every pair $1\le i<j\le t$ by Lemma 4.7 and the maximality, $\mathcal{P}_F=\{E_0,F_1,F_2,...,F_t\}$ is a valid partition of F and $|F|=\mathrm{val}(\mathcal{P}_F)$ holds.

Now consider an edge $e=(u,v)\in E\setminus F$. Since F is a maximal f_α-sparse subset of E, there is a set $X_e\subseteq F$ with $X_e{+}e\in\mathcal{C}(G)$ and $|X_e{+}e|>f_\alpha(X_e{+}e)$. Let $A=\{e\in E\setminus F\colon X_e\in\mathcal{C}(G)\}$ and $B=E\setminus(F\cup A)$.

For each $e\in A$, since X_e is f_α-sparse, we have $|X_e|=f_\alpha(X_e)=f_\alpha(X_e{+}e)$, which implies that X_e is f_α-tight and $X_e\subseteq F_i$ for some $1\le i\le t$. Choose such an F_i for each $e\in A$ and define $E_i=F_i\cup\{e\in A\colon F_i \text{ was chosen for } e\}$ for $1\le i\le t$. Then $\mathcal{P}=\{E_0,E_1,E_2,\dots,E_t\}$ is a valid partition of $E\setminus B$. Moreover, repeated applications of Lemma 4.9 imply $f_\alpha(F_i)=f_\alpha(E_i)$ for every $1\le i\le t$. Thus $\mathrm{val}(\mathcal{P})=\mathrm{val}(\mathcal{P}_F)=|F|$.

In order to make $\mathcal{P}$ to a valid partition of E, we update $\mathcal{P}$ by the following process. Consider any $e\in B$. Since X_e+e is connected but X_e is not, e is a bridge in G_{X_e+e} and X_e can be partitioned into two connected parts X_e^1 and X_e^2. Due to the f_α-sparsity, we have

$$\begin{aligned}(27)\qquad & k|V(X_e)|-\ell+\beta(X_e+e)=f_\alpha(X_e+e)<|X_e+e|\\ &=|X_e^1|+|X_e^2|+1\le k|V(X_e)|-2\ell+\beta(X_e^1)+\beta(X_e^2)+1,\end{aligned}$$

implying $\beta(X_e^1)+\beta(X_e^2)\ge\ell+\beta(X_e+e)$. On the other hand, by the monotonicity of β, $\beta(X_e^1)+\beta(X_e^2)\le\ell+\beta(X_e+e)$. Therefore we have $\beta(X_e^1)=\beta(X_e^2)=\beta(X_e+e)=\ell$, and (27) implies that X_e^1 and X_e^2 are f_α-tight. Hence each of X_e^1 and X_e^2 is contained in some $E_i\in\mathcal{P}\setminus\{E_0\}$.

If X_e^1 and X_e^2 are both contained in the same E_i, then we have $f_\alpha(E_i{+}e)=k|V(E_i+e)|=k|V(E_i)|=f_\alpha(E_i)$ by $\ell\ge\beta(E_i)\ge\beta(X_e^1)=\ell$. Hence we update $\mathcal{P}$ by replacing E_i with E_i+e, which keeps $\mathrm{val}(\mathcal{P})$.

If X_e^1 and X_e^2 are not contained in the same E_i, then without loss of generality assume that E_i contains X_e^i for $i=1,2$. We have $f_\alpha(E_1{\cup}E_2{+}e)=$

$k|V(E_1 \cup E_2 + e)| = k|V(E_1)| + k|V(E_2)| = f_\alpha(E_1) + f_\alpha(E_2)$ by $\ell \geq \beta(E_i) \geq \beta(X_e^i) = \ell$ for each $i = 1, 2$. Therefore we update $\mathcal{P}$ by removing E_1 and E_2 from $\mathcal{P}$ and inserting $E_1 \cup E_2 + e$. This again keeps $\mathrm{val}(\mathcal{P})$.

We perform the above modification one by one for each $e \in B$. Since each update keeps $\mathrm{val}(\mathcal{P})$, we finally get a valid partition $\mathcal{P}$ of E with $|F| = \mathrm{val}(\mathcal{P})$. This completes the proof. ∎

5. Checking the sparsity

Let k and ℓ be two integers with $k \geq 1$ and $0 \leq \ell \leq 2k-1$, and α be a polymatroidal function on 2^Γ. In this section we show how to check the f_α-sparsity of a given Γ-labeled graph (G, ψ) in polynomial time if ℓ is constant. This also gives an algorithm for checking the independence and computing the rank of the matroid induced by f_α. We assume that we are given an oracle that returns $\alpha(X)$ in polynomial time for each $X \subseteq \Gamma$.

We first give an algorithm to compute $f_\alpha(F)$ for a given $F \in \mathcal{C}(G)$. We need to show how to compute $\beta(F)$. To compute $\tilde{\alpha}(F)$, we fist take any spanning tree T in G_F, and compute the T-respecting equivalent ψ' by switching. Then $\psi'(F)$ generates $\langle F \rangle_{v,\psi'}$ for any $v \in V(F)$ (see, e.g., [7] for a detailed exposition), and hence $\tilde{\alpha}(F) = \alpha(\psi'(F))$. Thus $\tilde{\alpha}(F)$ can be computed in polynomial time.

To compute $\beta(F)$, it remains to check whether F is near-balanced. For this, we test whether a vertex $v \in V(F)$ can be a base or not as follows. We take a spanning tree T of G_F by extending a spanning forest of $G_F - v$, and let ψ' be a T-respecting equivalent gain function. Proposition 3.1 implies that v is a base for the near-balancedness of F if and only if F is unbalanced and there is a non-identity element $g \in \Gamma$ such that

- $\psi(e) = 1_\Gamma$ for $e \in F \setminus F_v$,
- for each fraction S of G_F at v, either $\psi(e) \in \{1_\Gamma, g\}$ or $\psi(e) \in \{1_\Gamma, g^{-1}\}$ for $e \in F_v \cap E(S)$,
- $\psi(e) \in \{g, g^{-1}\}$ for every $(L_v \cap F) \setminus L_v^\circ$.

Thus one can check whether v can be a base by computing a T-respecting equivalent gain function ψ'.

For checking f_α-sparsity, we need the following simple lemma. Recall that the (k,ℓ)-count matroid $\mathcal{M}_{k,\ell}(G)$ of G consists of the set of all (k,ℓ)-sparse edge sets in G as the independent set family. It is known and easy to check that a circuit in $\mathcal{M}_{k,\ell}(G)$ is always connected.

Lemma 5.1. *(G, ψ) is f_α-sparse if and only if G is $(k,0)$-sparse and $|C| \leq f_\alpha(C)$ for every nonempty $C \subseteq E(G)$ that is a circuit in $\mathcal{M}_{k,\ell'}(G)$ for some $1 \leq \ell' \leq \ell$.*

Proof. The necessity is trivial, and we prove the sufficiency. Suppose to the contrary that (G,ψ) is not f_α-sparse. Take any $F\in\mathcal{C}(G)$ such that $|F|>f_\alpha(F)$. Then $|F|>f_\alpha(F)\geq k|V(F)|-\ell$. On the other hand, since G is $(k,0)$-sparse, we have $|F|\leq k|V(F)|$. Therefore, there is an integer ℓ' with $1\leq\ell'\leq\ell$ such that $|F|=k|V(F)|-\ell'+1$. Since F is dependent in $\mathcal{M}_{k,\ell'}(G)$, F contains a circuit C in $\mathcal{M}_{k,\ell'}(G)$. Note that $k|V(F)|-\ell'-|F|=-1=k|V(C)|-\ell'-|C|$. Hence by the monotonicity of β, we get $0\leq f_\alpha(C)-|C|\leq f_\alpha(F)-|F|<0$, which is a contradiction. ∎

Based on Lemma 5.1 we have the following naive algorithm for checking f_α-sparsity:

1. Check whether G is $(k,0)$-sparse. If G is not $(k,0)$-sparse, then (G,ψ) is not f_α-sparse.
2. For each ℓ' with $1\leq\ell'\leq\ell$, enumerate all the circuits in $\mathcal{M}_{k,\ell'}(G)$ and check wether $|C|\leq f_\alpha(C)$ holds for each circuit C in $\mathcal{M}_{k,\ell'}(G)$. If there is a circuit C with $|C|>f_\alpha(C)$, then (G,ψ) is not f_α-sparse; otherwise it is f_α-sparse.

It is well-known that checking $(k,0)$-sparsity can be reduced to computing a maximum matching in an auxiliary bipartite graph of size $|V(G)|$, which can be done in $O(|V(G)|^{3/2})$ time (see, e.g., [4]). As for the second step, observe that the number of circuits in $\mathcal{M}_{k,\ell'}(G)$ is $O(|V(G)|^{\ell'-1})$. This can be seen as follows. If $\mathcal{M}_{k,\ell'}(G)$ is not connected (in the matroid sense), then the number of circuits in each connected component C is $O(|V(C)|^{\ell'-1})$ by induction and the sum over all components is $O(|V(G)|^{\ell'-1})$. Hence we may assume that $\mathcal{M}_{k,\ell'}(G)$ is connected, and the rank of $\mathcal{M}_{k,\ell'}(G)$ is $k|V(G)|-\ell'$. Since the size of the ground set is at most $k|V(G)|$ (as G is $(k,0)$-sparse), the rank of the dual of $\mathcal{M}_{k,\ell'}(G)$ is at most ℓ'. Therefore, the number of the hyperplanes in the dual is $O(|V(G)|^{\ell'-1})$, which in turn implies the claimed bound for the number of circuits.

It is known that all the circuits in a matroid can be enumerated in time polynomial in the size of the ground set and the number of the circuits [13], if a polynomial-time oracle for the rank function is available. In our case, the number of circuits is polynomial in $|V(G)|$ (assuming that ℓ is constant) and the rank of $\mathcal{M}_{k,\ell'}(G)$ can be computed in $O(|V(G)|^2)$ time (see, e.g., [1,8]). Therefore, the second step can also be done in polynomial time.

Developing a practical polynomial time algorithm whose time complexity is $O(|V(G)|^c)$ for some constant c irrelevant to ℓ is left as an open problem.

Acknowledgement. We thank anonymous referees for valuable comments. This work was supported by JSPS Postdoctoral Fellowships for Research

Abroad, JSPS Grant-in-Aid for Young Scientist (B) 24740058, and JSPS Grant-in-Aid for Scientific Research (B) 25280004.

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