Intertwining connectivities in representable matroids*

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Abstract

Let M be a representable matroid, and Q, R, S, T subsets of the ground set such that the smallest separation that separates Q from R has order k, and the smallest separation that separates S from T has order l. We prove that, if M is sufficiently large, then there is an element e such that in one of $M \backslash e$ and M/e both connectivities are preserved.

For matroids representable over a finite field we prove a stronger result: we show that we can remove e such that both a connectivity and a minor of M are preserved.

1 Introduction

For a matroid M on ground set E we define, as usual, the *connectivity function* λ_M by $\lambda_M(X) := \mathrm{rk}_M(X) + \mathrm{rk}_M(E - X) - \mathrm{rk}(M)$. For disjoint sets $S, T \subseteq E$, the *connectivity between S and T* is

$$\kappa_M(S,T) := \min\{\lambda_M(X) : S \subseteq X \subseteq E - T\}. \quad (1)$$

Geelen, in private communication, conjectured the following.

Conjecture 1.1. *There exists a function $c : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let M be a matroid, and let $Q, R, S, T \subseteq E(M)$ be sets of elements such that $Q \cap R = S \cap T = \emptyset$. Let $k := \kappa_M(Q,R)$ and $l := \kappa_M(S,T)$. If $|E(M) - (Q \cup R \cup S \cup T)| \geq c(k,l)$, then there exists an element $e \in E(M) - (Q \cup R \cup S \cup T)$ such that one of the following holds:*

(i) $\kappa_{M\backslash e}(Q,R) = k$ *and* $\kappa_{M\backslash e}(S,T) = l$;

(ii) $\kappa_{M/e}(Q,R) = k$ *and* $\kappa_{M/e}(S,T) = l$.

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In other words, for fixed Q, R, S, T, there is a finite number of minor-minimal matroids with the prescribed connectivities. This formulation is reminiscent of the definition of an *intertwine*, which is a minor-minimal matroid containing two prescribed *minors*. For that reason we speak of the intertwining of connectivities.

For graphs the result follows readily from Robertson and Seymour's Graph Minors Theorem [11]. In this paper we prove the conjecture for all representable matroids.

For matroids representable over a finite field we prove a stronger result:

Theorem 1.2. *There exists a function $c : \mathbb{N}^3 \to \mathbb{N}$ with the following property. Let q be a prime power, let M be a* GF(q)*-representable matroid, let N be a minor of M, let $S, T \subseteq E(M)$ be disjoint, and let $k := \kappa_M(S, T)$. If $|E(M) - (S \cup T \cup E(N))| > c(q, |E(N)|, k)$, then there exists an element $e \in E(M) - (S \cup T \cup E(N))$ such that at least one of the following holds:*

(i) $\kappa_{M\backslash e}(S, T) = k$ *and N is a minor of $M\backslash e$;*

(ii) $\kappa_{M/e}(S, T) = k$ *and N is a minor of M/e.*

For completeness, we later give a short proof that Theorem 1.2 is indeed a strengthening of Conjecture 1.1 for matroids representable over a fixed finite field.

By repeated use of this theorem, it is possible to bound the size of an intertwine of any fixed number of connectivities. This solves a weak form of the following problem:

Problem 1.3. Let $M = (S, \mathscr{I})$ be a matroid that is a gammoid. Give an upper bound, in terms of $|S|$, on the size of the directed graph needed to represent M as a gammoid.

In the case that M is represented by an (undirected) graph, Theorem 1.2 yields a very poor bound of $2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$ (a tower of twos of height $2^{|S|}$). Good upper bounds can potentially be useful in the study of parametrized complexity (c.f. [8]).

Our proof technique for Theorem 1.2 has been used previously in, for instance, [4, 6, 7]. For graphs it dates back at least to the work of Robertson and Seymour on graph minors (cf. [12]). In fact, Theorem 1.2 is a generalization of [6, Theorem 1.1] and [13, Theorem 13.3].

Theorem 1.2 becomes false when the dependence on q is removed. A counterexample is readily obtained from a construction of arbitrarily long blocking sequences in [6, Proposition 6.1]. It follows that different techniques are needed to prove Conjecture 1.1.

Our proof of Conjecture 1.1 for representable matroids uses a different approach, based on a suggestion by Geelen (private communication). Unfortunately, the proof uses a property of representable matroids that does not hold for general matroids.

The paper is organized as follows. In Section 2 we fix some terminology and state some easy lemmas. Section 3 contains results related to Tutte's Linking Theorem. The main result in that section shows that, if Conjecture 1.1 is false, there exist matroids with arbitrarily long sequences of nested separations. In Section 4 we prove Theorem 1.2, and in Section 5 we prove Conjecture 1.1 for all representable matroids.

2 Preliminaries

We will use the following elementary observation (cf. [10, 3]):

Lemma 2.1. *Let M be a matroid and let $(A, \{e\}, B)$ be a partition of $E(M)$. Then $e \in \mathrm{cl}_M(A)$ if and only if $e \notin \mathrm{cl}^*_M(B)$.*

It is well-known that the connectivity function is submodular:

Lemma 2.2. *Let M be a matroid, and let $X, Y \subseteq E(M)$. Then*

$$\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cap Y) + \lambda_M(X \cup Y).$$

The following lemmas are easily verified:

Lemma 2.3. *Let M be a matroid, let $X \subseteq E(M)$, and let N be a minor of M with $X \subseteq E(N)$. Then $\lambda_N(X) \leq \lambda_M(X)$.*

Lemma 2.4. *Let M be a matroid, let S, T be disjoint subsets of $E(M)$, and let N be a minor of M with $S \cup T \subseteq E(N)$. Then $\kappa_N(S, T) \leq \kappa_M(S, T)$.*

We introduce some terminology.

Definition 2.5. Let M be a matroid and let S, T be disjoint subsets of $E(M)$. A partition (A, B) of $E(M)$ is *$S - T$-separating of order $k + 1$* if $S \subseteq A$, $T \subseteq B$, and $\lambda_M(A) = k$. If B is implicit, we also say that A is *$S - T$-separating* of order $k + 1$.

If, moreover, $|A|, |B| \geq k + 1$ then (A, B) is an (exact) $(k+1)$*-separation* of M. Sometimes we will be sloppy and say that (A, B) is $S - T$ separating if $S \subseteq B$ and $T \subseteq A$.

Lemma 2.6. *Let M be a matroid, let $S, T \subseteq E(M)$ be disjoint subsets, and let $k := \kappa_M(S, T)$. If (A_1, B_1) and (A_2, B_2) are $S - T$-separating with $\lambda_M(A_1) = \lambda_M(A_2) = k$, then $(A_1 \cap A_2, B_1 \cup B_2)$ is $S - T$-separating of order $k + 1$.*

Proof. Clearly, $(A_1 \cap A_2, B_1 \cup B_2)$ and $(A_1 \cup A_2, B_1 \cap B_2)$ are $S - T$-separating. Since $\kappa_M(S, T) = k$, we must have $\lambda_M(A_1 \cap A_2) \geq k$ and $\lambda_M(A_1 \cup A_2) \geq k$. It follows from Lemma 2.2 that equality must hold. □

Finally, we will frequently use the following well-known result and its dual.

Lemma 2.7. *Let M be a matroid, let $S, T \subseteq E(M)$ be disjoint subsets, let $k := \kappa_M(S, T)$, and let $e \in E(M) - (S \cup T)$. A partition (A, B) of $E(M) - e$ is $S - T$-separating of order k in M/e if and only if $(A \cup e, B)$ is $S - T$-separating of order $k + 1$ in M with $e \in \mathrm{cl}_M(A) \cap \mathrm{cl}_M(B)$.*

3 Tutte's Linking Theorem

In [14], Tutte proved the following result, which can be seen to be a generalization of Menger's theorem to matroids (see [9, Section 8.5]):

Theorem 3.1. *Let M be a matroid and let S, T be disjoint subsets of $E(M)$. Then*

$$\kappa_M(S, T) = \max\{\lambda_N(S) : N \text{ minor of } M \text{ such that } E(N) = S \cup T\}. \quad (2)$$

Equivalently,

Theorem 3.2. *Let M be a matroid and let S, T be disjoint subsets of $E(M)$. For each $e \in E(M) - (S \cup T)$, at least one of the following holds:*

(i) $\kappa_{M\backslash e}(S, T) = \kappa_M(S, T)$*, or*

(ii) $\kappa_{M/e}(S, T) = \kappa_M(S, T)$.

We warn that we will henceforth use Theorem 3.1 without reference.

Definition 3.3. Let M be a matroid, let S, T be disjoint subsets of $E(M)$, and let $e \in E(M) - (S \cup T)$.

(*i*) If $\kappa_{M\backslash e}(S, T) = \kappa_M(S, T)$ then we say e is *deletable with respect to* (S, T).

(*ii*) If $\kappa_{M/e}(S, T) = \kappa_M(S, T)$ then we say e is *contractible with respect to* (S, T).

(*iii*) If e is both deletable and contractible then we say e is *flexible with respect to* (S, T).

We may omit the phrase "with respect to (S, T)" if it can be deduced from the context. We will mainly be concerned with non-flexible elements. The following theorem is the main result of this section:

Theorem 3.4. *Let M be a matroid, let S, T be disjoint subsets of $E(M)$, let $k := \kappa_M(S, T)$, and let $F \subseteq E(M) - (S \cup T)$ be a set of non-flexible elements. There exists an ordering $(f_1, f_2, \dots, f_t)$ of F and a sequence $(A_1, A_2, \dots, A_t)$ of subsets of $E(M)$, such that*

(i) A_i *is* $S - T$*-separating of order* $k + 1$ *for each* $i \in \{1, \dots, t\}$*;*

(ii) $A_i \subseteq A_{i+1}$ *for each* $i \in \{1, \dots, t - 1\}$*;*

(iii) $A_i \cap F = \{f_1, \dots, f_i\}$ *for each* $i \in \{1, \dots, t\}$*;*

(iv) $f_i \in \text{cl}_M(A_i - f_i) \cap \text{cl}_M(E(M) - A_i)$ *or* $f_i \in \text{cl}^*_M(A_i - f_i) \cap \text{cl}^*_M(E(M) - A_i)$.

We will need two lemmas to prove this theorem.

Lemma 3.5. *Let M be a matroid, let S, T be disjoint subsets of $E(M)$, let $k := \kappa_M(S, T)$, and let $e \in E(M) - (S \cup T)$ be non-contractible. If (A, B) is an $S - T$-separating partition of order $k + 1$ such that $e \in A$ and $|A|$ is minimum, then $e \in \text{cl}_M(A - e) \cap \text{cl}_M(B)$.*

Proof. By Lemma 2.7, there is an $S - T$-separating partition (A', B') of order $k + 1$ such that $e \in A'$ and $e \in \text{cl}_M(A' - e) \cap \text{cl}_M(B')$. Thus, A exists. By Lemma 2.6, $A \cap A'$ is $S - T$-separating of order $k + 1$. By minimality of A, it then follows that $A \subseteq A'$, and therefore $B \supseteq B'$. In particular, we have established that $e \in \text{cl}_M(B)$. Finally, if $e \notin \text{cl}_M(A - e)$, then $\lambda_M(A - e) = k - 1$, contradicting $\kappa_M(S, T) = k$. □

Lemma 3.6. *Let M be a matroid, let S,T be disjoint subsets of $E(M)$, let $k := \kappa_M(S,T)$, and let U be an $S-T$-separating set of order $k+1$. If $e \in E(M)-(T \cup U)$ is non-contractible with respect to (S,T), then e is non-contractible with respect to (U,T).*

Proof. First, observe that $\kappa_M(U,T) = k$. If the lemma is false, then there is an $S-T$-separating partition (A,B) of order k in M/e, yet $\kappa_{M/e}(U,T) = k$. In particular, $\lambda_{M/e}(U) = k$. By submodularity,

$$2k-1 = \lambda_{M/e}(A) + \lambda_{M/e}(U) \geq \lambda_{M/e}(U \cap A) + \lambda_{M/e}(U \cup A). \qquad (3)$$

Since $U \cup A$ is $U-T$-separating, we have $\lambda_{M/e}(U \cup A) \geq k$. Hence $\lambda_{M/e}(U \cap A) \leq k-1$. But $\lambda_M(U \cap A) = k$ since $U \cap A$ is $S-T$-separating. It follows that $e \in \mathrm{cl}_M(U \cap A)$, and in particular $e \in \mathrm{cl}_M(U)$. By Lemma 2.7, we cannot have $e \in \mathrm{cl}_M(E(M)-(U \cup e))$. Hence $u \in \mathrm{cl}^*_M(U)$ by Lemma 2.1. But then $\lambda_M(U \cup e) = k-1$, contradicting the fact that $\kappa_M(U,T) = k$. □

Proof of Theorem 3.4. We prove the result by induction on $|F|$, the case $|F| = 0$ being trivial. Suppose the result fails for a matroid M with subsets S,T,F as in the theorem. Let $k := \kappa_M(S,T)$ and $t := |F|$. For each $e \in F$, let (A_e, B_e) be $S-T$-separating of order $k+1$ with $e \in A_e$ and $|A_e|$ as small as possible. Let f be such that $|A_f| \leq |A_e|$ for all $e \in F$.

Claim 3.6.1. $A_f \cap F = \{f\}$.

Proof. Suppose $g \in A_f \cap F$ with $g \neq f$. By our choice of f, we must have that $A_g = A_f$ (using Lemma 2.6). Since g is not flexible, Lemma 3.5 implies that $(A_f - g, B_f \cup g)$ is $S-T$-separating of order $k+1$, contradicting minimality of $|A_f|$. □

By Lemma 3.6 we can apply the theorem inductively, replacing S by A_f and F by $F-f$, thus finding a sequence $(A_2, \ldots, A_t)$ of nested $A_f - T$-separating sets of order $k+1$. But now the sequence $(A_f, A_2, \ldots, A_t)$ satisfies all conditions of the theorem. □

We will use the following two facts:

Lemma 3.7. *Let M be a matroid, let S,T be disjoint subsets of $E(M)$, let $k := \kappa_M(S,T)$, and let $(A_1,B_1), \ldots, (A_t,B_t)$ be a sequence of nested $S-T$-separations of order $k+1$. Let (C,D) be a partition of $E(M)-(S \cup T)$ such that C is independent, D is coindependent, and $\lambda_{M/C \backslash D}(S) = k$. Let $i,j \in \{1,\ldots,t\}$ with $i<j$. Let $C' := C \cap (A_j - A_i)$, let $D' := D \cap (A_j - A_i)$, and let $M' := M/C' \backslash D'$. Then (A_i, B_j) is $S-T$-separating of order $k+1$ in M'. Moreover, $M'|A_i = M|A_i$ and $M'|B_j = M|B_j$.*

Proof. Let $M' := M/C' \backslash D'$. By definition of C and D, $\kappa_{M'}(S,T) = k$. By monotonicity of λ, $\lambda_{M'}(A_i) = k$. It follows from Lemma 2.7 that for all $e \in C'$, $e \notin \mathrm{cl}_M(A_i \cup (C' - \{e\})$ and $e \notin \mathrm{cl}_M(B_j \cup (C' - \{e\})$. From this the second claim follows. □

Lemma 3.8 (Geelen, Gerards, and Whittle [5, Lemma 4.7])**.** *Let M be a matroid, let S,T be disjoint subsets of $E(M)$, and let $k := \kappa_M(S,T)$. There exist sets $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_M(S_1,T_1) = k$.*

4 Proof of the result for finite fields

Let M be a rank-r matroid on ground set E. Write $M = M[D]$ if the $r \times E$ matrix D (over field $\mathbb{F}$) represents M. For $S \subseteq E$, denote by $D[S]$ the submatrix of D induced by the columns labeled by S, and denote by $\langle D[S]\rangle$ the vector space spanned by the columns of $D[S]$. To clean up notation we will write $\langle S\rangle$ for $\langle D[S]\rangle$ if D is clear from the context.

Recall that, if (A,B) is such that $\lambda_{M[D]}(A) = k$, then $\langle A\rangle \cap \langle B\rangle$ is a k-dimensional subspace of $\mathbb{F}^r$. Assume $\mathbb{F} = \mathrm{GF}(q)$. Denote by $M^+_{(A,B)}$ the matroid obtained from M by adding a copy of $\mathrm{PG}(k-1,q)$ to M, such that in the representation it is contained in $\langle A\rangle \cap \langle B\rangle$. Furthermore, $M^+_A := M^+_{(A,B)} \backslash B$ and $M^+_B := M^+_{(A,B)} \backslash A$. Now we can carry out row operations to get $M^+_{(A,B)} = M[D']$, with

$$D' = \begin{array}{c} \\ \left[\begin{array}{c|c|c} A & X & B \\ \hline D_1 & \begin{array}{c} 0 \\ \hline P \\ \hline 0 \end{array} & \begin{array}{c} 0 \\ \hline D_2 \end{array} \\ \hline 0 & & \end{array}\right] \end{array},$$

where P is a $k \times X$ matrix representing $\mathrm{PG}(k-1,q)$ (with elements labeled by X). We remark that $M^+_{(A,B)}$ is the generalized parallel connection of M^+_A and M^+_B along X (cf. [9, Section 11.4]). The following lemma follows easily from Lemma 3.7.

Lemma 4.1. *Let M be a $\mathrm{GF}(q)$-representable matroid, let S and T be disjoint subsets of $E(M)$ with $\kappa_M(S,T) = k$, and let (A,B) be $S-T$-separating of order $k+1$. Let (C,D) be a partition of $E(M) - (S \cup T)$ such that $\lambda_{M/C\backslash D}(S) = k$. Then $(M^+_{(A,B)}/C\backslash D)|X = M^+_{(A,B)}|X$.*

We repeat the main result, filling in an explicit value for the constant:

Theorem 4.2. *Let q be a prime power, let M be a $\mathrm{GF}(q)$-representable matroid on ground set E, let N be a minor of M on n elements, let $S, T \subseteq E$, and let $k := \kappa_M(S,T)$. If $|E - (S \cup T)| > n + 2(n+1)q^{n^2}$, then there exists an element $e \in E$ such that at least one of the following holds:*

(i) $\kappa_{M\backslash e}(S,T) = k$ and N is a minor of $M\backslash e$;

(ii) $\kappa_{M/e}(S,T) = k$ and N is a minor of M/e.

The proof is not hard, but unfortunately we could not avoid using rather involved notation. For that reason we give a rough sketch of the idea. Let M be a counterexample. First we construct a long sequence $(A_1,B_1),\ldots,(A_t,B_t)$ of nested $S-T$-separating partitions of order $k+1$. For each i we define the matroid M_i, obtained from $M^+_{B_i}$ by deleting or contracting the elements of $B_i - E(N)$ so that the minor N is preserved. Since each M_i will have the same number of elements, only a finite number of distinct represented matroids can arise. Since our matroid is sufficiently large it follows that, after suitably relabeling the new elements, $M_i = M_j$ for some $i < j$. This shows that the elements in $A_j - A_i$

can be removed in such a way that both N and the $S-T$-connectivity are preserved, which contradicts our choice of M.

Proof. Let q, M, N, n, S, T, and k be as stated, and assume $|E-(S\cup T)| > n+2(n+1)q^{n^2}$, yet no element can be removed keeping both the $S-T$-connectivity and the minor N. Let (C,D) be a partition of $E-(S\cup T)$ such that $\lambda_{M/C\backslash D}(S) = \kappa_M(S,T)$ and such that C is independent and D coindependent. Let (C_N, D_N) be a partition of $E-E(N)$ such that $N = M/C_N\backslash D_N$ and such that C_N is independent and D_N coindependent. By our assumption, $C\cap C_N = \emptyset$ and $D\cap D_N = \emptyset$.

Let $F := C\cup D - E(N)$, and let $t'' := |F|$. Then $t'' > 2(n+1)q^{n^2}$. By Theorem 3.4, there is a nested sequence $(A''_1,\dots,A''_{t''})$ of $S-T$-separating sets of order $k+1$ such that A''_i is a proper subset of A''_{i+1} for $i\in\{1,\dots,t''-1\}$. Let $(f''_1,\dots,f''_{t''})$ be the corresponding ordering of F. Consider the sequence $(A''_1\cap E(N),\dots,A''_{t''}\cap E(N))$. This sequence contains at most $n+1$ different elements. It follows that $(A''_1,\dots,A''_{t''})$ has a subsequence $(A'_1,\dots,A'_{t'})$ such that $A'_i\cap E(N) = A'_j\cap E(N)$ for all $i,j\in\{1,\dots,t'\}$, and such that $t'\geq t''/(n+1) > 2q^{n^2}$.

Let $(f'_1,\dots,f'_{t'})$ be the corresponding subsequence of F. Using duality if necessary we may assume that $|\{f'_1,\dots,f'_{t'}\}\cap C| \geq |\{f'_1,\dots,f'_{t'}\}\cap D|$. Let $(A_1,\dots,A_t)$ be a subsequence of $(A'_1,\dots,A'_{t'})$ such that $A_{i+1}-A_i$ contains an element of C for all $i\in\{1,\dots,t-1\}$, and such that $t\geq t'/2 > q^{n^2}$. For each $i\in\{1,\dots,t\}$, define $B_i := E-A_i$.

Let H be an $r\times E$ matrix over $\mathrm{GF}(q)$ representing M. Let $s := (q^k-1)/(q-1)$. For each i, let $W_i := \langle A_i\rangle \cap \langle B_i\rangle$, and let $X_i := \{x^i_1,\dots,x^i_s\}$ be a set of labels disjoint from E and disjoint from X_j for all $j\in\{1,\dots,t\}-\{i\}$. Let the $k\times X_1$ matrix P_1 be an arbitrary representation of $\mathrm{PG}(k-1,q)$ having ground set X_1.

For each $i\in\{1,\dots,t\}$, let M_i^+ be the matroid $M^+_{(A_i,B_i)}$ with the set X relabeled by X_i. Moreover, we assume this labeling was chosen such that, in $(M_1^+)_i^+/C\backslash D$, x^i_j is parallel to x^1_j for all $j\in\{1,\dots,s\}$ (where $(M_1^+)_i^+$ is defined in the obvious way). This can be done because of Lemma 4.1.

Now we define, for each i, a matroid N_i as follows: first set $N'_i := (M_i^+\backslash A_i)/(C_N\cap B_i)\backslash(D_N\cap B_i)$. Now N_i is obtained from N'_i by relabeling x^i_j by x^1_j. Let H_i be the corresponding representation matrix. Note that, for $i,j\in\{1,\dots,t\}$, $E(N_i) = E(N_j)\subseteq E(N)\cup X_1$. Hence $|E(N_i)|\leq n+s$. Since $X_i\subseteq\langle B_i\rangle$, we find that $\mathrm{rk}(N_i)\leq n$. Furthermore, for all $x\in X_1$, $H_i[x] = H_j[x]$. Hence there are at most $((q^n-1)/(q-1)+1)^n \leq q^{n^2}$ distinct representation matrices H_i. Since $t > q^{n^2}$, there exist $i,j\in\{1,\dots,t\}$ with $i<j$ such that $H_i = H_j$. But then $M/(B_i\cap C_N)\backslash(B_i\cap D_N)$ is equal to

$$\big(M/((A_j-A_i)\cap C)\backslash((A_j-A_i)\cap D)\big)/(B_j\cap C_N)\backslash(B_j\cap D_N),$$

using Lemmas 3.7 and 4.1. In particular, since $(A_j-A_i)\cap C\neq\emptyset$, there exists an $e\in C$ such that $\kappa_{M/e}(S,T) = k$ and M/e has N as minor, a contradiction. □

As promised, here is a proof of Conjecture 1.1 from Theorem 4.2 when M is $\mathrm{GF}(q)$-representable.

Proof of Conjecture 1.1 for GF(q)*-representable matroids.* Let $n := |Q \cup R|$ and set $c := n + 2(n+1)q^{n^2}$. Let (C, D) be a partition of $E - (Q \cup R)$ such that $\lambda_{M/C \backslash D}(Q) = k$. Now apply Theorem 4.2 with $N = M/C\backslash D$, S, and T. The result follows. □

5 Intertwining two connectivities

In this section we prove Conjecture 1.1 for all representable matroids. The key property we need for our proof is that we can add a point to the intersection of two non-skew flats. Formally:

Definition 5.1. A matroid M has the *intersection property* if for all flats $S, T \in E(M)$ such that $\sqcap(S, T) > 0$, there exist a matroid N and a non-loop element $e \in E(N)$ such that $N\backslash e = M$, and $e \in \mathrm{cl}_N(S) \cap \mathrm{cl}_N(T)$. In this case, we say that N is a *good extension* of M (with respect to S, T). A class of matroids $\mathscr{M}$ is *intersection-closed* if every $M \in \mathscr{M}$ has the intersection property, and $\mathscr{M}$ is closed under minors, duality, and good extensions.

Note that the class of representable matroids is evidently intersection-closed. The Vámos matroid shows that not all matroids have the intersection property. See [1] for more on matroids with the intersection property.

The restriction we use is reminiscent of the double-circuit property from the min-max theorem for matroid matching (see [2]). However, whereas the min-max theorem is false even for affine spaces, in our case the condition appears to be just an artifact of our proof. We remain hopeful that Conjecture 1.1 can be proven without this condition. We will now state and prove the main result.

Theorem 5.2. *There exists a function $c : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let M be a matroid in an intersection-closed family, and let $Q, R, S, T, F \subseteq E(M)$ be sets of elements such that $Q \cap R = S \cap T = \emptyset$ and $F \subseteq E(M) - (Q \cup R \cup S \cup T)$. Let $k := \kappa_M(Q, R)$ and $l := \kappa_M(S, T)$. If $|F| \geq c(k, l)$, then there exists an element $e \in F$ such that one of the following holds:*

(i) $\kappa_{M\backslash e}(Q, R) = k$ *and* $\kappa_{M\backslash e}(S, T) = l$;

(ii) $\kappa_{M/e}(Q, R) = k$ *and* $\kappa_{M/e}(S, T) = l$.

Proof. We prove that the result holds for $c(k, l) := 4^{k+l}$. We proceed by induction on $k + l$, noting that the base case where $k = 0$ or $l = 0$ is straightforward. Assume that the result holds for all k', l' with $k' + l' < k + l$, but that M, Q, R, S, T, F form a counterexample. Possibly after relabeling we may assume $k \leq l$. By Lemma 3.8 we can assume that $|S| = |T| = l$, and that S and T are independent sets. Furthermore, we can assume that for each $e \in F$ either $\kappa_{M\backslash e}(Q, R) < k$ or $\kappa_{M/e}(Q, R) < k$.

Claim 5.2.1. *There exists a $Q-R$ separating partition (A, B) with $\lambda(A) = k$, such that $A \cap S \neq \emptyset$, $A \cap T \neq \emptyset$, $|A \cap (S \cup T)| \geq l$, and $|B \cap F| \geq \lfloor \frac{1}{2} c(k, l) \rfloor$.*

Proof. Let $(A_1,\dots,A_t)$ be the nested sequence of $Q-R$ separating sets from Theorem 3.4, let $(B_1,\dots,B_t)$ be their complements, and let $(f_1,\dots,f_t)$ be the corresponding ordering of F. Let $i := \lfloor t/2 \rfloor$. First we show that one of A_i and B_i meets both of S and T. Indeed: otherwise we have (possibly after swapping S and T) that $S \subseteq A_i$ and $T \subseteq B_i$. In that case (A_i, B_i) is $S-T$ separating with $\lambda_M(A_i) = k$. It follows that $k = l$. Assume f_i is non-contractible with respect to (Q,R). Then $\lambda_{M/f_i}(B_i) = k-1$, and therefore f_i is also non-contractible with respect to (S,T), so the theorem holds with $e = f_i$.

Hence, possibly after exchanging the sequences $(A_1,\dots,A_t)$ and $(B_1,\dots,B_t)$, we can assume $A_i \cap S \neq \emptyset$ and $A_i \cap T \neq \emptyset$. If $|A_i \cap (S \cup T)| < l$ then $|B_i \cap (S \cup T)| > l$, and therefore $(A,B) = (B_i, A_i)$ is a partition as desired; otherwise we simply take $(A,B) = (A_i, B_i)$. □

If necessary, we relabel Q and R so that $Q \subseteq A$ and $R \subseteq B$. Define

$$S_1 := A \cap S \qquad T_1 := A \cap T$$
$$S_2 := B \cap S \qquad T_2 := B \cap T.$$

Also define $F_2 := B \cap F$. We try to remove the elements from A while preserving the $S-T$ connectivity. Let $N_0 := M$, and order the elements of $A - (S_1 \cup T_1)$ arbitrarily as $a_1,\dots,a_u$. For $i = 1,2,\dots,u$ define N_i as follows. If $\kappa_{N_{i-1}\backslash a_i}(S,T) = l$ and $a_i \notin \mathrm{cl}^*_{N_{i-1}}(B)$, then $N_i := N_{i-1}\backslash a_i$. Else, if $\kappa_{N_{i-1}/a_i}(S,T) = l$ and $a_i \notin \mathrm{cl}_{N_{i-1}}(B)$, then $N_i := N_{i-1}/a_i$. Otherwise $N_i := N_{i-1}$. Observe that $\kappa_{N_u}(S,T) = l$ and $\kappa_{N_u}(A \cap E(N_u), R) = \lambda_{N_u}(B) = k$. We distinguish two cases.

Case I: $\sqcap_{N_u}(S_1, T_1) > 0$**.** Since N_u is a member of an intersection-closed family, we can find a matroid N^+ in this family with a non-loop element s such that $N^+\backslash s = N_u$, and $s \in \mathrm{cl}_{N^+}(S_1) \cap \mathrm{cl}_{N^+}(T_1)$. We distinguish two subcases:

Case Ia: $s \notin \mathrm{cl}_{N^+}(B)$**.** Let $N := N^+/s$, and define $Q' := A \cap E(N)$. Then $\kappa_N(S,T) = l-1$ and $\kappa_N(Q',R) = k$. Since $|F_2| \geq c(k,l-1)$, by induction we can find an element $e \in F_2$ such that either $\kappa_{N/e}(S,T) = l-1$ and $\kappa_{N/e}(Q',R) = k$, or $\kappa_{N\backslash e}(S,T) = l-1$ and $\kappa_{N\backslash e}(Q',R) = k$. We assume the former, and remark that the proof for the latter case is similar.

Claim 5.2.2. *$\kappa_{M/e}(Q,R) = k$ and $\kappa_{M/e}(S,T) = l$.*

Proof. Suppose $\kappa_{M/e}(Q,R) < k$, that is, e is non-contractible with respect to (Q,R). By Lemma 3.6, e is also non-contractible with respect to (A,R) in M. But (A,B) is $Q'-R$ separating, so we must have $\lambda_{M/e}(A) = k$, a contradiction.

Next, let C,D be such that C is independent in N, $e \in C$ and, in $N_\infty := N/C\backslash D$, we have $E(N_\infty) = S \cup T$ and $\lambda_{N_\infty}(S) = l-1$. Since C is independent in N^+/s, it follows that s is not a loop in N^+/C. Let $N^+_\infty := N^+/C\backslash D$. Since $s \in \mathrm{cl}_{N^+_\infty}(S) \cap \mathrm{cl}_{N^+_\infty}(T)$, we must have that $\lambda_{N^+_\infty\backslash s}(S) = l$. It follows that $\kappa_{M/e}(S,T) = l$ as desired. □

Case Ib: $s \in \mathrm{cl}_{N^+}(B)$**.** Again we define $Q' := A \cap E(N)$. Let $(A_1, \ldots, A_{t'})$ be the nested sequence of $Q' - R$ separating sets in N^+ from Theorem 3.4 (applied to N, Q', R, and F_2), let $(B_1, \ldots, B_{t'})$ be their complements, and let $(f_1, \ldots, f_{t'})$ be the corresponding ordering of F_2. Let $j := c(k-1, l-1)$. If $s \notin \mathrm{cl}_{N^+}(B_j)$ then we apply the arguments from Case (Ia) with $A_j \cap E(N)$ replacing Q', B_j replacing B, and $F \cap B_j$ replacing F_2. Otherwise, let $N := N^+/s$, define $R' := B_j$ and $F_2' := F_2 - B_j$. We have $\kappa_N(Q', R') = k - 1$ and $\kappa_N(S, T) = l - 1$. Since $|F_2'| \geq c(k-1, l-1)$, we find by induction an element $e \in F_2'$ such that either $\kappa_{N/e}(Q', R') = k - 1$ and $\kappa_{N/e}(S, T) = l - 1$, or $\kappa_{N \backslash e}(Q', R') = k - 1$ and $\kappa_{N \backslash e}(S, T) = l - 1$. We assume the latter, and remark that the proof in the former case is similar.

Claim 5.2.3. *$\kappa_{M \backslash e}(Q, R) = k$ and $\kappa_{M \backslash e}(S, T) = l$.*

Proof. Suppose $e = f_i' \in F_2'$ is non-deletable with respect to (Q, R). Then $e \in \mathrm{cl}^*_{N_u}(B_{i'})$, so $\lambda_{N_u \backslash e}(B_{i'}) = k - 1$. But $s \in \mathrm{cl}_{N^+}(B_{i'}) \cap \mathrm{cl}_{N^+}(A_{i'} - e)$, so we must have $\lambda_{N^+ \backslash e}(B_{i'}) = k - 1$. But then $\lambda_{N^+ \backslash e/s}(B_{i'}) = k - 2$, contradicting our choice of e. Hence e is deletable with respect to (Q, R).

The proof that $\kappa_{M \backslash e}(S, T) = l$ is the same as before and we omit it. □

Case II: $\sqcap_{N_u}(S_1, T_1) = \sqcap^*_{N_u}(S_1, T_1) = 0$**.** By dualizing if necessary, we may assume there is an element $e \in \mathrm{cl}_{N_u}(A) \cap \mathrm{cl}_{N_u}(B) \cap F$, i.e. an element that is deletable with respect to (Q, R) in M. We assume $e \in A$ (replacing (A, B) by $(A \cup e, B - e)$ otherwise).

Claim 5.2.4. $e \in \mathrm{cl}_{N_u}(S_1 \cup T_1)$.

Proof. First we show that $\mathrm{cl}^*_{N_u}(B) - (S_1 \cup T_1)$ spans $S_1 \cup T_1$. Suppose not, and let $X := (S_1 \cup T_1) - \mathrm{cl}^*_{N_u}(B)$. By construction of N_u, all remaining elements are in $\mathrm{cl}_{N_u}(B)$, so we have that $N_u \backslash X$ has lower rank than N_u. Hence X contains a cocircuit. But this contradicts the fact that S_1 and T_1 are coskew.

Now pick $B' := \mathrm{cl}^*_{N_u}(B) - (S_1 \cup T_1 \cup e)$ and $A' := A - B'$. Then $k' := \lambda_{N_u}(A') \leq k$. But since $S_1 \cup T_1 \cup e \subseteq A'$ and $S_1 \cup T_1 \cup e \subseteq \mathrm{cl}_{N_u}(B')$, we must have that $\mathrm{rk}_{N_u}(S_1 \cup T_1 \cup e) \leq k' \leq k \leq l$. Note that $|S_1 \cup T_1| \geq l$ and, since S_1 and T_1 are skew, $\mathrm{rk}_{N_u}(S_1 \cup T_1) \geq l$. It follows that $k' = k = l$, and therefore $e \in \mathrm{cl}_{N_u}(S_1 \cup T_1)$ as desired. □

Similar to before, we define $Q' := A \cap E(N_u) - \{e\}$. Let $(A_1, \ldots, A_{t'})$ be the nested sequence of $Q' - R$ separating sets in N_u from Theorem 3.4 (applied to Q', R, and F_2), let $(B_1, \ldots, B_{t'})$ be their complements, and let $(f_1, \ldots, f_{t'})$ be the corresponding ordering of F_2. Let $j := c(k-1, l)$. Again we distinguish two cases.

Case IIa: $e \notin \mathrm{cl}_{N_u}(B_j)$**.** Let N_v be obtained from N_u by contracting e and removing the other elements from A_j according to the same rules used to obtain N_u. We can then apply the arguments of Case I to N_v (with A_j replacing A and B_j replacing B), observing that $|F \cap B_j| \geq 2(c(k-1, l-1) + c(k, l-1))$.

Case IIb: $e \in \mathrm{cl}_{N_u}(B_j)$**.** Let $N := N_u/e$, and define $R' := B_j$. By induction we find an element $f \in \{f_1,\ldots,f_j\}$ such that either $\kappa_{N/f}(Q',R') = k-1$ and $\kappa_{N/f}(S,T) = l$, or $\kappa_{N\backslash f}(Q',R') = k-1$ and $\kappa_{N\backslash f}(S,T) = l$. As before, in the former case we have $\kappa_{M/f}(Q,R) = k$ and $\kappa_{M/f}(S,T) = l$ and in the latter case we have $\kappa_{M\backslash f}(Q,R) = k$ and $\kappa_{M\backslash f}(S,T) = l$. This completes the proof of the theorem. □

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