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AN ALGEBRAIC FRAMEWORK FOR DISCRETE TOMOGRAPHY: REVEALING THE STRUCTURE OF DEPENDENCIES*

ARJEN STOLK† AND K. JOOST BATENBURG‡

Abstract. Discrete tomography is concerned with the reconstruction of images that are defined on a discrete set of lattice points from their projections in several directions. The range of values that can be assigned to each lattice point is typically a small discrete set. In this paper we present a framework for studying these problems from an algebraic perspective, based on ring theory and commutative algebra. A principal advantage of this abstract setting is that a vast body of existing theory becomes accessible for solving discrete tomography problems. We provide proofs of several new results on the structure of dependencies between projections, including a discrete analogon of the well-known Helgason–Ludwig consistency conditions from continuous tomography.

Key words. discrete tomography, reconstruction of integer-valued functions, dependencies between line sums

AMS subject classifications. Primary, 94A12; Secondary, 94C10, 11H71

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1. Introduction.

Discrete tomography (DT) is concerned with the reconstruction of discrete images from their projections. According to [16, 17], the field of DT deals with the reconstruction of images from a small number of projections, where the set of pixel values is known to have only a few discrete values. On the other hand, when the field of DT was founded by Shepp in 1994, the main focus was on the reconstruction of (usually binary) images for which the domain is a discrete set, which seems to be more natural as a characteristic property of DT. The number of pixel values may be as small as two, but reconstruction problems for more values are also considered. In this paper, we follow the latter definition of DT.

Most of the literature on DT focuses on the reconstruction of lattice images that are defined on a discrete set of points, typically a subset of $\mathbb{Z}^2$. An image is formed by assigning a value to each lattice point. The range of these values is usually restricted to a small, discrete set. *Projections* of an image are obtained by summation of the point values along sets of parallel discrete lines. For an individual line, such a sum is often referred to as the *line sum*.

DT problems have been studied in various fields of mathematics, including combinatorics, discrete mathematics, and combinatorial optimization. An overview of known results is given at the end of section 2 of [10]. Already in the 1950s, both Ryser [24] and Gale [9] considered the combinatorial problem of reconstructing a binary matrix from its row and column sums. They provided existence and uniqueness conditions, as well as concrete reconstruction algorithms. DT emerged as a field of research in the 1990s, motivated by applications in atomic resolution electron microscopy [25, 19, 18]. Since that time, many fundamental results on the existence,

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http://www.siam.org/journals/sidma/24-3/76669.html
†Mathematical Institute, University of Leiden, 2333 CA Leiden, The Netherlands (stzvlkx@gmail.com).
‡Vision Lab, University of Antwerp, B-2610 Wilrijk, Belgium (joost.batenburg@ua.ac.be).

uniqueness, and stability of solutions have been obtained, as well as a variety of proposed reconstruction algorithms.

Besides purely combinatorial properties, integer numbers play an important role throughout DT, due to their close connection with the concepts of lattice lines and line sums. A link with the field of algebraic number theory was established in [11], where Gardner and Gritzmann used Galois theory and p-adic valuations to prove that convex lattice sets are uniquely determined by their projections in certain finite sets of directions. Hajdu and Tijdeman described in [14] how a powerful extension of the binary tomography problem is obtained by considering images for which each point is assigned a value in $\mathbb{Z}$. The fact that both the image values and the line sums are in $\mathbb{Z}$ allows for the application of ring theory, and in particular the Chinese remainder theorem, for characterizing the set of switching components: images for which the projections in all given lattice directions are 0. Their theory for the extended problem leads to new insights in the binary reconstruction problem as well, as any binary solution must also be a solution of the extended problem, and the binary solutions can be characterized as the solutions of the extended problem that have minimal Euclidean norm.

More recently, techniques from algebra and algebraic number theory were used to obtain DT results on stability [1], a link between DT and the Prouhet–Tarry–Escott problem from number theory [2], and the reconstruction of quasi crystals [4, 13].

In this paper we present a comprehensive framework for the treatment of DT problems from an algebraic perspective, based on general ring theory and commutative algebra. Modern algebra is a mature mathematical field that provides a framework in which a wide range of problems can be described, analyzed, and solved. An important advantage of this abstract setting is that a vast body of existing theory becomes accessible for solving DT problems. Based on our algebraic framework, we provide proofs of several new results on the structure of dependencies between the projections, including a discrete analogon of the well-known Helgason–Ludwig consistency conditions from continuous tomography.

A principal aim of this paper is to give additional connections between the study of DT in the fields of combinatorics and classical number theory on one side and the proposed abstract algebraic model on the other side. To this end, the definitions and results we describe within our algebraic model will be followed by concrete examples, illustrating their correspondences with existing results and concepts.

This paper is organized as follows. In section 2 the basic DT problems are introduced in a combinatorial setting. In section 2.2 we recall an example from the literature. Section 3 introduces the same concepts, but this time in our proposed algebraic framework. We also derive some basic properties linking combinatorial notions to notions within the framework. Sections 4 and 5 set up the algebraic theory for images defined on $\mathbb{Z}^2$ (the *global* case). In section 6 we revisit the example from section 2.2 from an algebraic perspective.

In the next sections, the attention is shifted toward images that are defined on a *subset* of $\mathbb{Z}^2$. Section 7 introduces a relative setup, where a DT problem on a particular domain is related to a problem on a subset of that domain. In sections 8 and 9, we apply this relation to completely describe the structure of line sums for finite convex sets. The appendix collects some algebraic results used in this paper.

2. Classical definitions and problems. In this section we provide an overview of several important problems in DT, within their original combinatorial context. For the most part, we follow the basic terminology from [16].

Let $K \subset \mathbb{Z}$. We will call the elements of K *colors*. In DT, we often have $K = \{0, 1\}$. Note that K does not have to be finite. A nonzero vector $v = (a, b) \in \mathbb{Z}^2$ such that $a \geq 0$ is called a *lattice direction*. If a and b are coprime, we call v a *primitive lattice direction*. The set of all lattice directions is denoted by $\mathcal{V}$. For any $t \in \mathbb{Z}^2$, the set $\ell_{v,t} = \{\lambda v + t \,|\lambda \in \mathbb{Z}\}$ is called a *lattice line parallel to* v. The set of all lattice lines parallel to v is denoted by $\mathcal{L}_v$. A function $f : \mathbb{Z}^2 \to K$ with finite support is called a *table*. The set of all tables is denoted by $\mathcal{F}$. We prefer using the word table over the more common *image*, as the latter is also used to denote the image of a map.

Definition 2.1. *Let $f \in \mathcal{F}$ and $v \in \mathcal{V}$. The function $P_v(f) : \mathcal{L}_v \to \mathbb{Z}$ defined by*

$$P_v(f)(\ell) = \sum_{x \in \ell} f(x)$$

is called the projection of f in the direction v.

The values $P_v(f)(\ell)$ are usually called *line sums*. For $v \in \mathcal{V}$, we denote the set of all functions $\mathcal{L}_v \to \mathbb{Z}$ by L_v (the *potential line sums for direction* v).

For a finite ordered set $D = \{v_1, \dots, v_k\} \subset \mathcal{V}$ of distinct primitive lattice directions, we define the *projection* of f along D by

$$P_D(f) = P_{v_1}(f) \oplus \cdots \oplus P_{v_k}(f),$$

where $\oplus$ denotes the direct sum. The map P_D is called the *projection map*. Put $L_D = L_{v_1} \oplus \cdots \oplus L_{v_k}$, the set of *potential line sums for directions* D.

Most problems in DT deal with the reconstruction of a table f from its projections in a given set of lattice directions. It is common that a set $A \subset \mathbb{Z}^2$ is given, such that the support of f must be contained in A. We call the set A the *reconstruction grid*. Put $\mathcal{A} = \{f \in \mathcal{F} : x \notin A \implies f(x) = 0\}$.

As in Chapter 1 of [16], we introduce three basic problems of DT: consistency, reconstruction, and uniqueness.

Problem 1 (consistency). *Let K and A be given. Let $D = \{v_1, \dots, v_k\} \subset \mathcal{V}$ be a finite set of distinct primitive lattice directions and let $p \in L_D$ be a given map of potential line sums. Does there exist a table $f \in \mathcal{A}$ such that $P_D(f) = p$?*

Problem 2 (reconstruction). *Let K and A be given. Let $D = \{v_1, \dots, v_k\} \subset \mathcal{V}$ be a finite set of distinct primitive lattice directions and let $p \in L_D$ be a given map of potential line sums. Construct a table $f \in \mathcal{A}$ such that $P_D(f) = p$, or decide that no such table exists.*

Problem 3 (uniqueness). *Let K and A be given. Let $D = \{v_1, \dots, v_k\} \subset \mathcal{V}$ be a finite set of distinct primitive lattice directions and let $p \in L_D$ be a given map of potential line sums. Let $f \in \mathcal{A}$ such that $P_D(f) = p$. Is there another solution $g \neq f$ with $P_D(g) = p$?*

In the most common reconstruction problem in the DT literature, A is a finite rectangular set of points and $K = \{0, 1\}$. In this case, a table f is usually considered as a rectangular binary matrix. For the case $D = \{(1, 0), (0, 1)\}$, the three basic problems were solved by Ryser in the 1950s. It was proved by Gardner, Gritzmann, and Prangenberg that the reconstruction problem for more than two lattice directions is NP-hard [12]. Several variants of the reconstruction problem that make additional assumptions about the table f, such as convexity or periodicity, can be solved effectively if more projections are given [5, 8].

Hajdu and Tijdeman considered the case that A is a rectangular set and $K = \mathbb{Z}$. They showed that the resulting problems are strongly connected to the binary case: if the reconstruction problem for $K = \mathbb{Z}$ has a binary solution, the set of binary solutions is exactly the set of tables over $\mathbb{Z}$ for which the Euclidean norm is minimal. In [14], they characterized the set of *switching components*, tables for which the projection is 0 in all given lattice directions. In particular, this provides a (partial) solution for the uniqueness problem, which also has consequences for the case $K = \{0, 1\}$.

2.1. Dependencies. The theory of Hajdu and Tijdeman also provides insight into the *dependencies* between the projections of a table; see Definition 2.2.

If the reconstruction grid A is finite, the set of lines along directions in D intersecting with A is also finite. Denote the number of such lines by $n(A, D)$. A map $p \in L_D$ of potential line sums can now be represented by an $n(A, D)$-dimensional vector over $\mathbb{Z}$, where we consider only the line sums for lines that intersect with A. In the remainder of this section, we use this representation for the projection of a table.

DEFINITION 2.2 (dependency). *Let $A \subset \mathbb{Z}^2$ be a finite reconstruction grid. Let $D \subset \mathcal{V}$ be a finite set of distinct primitive lattice directions. A* dependency *is a vector $c \in \mathbb{Z}^{n(A,D)}$ such that for all $f \in \mathcal{F} : P_D(f) \cdot c = 0$, where $\cdot$ denotes the vector inner product.*

The vector c is called the *coefficient vector* of the dependency. Intuitively, dependencies are relations that must always hold between the set of projections of an object. The simplest such relation corresponds to the fact that for all lattice directions $v_1, v_2 \in \mathcal{V}$,

$$\sum_{\ell \in \mathcal{L}_{v_1}} P_{v_1}(f)(\ell) = \sum_{\ell \in \mathcal{L}_{v_2}} P_{v_2}(f)(\ell) = \sum_{x \in \mathcal{A}} f(x).$$

More complex dependencies can be formed between sets of three or more projections. We call a set of dependencies *independent* if the corresponding coefficient vectors are linearly independent. Note that the dependencies form a linear subspace of $\mathbb{Z}^{n(A,D)}$.

2.2. Example. In [14], the dependencies were systematically investigated for the case $K = \mathbb{Q}$, $A = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i < m, 0 \leq j < n\}$, and $D = \{(1, 0), (0, 1), (1, 1), (1, -1)\}$. Put

$$\begin{aligned}
r_j &= \sum_{i=0}^{m-1} f(i,j), && 0 \leq j \leq n-1, && \text{the row sums,}\\
c_i &= \sum_{j=0}^{n-1} f(i,j), && 0 \leq i \leq m-1, && \text{the column sums,}\\
t_h &= \sum_{\substack{j=i+h\\(i,j)\in A}} f(i,j), && -m+1 \leq h < n, && \text{the diagonal sums,}\\
u_h &= \sum_{\substack{j=-i+h\\(i,j)\in A}} f(i,j), && 0 \leq h < m+n-1, && \text{the antidiagonal sums.}
\end{aligned}$$

Then the following seven dependencies hold for the line sums:

$$\begin{aligned}
\sum_{j=0}^{n-1} r_j = \sum_{i=0}^{m-1} c_i &= \sum_{h=-m+1}^{n-1} t_h = \sum_{h=0}^{m+n-2} u_h, \\
\sum_{\substack{h=-m+1 \\ h \text{ is odd}}}^{n-1} t_h &= \sum_{\substack{h=0 \\ h \text{ is odd}}}^{m+n-2} u_h, \\
-\sum_{j=0}^{n-1} j r_j + \sum_{i=0}^{m-1} i c_i &= \sum_{h=-m+1}^{n-1} h t_h, \\
\sum_{j=0}^{n-1} j r_j + \sum_{i=0}^{m-1} i c_i &= \sum_{h=0}^{m+n-2} h u_h, \\
2\sum_{j=0}^{n-1} j^2 r_j + 2\sum_{i=0}^{m-1} i^2 c_i &= \sum_{h=-m+1}^{n-1} h^2 t_h + \sum_{h=0}^{m+n-2} h^2 u_h.
\end{aligned}$$

If A is sufficiently large, these dependencies form an independent set. It was shown in [14] that these relations form a basis of the space of all dependencies over $\mathbb{Q}$. Although Hajdu and Tijdeman described the complete set of dependencies for this particular set of directions, they did not provide a characterization of dependencies for general sets of directions. They derived a formula for the dimension of the space of dependencies for any rectangular set A and any set of directions.

Several properties of the given example deserve further attention. The coefficients of the vectors describing the dependencies have the structure of polynomials in i, j, and h. The degree of these polynomials is at most two (for the last dependency), and this degree appears to increase along with the number of directions. In particular, the maximum degree of the polynomials describing the coefficients in this example is two for the dependency involving all four directions, whereas the maximum degree for a dependency involving any subset of three directions is one, and the maximum degree for the pairwise dependencies is zero.

For this set D, all of the seven independent dependencies can be defined for the case $A = \mathbb{Z}^2$, such that for smaller reconstruction grids the same relations hold, restricted to the lines intersecting A. In this paper, we will denote such dependencies by the term *global dependencies*.

For other sets of directions, such as $D = \{(1,1),(1,2)\}$, there can also be dependencies such as the one shown in Figure 2.1. Two corner points of the reconstruction grid belong to a line in both directions, leading to trivial dependencies between the corresponding line sums. Such dependencies depend on the shape of the reconstruction grid and cannot be extended to dependencies on $A = \mathbb{Z}^2$. We refer to such dependencies as *local dependencies*. An analysis of the dependencies for the case of a rectangular reconstruction grid A is given in [27].

There is a strong analogy between the concept of dependencies between line sums in DT, and so-called *consistency conditions* in continuous tomography. Ludwig [21] and Helgason [15] described a set of relations between the projections of a continuous function f defined on $\mathbb{R}^2$. Moreover, if a set of one-dimensional functions satisfies these relations, this is also a *sufficient* condition for correspondence to a projected function. The first such relation represents the fact that the total *mass* of the scanned object (expressed as an area integral of f) should equal the total mass in each of the

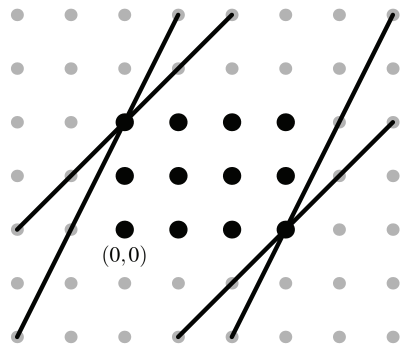


FIG. 2.1. *At corners of the reconstruction grid, there can be local dependencies between line sums in two or more directions.*

projections (expressed as an integral of each projection function). The second relation represents the fact that the center of mass of the object corresponds to the center of mass of the projections. Ludwig and Helgason describe an infinite set of such relations, of increasing order.

In the remainder of this paper, we provide a characterization of the dependencies between projections in DT, based on our algebraic framework. As dependencies indicate relations that must hold for any set of projections, they provide a necessary condition for the consistency problem. We prove that for a particular class of DT problems, a set of projections satisfies the dependency relations *if and only if* it corresponds to a table. This leads to a discrete analogon of the consistency conditions from continuous tomography.

3. Algebraic framework. In this section we introduce the basic concepts and definitions used in our algebraic formulation of DT. For a thorough introduction to terminology and concepts of algebra, we refer the reader to [20]. The appendix of this paper covers some of the properties used in detail.

First, the *line sum map*, which maps a table to its corresponding line sums, is defined in algebraic terms. We will prove (Lemma 3.4) that if the cokernel of the *line sum map* has a simple structure, the space of dependencies also has this structure. The section concludes with an example demonstrating that if the cokernel does not have this structure, the result does not apply.

Let $A \subset \mathbb{Z}^2$ be nonempty and let k be a commutative ring that is not the zero ring. We let

$$T(A,k) = k^{(A)} = \{f : A \to k \mid f(x) = 0 \text{ for all but finitely many } x \in A\}$$

be the space of k-valued *tables* on A. It is a free k-module with a basis indexed by the elements of A. We will identify the elements of A with the elements of this basis.

Let $d \in \mathbb{Z}^2 \setminus \{0\}$ be a direction and $p \in \mathbb{Z}^2$ be a point. Recall that the *(lattice) line* through p in the direction d is the set $\{p + \lambda d \mid \lambda \in \mathbb{Z}\}$. Two points p and q are on the same line in direction d precisely if they differ by an integer multiple of d. The quotient group $\mathbb{Z}^2/\langle d\rangle$ therefore parametrizes all the lines in the direction d. For $A \subset \mathbb{Z}^2$ write $\mathcal{L}_d(A)$ for the image of A in $\mathbb{Z}^2/\langle d\rangle$, i.e., the set of lines in the direction d that intersect A.

We call $(a,b) \in \mathbb{Z}^2 \setminus \{0\}$ with $\gcd(a,b) = 1$ a *primitive* direction. Whenever d is a primitive direction, the quotient $\mathbb{Z}^2/\langle d\rangle$ is isomorphic to $\mathbb{Z}$. This means we can label the lines in direction d with integers, starting with 0 for the line through the origin.

We fix once and for all pairwise independent directions $d_1, \ldots, d_t \in \mathbb{Z}^2 \setminus \{0\}$ and write $\mathcal{L}_i(A) = \mathcal{L}_{d_i}(A)$ for the lines in direction d_i that meet A. Let

$$L_i(A,k) = k^{(\mathcal{L}_i(A))}$$

be the space of *potential line sums* in direction d_i and let

$$L(A,k) = \bigoplus_{i=1}^{t} L_i(A,k)$$

be the full space of potential line sums. These are all free k-modules. A basis for $L_i(A,k)$ is given by $\mathcal{L}_i(A)$, and so a basis for $L(A,k)$ is given by $\mathcal{L}(A) := \coprod_{i=1}^{t} \mathcal{L}_i(A)$.

DEFINITION 3.1. *The* line sum map

$$\sigma_{A,k} : T(A,k) \longrightarrow L(A,k)$$

is defined as the k-linear map that sends $x \in A$ to the vector $(\ell_i)_{i=1}^{t}$, where $\ell_i \in \mathcal{L}_i(A)$ is the line in direction d_i through x.

The line sum map is the direct sum of the component maps $\sigma_{i,A,k} : T(A,k) \to L_i(A,k)$.

The kernel of the line sum map,

$$\ker(\sigma_{A,k}) = \{t \in T(A,k) \mid \sigma_{A,k}(t) = 0\},$$

identifies the space of *switching components* of the DT problem: two tables have the same vector of line sums if and only if they differ by an element of $\ker(\sigma_{A,k})$. We will use the cokernel

$$\operatorname{cok}(\sigma_{A,k}) = L(A,k) \big/ \operatorname{im}(\sigma_{A,k})$$

to gain insight into the structure of the set of possible line sums of tables within the full space of potential line sums. In particular, the cardinality of the cokernel "measures" the difference between these sets.

DEFINITION 3.2. *A k-linear* dependency *between line sums is a k-linear map*

$$r : L(A,k) \longrightarrow k$$

such that $r \circ \sigma_{A,k}$ is the zero map.

Note that such a map gives rise to a map $\bar{r} : \operatorname{cok}(\sigma_{A,k}) \to k$ and that, conversely, any k-linear map $\operatorname{cok}(\sigma_{A,k}) \to k$ gives rise to a dependency. In other words, there is an inclusion

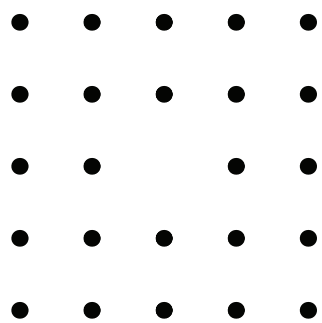
$$\operatorname{Hom}_k(\operatorname{cok}(\sigma_{A,k}), k) \subset \operatorname{Hom}_k(L(A,k), k)$$

whose image is precisely the set of dependencies. We will write $\operatorname{Dep}(A,k)$ for this subspace.

Remark 3.3. The natural map

$$\begin{array}{rccc} W : & \operatorname{Hom}_k(L(A,k),k) & \longrightarrow & \{c : \mathcal{L}(A) \to k\} \\ & \phi & \longmapsto & [\ell \mapsto \phi(\ell)] \end{array}$$

is a bijection.



			1	−1
	−1	1	−1	1
	1		1	
1	−1	1	−1	
−1	1			

FIG. 3.1. *Left:* 5×5 *grid where the center point has been removed. Right: example of a table on this grid where the dependencies do not provide sufficient conditions for consistency.*

For a $\phi \in \operatorname{Hom}_k(L(A,k),k)$ we can think of $W(\phi)$ as the *weight* that ϕ assigns to each line in $\mathcal{L}(A)$. For dependencies this corresponds to the concept of a *coefficient vector* introduced in section 2.1. If $r \in \operatorname{Dep}(A,k)$ is a dependency, then $W(r)$ corresponds to the vector c from Definition 2.2.

The next lemma gives an example of the link between algebraic properties of the cokernel and questions concerning the DT problem.

LEMMA 3.4. *Let* $A \subset \mathbb{Z}^2$ *and let* k *be a commutative ring that is not the zero ring. Suppose that* $\operatorname{cok}(\sigma_{A,k})$ *is a free* k*-module of finite rank* n*. Then* $\operatorname{Dep}(A,k)$ *is also a free* k*-module of rank* n*, and for any* $l \in L(A,k)$ *we have* $l \in \operatorname{im}(\sigma_{A,k})$ *if and only if* $d(l) = 0$ *for all* $d \in \operatorname{Dep}(A,k)$*.*

Proof. Let $c_1, \dots, c_n$ be a basis for $\operatorname{cok}(\sigma_{A,k})$. We can write any $x \in \operatorname{cok}(\sigma_{A,k})$ uniquely as $x_1c_1 + \cdots + x_nc_n$. The maps $e_i : x \mapsto x_i$ are elements of $\operatorname{Dep}(A,k) = \operatorname{Hom}_k(\operatorname{cok}(\sigma_{A,k}), k)$. We claim that the e_i are a basis for $\operatorname{Dep}(A,k)$. Let r be in $\operatorname{Dep}(A,k)$. For any $x = \sum x_ic_i$ in $\operatorname{cok}(\sigma_{A,k})$ we have

$$r(x) = d\left(\sum x_ic_i\right) = \sum x_ir(c_i).$$

Put $r_i = r(c_i)$. Then we have $r = \sum r_ie_i$. So the e_i generate $\operatorname{Dep}(A,k)$. Note that the r_i are uniquely determined by r. We conclude that the e_i are a basis of $\operatorname{Dep}(A,k)$.

Note that for all $x \in \operatorname{cok}(\sigma_{A,k})$, we have $x = \sum e_i(x)c_i$, so if $d(x) = 0$ for all $d \in \operatorname{Dep}(A,k)$, then $x = 0$. When we apply this to $x = \bar{l}$ for some $l \in L(A,k)$, we see that $r(l) = 0$ for all $r \in \operatorname{Dep}(A,k)$ if and only if $\bar{l} = 0$, i.e., $l \in \operatorname{im}(\sigma_{A,k})$. ∎

The lemma that we have just proved can be interpreted as follows. Whenever we find for some A that $\operatorname{cok}(\sigma_{A,k})$ is a free k-module of finite rank, we have the following: *a vector of potential line sums comes from a table precisely if it satisfies all dependencies.* As the space of dependencies is also free and of finite rank, it in fact suffices to check finitely many dependencies.

To further motivate the interest of such freeness results, we conclude this section with an example where it fails. In this example, the cokernel is not free, and, as a consequence, the dependencies do not give sufficient conditions for consistency.

Let A be the grid as shown in Figure 3.1 (left), a 5×5 square with the center point removed. We take projections in the directions $(0,1)$, $(1,0)$, $(1,2)$, and $(2,1)$. For the ring k we take $\mathbb{Z}$, the integers.

Something interesting happens when we consider the table shown on the right of Figure 3.1. Its line sums in the given directions are all 0, with the exception of the ones going through the missing center point, which are all 2. Write v for this vector in $L(A,k)$.

Now consider $\frac{1}{2}v$: four 1's for the lines through the center, and 0's everywhere else. One checks easily by hand that this vector is *not* in the image of the projection map. No table with coefficients in $\mathbb{Z}$ gives rise to these line sums.

Any dependency for this configuration will send v to 0 as it is in the image of the projection map. But then $\frac{1}{2}v$ also goes to 0, by linearity. However, $\frac{1}{2}v$ is not in the image of the projection map. So here we have an example where satisfying all dependencies is not sufficient to ensure a vector is in the image of the projection map.

To understand this example and how it was constructed, the reader may want to see how the proof of Theorem 8.3 fails for this particular A.

4. The global case. In this section we consider the case $A = \mathbb{Z}^2$. We will show that in this case, the objects defined in the previous section have the structure of rings and modules and their homomorphisms. This allows us to completely describe the kernel and cokernel of the line sum map.

According to Lemma 3.4, the space $\mathrm{Dep}\,(A,k)$ can be characterized if $\mathrm{cok}\,(\sigma_{A,k})$ is a free k-module of finite rank. We will demonstrate (Theorem 4.2) that this property is satisfied for the case $A = \mathbb{Z}^2$ and, in addition, give a characterization of the switching components for this case, similar to the work of Hajdu and Tijdeman in [14]. This results in a strong statement concerning the consistency problem for the case $A = \mathbb{Z}^2$: a set of line sums corresponds to a table if and only if it satisfies a certain number of independent dependencies (Corollary 4.3).

The following three k-modules are isomorphic in a natural way:

$$T(\mathbb{Z}^2,k) \cong k[\mathbb{Z}^2] \cong k[u,u^{-1},v,v^{-1}].$$

For some basic properties of group rings such as $k[\mathbb{Z}^2]$, see the appendix. The isomorphisms are

$$\begin{array}{ccc} T(\mathbb{Z}^2,k) & \longrightarrow & k[\mathbb{Z}^2], \\ [c:\mathbb{Z}^2 \to k] & \longmapsto & \displaystyle\sum_{x\in\mathbb{Z}^2} c(x)x \end{array}$$

and

$$\begin{array}{ccc} k[\mathbb{Z}^2] & \longrightarrow & k[u,u^{-1},v,v^{-1}], \\ \displaystyle\sum_{x\in\mathbb{Z}^2} \lambda_x x & \longmapsto & \displaystyle\sum_{(a,b)\in\mathbb{Z}^2} \lambda_{(a,b)} u^a v^b. \end{array}$$

Note that $k[\mathbb{Z}^2]$ and $k[u,u^{-1},v,v^{-1}]$ are both k-algebras and that the second isomorphism is an isomorphism of k-algebras. We also view $T(\mathbb{Z}^2,k)$ as a k-algebra via these isomorphisms.

In the same way there is a natural isomorphism of k-modules,

$$L_i(\mathbb{Z}^2,k) \cong k\left[\mathbb{Z}^2/\langle d_i\rangle\right],$$

which puts a ring structure on the spaces of potential line sums. By Lemma A.2 we have an isomorphism $k[\mathbb{Z}^2/\langle d_i\rangle] \cong k[\mathbb{Z}^2]/(d_i-1)$. Viewed in this way, the line sum map $\sigma_{i,\mathbb{Z}^2,k} : T(\mathbb{Z}^2,k) \to L_i(\mathbb{Z}^2,k)$ is the quotient map

$$k[\mathbb{Z}^2] \to k[\mathbb{Z}^2]/(d_i-1).$$

Taking sums, we find a k-algebra structure on $L(\mathbb{Z}^2,k)$ such that the line sum map $\sigma_{\mathbb{Z}^2,k} : T(\mathbb{Z}^2,k) \to L(\mathbb{Z}^2,k)$ is a k-algebra map which is the direct sum of quotient maps. We will now study the structure of these quotient maps from an algebraic perspective using the ideas outlined in the last part of the appendix.

LEMMA 4.1. *Let $d, e \in \mathbb{Z}^2$ be independent directions. Then $d-1$ is weakly coprime (see Definition* A.6 *in the appendix) to $e-1$ in $k[\mathbb{Z}^2]$.*

Proof. By Lemma A.2 we can see $k[\mathbb{Z}^2]/(d-1)$ as the group ring $k[\mathbb{Z}^2/\langle d\rangle]$. Suppose we have

$$f = \sum_{x \in \mathbb{Z}^2/\langle d\rangle} f_x x \in k\left[\mathbb{Z}^2/\langle d\rangle\right]$$

such that $(e-1)f = 0$. When we expand

$$0 = (e-1)f = \sum_{x \in \mathbb{Z}^2/\langle d\rangle} (f_{x-e} - f_x)x,$$

we see that $f_{x+ke} = f_x$ for all $x \in \mathbb{Z}^2/\langle d\rangle$ and $k \in \mathbb{Z}$. As d and e are independent, all $x + ke$ are different in $\mathbb{Z}^2/\langle d\rangle$. We conclude that we must have $f_x = 0$ for all $x \in \mathbb{Z}^2/\langle d\rangle$, as only finitely many coefficients of f are nonzero. ◻

THEOREM 4.2. *The kernel of $\sigma_{\mathbb{Z}^2,k}$ is given by*

$$\ker\left(\sigma_{\mathbb{Z}^2,k}\right) = (d_1 - 1)\cdots(d_t - 1)k[\mathbb{Z}^2].$$

The cokernel $\operatorname{cok}\left(\sigma_{\mathbb{Z}^2,k}\right)$ *is a free k-module of rank*

$$\sum_{1 \le i < j \le t} |\det(d_i, d_j)|.$$

Proof. By Lemma 4.1, $d_i - 1$ is weakly coprime to $d_j - 1$ in $k[\mathbb{Z}^2]$ whenever $i \neq j$. So we can apply Theorem A.9 to the map

$$\sigma_{\mathbb{Z}^2,k} : k[\mathbb{Z}^2] \longrightarrow \bigoplus_{i=0}^{t} \mathbb{Z}^2/_{d_i - 1}.$$

This immediately gives us the formula for the kernel given in the theorem. For the cokernel, we note that by Lemma A.2 we have

$$k[\mathbb{Z}^2]/(d_i - 1, d_j - 1) = k\left[\mathbb{Z}^2/\langle d_i, d_j\rangle\right],$$

which is a free k-module of rank $|\det(d_i, d_j)|$. In particular, all the successive quotients of the filtration on the cokernel are free k-modules. Therefore all the quotients are split (see, e.g., [20, Chap. III.3, Prop. 3.2]), and we conclude that

$$\operatorname{cok}\left(\sigma_{\mathbb{Z}^2,k}\right) \cong \bigoplus_{1 \le i < j \le t} k\left[\mathbb{Z}^2/\langle d_i, d_j\rangle\right]. \qquad ◻$$

This result leads to a partial discrete analogon of the Helgason–Ludwig consistency conditions from continuous tomography, which involves a finite set of directions and provides a necessary and sufficient condition for consistency of a vector of potential line sums.

COROLLARY 4.3. *A vector of potential line sums in $L(\mathbb{Z}^2, k)$ comes from a table in $T(\mathbb{Z}^2, k)$ if and only if it satisfies all dependencies. Moreover, we only have to check this for a set of $\sum_{1 \le i < j \le t} |\det(d_i, d_j)|$ independent dependencies.*

Proof. This is an easy consequence of two earlier results: Theorem 4.2 shows that we can apply Lemma 3.4 to the global cokernel. ∎

Note that the Helgason–Ludwig conditions from continuous tomography concern the consistency of an *infinite* set of projections.

Looking at the example in section 2.2, we compute $\sum_{1 \le i < j \le 4} |\det(d_i, d_j)| = 7$. This tells us that the list of seven independent dependencies we had is complete, in the sense that at least when k is a field, they will form a basis of $\operatorname{Dep}(\mathbb{Z}^2, k)$.

For a full discrete analogon of the continuous consistency conditions, one should also provide a characterization of the structure of the individual dependencies. The next section provides additional insight into the coefficient structure of the dependencies.

5. The global line sum map as an extension of rings.

We now focus our attention more on the ring theoretic aspect of the line sum map for the case $A = \mathbb{Z}^2$. The main result of this section is a full characterization of the actual *coefficients* of the dependencies (Theorem 5.2), which is necessary to make practical use of the developed theory, for example, when checking a set of line sums for errors.

We can view $L(\mathbb{Z}^2, k)$ as an extension of its subring $\operatorname{im}(\sigma_{\mathbb{Z}^2,k})$. Both of these rings have relative dimension 1 over k. This is a situation that has been extensively studied because of its relation to algebraic number theory (see, e.g., [3]) and singular curves in algebraic geometry (see, e.g., [26]). An interesting attribute of such an extension is its *conductor*: the largest ideal of $L(\mathbb{Z}^2, k)$ that is also an ideal of $\operatorname{im}(\sigma_{\mathbb{Z}^2,k})$.

LEMMA 5.1. *Put $D_i = \prod_{j \ne i}(d_j - 1)$. The conductor of $L(\mathbb{Z}^2, k)$ over $\operatorname{im}(\sigma_{\mathbb{Z}^2,k})$ is given by*

$$\mathfrak{f}_k = \overline{D_1} k[\mathbb{Z}^2]\big/_{d_1 - 1} \oplus \cdots \oplus \overline{D_t} k[\mathbb{Z}^2]\big/_{d_t - 1}.$$

Proof. Note that D_i reduces to 0 in $k[\mathbb{Z}^2]/(d_j - 1)$ for all $j \ne i$. We conclude that the ideal $(D_1, \ldots, D_t)$ of $k[\mathbb{Z}^2]$ is mapped by $\sigma_{\mathbb{Z}^2,k}$ onto $\mathfrak{f}_k$. In particular, this implies that $\mathfrak{f}_k$ is indeed an $\operatorname{im}(\sigma_{\mathbb{Z}^2,k})$ ideal.

Conversely, suppose $I \subset \operatorname{im}(\sigma_{\mathbb{Z}^2,k})$ is an ideal that is also closed under multiplication by $L(\mathbb{Z}^2, k)$. We want to show that $I \subset \mathfrak{f}_k$. Let $x = (x_1, \ldots, x_t) \in I$. As I is an $L(\mathbb{Z}^2, k)$ ideal, we must also have $(0, \ldots, x_i, \ldots, 0) \in I$. As $I \subset \operatorname{im}(\sigma_{\mathbb{Z}^2,k})$, there is an $\tilde{x}_i \in k[\mathbb{Z}^2]$ such that $\sigma_{\mathbb{Z}^2,k}(\tilde{x}_i) = (0, \ldots, x_i, \ldots, 0)$. We have $x = \sigma_{\mathbb{Z}^2,k}(\tilde{x}_1 + \cdots + \tilde{x}_t)$, so we are done if we can show that $\tilde{x}_i$ is a multiple of D_i for all i.

To show this, we apply Theorem 4.2 to the directions d_j with $j \ne i$. Note that $\tilde{x}_i$ maps to 0 under the line sum map in this case. The theorem tells us that the kernel of this map is generated by D_i, so that $\tilde{x}_i$ is indeed a multiple of D_i for all i. ∎

Note that the quotient module $L(\mathbb{Z}^2, k)/\mathfrak{f}_k$ is a free k-module that has dimension $\sum_{i \ne j} |\det(d_i, d_j)|$. This is twice the dimension of $\operatorname{cok}(\sigma_{\mathbb{Z}^2,k}) = L(\mathbb{Z}^2, k)/\operatorname{im}(\sigma_{\mathbb{Z}^2,k})$. We see that $\operatorname{im}(\sigma_{\mathbb{Z}^2,k})$ sits precisely in the middle between $L(\mathbb{Z}^2, k)$ and $\mathfrak{f}_k$. This is not a surprise; it happens in this situation whenever the rings are "sufficiently nice," e.g., when they are Gorenstein rings (see [6] for more information about these rings).

We have not yet fully explored the implications of this ring theoretic view for the structure of $\operatorname{cok}(\sigma_{\mathbb{Z}^2,k})$, but we believe it warrants further investigation. To illustrate its use, we will derive the following result on the coefficient functions of dependencies in $\mathrm{Dep}\left(\mathbb{Z}^2, k\right)$.

For the remainder of this section, we assume that all the d_i are primitive directions. This means that $\mathbb{Z}^2/\langle d_i\rangle$ is isomorphic to $\mathbb{Z}$. For the rest of this section we also fix isomorphisms $\mathbb{Z}^2/\langle d_i\rangle \cong \mathbb{Z}$. What this means is that the lines in each direction d_i can be numbered in sequence. The choice of isomorphisms comes down to picking whether we number from left to right or the other way around.

Recall from Remark 3.3 that a dependency $r \in \mathrm{Dep}\left(Z^2, k\right)$ can be represented by a function $W(r)$ from $\mathcal{L}(\mathbb{Z}^2)$ to k. From the choices we have just made, $\mathcal{L}(\mathbb{Z}^2)$ is identified with t copies of $\mathbb{Z}$. This means that we can represent a dependency by a set of t two-sided infinite sequences,

$$W_i(r) : \mathbb{Z} \longrightarrow k.$$

THEOREM 5.2. *There are positive integers $s_1, \dots, s_t$ and $c_{i,j} \in \mathbb{Z}$ for $i = 1, \dots, t$ and $j = 1, \dots s_i$, with $c_{i,1} = \pm 1$ and $c_{i,s_i} = \pm 1$ for $i = 1, \dots, t$ such that the following holds. For every commutative ring k that is not the zero ring, for every dependency $r \in \mathrm{Dep}\left(\mathbb{Z}^2, k\right)$, and for $i = 1, \dots, t$, the two-sided infinite sequence $W_i(r)$ defined above satisfies the linear recurrence relation*

$$\forall n \in \mathbb{Z} : \sum_{j=1}^{s_i} c_{i,j} [W_i(r)](n + j) = 0.$$

Proof. The isomorphism $\mathbb{Z}^2/\langle d_i\rangle \cong \mathbb{Z}$ gives rise to an isomorphism

$$k[\mathbb{Z}^2]/_{d_i - 1} \cong k\left[Z^2/_{\langle d_i\rangle}\right] \cong k[x, x^{-1}]$$

of $L_i(\mathbb{Z}^2, k)$ with the Laurent polynomial ring $k[x, x^{-1}]$. Write $\overline{D_i} = \sum_j a_j x^j$ in $k[x, x^{-1}]$.

Let $r \in \mathrm{Dep}\left(\mathbb{Z}^2, k\right)$ be a dependency. We consider the map $r_i : L_i(\mathbb{Z}^2, k) \to k$ induced by r. As $\mathrm{im}\left(\sigma_{\mathbb{Z}^2,k}\right)$ is in the kernel of r, we have $\mathfrak{f}_k \subset \ker(r)$. As $x \in k[x, x^{-1}]$ is a unit, we see that $x^n\overline{D_i}$ must be in $\ker(r_i)$ for all integers n.

From the definition of $W_i(r)$, we know that $r_i(x^n) = [W_i(r)](n)$. Using this we see that for all $n \in \mathbb{Z}$

$$0 = r_i(x^n\overline{D_i}) = r_i\Big(\sum_j a_j x^{n+j}\Big) = \sum_j a_j [W_i(r)](n + j).$$

What remains to be shown is that the coefficients a_j of $\overline{D_i}$ are integers that do not depend on k and that the leading and trailing coefficients are ± 1. Note that these properties hold for a product if they hold for the factors. Moreover, we have

$$\overline{D_i} = \prod_{j \neq i} (x^{\det(d_i, d_j)} - 1),$$

and the properties clearly hold for the polynomials $x^* - 1$ in $k[x, x^{-1}]$. ∎

The upshot of the outer coefficients being ± 1 is that the recurrence relations can always be used to uniquely determine the sequences from any sufficiently large set of consecutive coefficients.

6. An example. We revisit the example from [14] that was discussed in section 2.2. It concerns the directions $d_1 = (1, 0)$, $d_2 = (0, 1)$, $d_3 = (1, 1)$, and $d_4 = (1, -1)$. For simplicity, we take $k = \mathbb{Q}$, but we will make some comments on how to deal with the case $k = \mathbb{Z}$.

We identify $T(\mathbb{Z}^2, k)$ with $k[x, x^{-1}, y, v^{-1}]$. Note that for each i, we have $\mathbb{Z}^2/\langle d\rangle \cong \mathbb{Z}$. We pick isomorphisms $L_i(\mathbb{Z}^2, k) = k[z, z^{-1}]$ in such a way that the components of the line sum map are the maps $k[x, x^{-1}, y, y^{-1}] \to k[z, z^{-1}]$ given by

i	map	$x \mapsto$	$y \mapsto$
1	r	1	z
2	c	z	1
3	t	z	z^{-1}
4	u	z	z

The line sum map is given by

$$\sigma = (r, c, t, u) : k[x, x^{-1}, y, y^{-1}] \longrightarrow \left(k[z, z^{-1}]\right)^4 .$$

The maps r, c, t, and u are related to the line sums described in section 2.2 in a straightforward manner. Let f and the r_i, c_i, t_i, and u_i be as in that section. Put $F = \sum_{i,j} f(i, j)x^i y^j$. Then we have $r(F) = \sum_i r_i z^i$ and likewise for the other maps.

We compute

$$\begin{aligned} D_1 &= (y - 1)(xy - 1)(xy^{-1} - 1), & r(D_1) &= -z^{-1}(z - 1)^3,\\ D_2 &= (x - 1)(xy - 1)(xy^{-1} - 1), & c(D_2) &= (z - 1)^3,\\ D_3 &= (x - 1)(y - 1)(xy^{-1} - 1), & t(D_3) &= (z - 1)^3(z + 1),\\ D_4 &= (x - 1)(y - 1)(xy - 1), & u(D_4) &= (z - 1)^3(z + 1). \end{aligned}$$

Let $M = M_1 \oplus \cdots \oplus M_4$ be the quotient vector space

$$M = \frac{k[z, z^{-1}]}{r(D_1)} \oplus \frac{k[z, z^{-1}]}{c(D_2)} \oplus \frac{k[z, z^{-1}]}{t(D_3)} \oplus \frac{k[z, z^{-1}]}{u(D_4)},$$

and let $\pi = (\pi_1, \ldots, \pi_4)$ be the quotient map $(k[z, z^{-1}])^4 \to M$. As discussed in the previous section, there is a surjective map $M \to \operatorname{cok}(\sigma)$. This means we can realize $\operatorname{Dep}\left(\mathbb{Z}^2, k\right)$ as a subspace of $\operatorname{Hom}(M, k)$.

A basis for $\operatorname{Hom}\left(k[z, z^{-1}]/(z - 1)^3, k\right)$ is given by the maps

$$v_1 : z^i \mapsto 1, \qquad v_2 : z^i \mapsto i, \qquad v_3 : z^i \mapsto i^2.$$

Let $e : \mathbb{Z} \mapsto \mathbb{Z}$ be the map that sends n to 0 if n is odd, and to 1 if n is even. A basis for $\operatorname{Hom}\left(k[z, z^{-1}]/(z - 1)^3(z + 1), k\right)$ is given by

$$w_1 : z^i \mapsto e(i), \qquad w_2 : z^i \mapsto 1 - e(i), \qquad w_3 : z^i \mapsto i, \qquad w_4 : z^i \mapsto i^2.$$

These maps together give a basis for $\operatorname{Hom}(M, k)$ consisting of 14 elements:

- $v_{1,1}$, $v_{1,2}$, and $v_{1,3}$ acting on the first coordinate;
- $v_{2,1}$, $v_{2,2}$, and $v_{2,3}$ acting on the second coordinate;
- $w_{1,1}, \ldots, w_{1,4}$ acting on the third coordinate; and
- $w_{2,1}, \ldots, w_{2,4}$ acting on the fourth coordinate.

These maps correspond to the sums of line sums that also come up in section 2.2. For example, $v_{1,1}$ sends F to $\sum_i r_i$, and $w_{2,3}$ sends F to $\sum_i i^2 u_i$.

The dependencies form a subvector space of Hom (M, k) of dimension 7. What we still have to do is to determine which linear combinations of $v_{i,j}$'s and $w_{i,j}$'s correspond to dependencies. One way to do this is to write down the restrictions coming from the fact that tables of the form $x^i y^j$ must be sent to 0 by a dependency. We will see in section 8 that we have only to check finitely many such tables before we have a complete set of restrictions.

Another way to find these restrictions is to consider the compositions of the v's and w's with $\pi \circ \sigma$, i.e., the maps they induce in Hom $\left(k[x, x^{-1}, y, y^{-1}], k\right)$. The dependencies are precisely those relations that go to 0 under this composition. The maps we obtain in this way are

map	$x^i y^j \mapsto$	map	$x^i y^j \mapsto$
$v_{1,1}$	1	$v_{2,1}$	1
$v_{1,2}$	i	$v_{2,1}$	j
$v_{1,3}$	i^2	$v_{2,1}$	j^2
$w_{1,1}$	$e(i-j)$	$w_{2,1}$	$e(i+j)$
$w_{1,2}$	$1-e(i-j)$	$w_{2,2}$	$1-e(i+j)$
$w_{1,3}$	$i-j$	$w_{2,3}$	$i+j$
$w_{1,3}$	$(i-j)^2$	$w_{2,3}$	$(i+j)^2$

From this table, one easily reads off a basis for the dependencies. For example, we can take

$$\begin{aligned} v_{1,1} = v_{2,1} = w_{1,1} + w_{1,2} &= w_{2,1} + w_{2,2}, \\ w_{1,1} &= w_{2,1}, \\ v_{1,2} - v_{2,2} &= w_{1,3}, \\ v_{1,2} + v_{2,2} &= w_{2,3}, \\ 2v_{1,3} + 2v_{2,3} &= w_{1,4} + w_{2,4}. \end{aligned}$$

These correspond to the dependencies described in section 2.2.

If we want to write down a basis for the dependencies not over $\mathbb{Q}$ but over $\mathbb{Z}$ or some other ring, we have to be a little more careful. The maps $v_1, \ldots, v_3$ do not form a basis of Hom $\left(k[z, z^{-1}]/(z-1)^3, k\right)$ if $k = \mathbb{Z}$. The map sending z^i to $\frac{1}{2}i(i-1)$ is in this module, but it is equal to $\frac{1}{2}(v_3 - v_2)$, which is not a $\mathbb{Z}$-linear combination of the v's.

A basis that works regardless of the ring k is found as follows. Note that

$$k[z, z^{-1}]\big/(z-1)^3 = k \cdot 1 \oplus k \cdot z \oplus k \cdot z^2.$$

This choice of a basis also gives a basis for the k-dual. This basis works independently of k. The price we pay for this more general approach is that the formulas that come out are not as nice, making it harder to find the dependencies by hand. The linear algebra involved does not become more difficult.

7. The comparison sequence. We now turn our attention to cases where A is a true subset of $\mathbb{Z}^2$. Let $A \subset B \subset \mathbb{Z}^2$. Our aim in this section is to compare the kernels and cokernels of $\sigma_{A,k}$ and $\sigma_{B,k}$. The key result, Lemma 7.1, reflects the approach that will be followed in the next sections to carry the results for $A = \mathbb{Z}^2$ over to finite convex subsets of $\mathbb{Z}^2$.

Put $T(B/A,k) = k^{(B\setminus A)}$ and $L(B/A,k) = \bigoplus_{i=1}^{t} k^{(\mathcal{L}_i(B)\setminus\mathcal{L}_i(A))}$. Looking at the bases for the spaces involved, it is clear that there are direct sum decompositions $T(B,k) = T(A,k) \oplus T(B/A,k)$ and $L(B,k) = L(A,k) \oplus L(B/A,k)$.

This means we can represent $\sigma_{B,k}$ as a two-by-two matrix of k-linear maps,

$$\sigma_{B,k} = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

where $p : T(A,k) \to L(A,k)$, $q : T(B/A,k) \to L(A,k)$, $r : T(A) \to L(B/A,k)$, and $s : T(B/A,k) \to L(B/A,k)$ are the restrictions and projections of $\sigma_{B,k}$ to the appropriate subspaces. The usual matrix multiplication rule

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

holds when we have $x \in T(A,k)$, $y \in T(B/A,k)$, $u \in L(A,k)$, and $v \in L(B/A,k)$ such that $\sigma_{B,k}(x \oplus y) = u \oplus v$.

As $L(B/A,k)$ consists precisely of those lines through B that do not intersect A, we have $r = 0$. Similarly, p is just the map sending tables on A to their line sums, so $p = \sigma_{A,k}$. The other two maps, q and s, encode interesting information about the relative situation, so we will give them more descriptive names:

$$\sigma_{B/A,k} : T(B/A,k) \longrightarrow L(B/A,k) \quad \text{(the } \textit{relative line sum map}\text{)}$$

and

$$\delta_{B/A,k} : T(B/A,k) \longrightarrow L(A,k) \quad \text{(the } \textit{interference map}\text{)}.$$

LEMMA 7.1 (the comparison sequence). *There is a long exact sequence*

$$\begin{aligned} 0 &\to \ker(\sigma_{A,k}) \to \ker(\sigma_{B,k}) \to \ker\left(\sigma_{B/A,k}\right) \\ &\to \operatorname{cok}(\sigma_{A,k}) \to \operatorname{cok}(\sigma_{B,k}) \to \operatorname{cok}\left(\sigma_{B/A,k}\right) \to 0. \end{aligned}$$

The map $\overline{\delta_{B/A,k}} : \ker\left(\sigma_{B/A,k}\right) \to \operatorname{cok}(\sigma_{A,k})$ *comes from the interference map* $\delta_{B/A,k}$ *defined above.*

Proof. This is an application of the snake lemma (see, e.g., [20, Chap. III.9, Lemma 9.1.]). □

The extension B/A is called *noninterfering* if it satisfies the following (equivalent) conditions:

1. The map $\overline{\delta_{B/A,k}}$ is the zero map.
2. The map $\ker(\sigma_{B,k}) \to \ker\left(\sigma_{B/A,k}\right)$ is surjective.
3. The map $\operatorname{cok}(\sigma_{A,k}) \to \operatorname{cok}(\sigma_{B,k})$ is injective.

8. Finite, convex *A*. Recall that according to Lemma 3.4, the space $\operatorname{Dep}(A,k)$ can be characterized if $\operatorname{cok}(\sigma_{A,k})$ is a free k-module of finite rank. We have already seen that this property holds for the case $A = \mathbb{Z}^2$. In this section, we will demonstrate that this property also holds if A is a finite, convex subset of $\mathbb{Z}^2$ (Theorem 8.3).

We briefly recall some basic notions about convex sets and explain how we want to use them. Convex sets have been extensively studied in relation to linear programming and optimization; see, for example, [23, Chap. 2] and [7, Chap. 2].

A subset $C \subset \mathbb{R}^2$ is called *convex* if for any $x, y \in C$ the line segment between x and y is completely contained in C. The *convex hull* of a subset $S \subset \mathbb{R}^2$ is the smallest convex subset C of $\mathbb{R}^2$ containing S. We write $H(S)$ for the convex hull of S. We call $A \subset \mathbb{Z}^2$ convex if $A = H(A) \cap \mathbb{Z}^2$.

We call $C \subset \mathbb{R}^2$ a *convex polygon* if $C = H(S)$ for some finite $S \subset \mathbb{R}^2$. The set of *corners* of a convex polygon C is the smallest set S such that $H(S) = C$.

Let $C_1, C_2 \subset \mathbb{R}^2$ be convex polygons. Then

$$C_1 + C_2 = \{c_1 + c_2 \mid c_1 \in C_1, c_2 \in C_2\}$$

is also a convex polygon. Let s be a corner of $C_1 + C_2$. Then s can be written in a unique way as $s_1 + s_2$ with $s_1 \in C_1$ and $s_2 \in C_2$. Moreover, s_1 and s_2 are corners of C_1 and C_2, respectively.

Let $f \in k[\mathbb{Z}^2]$ and write $f = \sum_{x \in \mathbb{Z}_2} f_x x$. Then the *support* of f is the set

$$\operatorname{supp} f = \left\{x \in \mathbb{Z}^2 \mid f_x \neq 0\right\}.$$

Note that $\operatorname{supp} f$ is always a finite set. The *polygon* of f is

$$P(f) = H(\operatorname{supp} f).$$

It is a convex polygon. Let s be a corner of $P(f)$; then we say that s is a *strong* corner of $P(f)$ if f_s is not a zero divisor. We say that f *has strong corners* if all corners of $P(f)$ are strong.

LEMMA 8.1. *Let $f, g \in k[\mathbb{Z}^2]$ and suppose that f has strong corners. Then*

$$P(fg) = P(f) + P(g).$$

If g also has strong corners, then fg has strong corners.

Proof. The inclusion $P(fg) \subset P(f) + P(g)$ is obvious. For the other inclusion, suppose that s is a corner of $P(f) + P(g)$. Then the coefficient of fg at s is

$$\sum_{a+b=s} f_a g_b = f_{s_f} g_{s_g},$$

where s_f and s_g are the unique corners of $P(f)$ and $P(g)$, respectively, such that $s = s_f + s_g$. We see that this coefficient is nonzero as f_{s_f} is not a zero divisor, so $s \in P(fg)$. This shows that $P(f) + P(g) \subset P(fg)$. Moreover, if g also has strong corners, g_{s_g} is also not a zero divisor and so $f_{s_f} g_{s_g}$ is not a zero divisor. ∎

LEMMA 8.2. *The generator of* $\ker\left(\sigma_{\mathbb{Z}^2,k}\right)$,

$$D = (d_1 - 1) \cdots (d_t - 1),$$

has strong corners. Moreover, $\Delta = P(D)$ does not depend on k.

Proof. The polygon of $d_i - 1$ is a 2-gon with coefficients ± 1 at the corners, so $d_i - 1$ has strong corners. The previous lemma then implies that D has strong corners.

Let $D_{\mathbb{Z}} = (d_1 - 1) \cdots (d_t - 1) \in \mathbb{Z}[\mathbb{Z}^2]$; then D is the image of $D_{\mathbb{Z}}$ under the natural map $\mathbb{Z}[\mathbb{Z}^2] \to k[\mathbb{Z}^2]$. Note that the corners of $D_{\mathbb{Z}}$ will have coefficients ± 1, as this is true for all the factors $d_1 - 1$. This means that $P(D) = P(D_{\mathbb{Z}})$ does not depend on k, as ± 1 never maps to 0 in k. ∎

THEOREM 8.3. *Let $A \subset \mathbb{Z}^2$ be finite and convex. Then* $\ker(\sigma_{A,k})$ *and* $\operatorname{cok}(\sigma_{A,k})$ *are free k-modules of finite rank. The ranks of these modules do not depend on k.*

Proof. Note that $\sigma_{A,k}$ is the restriction of $\sigma_{\mathbb{Z}^2,k}$ to A, and so we have

$$\ker(\sigma_{A,k}) = \ker\left(\sigma_{\mathbb{Z}^2,k}\right) \cap T(A,k).$$

Using this, we compute

$$\begin{aligned}\ker(\sigma_{A,k}) &= \ker\left(\sigma_{\mathbb{Z}^2,k}\right) \cap T(A,k)\\ &= Dk[\mathbb{Z}^2] \cap T(A,k)\\ &= \left\{f \in Dk[\mathbb{Z}^2] \mid \operatorname{supp} f \subset A\right\}\\ &= \left\{f \in Dk[\mathbb{Z}^2] \mid P(f) \subset H(A)\right\}\\ &= \left\{fD \mid f \in k[\mathbb{Z}^2], P(fD) \subset H(A)\right\}\\ &= \left\{fD \mid f \in k[\mathbb{Z}^2], P(f) + \Delta \subset H(A)\right\}.\end{aligned}$$

The latter is clearly a free k-module of finite rank with a basis indexed by the $x \in \mathbb{Z}^2$ such that $x + \Delta \subset H(A)$. By Lemma 8.2, this basis is independent of k. Therefore the rank of $\ker(\sigma_{A,k})$ does not depend on k.

This proves the result for the kernel. The result for the cokernel now follows from algebraic generalities. It suffices to show that $\operatorname{cok}(\sigma_{A,\mathbb{Z}})$ is a free $\mathbb{Z}$-module of finite rank, as taking cokernels commutes with taking tensor products (see, e.g., [20, Chap. XVI.2, Prop. 2.6]). Since it is clearly finitely generated, we must show that it is torsion-free [20, Chap. I.8, Thm. 8.4]. We do this by comparing the ranks over $\mathbb{F}_p$ for p prime to the rank over $\mathbb{Z}$.

From the sequence

$$0 \to \ker(\sigma_{A,\mathbb{Z}}) \to T(A,\mathbb{Z}) \to L(A,\mathbb{Z}) \to \operatorname{cok}(\sigma_{A,\mathbb{Z}}) \to 0,$$

we see that

$$\operatorname{rk}_{\mathbb{Z}}(\operatorname{cok}(\sigma_{A,\mathbb{Z}})) = \operatorname{rk}_{\mathbb{Z}}(\ker(\sigma_{A,\mathbb{Z}})) - \#A + \sum_{i=1}^{t} \#\mathcal{L}_i(A).$$

In the same way, we have for any prime p

$$\dim_{\mathbb{F}_p}\left(\operatorname{cok}\left(\sigma_{A,\mathbb{F}_p}\right)\right) = \dim_{\mathbb{F}_p}\left(\ker\left(\sigma_{A,\mathbb{F}_p}\right)\right) - \#A + \sum_{i=1}^{t} \#\mathcal{L}_i(A).$$

By the result about the kernel, we know that $\operatorname{rk}_{\mathbb{Z}}(\ker(\sigma_{A,\mathbb{Z}})) = \dim_{\mathbb{F}_p}\left(\ker\left(\sigma_{A,\mathbb{F}_p}\right)\right)$. Using the formulas above, this implies

$$\operatorname{rk}_{\mathbb{Z}}(\operatorname{cok}(\sigma_{A,\mathbb{Z}})) = \dim_{\mathbb{F}_p}\left(\operatorname{cok}\left(\sigma_{A,\mathbb{F}_p}\right)\right).$$

But if $\operatorname{cok}(\sigma_{A,\mathbb{Z}})$ has any p-torsion, the $\mathbb{F}_p$-dimension would be strictly bigger. We conclude that $\operatorname{cok}(\sigma_{A,\mathbb{Z}})$ is torsion-free. ∎

Similar to the global case ($A = \mathbb{Z}^2$), this result allows us to state a necessary and sufficient condition for consistency of a vector of potential line sums in the case of finite convex A.

COROLLARY 8.4. *Let $A \subset \mathbb{Z}^2$ be finite and convex. A vector of potential line sums in $L(A,k)$ comes from a table in $T(A,k)$ if and only if it satisfies all dependencies.*

Proof. Theorem 8.3 shows that we can apply Lemma 3.4 to the cokernel of the line sum map. ∎

9. Local and global dependencies. This section deals with a subdivision of the dependencies for finite convex A into *global dependencies* that can be extended to dependencies on $\mathbb{Z}^2$, and *local dependencies* that are inherent to the finite extent of A. We first show that if A satisfies an additional shape property, called *roundedness*, all dependencies are global (Theorem 9.2). The next result, Theorem 9.3, states that for finite convex A, $\mathrm{Dep}\,(A,k)$ can be composed as a direct sum of global and local dependencies in a natural way.

Let $A \subset \mathbb{Z}^2$. From the comparison sequence (Lemma 7.1) we have a map $\mathrm{cok}\,(\sigma_{A,k}) \to \mathrm{cok}\,\left(\sigma_{\mathbb{Z}^2,k}\right)$. This map induces a map on the k-duals

$$\mathrm{Dep}\,\left(\mathbb{Z}^2,k\right) \longrightarrow \mathrm{Dep}\,(A,k)\,.$$

We call the image of this map the *global dependencies* on A. When this map is injective, the dependencies on $\mathbb{Z}^2$ all restrict to different dependencies on A. Our intuition is that this should happen whenever A is "sufficiently large."

LEMMA 9.1. *Suppose there is an $x \in \mathbb{Z}^2$ such that $x + \Delta \subset H(A)$. Then $\mathrm{cok}\,(\sigma_{A,k}) \to \mathrm{cok}\,\left(\sigma_{\mathbb{Z}^2,k}\right)$ is surjective, and so*

$$\mathrm{Dep}\,\left(\mathbb{Z}^2,k\right) \longrightarrow \mathrm{Dep}\,(A,k)$$

is injective.

The *geometric line* through $p \in \mathbb{R}^2$ in the direction $d \in \mathbb{R}^2 \setminus \{0\}$ is the set

$$\{p + \lambda d \mid \lambda \in \mathbb{R}\} \cap \mathbb{Z}^2,$$

provided that this set contains at least two points.

Let $d = (a,b) \in \mathbb{Z}^2 \setminus \{0\}$ and put $g = \gcd\,(a,b)$. Then any geometric line in direction d is the union of g lines. If l is a geometric line in direction d and $p,q \in H(l)$ are at least $|d|$ apart, then the line segment from p to q contains at least one point of every line through l.

Proof of Lemma 9.1. Without loss of generality we restrict ourselves to $A = \Delta \cap \mathbb{Z}^2$. We want to show that for any $l \in L(\mathbb{Z}^2,k)$, there is an $l' \in L(A,k)$ that maps to the same element in $\mathrm{cok}\,\left(\sigma_{\mathbb{Z}^2,k}\right)$. That is, we must show

$$L(\mathbb{Z}^2,k) = \mathrm{im}\,\left(\sigma_{\mathbb{Z}^2,k}\right) + L(A,k).$$

Recall that the conductor

$$\mathfrak{f}_k = \overline{D_1}k[\mathbb{Z}^2]\big/d_1 - 1 \oplus \cdots \oplus \overline{D_t}k[\mathbb{Z}^2]\big/d_t - 1$$

is the largest $L(\mathbb{Z}^2,k)$ ideal that is contained in $\mathrm{im}\,\left(\sigma_{\mathbb{Z}^2,k}\right)$. It is therefore sufficient to show that $L(\mathbb{Z}^2,k) = \mathfrak{f}_k + L(A,k)$ or, equivalently, that

$$k^{(\mathcal{L}_i(A))} \longrightarrow k[\mathbb{Z}^2]\big/(d_i - 1, D_i)$$

is surjective for all i.

Let l be a geometric line in direction d_i such that $H(l)$ intersects Δ. As we have $\Delta = P(D_i) + P(d_i - 1)$, the intersection is a segment of width at least $|d_i|$, so every line in the direction d_i that lies in l is in $\mathcal{L}_i(A)$. Let $S \subset \mathbb{Z}^2$ be the union of all the lines in $\mathcal{L}_i(A)$.

Note that $P(D_i)$ does not have a side parallel to d_i, as all the directions are pairwise independent. It follows that $P(D_i)$ has maximal points in the directions orthogonal to d_i. These points are necessarily corners. The coefficients on these corners are ± 1. It follows that for any $f \in k[\mathbb{Z}^2]$, there is a $g \in k[\mathbb{Z}^2]$ such that $\operatorname{supp} g \subset S$ and $f - g \in D_i k[\mathbb{Z}^2]$.

By the above, this implies that

$$k^{(\mathcal{L}_i(A))} + \overline{D_i} k[\mathbb{Z}^2] \big/ d_i - 1 = k[\mathbb{Z}^2] \big/ d_i - 1,$$

and so

$$k^{(\mathcal{L}_i(A))} \longrightarrow k[\mathbb{Z}^2] \big/ (d_i - 1, D_i)$$

is surjective. ∎

Let A be finite and convex. We define the *rounded part* of A to be the subset

$$A' = \left(\bigcup (x + \Delta) \right) \cap \mathbb{Z}^2,$$

where the union runs over all $x \in \mathbb{Z}^2$ such that $x + \Delta \subset H(A)$. We call A *rounded* if it is nonempty and $A' = A$.

THEOREM 9.2. *Let A be finite, convex, and rounded. Then* $\operatorname{cok}(\sigma_{A,k})$ *is equal to* $\operatorname{cok}\left(\sigma_{\mathbb{Z}^2,k}\right)$*, and so we have*

$$\operatorname{Dep}(A, k) = \operatorname{Dep}\left(\mathbb{Z}^2, k\right).$$

Proof. Note that by Lemma 9.1 the map

$$\operatorname{cok}(\sigma_{A,k}) \longrightarrow \operatorname{cok}\left(\sigma_{\mathbb{Z}^2,k}\right)$$

is surjective, so we just have to show it is injective. The strategy for this is to construct

$$A = A_0 \subset A_1 \subset A_2 \subset \cdots$$

such that A_{i+1}/A_i is noninterfering for all $i \geq 0$ and $\bigcup_{i \geq 0} A_i$ is all of $\mathbb{Z}^2$. Suppose that $l \in L(A, k)$ such that $l = \sigma_{\mathbb{Z}^2,k}(t)$ for some $t \in T(\mathbb{Z}^2, k)$. Then $t \in T(A_i, k)$ for some i, so l maps to 0 in $\operatorname{cok}(\sigma_{A_i,k})$. By the noninterference, $\operatorname{cok}(\sigma_{A,k})$ maps injectively to $\operatorname{cok}(\sigma_{A_i,k})$, so it follows that l maps to 0 in $\operatorname{cok}(\sigma_{A,k})$, as required.

Pick a point p in the interior of $H(A)$ in a sufficiently general position (we will make this more precise later). For $\lambda \in \mathbb{R}_{\geq 1}$, let $H(\lambda)$ be the point multiplication of the set $H(A)$ with factor λ and center p. Let $A(\lambda) = H(\lambda) \cap \mathbb{Z}^2$. Note that the union of all $H(\lambda)$ is the entire plane, so we have

$$\bigcup_{\lambda \geq 1} A(\lambda) = \mathbb{Z}^2.$$

As $\mathbb{Z}^2 \subset \mathbb{R}^2$ is countable and discrete, the set of λ's such that

$$A(\lambda) \neq \bigcup_{1 \leq \mu < \lambda} A(\mu)$$

is a countable and discrete subset of $\mathbb{R}_{\geq 1}$. Let $(\lambda_i)_{i=0}^{\infty}$ be the sequence of these λ's in increasing order. Put $A_i = A(\lambda_i)$.

For all $\lambda \in \mathbb{R}_{\geq 1}$ one sees that

$$\bigcup_{1 \leq \mu < \lambda} H(\mu)$$

is the boundary of $H(\lambda)$. Therefore, any point in $A_{i+1} \setminus A_i$ is on the boundary of $H(\lambda_i)$. This means that these points lie on finitely many line segments: the edges of the polygon $H(\lambda_{i+1})$.

In fact, by choosing the point p outside a countable union of lines, one can ensure that for every i there is a single edge l_i of the polygon $H(\lambda_{i+1})$ such that all the points in $A_{i+1} \setminus A_i$ lie on that edge.

Suppose that l_i does not lie in one of the directions $d_1, \ldots, d_t$. Then Δ has a maximal point m in the direction orthogonal to l_i, which is a corner, and so the corresponding coefficient of D is ± 1. Let $p \in A_{i+1} \setminus A_i$. As A is rounded, the translate of Δ such that m coincides with p is contained entirely in A_{i+1}. It follows that the map

$$\ker\left(\sigma_{A_{i+1},k}\right) \longrightarrow k^{(A_{i+1} \setminus A_i)}$$

is surjective, so A_{i+1}/A_i is noninterfering.

Suppose that l_i lies in the direction d_j. The edge of $H(A)$ in direction d_j is at least $|d_j|$ long, as A is rounded. So the edge l_i of $H(\lambda_{i+1})$ has length $\lambda_{i+1}|d_j| > |d_j|$. Therefore, every line in the direction d_j that lies inside the geometric line containing l_i meets A_{i+1}. Note that Δ has an edge in direction d_j and that the intersection of $\operatorname{supp} D$ with the geometric line through that edge consists precisely of the two corner points, both of which have coefficient ± 1. These two points are adjacent points within the same line on that geometric line. As A is rounded, every translate of Δ, such that the edge in direction d_j lies between on l_i, lies completely within $H(A)$. From these observations we can conclude that

$$\sigma_{A_{i+1}/A_i,k} : T(A_{i+1}/A_i, k) \longrightarrow L(A_{i+1}/A_i, k)$$

is onto and that its kernel is generated by the intersections of the correct translates of D with $T(A_{i+1}/A_i, k)$. Therefore the map

$$\ker\left(\sigma_{A_{i+1},k}\right) \longrightarrow \ker\left(\sigma_{A_{i+1}/A_i,k}\right)$$

is onto; that is, A_{i+1}/A_i is noninterfering. □

THEOREM 9.3. *Let A be finite and convex and suppose that A' is nonempty. Then* Dep (A, k) *decomposes in a natural way as a direct sum*

$$\operatorname{Dep}(A,k) = \operatorname{Dep}\left(\mathbb{Z}^2, k\right) \oplus \operatorname{Hom}_k\left(\operatorname{cok}\left(\sigma_{A/A',k}\right), k\right).$$

We call the second summand the local dependencies *on A.*

Proof. From the comparison sequence (Lemma 7.1) for A/A' we have

$$\operatorname{cok}\left(\sigma_{A',k}\right) \underset{f_{A/A'}}{\longrightarrow} \operatorname{cok}\left(\sigma_{A,k}\right) \longrightarrow \operatorname{cok}\left(\sigma_{A/A',k}\right) \longrightarrow 0.$$

Lemma 9.1 shows that $f_A : \operatorname{cok}(\sigma_{A,k}) \to \operatorname{cok}\left(\sigma_{\mathbb{Z}^2,k}\right)$ is surjective and Theorem 9.2 shows that $f_{A'} : \operatorname{cok}(\sigma_{A',k}) \to \operatorname{cok}\left(\sigma_{\mathbb{Z}^2,k}\right)$ is bijective. Note that $f_{A'} = F_A \circ f_{A/A'}$. We conclude that $f_{A/A'}$ is injective (so A/A' is noninterfering) and that $f_{A'}^{-1} \circ f_A$ is a splitting map of $f_{A/A'}$. It follows that

$$\operatorname{cok}\left(\sigma_{A,k}\right) = \operatorname{cok}\left(\sigma_{A',k}\right) \oplus \operatorname{cok}\left(\sigma_{A/A',k}\right).$$

This implies the required result (recall that Dep $(A', k) = \operatorname{Dep}(\mathbb{Z}^2, k)$). □

10. Conclusions. To conclude this paper, we summarize the main results obtained within our algebraic framework and their interpretation from the classical combinatorial perspective.

Lemma 3.4 relates an algebraic property of the cokernel of the line sum map to the consistency problem. Theorem 4.2 states that for the case $A = \mathbb{Z}^2$, the cokernel actually satisfies this property. In addition, a characterization of the switching components is provided for this case. This results in a strong statement concerning the consistency problem for the case $A = \mathbb{Z}^2$: a set of line sums corresponds to a table if and only if it satisfies a certain number of independent dependencies (Corollary 4.3). In section 5, properties are derived on the structure of the coefficients in the separate dependencies, resulting in an explicit characterization of the coefficients for the case $A = \mathbb{Z}^2$ (Theorem 5.2). Section 6 relates the material from sections 3–5 to the example from the combinatorial DT literature given in section 2.2.

The next sections, starting with section 7, focus on cases where A is a true subset of $\mathbb{Z}^2$. A relative setup is introduced in section 7, where a DT problem on a particular domain is related to a problem on a subset of that domain. In sections 8 and 9, this relation is applied to describe the structure of line sums for finite convex sets. Corollary 4.3 provides a necessary and sufficient condition for consistency in the case of a finite, convex reconstruction domain. Theorem 9.2 shows that if A is finite, convex, and rounded, the dependencies are exactly those that also apply to the global case $A = \mathbb{Z}^2$. Finally, Theorem 9.3 considers the decomposition of the dependencies for the general finite convex case into global and local dependencies.

The results on the structure of dependencies between the line sums in DT problems can be viewed either as a collection of new research results or as an illustration of the power of applying ring theory and commutative algebra to this combinatorial problem. We expect that a range of additional results can be obtained within the context of this algebraic framework.

Appendix. Tools from algebra.

A.1. Group rings. We begin by recalling some results on group rings. See, for example, [20, Chap. II.3] for a short introduction or [22] for more results on these rings.

DEFINITION A.1. *Let k be a commutative ring and let G be a group. The* group ring $k[G]$ *is the k-algebra which as a k-module is free with basis G,*

$$k[G] = \bigoplus_{g \in G} k[g],$$

and whose multiplication is given by

$$\begin{aligned} [g] \cdot [h] &= [gh] \qquad \text{for all } \ g, h \in G, \\ [g] \cdot \lambda &= \lambda[g] \qquad \text{for all } \ g \in G, \ \lambda \in k. \end{aligned}$$

When there is no confusion possible we will drop the brackets around elements of G, writing a typical element of $k[G]$ simply as $\sum_{g \in G} \lambda_g g$ with $\lambda_g = 0$ for almost all $g \in G$.

A ring homomorphism $k \longrightarrow k'$ induces a unique ring homomorphism

$$k[G] \longrightarrow k'[G].$$

A group homomorphism $G \longrightarrow H$ induces a unique k-algebra homomorphism

$$k[G] \longrightarrow k[H].$$

LEMMA A.2. *Let G be a group and let N be a normal subgroup. Let I_N be the ideal of $k[G]$ generated by all elements of the form $n-1$ with $n \in N$. Then there is a short exact sequence*

$$0 \longrightarrow I_N \longrightarrow k[G] \longrightarrow k[G/N] \longrightarrow 0.$$

A.2. Filtrations. We continue with some generalities on filtrations.

DEFINITION A.3. *Let R be a commutative ring and let M be an R-module. A* filtration *of M is a collection of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M.$$

The quotient modules M_{i+1}/M_i are called the successive quotients *of the filtration.*

LEMMA A.4. *Let R be a commutative ring and let M' and M'' be filtered R-modules. Suppose we have a short exact sequence*

$$0 \to M' \to M \to M'' \to 0.$$

Then M admits a filtration whose successive quotients are those of M' followed by those of M''.

LEMMA A.5. *Let R be a commutative ring, let A, B, and C be R-modules, and suppose that $f : A \to B$ and $g : B \to C$ are injective morphisms. Then there is a short exact sequence*

$$0 \to \operatorname{cok}(f) \to \operatorname{cok}(gf) \to \operatorname{cok}(g) \to 0.$$

A.3. Weak coprimality. The rest of this appendix is devoted to a generalization of the concept of coprimality and the Chinese remainder theorem.

DEFINITION A.6. *Let R be a commutative ring and let $f, g \in R$. We say that f is* weakly coprime to *g if multiplication by f is an injective map on R/g.*

The common notion of coprimality, namely, that the ideal (f, g) generated by f and g be all of R, implies that multiplication by f is a *bijective* map on R/g.

LEMMA A.7. *Let R be a commutative ring and let $f, g \in R$ such that f is weakly coprime to g. Then there is a short exact sequence*

$$0 \to R/fg \to R/f \oplus R/g \to R/(f,g) \to 0.$$

Proof. The proof follows by straightforward verification. ∎

If two elements are coprime in the common (strong) sense, then in the sequence above we have $R/(f, g) = 0$, so the first map is an isomorphism. This fact is commonly referred to as the Chinese remainder theorem.

LEMMA A.8. *Let R be a commutative ring and let f_1, f_2, and g be in R. Suppose that f_1 and f_2 are weakly coprime to g. Then there is a short exact sequence*

$$0 \to R/(f_1, g) \to R/(f_1 f_2, g) \to R/(f_2, g) \to 0.$$

Proof. Apply Lemma A.5 to the multiplication by f_1 and f_2 maps on R/g. This completes the proof. ∎

THEOREM A.9 (weak Chinese remainder theorem). *Let R be a commutative ring and let $x_1, \ldots, x_t \in R$ have the property that x_i is weakly coprime to x_j whenever $i < j$. Then the natural map*

$$\phi : R/x_1 \cdots x_t \longrightarrow R/x_1 \oplus \cdots \oplus R/x_t$$

is injective. Its cokernel admits a filtration whose successive quotients are isomorphic to $R/(x_i, x_j)$ *for* $1 \leq i < j \leq t$.

Proof. We proceed by induction on t. For $t = 2$ the result is that of Lemma A.7. Let $t \geq 3$ and assume that the theorem holds for any smaller number of x_i's.

We write ϕ as a composition of two maps. Let ϕ_1 be the natural map

$$\phi_1 : R/x_1 \cdots x_t \longrightarrow R/x_1 \cdots x_{t-1} \oplus R/x_t,$$

and let ϕ_2 be the natural map

$$\phi_2 : R/x_1 \cdots x_{t-1} \longrightarrow R/x_1 \oplus \cdots \oplus R/x_{t-1}.$$

Then we have $\phi = (\phi_2 \oplus \mathrm{id}_{R/x_t}) \circ \phi_1$.

Note that $x_1 \cdots x_{t-1}$ is weakly coprime to x_t as a composition of injective maps is again injective. So Lemma A.7 applies to ϕ_1. In particular, ϕ_1 is injective. By the induction hypothesis, ϕ_2 is also injective. We conclude that ϕ is injective.

By Lemma A.7 the cokernel of ϕ_1 is $R/(x_1 \cdots x_{t-1}, x_t)$. By repeatedly applying Lemma A.8, this module admits a filtration whose successive quotients are $R/(x_i, x_t)$ for $1 \leq i \leq t-1$.

Furthermore, we have $\operatorname{cok}\left(\phi_2 \oplus \mathrm{id}_{R/x_t}\right) = \operatorname{cok}(\phi_2)$, which by the induction hypothesis has a filtration whose successive quotients are isomorphic to $R/(x_i, x_j)$ with $1 \leq i < j \leq t-1$.

We apply Lemma A.5 to the maps ϕ_1 and $\phi_2 \oplus \mathrm{id}_{R/x_t}$ and conclude that the cokernel of ϕ has the required filtration. ∎

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