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On Shortest T-Joins and Packing T-Cuts

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We give a class of graphs with the property that for each even set T of nodes in G the minimum length of a T-join is equal to the maximum number of pairwise edge disjoint T-cuts. Our class contains the bipartite and the series-parallel graphs for which this property was derived earlier by Seymour.

1. INTRODUCTION

Let G be an undirected connected graph. For each $v \in V(G)$, the node set of G, and each $F \subseteq E(G)$, the edge set of G, we define $d_F(v)$ to be the number of edges in F incident with v. If $T \subseteq V(G)$, then we call a set $F \subseteq E(G)$ a *T-join* if $T = \{v \in V(G) \mid d_F(v)$ is odd$\}$. Throughout this paper we shall always assume $|T|$ to be even. We denote the minimum cardinality of a T-join in G by $\tau_T(G)$. For $U \subseteq V(G)$ we define $\delta(U) := \{uv \in E(G) \mid u \in U, v \notin U\}$. Such a set is called a *coboundary*. If $u \in V(G)$ then $\delta(u) := \delta(\{u\})$. If $|U \cap T|$ is odd we call $\delta(U)$ a *T-cut*. The maximum number of pairwise edge disjoint T-cuts is denoted by $\nu_T(G)$. Since, obviously, each T-join has at least one edge in common with each T-cut, the following, well-known, inequality holds:

$$\nu_T(G) \leqslant \tau_T(G). \tag{1.1}$$

In this paper we consider sufficient conditions for equality in (1.1). In particular, we give a class of graphs for which (1.1) holds with equality for each even cardinality subset T of $V(G)$. For convenience we baptize such a graph a *Seymour graph*. Seymour proved the following two results:

A connected bipartite graph is a Seymour graph [18]; (1.2)

A connected series-parallel graph is a Seymour graph [17]. (1.3)

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(A graph G is called *series-parallel* if no subgaph of G is homeomorph with K_4, the complete graph with four nodes.) It should be noted that (1.3) is a very special, simple, case of Seymour's deep result on binary clutters with the max-flow min-cut property [17]. The two sufficient conditions for graphs to be Seymour graphs contained in (1.2) and (1.3) are of quite different natures: bipartiteness is a parity condition (all circuits are even), whereas series-parallelism is a topological condition (no homeomorph of K_4 as a subgraph). The result of this paper is the following theorem, which unifies these two conditions by one weaker condition.

THEOREM 1.1. *Let G be an undirected connected graph. If G contains neither an odd-K_4 nor an odd-prism, then G is a Seymour graph.*

(We prove this result later, in Section 3.)

Here an *odd-K_4* and an *odd-prism* are graphs as depicted in Fig. 1. Wriggled lines stand for pairwise openly disjoint paths, while *odd*, *even* indicate that the corresponding faces are bounded by odd circuits, even circuits, respectively.

It is straightforward to see that neither bipartite graphs nor series-parallel graphs contain an odd-K_4 or an odd-prism. So Theorem 1.1 implies (1.1) as well as (1.3). But in addition Theorem 1.1 gives Seymour graphs which are not series-parallel and not bipartite (e.g., the graph shown in Fig. 2a). The two forbidden configurations odd-K_4 and odd-prism are motivated by the fact that $\nu_{V(G)}(G) \neq \tau_{V(G)}(G)$ if $G = K_4$ or G is the triangular prism (Fig. 2b).

The organization of this paper is as follows. In Section 2 we give some preliminary results useful in proving Theorem 1.1. In particular we give a decomposition result for graphs with no odd-K_4 and no odd-prism. The proof of Theorem 1.1 is given in Section 3. We conclude Section 1 with a few remarks.

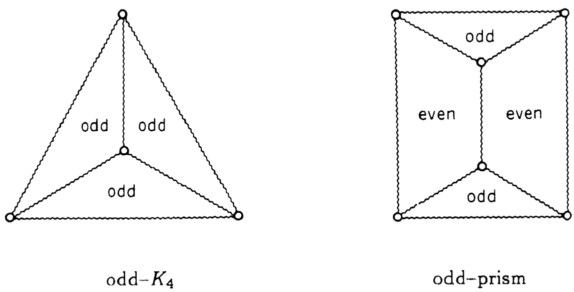

FIGURE 1

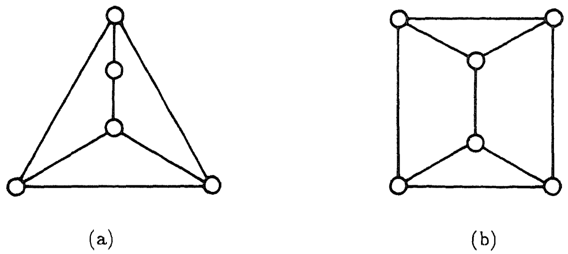

FIGURE 2

Remarks. (i) The condition in Theorem 1.1 is not a necessary conditions since the graph shown in Fig. 3a is an odd-K_4 as well as a Seymour graph. However, from Theorem 1.1 one can derive:

Let G be a connected graph. Then the following are equivalent:

($*$) G contains no odd-K_4 and no odd-prism;

($**$) For each weight function $w \in \mathbb{Z}_+^{E(G)}$ with the property that $\sum_{e \in E(C)} w_e$ is even if and only if C is an even circuit in G, we have:

for each even $T \subseteq V(G)$ the minimum weight of a T-join with respect to w is equal to the maximum size of a w-packing of T-cuts. (1.4)

(A *w-packing of T-cuts* is a family $B_1, ..., B_t$ of T-cuts (repetition allowed) such that each $e \in E(G)$ is in at most w_e members of that family. The *size* of a family $B_1, ..., B_t$ is t.

Note that the class of graphs for which the min–max relation in (1.4)($**$) holds for each T and for each weight function w (not necessarily satisfying the parity condition in (1.4)($**$)) is the class of series-parallel graphs.

The graph shown in Fig. 3b is a Seymour graph. But the graph obtained by deleting the edge marked e is not a Seymour graph. This example, due

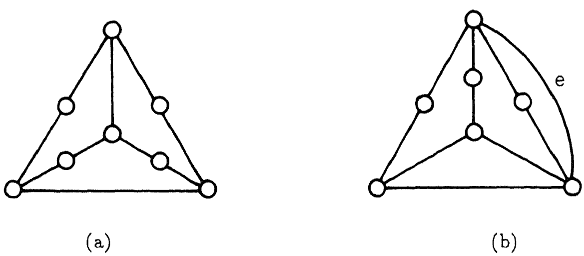

FIGURE 3

to one of the referees, shows that a characterization of Seymour graphs in terms of forbidden subgraphs (like (1.4)) does not exist.

(ii) Middendorf and Pfeiffer [11] have shown that it is $\mathcal{NP}$-complete to decide whether a given graph G with a given subset T of nodes satisfies (1.1) with equality. As far as I know the complexity of the class of Seymour graphs is open. I do not even know whether it is in $\mathcal{NP} \cup$ co-$\mathcal{NP}$.

(iii) From (1.1) one easily derives a min–max relation for $\tau_T(G)$ in arbitrary graphs and for arbitrary T. This min–max relation, derived by Lovász [8], is:

> Let G be a connected graph, and let T be an even subset of $V(G)$. Then $2\tau_T(G)$ is equal to the maximum cardinality of a 2-packing of T-cuts. (1.5)

The min–max relation $\tau_T(G) = \nu_T(G)$ is particularly relevant for multicommodity flows in planar graphs (cf. [18]). Besides the already mentioned references on T-joins and T-cuts, there are quite a few others: Edmonds [1], Edmonds and Johnson [2, 3], Frank, Sebő, and Tardos [4], Korach [6], Korach and Penn [7], Mei Gu Guan [10], Sebő [12–16].

2. Preliminaries

Signed Graphs

The proof of Theorem 1.1 given in Section 3 makes use of a "decomposition" result for graphs with no odd-K_4 and no odd-prism (Theorem 2.2). It is convenient to state and prove this result in terms of "signed" graphs. A *signed graph* is a pair (G, Σ), where $\Sigma \subseteq E(G)$ of G. The edges in Σ are called *odd*, the other edges *even*. A circuit C in G is called *odd* (*even*, respectively) if $|\Sigma \cap E(C)|$ is odd (even, respectively). We call a signed graph *bipartite* if $\Sigma = \delta(U)$ for some $U \subseteq V(G)$. For example, $(G, \emptyset)$ is bipartite. Moreover, $(G, E(G))$ is bipartite if and only if G is a bipartite graph in the usual sense. It is easy to see that a signed graph is bipartite if and only if it contains no odd circuits. Let (G, Σ) be a signed graph, and let $U \subseteq V(G)$. Obviously (G, Σ) and $(G, \Sigma \Delta \delta(U)))$ have the same collection of odd circuits (Δ denotes the set-theoretic *symmetric difference*). We call the operation $\Sigma \to \Sigma \Delta \delta(U)$ *resigning* (*on* U). We say that (G, Σ) *reduces to* (G', Σ') if (G', Σ') can be obtained from (G, Σ) by a series of the following operations:

— deleting an edge from G (and from Σ);
— contracting an even edge in G;
— resigning;
— deleting a vertex from G.

The notions *odd-*K_4 and *odd-prism* can be extended easily to signed graphs. We do this by saying that the word *odd* (*even*, respectively) in Fig. 1 indicates that the corresponding face is bounded by an odd circuit (even circuit, respectively) in (G, Σ). The signed graph $(K_4, E(K_4))$ will be denoted by $\tilde{K}_4$.

The following is easy to prove.

Let (G, Σ) be a signed graph. Then (G, Σ) contains an odd-K_4 as a subgraph if and only if (G, Σ) reduces to $\tilde{K}_4$. (2.1)

Let $C = (V(C), E(C))$ be a circuit in G. Then we say that e, $f \in E(G) \setminus E(C)$ are equivalent if there exists a path $u_0, u_1, ..., u_t, u_{t+1}$ in G with $u_1, ..., u_t \notin V(C)$ and $u_0 u_1 = e$, $u_t u_{t+1} = f$. The equivalence classes of this equivalence relation are called the *bridges* of C. (In particular, a chord uv of C (i.e., $u, v \in V(C)$, $uv \in E(G) \setminus E(C)$), forms a bridge of C.) We call C *non-separating* if C has at most one bridge.

We first show a technical lemma, which will be used in the proof of the decomposition theorem, Theorem 2.2 (cf. Lemma in [5]).

LEMMA 2.1. *Let (G, Σ) be a signed graph with no odd-K_4 as a subgraph, and with no one-node cutset. Let C be a non-separating odd circuit in G with $C \neq G$. If C satisfies:*

(i) *$V(C) \cap V(C') \neq \emptyset$ for each odd circuit C' in (G, Σ);*

(ii) *C contains at least three nodes with degree at least three,*

then C has a unique subgraph I_C such that:

(i′) *I_C is a path, $V(I_C) \neq \emptyset$;*

(ii′) *any odd circuit C' in (G, Σ) contains I_C as a subgraph;*

(iii′) *there exists an odd circuit C' in (G, Σ) such that $V(C) \cap V(C') = V(I_C)$ and $E(C) \cap E(C') = E(I_C)$.*

Proof. Clearly $V(G) \setminus V(C) \neq \emptyset$. (If $V(G) = V(C)$, then C has exactly one chord, uv say, as $C \neq G$ and C is non-separating. This violates (ii).) Let B be a tree spanning $V(G) \setminus V(C)$ (which exists, as C is non-separating). Now delete all the edges with both endpoints in $V(G) \setminus V(C)$ which are not in B. Resign such that $\Sigma \cap E(B) = \emptyset$, and then contract the edges in B. As the edges contained in $V(G) \setminus V(C)$ form a bipartite graph (by condition (i)), each odd circuit in the original signed graph contains an odd circuit in the reduced signed graph. Conversely each odd circuit in the contracted signed graph is contained in an odd circuit of the original signed graph. By (2.1) the contracted graph contains no odd-K_4. Hence we may assume that (G, Σ) is the contracted graph, i.e., $V(G) = V(C) \cup \{w\}$ for some node w.

Let C' be an odd circuit in G which has a minimum number of edges in common with C. Define I_C by $V(I_C) = V(C) \cap V(C')$ and $E(I_C) = E(C) \cap E(C')$. Obviously I_C satisfies (i′) and (iii′). Suppose (ii′) is not satisfied by I_C. Let C'' be an odd circuit not containing I_C. By the minimality of $|E(C') \cap E(C)|$, we have that $E(C') \cap E(C) \cap E(C'') = \varnothing$. Now there are five possibilities, indicated in Fig. 4. In each of them, (G, Σ) contains an odd-K_4. The existence of edge wv in the rightmost figure follows from (ii). This proves the existence of I_C. The uniqueness of I_C is obvious. ∎

Remark. Note that in Lemma 2.1 it might be the case that $|V(I_C)| = 1$ and $E(I_C) = \varnothing$.

Decomposition

A *k-split* of G is a pair G_1, G_2 of subgraphs of G, such that $V(G_1) \cup V(G_2) = V(G)$, $|V(G_1) \cap V(G_2)| \leqslant k$; $E(G_1) \cup E(G_2) = E(G)$, $E(G_1) \cap E(G_2) = \varnothing$, and $|E(G_1)|, |E(G_2)| \geqslant k$. If both $E(G_1)$ and $E(G_2)$ contain an odd circuit in (G, Σ) we call the k-split *strong*.

We call a signed graph (G, Σ) *almost-bipartite* if there exists a node $u \in V(G)$ which is on each odd circuit in (G, Σ).

The following theorem shows that signed graphs containing no odd-K_4 and no odd-prism are essentially almost bipartite.

THEOREM 2.2. *Let (G, Σ) be a signed graph with no odd-K_4 and no odd-prism. If G is simple (i.e., has no loops and parallel edges), then one of the following holds:*

(i) *(G, Σ) has a 1-split;*

(ii) *(G, Σ) has a strong 2-split;*

(iii) *(G, Σ) is almost bipartite.*

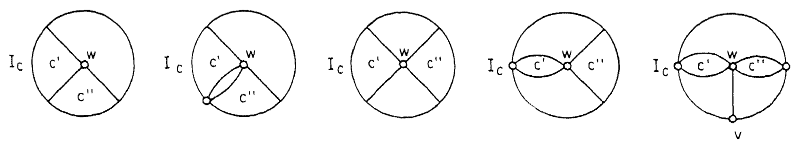

FIGURE 4

Proof. Let (G, Σ) satisfy the conditions of the theorem, without satisfying (i) or (ii). We prove that G is almost bipartite.

CLAIM 1. *There are no two node disjoint odd circuits.*

Proof of Claim 1. Suppose to the contrary that C_1 and C_2 are odd circuits with $V(C_1) \cap V(C_2) = \emptyset$. Obviously $|V(C_i)| \geqslant 3$ for $i = 1, 2$ (as G is simple). Since (i) and (ii) are not satisfied, Menger's theorem [9] yields the existence of three paths P_1, P_2, and P_3 from C_1 to C_2 such that $V(P_i) \cap V(P_j) = \emptyset$ (i, $j = 1, 2, 3$, $i \neq j$). It is easy to see that C_1, C_2, P_1, P_2, and P_3 together form an odd-prism or contain an odd-K_4. This is a contradiction. *End of proof of Claim* 1

For each odd circuit C in (G, Σ) and each bridge $B \subseteq E(G)$ of C there exists a unique path $I_C(B)$ on C with the following properties:

— there exists an odd circuit C' such that $E(C') \subseteq E(C) \cup B$; $V(C) \cap V(C') = V(I_C(B))$ and $E(C) \cap E(C') = E(I_C(B))$;

— each odd circuit C' with $E(C') \subseteq E(C) \cup B$ satisfies $V(C) \cap V(C') \supseteq V(I_C(B))$ and $E(C) \cap E(C') \supseteq E(I_C(B))$.

Indeed if C contains at least three nodes with degree at least three, this follows from Claim 1 and Lemma 2.1. If C contains at most two nodes of degree at least three, this follows from the fact that (G, Σ) has no 1-split and no strong 2-split.

Now choose an odd circuit $\tilde{C}$ and a bridge $\tilde{B}$ of $\tilde{C}$, such that $I_{\tilde{C}}(\tilde{B})$ has a minimal number of edges, among all $I_C(B)$ (over all odd circuits C, and bridges B of C). Let $\tilde{u}$ be an endpoint of $I_{\tilde{C}}(\tilde{B})$.

Claim 2. *$\tilde{u} \in V(I_{\tilde{C}}(B))$ for each bridge B of $\tilde{C}$.*

Proof of Claim 2. Suppose to the contrary that $\tilde{u} \notin V(I_{\tilde{C}}(B))$ for some bridge B of $\tilde{C}$. Since $I_{\tilde{C}}(\tilde{B})$ is minimal, $V(I_{\tilde{C}}(B)) \setminus V(I_{\tilde{C}}(\tilde{B})) \neq \emptyset$. Let $u \in V(I_{\tilde{C}}(B)) \setminus V(I_{\tilde{C}}(\tilde{B}))$.

Let $\hat{C}$ be an odd circuit, with $E(\hat{C}) \subseteq E(\tilde{C}) \cup \tilde{B}$, $V(\hat{C}) \cap V(\tilde{C}) = V(I_{\tilde{C}}(\tilde{B}))$, and $E(\hat{C}) \cap E(\tilde{C}) = E(I_{\tilde{C}}(\tilde{B}))$. Similarly, let C be an odd circuit, with $E(C) \subseteq E(\tilde{C}) \cup B$, $V(C) \cap V(\tilde{C}) = V(I_{\tilde{C}}(B))$, and $E(C) \cap E(\tilde{C}) = E(I_{\tilde{C}}(B))$. Obviously $u \notin V(\hat{C})$. Let $\hat{B}$ be the bridge of $\hat{C}$ containing u. Then $E(C)$ is contained in $\hat{B} \cup E(\hat{C})$. So $V(I_{\hat{C}}(\hat{B})) \subseteq V(\hat{C}) \cap V(C) \subseteq V(I_{\tilde{C}}(\tilde{B})) \setminus \{\tilde{u}\}$, contradicting the minimality of $I_{\tilde{C}}(\tilde{B})$. *End of proof of Claim* 2

It is an easy exercise to derive from Claim 2 that each odd circuit in (G, Σ) contains $\tilde{u}$. So (G, Σ) is almost bipartite. ∎

3. Proof of Theorem 1.1

In proving Theorem 1.1 we apply the results in Section 2 to the signed graph $(G, E(G))$. (If the signing Σ is not explicitly given we allways assume $\Sigma = E(G)$.)

Let G be a connected graph. Then we have $\nu_T(G) = \nu_T(G)$ for every even subset T of $V(G)$ if and only if

for each $w \in \{-1, 1\}^{E(G)}$ such that $\sum_{e \in E(C)} w_e \geqslant 0$ for each circuit C in G there exists a collection of edge disjoint coboundaries $\delta(U_{\tilde{e}})$, $\tilde{e} \in \{e \in E(G) \mid w_e = -1\}$ $(=: F_w)$, such that $\tilde{e} \in \delta(U_{\tilde{e}})$ for each $\tilde{e} \in F_w$. (3.1)

(The proof of this equivalence is easy and is left to the reader (cf. [15, 16]). Let G be a graph such that $(G, E(G))$ contains no odd-K_4 and no odd-prism, and such that Theorem 1.1 is correct for all graphs with fewer edges than G. We prove that (3.1) holds for G. So let $w \in \{-1, 1\}^{E(G)}$ such that

$$\sum_{e \in E(C)} w_e \geqslant 0 \qquad \text{for each circuit } C \text{ in } G. \tag{3.2}$$

We consider the three cases of Theorem 2.2.

Case I. *G has a one node cutset, $\{u\}$ say.* It is not hard to see that now a packing with coboundaries, as meant in (3.1), is obtained by taking the union of such packings in each of the sides of the cutset $\{u\}$.

Case II. *G is two-connected, and has a strong 2-split.* So G has two non-bipartite subgraphs G_1 and G_2 such that $V(G_1) \cup V(G_2) = V(G)$, $|V(G_1) \cap V(G_2)| = 2$ ($V(G_1) \cap V(G_2) = \{u, v\}$ say), $E(G_1) \cup E(G_2) = E(G)$, and $E(G_1) \cap E(G_2) = \varnothing$. For $i = 1, 2$, let α_i be the length, with respect to w, of the shortest uv-path in G_i. By (3.2), $\alpha_1 + \alpha_2 \geqslant 0$. Hence we may assume $\alpha_2 \geqslant 0$.

Construct $\tilde{G}_1$ from G_1 by adding to G_1 a uv-path, P say, such that $|E(P)| = \alpha_2$. (If $\alpha_2 = 0$, identify u and v and call the new node u again.) Define $w^1 \in \{-1, 1\}^{E(\tilde{G}_1)}$ by $w^1_e = 1$ if $e \in E(P)$ and $w^1_e = w_e$ if $e \in E(G_1)$. Now $(\tilde{G}_1, E(\tilde{G}_1))$ contains neither an odd-K_4, nor an odd-prism. (Indeed, there exists a uv-path Q in G_2 with $|E(Q)| \equiv \alpha_2 = |E(P)|$ (modulo 2).) Moreover $\tilde{G}_1$ contains no negatively weighted circuits with respect to w^1. So, there exists a collection $\{\delta(U_e) \mid e \in T_{w^1}\}$ of coboundaries in $\tilde{G}_1$, satisfying (3.1) with respect to w^1. We may assume $u \notin U_e$ for each $e \in F_{w^1}$. Define $Z := \{e \in F_{w^1} \mid \delta(U_e) \cap E(P) \neq \varnothing\}$, and $\rho := |Z|$.

Next we construct $\tilde{G}_2$ from G_2 by adding a uv-path Q to G_2 with $|E(Q)| = \rho$. (If $\rho = 0$, identify u and v, and call the new node u again.)

CLAIM 1. *$(\tilde{G}_2, E(\tilde{G}_2))$ contains neither an odd-K_4 nor an odd-prism.*

Proof of Claim 1. As G_1 is non-bipartite, and G is two-connected there exists in G_1 an even uv-path, as well as an odd uv-path.

End of proof of Claim 1

Define $w^2 \in \{-1, 1\}^{E(\tilde{G}_2)}$ by $w^2_e = -1$ if $e \in E(Q)$, and $w^2_e = w_e$ if $e \in E(G_2)$. There are no negative weighted circuits with respect to w^2 in $\tilde{G}_2$. (Note that $\rho \leqslant \alpha_2$, and hence $-\rho + \alpha_2 \geqslant 0$.) So as $\tilde{G}_2$ has fewer edges than G, there exists a collection $\{\delta(V_e) \mid e \in F_{w^2}\}$ of coboundaries in $\tilde{G}_2$ in the sense of (3.1) with respect to w^2. We may assume $u \notin \delta(V_e)$ for each $e \in F_{w^2}$.

If $\rho \neq 0$ let π be some bijection from Z to $E(Q)$. Now it is easy to see that

$$\{\delta(U_e) \mid e \in F_{w^1} \backslash Z\} \cup \{\delta(V_e) \mid e \in F_{w^2} \backslash E(Q)\} \cup \{\delta(U_e \cup V_{\pi(e)}) \mid e \in Z\}$$

(or if $\rho = 0$: $\{\delta(U_e) \mid e \in F_{w^1}\} \cup \{\delta(V_e) \mid e \in F_{w^2}\}$) is a collection of coboundaries in G, satisfying (3.1) with respect to w.

Case III. *G is almost bipartite.* Let $u \in V(G)$ such that $G|(V(G)\backslash u)$ is bipartite, with bipartition U_1, U_2, say. Define a bipartite graph $\tilde{G}$ as follows:

$$V(\tilde{G}) = (V(G)\backslash\{u\}) \cup \{u_1, u_2\};$$
$$E(\tilde{G}) = (E(G)\backslash\delta(u)) \cup \{vu_i \mid v \in U_i, vu \in E(G), i = 1, 2\} \cup \{u_1u_2\},$$

and $\tilde{w}_e \in \{-1, 1\}^{E(\tilde{G})}$ by

$$\tilde{w}_e = \begin{cases} w_e & \text{if } e \in E(G)\backslash\delta(u); \\ w_{vu} & \text{if } e = vu_i;\, v \in V(G)\backslash\{u\};\, i = 1, 2; \\ -1 & \text{if } e = u_1u_2. \end{cases}$$

CLAIM 2. $\sum_{e \in E(C)} \tilde{w}_e \geqslant 0$ *for all circuits C in $\tilde{G}$.*

Proof of Claim 2. Suppose to the contrary that $\sum_{e \in E(C)} \tilde{w}_e < 0$ for a circuit C in $\tilde{G}$. Obviously the edges in $E(C)\backslash\{u_1u_2\}$ give a circuit $\tilde{C}$ in G, hence $u_1u_2 \in E(C)$. But this means that $\tilde{C}$ is odd in G, and so $\sum_{e \in E(C)} \tilde{w}_e = -1 + \sum_{e \in E(\tilde{C})} w_e \geqslant -1 + 1 = 0$. Contradiction.

End of proof of Claim 2

Since $\tilde{G}$ is bipartite, (1.2) yields the existence of a collection $\{\delta(U_e) \mid e \in F_{\tilde{w}}\}$ of coboundaries as meant in (3.1) with respect to $\tilde{w}$ in G. We may assume $u_1 \notin U_e (e \in F_{\tilde{w}})$. But now $\{\delta(U_e) \mid e \in F_w\}$ is a desired collection of coboundaries with respect to w in G. ∎

Remark. Case III in the proof above was derived independently by D. Wagner.

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