A note on the gap between rank and border rank

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Abstract

We study the tensor rank of a certain algebra. As a result we find a sequence of tensors with a large gap between rank and border rank, and thus a counterexample to a conjecture of Rhodes. We also obtain a new lower bound on the tensor rank of powers of the generalized W-state.

Keywords. Tensor rank, border rank, algebraic complexity theory, quantum information theory.

1. Introduction

Let $V_1, \ldots, V_k$ be finite-dimensional complex vector spaces and let $V \coloneqq V_1 \otimes \cdots \otimes V_k$ be the space of k-*tensors*. A tensor of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_k \in V$ is called *simple*. The *rank* of a tensor $t \in V$ is the smallest number r such that t can be written as a sum of r simple tensors. The *border rank* $\underline{\mathrm{R}}(t)$ of t is the smallest number r such that t is the limit of a sequence of tensors in V of rank at most r. Clearly, $\underline{\mathrm{R}}(t) \leq \mathrm{R}(t)$. The following problem motivates this note.

Problem. What is the maximal rank of a tensor t in $(\mathbf{C}^n)^{\otimes k}$ that has border rank n?

Our main result is the following.

Theorem 10. Let $k \geq 3$. There exists an explicit sequence $(t_{k,n})_n$ of tensors $t_{k,n} \in (\mathbf{C}^{2^n})^{\otimes k}$ of border rank 2^n such that,

$$\frac{\mathrm{R}(t_{k,n})}{\underline{\mathrm{R}}(t_{k,n})} \geq k - \frac{o(2^n)}{2^n}.$$

We obtain Theorem 10 by applying a tensor rank lower bound of Bläser to the tensor corresponding to the algebra $A_{d,n} \coloneqq \mathbf{C}[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d)$ of n-variate complex polynomials modulo the dth power of each variable. In the process, we also find the following lower bound for the tensor rank of tensor powers of the generalized W-state $W_k \coloneqq |10\cdots0\rangle + |01\cdots0\rangle + \cdots + |0\cdots01\rangle \in (\mathbf{C}^2)^{\otimes k}$, which slightly improves the lower bound $\mathrm{R}(W_k^{\otimes n}) \geq (k-1)\cdot 2^n - k + 2$ of Chen et al. [CCD+10].

Theorem 9. Let $k \geq 3$, $n \geq 1$. For any integer $m \geq 1$,

$$\begin{aligned}\mathrm{R}(W_k^{\otimes n}) &\geq (k-1)\cdot 2^n + \sum_{i=0}^{2m-2} \binom{n}{i} - 2\sum_{i=0}^{m-1}\binom{n}{i} - k + 3 \\ &\geq k \cdot 2^n - o(2^n).\end{aligned}$$

We note that it is a major open problem to find explicit tensors $t \in (\mathbf{C}^n)^{\otimes 3}$ with $\mathrm{R}(t) \geq (3+\varepsilon)n$ for some $\varepsilon > 0$ [Blä14].

Related work. De Silva and Lim show that for a 3-tensor t the difference $\underline{\mathrm{R}}(t) - \mathrm{R}(t)$ can be arbitrarily large [DSL08]. However, their result only implies a lower bound of 3/2 on the maximal ratio $\mathrm{R}(t)/\underline{\mathrm{R}}(t)$ for t a 3-tensor.

Allman et al. give explicit tensors K_n in $\mathbf{C}^n \otimes \mathbf{C}^n \otimes \mathbf{C}^n$ of border rank n and rank $2n-1$ [AJRS13]; a rank to border rank ratio that converges to 2. They provide references to other tensors with similar rank and border rank behaviour. We note that the tensor K_n is essentially the tensor of the algebra $\mathbf{C}[x]/(x^n)$. It was conjectured by Rhodes that the rank of a tensor in

$\mathbf{C}^n \otimes \mathbf{C}^n \otimes \mathbf{C}^n$ of border rank n is at most $2n-1$ [BB13, Conjecture 0]. Theorem 10 shows that this conjecture is false.

Independently of the author and with different techniques, Landsberg and Michałek have recently constructed a sequence of 3-tensors $T_{\mathrm{biggap},m}$ with a ratio of rank to border rank converging to 5/2, thus also disproving the above conjecture [LM15].

Finally, as is also mentioned in [LM15], we note that for any $k \geq 3$, the tensor $W_k \in (\mathbf{C}^2)^{\otimes k}$ has border rank 2 and rank k, thus giving a rank to border rank ratio of $k/2$, see the proof of Theorem 10.

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2. The algebra $A_{d,n}$

Many examples of interesting 3-tensors come from algebras. A complex algebra is a complex vector space V together with a multiplication defined by a bilinear map $\phi : V \times V \to V$. We can naturally view ϕ as a tensor in $V \otimes V \otimes V$ by

$$\phi \mapsto \sum_{i,j,k} e_k^*(\phi(e_i, e_j))\, e_i \otimes e_j \otimes e_k,$$

where (e_i) is a basis of V, and hence we can speak about the tensor rank and border rank of an algebra. We will study the algebra

$$A_{d,n} \coloneqq (\mathbf{C}[x]/(x^d))^{\otimes n} = \mathbf{C}[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d),$$

of n-variate complex polynomials modulo the dth power of each variable. There are many results on the tensor rank and border rank of algebras, in particular of the algebra of $n \times n$ matrices, for which we refer to [BCS97] and [Blä00]. For results on the tensor rank and border rank of general tensors we refer to [Lan12].

2.1. Border rank

A tensor t in $V_1 \otimes \cdots \otimes V_k$ is called 1-*concise* if there does not exist a proper subspace $U_1 \subsetneq V_1$ such that $t \in U_1 \otimes V_2 \otimes \cdots \otimes V_k$. Similarly, we define i-conciseness for $i \in \{2, \ldots, k\}$. A tensor is called *concise* if it is i-concise for all i. We can think of a concise tensor as a tensor that "uses" all dimensions of the local spaces V_i. Tensors of algebras with a unit element are concise. For a concise tensor t in $V_1 \otimes \cdots \otimes V_k$ the border rank is at least $\max_i \dim V_i$ [BCS97, Lemma 15.23]. The following proposition is a direct consequence of the well-known fact that $\underline{\mathrm{R}}(\mathbf{C}[x]/(x^d)) = d$ [BCS97, Example 15.20].

Proposition 1. $\underline{\mathrm{R}}(A_{d,n}) = d^n$.

Proof. The tensor $A_{d,n} \in \mathbf{C}^{d^n} \otimes \mathbf{C}^{d^n} \otimes \mathbf{C}^{d^n}$ is concise. Therefore, $\underline{\mathrm{R}}(A_{d,n}) \geq d^n$. On the other hand, border rank is submultiplicative under tensor products, so $\underline{\mathrm{R}}(A_{d,n}) = \underline{\mathrm{R}}((\mathbf{C}[x]/(x^d))^{\otimes n}) \leq \underline{\mathrm{R}}(\mathbf{C}[x]/(x^d))^n = d^n$. □

2.2. Rank upper bound

It is well-known that upper bounds on border rank imply upper bounds on rank. Proposition 1 implies the following upper bound on $\mathrm{R}(A_{d,n})$. We will not use the upper bound later, but it will provide some context for the lower bound of Corollary 7.

Proposition 2. $\mathrm{R}(A_{d,n}) \leq (nd+1)d^n$.

Proof. The statement follows from the proof of Theorem 5 in [VC13], using that the error degree in the generation of the d-th unit tensor to $A_{d,n}$ is d [BCS97, Example 15.20]. □

2.3. Rank lower bound

Our main tool for proving lower bounds is the following lower bound for the tensor rank of algebras. Let A be a finite-dimensional associative unital algebra over $\mathbf{C}$. The *nilradical* of A is the sum of all nilpotent left-ideals in A.

Theorem 3 ([Blä00, Theorem 7.4])**.** *Let A be a finite-dimensional complex associative unital algebra over and let N be the nilradical of A. For any integer $m \geq 1$,*

$$\mathrm{R}(A) \geq \dim(A) - \dim(N^{2m-1}) + 2\dim(N^m).$$

We will apply Theorem 3 to the algebra $A_{d,n}$. Let's first look at a small example.

Example 4. Consider the algebra $A \coloneqq A_{2,2} = \mathbf{C}[x_1, x_2]/(x_1^2, x_2^2)$ of dimension 4. The elements of A are of the form $\alpha + \beta x_1 + \gamma x_2 + \delta x_1 x_2$ for $\alpha, \beta, \gamma, \delta \in \mathbf{C}$. The nilradical $N \subset A$ is the subspace spanned by x_1, x_2 and $x_1 x_2$ and hence has dimension 3. The square of the nilradical N^2 is spanned by $x_1 x_2$ and hence has dimension 1. Theorem 3 thus gives $\mathrm{R}(A) \geq 4 - 3 + 2 \cdot 3 = 7$.

To get a handle on the dimension of powers of the radical of $A_{d,n}$ we use the following two lemma's. Let $f(b,c,d)$ be the number of ways to put b balls into c containers with at most $d-1$ balls per container. This equals the number of monomials of degree b in $\mathbf{C}[x_1, \ldots, x_c]/(x_1^d, \ldots, x_c^d)$.

Lemma 5. *For integers $b \geq 1$, $c \geq 1$, $d \geq 2$,*

$$f(b,c,d) = \sum_{i=0}^{\min\{c, \lfloor b/d \rfloor\}} (-1)^i \binom{c}{i} \binom{b+c-1-i\cdot d}{c-1}.$$

In particular, $f(b,c,2) = \binom{c}{b}$.

Proof. Let $X \coloneqq \{$ways to put b balls into c containers$\}$ and for $j \in [c]$ let $A_j \coloneqq \{$ways to put b balls in c containers such that container j has at least d balls$\} \subset X$. By the inclusion-exclusion principle [Juk11, Proposition 1.13], the number of elements of X which lie in none of the subsets A_j is

$$\sum_{I \subseteq \{1,\ldots,c\}} (-1)^{|I|} \left| \cap_{j\in I} A_j \right| = \sum_{I \subseteq \{1,\ldots,c\}} (-1)^{|I|} \binom{b+c-1-|I|\cdot d}{c-1}.$$

Now use that there are $\binom{c}{|I|}$ subsets of size $|I|$ in $\{1, \ldots, c\}$. The statement about the special case $d = 2$ follows immediately from the definition. □

Lemma 6 ([FG06, Lemma 16.19])**.** *Let $n \geq 1$ and $0 < \varepsilon \leq 1/2$. Then*

$$\sum_{i=0}^{\lfloor \varepsilon n \rfloor} \binom{n}{i} \leq 2^{H(\varepsilon)n},$$

where $H(\varepsilon) \coloneqq -\varepsilon \log_2(\varepsilon) - (1-\varepsilon)\log_2(1-\varepsilon)$ is the binary entropy of ε. In particular, if $\varepsilon < \frac{1}{2}$, then $\sum_{i=0}^{\lfloor \varepsilon n \rfloor} \binom{n}{i}$ is $o(2^n)$.

Corollary 7. *Let $n \geq 1, d \geq 2$ be integers. Then $\mathrm{R}(A_{d,n}) \geq 2\, d^n + g(d,n)$, where*

$$g(d,n) \coloneqq \max_{m\geq 1} \sum_{b=0}^{2m-2} f(b,n,d) - 2\sum_{b=0}^{m-1} f(b,n,d).$$

In particular, for any $n, m \geq 1$,

$$\mathrm{R}(A_{2,n}) \geq 2 \cdot 2^n + \sum_{i=0}^{2m-2} \binom{n}{i} - 2\sum_{i=0}^{m-1} \binom{n}{i} \geq 3 \cdot 2^n - o(2^n).$$

Proof. The nilradical of our algebra $A_{d,n}$ is the ideal $N := (x_1, \ldots, x_n) \subset A_{d,n}$, that is, N is the subspace of $A_{d,n}$ of elements with zero constant term. The mth power N^m is the subspace spanned by monomials of degree at least m, hence the dimension of N^m equals $d^n - \sum_{b=0}^{m-1} f(b,n,d)$ when $m \leq n(d-1)$ and 0 when $m > n(d-1)$. Theorem 3 then gives, for any $1 \leq m$ such that $2m-2 \leq n(d-1)$,

$$\begin{aligned} \mathrm{R}(A_{d,n}) &\geq d^n - \Big(d^n - \sum_{b=0}^{2m-2} f(b,n,d)\Big) + 2\Big(d^n - \sum_{b=0}^{m-1} f(b,n,d)\Big) \\ &= 2d^n + \sum_{b=0}^{2m-2} f(b,n,d) - 2\sum_{b=0}^{m-1} f(b,n,d). \end{aligned}$$

Now let $d = 2$. Then, as mentioned in Lemma 5, $f(b,n,d)$ is simply $\binom{c}{b}$. It remains to show that $g(2,n) \geq 2^n - o(2^n)$. For this we use the upper bound from Lemma 6. If $2m-2 \leq n$, then

$$\sum_{i=0}^{2m-2} \binom{n}{i} = 2^n - \sum_{i=2m-1}^{n} \binom{n}{i} = 2^n - \sum_{i=0}^{n-(2m-2)} \binom{n}{i}$$

so then

$$g(2,n) = 2^n - \sum_{i=0}^{n-(2m-2)} \binom{n}{i} - 2\sum_{i=0}^{m-1} \binom{n}{i}.$$

One checks that for n large enough there exists an integer $m \geq 1$ such that $(n-(2m-2))/n < \frac{1}{2}$ and $(m-1)/n < \frac{1}{2}$. Therefore, by Lemma 6,

$$g(2,n) \geq 2^n - 2^{H((n-(2m-2))/n)n} - 2 \cdot 2^{H((m-1)/n)n} = 2^n - o(2^n). \qquad \square$$

In the table below we list some values of the lower bound of Corollary 7.

d	2	3	4	5	6
n					
1	**3**	**5**	**7**	**9**	**11**
2	**7**	18	33	53	78
3	15	57	142	285	501
4	33	182	601	1509	3166
5	68	576	2507	7824	19782
6	141	1773	10356	40329	121971

Table 1: Lower bounds for $\mathrm{R}(A_{d,n})$ from Corollary 7. The bold numbers are known to be sharp.

3. Generalized W-state

In quantum information theory, the generalized W-state is the tensor

$$\begin{aligned} W_k &:= |10\cdots0\rangle + |01\cdots0\rangle + \cdots + |0\cdots01\rangle \\ &= e_1 \otimes e_0 \otimes \cdots \otimes e_0 + e_0 \otimes e_1 \otimes \cdots \otimes e_0 + \cdots + e_0 \otimes \cdots \otimes e_0 \otimes e_1 \quad \in (\mathbf{C}^2)^{\otimes k}. \end{aligned}$$

It is not hard to check that, in a particular basis, the tensor of the algebra $A_{2,1} = \mathbf{C}[x]/(x^2)$ equals W_3. Therefore, $\mathrm{R}(A_{2,n}) = \mathrm{R}(W_3^{\otimes n})$. By the following proposition, lower bounds for $\mathrm{R}(W_3^{\otimes n})$ give lower bounds for $\mathrm{R}(W_k^{\otimes n})$.

Proposition 8 ([CCD+10])**.** $\mathrm{R}(W_k^{\otimes n}) \geq \mathrm{R}(W_3^{\otimes n}) + (k-3)(2^n - 1)$.

Theorem 9. $\mathrm{R}(W_k^{\otimes n}) \geq (k-1)2^n + g(2,n) - (k-3) = k \cdot 2^n - o(2^n)$.

Proof. Combine Proposition 8 with Corollary 7 for $A_{2,n}$. □

Chen et al. give the lower bound $\mathrm{R}(W_k^{\otimes n}) \geq (k-1)2^n - k + 2$, which they obtain by combining the lower bound $\mathrm{R}(A_{2,n}) \geq 2^{n+1} - 1$ with Proposition 8 [CCD+10]. Since $2 \cdot 2^n + g(2,n) \geq 2^{n+1} - 1$, the lower bound of Theorem 9 improves the lower bound of Chen et al. The best upper bound so far is $\mathrm{R}(W_k^{\otimes n}) \leq (n(k-1)+1)2^n$ [VC13]. Below we list some values of the lower bound of Theorem 9. One can show that $\mathrm{R}(W_3^{\otimes 3}) = 16$ by showing that the algebra $(\mathbf{C}[x]/(x^2))^{\otimes 3}$ is not of *minimal rank*. (We refer to [BCS97] for the theory of algebras of minimal rank.) Therefore, the bounds in the third column are not sharp.

n	1	2	3	4	5	6	7	8	9	10
k										
3	**3**	**7**	15	33	68	141	297	601	1230	2544
4	**4**	**10**	22	48	99	204	424	856	1741	3567
5	**5**	**13**	29	63	130	267	551	1111	2252	4590
6	**6**	**16**	36	78	161	330	678	1366	2763	5613
7	**7**	**19**	43	93	192	393	805	1621	3274	6636
8	**8**	**22**	50	108	223	456	932	1876	3785	7659
9	**9**	**25**	57	123	254	519	1059	2131	4296	8682
10	**10**	**28**	64	138	285	582	1186	2386	4807	9705

Table 2: Lower bounds for $\mathrm{R}(W_k^{\otimes n})$ from Theorem 9. The bold numbers are known to be sharp [CCD+10].

4. Gap between rank and border rank

A direct consequence of Theorem 9 is the following statement about the maximal gap between rank and border rank.

Theorem 10. *Let $k \geq 3$. There exists an explicit sequence $(t_{k,n})_n$ of concise tensors $t_{k,n} \in (\mathbf{C}^{2^n})^{\otimes k}$ of border rank 2^n such that,*

$$\frac{\mathrm{R}(t_{k,n})}{\underline{\mathrm{R}}(t_{k,n})} \geq k - \frac{o(2^n)}{2^n}.$$

Proof. Let $t_{k,n} = W_k^{\otimes n}$. The tensor $W_k \in (\mathbf{C}^2)^{\otimes k}$ is concise and tensor powers of concise tensors are concise. Therefore, $W_k^{\otimes n} \in (\mathbf{C}^{2^n})^{\otimes k}$ is a concise tensor of minimal border rank, that is, $\underline{\mathrm{R}}(W_k^{\otimes n}) = 2^n$. By Theorem 9, therefore,

$$\frac{\mathrm{R}(W_k^{\otimes n})}{\underline{\mathrm{R}}(W_k^{\otimes n})} \geq k - \frac{o(2^n)}{2^n}.$$ □

References

[AJRS13] Elizabeth S. Allman, Peter D. Jarvis, John A. Rhodes, and Jeremy G. Sumner. Tensor rank, invariants, inequalities, and applications. *SIAM Journal on Matrix Analysis and Applications*, 34(3):1014–1045, 2013.

[BB13] Edoardo Ballico and Alessandra Bernardi. Stratification of the fourth secant variety of veronese varieties via the symmetric rank. *Advances in Pure and Applied Mathematics*, 4(2):215–250, 2013.

[BCS97] Peter Bürgisser, Michael Clausen, and M. Amin Shokrollahi. Algebraic complexity theory, volume 315 of Grundlehren der Mathematischen Wissenschaften, 1997.

[Blä00] Markus Bläser. Lower bounds for the bilinear complexity of associative algebras. *computational complexity*, 9(2):73–112, 2000.

[Blä14] Markus Bläser. Explicit tensors. In *Perspectives in Computational Complexity*, pages 117–130. Springer, 2014.

[CCD+10] Lin Chen, Eric Chitambar, Runyao Duan, Zhengfeng Ji, and Andreas Winter. Tensor rank and stochastic entanglement catalysis for multipartite pure states. *Physical review letters*, 105(20):200501, 2010.

[DSL08] Vin De Silva and Lek-Heng Lim. Tensor rank and the ill-posedness of the best low-rank approximation problem. *SIAM Journal on Matrix Analysis and Applications*, 30(3):1084–1127, 2008.

[FG06] Jörg Flum and Martin Grohe. Parameterized complexity theory, volume XIV of texts in theoretical computer science. An EATCS series, 2006.

[Juk11] Stasys Jukna. *Extremal combinatorics: with applications in computer science*. Springer Science & Business Media, 2011.

[Lan12] J. M. Landsberg. *Tensors: geometry and applications*, volume 128. American Mathematical Soc., 2012.

[LM15] J. M. Landsberg and M. Michałek. Abelian Tensors. *ArXiv e-prints*, April 2015.

[VC13] Peter Vrana and Matthias Christandl. Asymptotic entanglement transformation between W and GHZ states. *arXiv preprint arXiv:1310.3244*, 2013.