A Highly Efficient Pricing Method for European-Style Options Based on Shannon Wavelets

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Abstract In the search for robust, accurate and highly efficient financial option valuation techniques, we present here the SWIFT method (Shannon Wavelets Inverse Fourier Technique), based on Shannon wavelets. SWIFT comes with control over approximation errors made by means of sharp quantitative error bounds. The nature of the local Shannon wavelets basis enables us to adaptively determine the proper size of the computational interval. Numerical experiments on European-style options confirm the bounds, robustness and efficiency.

1 Introduction

European options are financial derivatives, governed by the solution of an integral, the so-called discounted expectation of a final condition, i.e., the pay-off function. A strain of literature dealing with highly efficient pricing of these contracts already exists, where the computation often takes place in Fourier space. For the computation of the expectation we require knowledge about the probability density function governing the stochastic asset price process, which is typically not available for relevant price processes. Methods based on quadrature and the Fast Fourier Transform (FFT) [1, 6, 7], methods based on Fourier cosine expansions [4, 12] and methods based on wavelets [8, 9] have therefore been developed because for relevant *log-asset price processes* the characteristic function appears to be available. The characteristic function is defined as the Fourier transform of the density function. In this paper, we will explore the potential of Shannon wavelets [2] for

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the valuation of European-style options, which is also based on the availability of the characteristic function. We will call the resulting numerical wavelets technique *"SWIFT"* (Shannon Wavelet Inverse Fourier Technique). Further details on the method can be found in [10].

The pricing of European options in computational finance is governed by the numerical solution of partial differential, or partial integro-differential, equations. The corresponding solution, being the option value at time t, can also be found by means of the Feynman–Kac formula as a discounted expectation of the option value at final time $t = T$, the so-called pay-off function. Here, we consider this *risk-neutral option valuation formula*,

$$v(x,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left(v(y,T)|x\right) = e^{-r(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)\,dy, \tag{1}$$

where v denotes the option value, T is the maturity, t the initial date, $\mathbb{E}^{\mathbb{Q}}$ the expectation operator under the risk-neutral measure $\mathbb{Q}$, x and y are state variables at time t and T, respectively, $f(y|x)$ is the probability density of y given x, and r is the deterministic risk-neutral interest rate.

Whereas f is typically not known, the characteristic function of the log-asset price is often available (sometimes in closed-form), as the Fourier transform of f. We represent the option values as functions of the scaled log-asset prices, and denote these prices by $x = \ln(S_t/K)$ and $y = \ln(S_T/K)$, with S_t the underlying price at time t and K the strike price.

The pay-off $v(y,T)$ for European options in log-asset space is then given by

$$v(y,T) = \left[\alpha \cdot K\left(e^{y} - 1\right)\right]^{+}, \tag{2}$$

with $\alpha = 1$ for a call, and $\alpha = -1$ for a put.

2 SWIFT

The strategy to follow to determine the price of the option consists of approximating the density function f in (1) by means of a *finite combination of Shannon scaling functions* and recovering the coefficients of the approximation from its Fourier transform.

Let us consider the probability density function f in (1) and its Fourier transform,

$$\hat{f}(w) = \int_{\mathbb{R}} e^{-iwy} f(y|x)\,dy. \tag{3}$$

Following the wavelets theory from [3], the function f can be approximated at a level of resolution m, i.e.,

$$f(y|x) \approx \mathcal{P}_m f(y|x) = \sum_{k\in\mathbb{Z}} c_{m,k}(x)\phi_{m,k}(y), \tag{4}$$

where $\mathcal{P}_m f$ converges to f in $L^2(\mathbb{R})$, that is, $\|f - \mathcal{P}_m f\|_2 \to 0$, when $m \to +\infty$, and where $\phi_{m,k}(y) = 2^{m/2}\phi(2^m y - k)$, $\phi(y) = \mathrm{sinc}(y)$, $c_{m,k}(x) = \langle f, \phi_{m,k}\rangle$, and $< f, g >= \int_{\mathbb{R}} f(y|x)\overline{g(y)}\,dy$ denotes the inner product in $L^2(\mathbb{R})$ (the bar denoting complex conjugation).

Lemma 1 in [10] guarantees that the infinite series in (4) is well-approximated by a finite summation without loss of considerable density mass,

$$\mathcal{P}_m f(y|x) \approx f_m(y|x) := \sum_{k=k_1}^{k_2} c_{m,k}(x)\phi_{m,k}(y|x), \tag{5}$$

for certain accurately chosen values k_1 and k_2.

2.1 Density Coefficients

We compute the coefficients in expression (5) by considering

$$c_{m,k}(x) = \langle f, \phi_{m,k}\rangle = \int_{\mathbb{R}} f(y|x)\overline{\phi}_{m,k}(y)\,dy = 2^{m/2}\int_{\mathbb{R}} f(y|x)\phi(2^m y - k)\,dy. \tag{6}$$

Using the classical *Vieta formula* [5], the cardinal sinus can be expressed as an infinite product, i.e.,

$$\mathrm{sinc}(t) = \prod_{j=1}^{+\infty} \cos\left(\frac{\pi t}{2^j}\right). \tag{7}$$

If we truncate this infinite product to a finite product with J factors, then, thanks to the cosine product-to-sum identity [11], we have

$$\prod_{j=1}^{J} \cos\left(\frac{\pi t}{2^j}\right) = \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos\left(\frac{2j-1}{2^J}\pi t\right). \tag{8}$$

By (7) and (8), the sinc function can thus be approximated as

$$\text{sinc}(t) \approx \text{sinc}^*(t) := \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos\left(\frac{2j-1}{2^J}\pi t\right). \tag{9}$$

If we replace the function ϕ in (6) by its approximation (9) then,

$$c_{m,k}(x) \approx c^*_{m,k}(x) := \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \int_{\mathbb{R}} f(y|x) \cos\left(\frac{2j-1}{2^J}\pi(2^m y - k)\right) dy.$$

Taking into account that $\Re\left(\hat{f}(w)\right) = \int_{\mathbb{R}} f(y|x)\cos(wy)\,dy$ in expression (3) (where $\Re(z)$ denotes the real part of z), and observing that

$$\hat{f}(w)e^{ik\pi\frac{2j-1}{2^J}} = \int_{\mathbb{R}} e^{-i\left(wy - \frac{k\pi(2j-1)}{2^J}\right)} f(y|x)\,dy,$$

we end up with the following expression for computing the density coefficients,

$$c_{m,k}(x) \approx c^*_{m,k}(x) = \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \Re\left[\hat{f}\left(\frac{(2j-1)\pi 2^m}{2^J}\right) e^{\frac{ik\pi(2j-1)}{2^J}}\right].$$

2.2 Pay-off Coefficients

The pay-off functions for European call or put options have been given in equation (2). We truncate the infinite integration range in (1) to a finite domain $\mathcal{I}_m = [\frac{k_1}{2^m}, \frac{k_2}{2^m}]$, which gives,

$$v(x,t) = e^{-r(T-t)} \int_{\mathbb{R}} v(y,T) f(y|x)\,dy \approx v_1(x,t) = e^{-r(T-t)} \int_{\mathcal{I}_m} v(y,T) f(y|x)\,dy.$$

If we now replace f by its approximation f_m, we find

$$\begin{aligned} v_1(x,t) = e^{-r(T-t)} \int_{\mathcal{I}_m} v(y,T) f(y|x)\,dy &\approx v_2(x,t) = e^{-r(T-t)} \int_{\mathcal{I}_m} v(y,T) f_m(y|x)\,dy \\ &= e^{-r(T-t)} \sum_{k=k_1}^{k_2} c_{m,k}(x) \cdot V^{\alpha}_{m,k}, \end{aligned}$$

with the pay-off coefficients $V^{\alpha}_{m,k} := \int_{\mathcal{I}_k} v(y,T)\phi_{m,k}(y)\,dy$. Then, let us define, $\bar{k}_1 := \max(k_1, 0)$. The pay-off coefficients for a European call option are computed as follows,

$$V^1_{m,k} \approx V^{1,*}_{m,k} := \begin{cases} \frac{K2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[I_1\left(\frac{\bar{k}_1}{2^m}, \frac{k_2}{2^m}\right) - I_2\left(\frac{\bar{k}_1}{2^m}, \frac{k_2}{2^m}\right)\right], & \text{if } k_2 > 0, \\ 0, & \text{if } k_2 \leq 0, \end{cases}$$

where

$$I_1(a,b) = \frac{C_j 2^m}{1 + (C_j 2^m)^2}\Bigg[e^b \sin(C_j(2^m b - k)) - e^a \sin(C_j(2^m a - k)) + \frac{1}{C_j 2^m}\left(e^b \cos(C_j(2^m b - k)) - e^a \cos(C_j(2^m a - k))\right)\Bigg],$$

and

$$I_2(a,b) = \frac{1}{C_j 2^m}\left(\sin(C_j(2^m b - k)) - \sin(C_j(2^m a - k))\right), \quad C_j = \frac{2j-1}{2^J}\pi.$$

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