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On Cutting Planes and Matrices

A. M. H. GERARDS

ABSTRACT. Continuing the work of Chvátal and Gomory, Schrijver proved that any rational polyhedron $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leqq \mathbf{b}\}$ has finite Chvátal rank. This was extended by Cook, Gerards, Schrijver, and Tardos, who proved that in fact this Chvátal rank can be bounded from above by a number only depending on $\mathbf{A}$, hence independent of $\mathbf{b}$. The aim of this note is to show that the latter result can be proved quite easily from the result of Chvátal and Schrijver.

Introduction. Consider a *rational polyhedron* P, i.e., $P = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}\mathbf{x} \leqq \mathbf{b}\}$ with $\mathbf{A} \in \mathbf{Z}^{m \times n}$, $\mathbf{b} \in \mathbf{Z}^m$. A *cutting plane for* P is an inequality

$$\mathbf{c}^{\mathrm{T}}\mathbf{x} \leq \lfloor \delta \rfloor,$$

with

$$\mathbf{c} \in \mathbf{Z}^n \quad \text{and} \quad \delta \geqq \max\{\mathbf{c}^{\mathrm{T}}\mathbf{x} \mid \mathbf{x} \in P\}.$$

The set of vectors satisfying all cutting planes for P is denoted by P'. Obviously, P' satisfies

$$(1) \qquad P_I \subseteq P' \subseteq P,$$

where $P_I :=$ convex hull $(P \cap \mathbf{Z}^n)$. Moreover, P' is a polyhedron again (Schrijver [5]) and satisfies

$$(2) \qquad P = P' \Leftrightarrow P = P_I.$$

(1) and (2) suggest the following procedure to get a system of inequalities $\mathbf{M}\mathbf{x} \leqq \mathbf{d}$ such that $P_I = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{M}\mathbf{x} \leqq \mathbf{d}\}$. Namely, define

$$(3) \qquad P^{(0)} := P; \qquad P^{(i)} := (P^{(i-1)})' \quad \text{for } i = 1, 2, \ldots.$$

From (1) and (2) we get

$$(4) \qquad \begin{aligned} &P = P^{(0)} \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \cdots \supseteq P^{(i)} \supseteq \cdots \supseteq P_I, \\ &P^{(i)} = P^{(i-1)} \Leftrightarrow P^{(i)} = P_I \quad (i = 1, 2, \ldots). \end{aligned}$$

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Schrijver [5] proved that

(5) for each rational polyhedron P there exists a $t \in \mathbf{N}$ such that $P^{(t)} = P_I$.

Cook, Gerards, Schrijver, and Tardos [3] extended this result by proving that

(6) for each matrix $\mathbf{A} \in \mathbf{Z}^{m\times n}$, there exists a $t \in \mathbf{N}$, such that for each $\mathbf{b} \in \mathbf{Z}^m$ we have that
$$\{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}\mathbf{x} \leqq \mathbf{b}\}^{(t)} = \{x \in \mathbf{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}_I .$$

The aim of this note is to present a short proof of (6) using (5).

REMARKS.

(i) The procedure described above can be considered as a polyhedral version of Gomory's cutting-plane method for integer linear programming (Gomory [4]). Chvátal [2] proved (5), for the case that P is bounded in $\mathbf{R}^n$.

(ii) As C. Blair observed, (6) is equivalent to the result, due to Blair and Jeroslow [1], that "each integer programming value function is a Gomory function." For a discussion see Cook, Gerards, Schrijver, and Tardos [3].

(iii) In fact, Cook, Gerards, Schrijver, and Tardos [3] proved that t in (6) can be taken equal to $2^{n^3+1} n^{5n} \Delta(\mathbf{A})^{n+1}$, where $\Delta(\mathbf{A})$ denotes the maximum of the absolute values of the subdeterminants of $\mathbf{A}$. Since the proof of (6) given below relies on (5), it cannot be expected to give such an explicit bound.

PROOF OF (6). Let $\mathbf{A} \in \mathbf{Z}^{m\times n}$ and assume that $\mathbf{A}$ violates (6). This implies the existence of a sequence

(7) $\{\mathbf{b}_i, \mathbf{w}_i, \alpha_i\}_{i\in\mathrm{N}}$ with $\mathbf{b}_i \in \mathbf{Z}^m$, $\mathbf{w}_i \in \mathbf{Z}^n$, $\alpha_i \in \mathbf{Z}$ for $i \in \mathbf{N}$

such that

(8) for each $i \in \mathbf{N}$, $\mathbf{w}_i^{\mathrm{T}}\mathbf{x} \leqq \alpha_i$ is valid for $(P_i)_I$, but not valid for $(P_i)^{(i)}$, where $P_i := \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}\mathbf{x} \leqq \mathbf{b}_i\}$.

In the sequel we often use the following fact, which trivially follows from (4).

(9) (8) is invariant under taking subsequences of (7).

By (9), it is obvious that we only need to consider one of the following two cases:

Case 1: $P_i \neq \emptyset = (P_i)_I$ for each $i \in \mathbf{N}$;

Case 2: $(P_i)_I \neq \emptyset$ for each $i \in \mathbf{N}$.

(Indeed, by (8) none of the P_i is empty, so (7) has to have a subsequence satisfying one of the two possibilities above.)

We settle the cases separately.

Case 1: (8) is invariant under translation of the polyhedra P_i over an integral vector $\mathbf{x}_i$ (i.e., replacing $\mathbf{b}_i$ by $\mathbf{b}_i + \mathbf{A}\mathbf{x}_i$). So we may assume that each P_i contains a vector in $\{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{0} \leqq \mathbf{x} \leqq \mathbf{1}\}$. This means that the "component sequences" $\{(b_i)_j\}_{i\in\mathbf{N}}$ are bounded from below for $j = 1, \dots, m$. Hence we may assume (by (9) and by renumbering indices j) that there exists a constant vector $\mathbf{c} = [c_1, \dots, c_k]^{\mathrm{T}}$ such that

(10) $\quad (b_i)_j = c_j$ for $i \in \mathbf{N}$ and $j = 1, \dots, k$, and

(11) $\quad \{(b_i)_j\}$ is strictly increasing for $j = k+1, \dots, m$.

Split each system $\mathbf{A}\mathbf{x} \leqq \mathbf{b}_i$ into the two subsystems $\mathbf{A}_1\mathbf{x} \leqq \mathbf{c}$ and $\mathbf{A}_2\mathbf{x} \leqq \mathbf{d}_i$ ($\mathbf{d}_i := [(b_i)_{k+1}, \dots, (b_i)_m]^{\mathrm{T}}$) and set $Q := \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}_1\mathbf{x} \leqq \mathbf{c}\}$. Let $t \in \mathbf{N}$ such that $Q^{(t)} = Q_I$ (t exists by (5)). For $i > t$ we have that $\mathbf{w}_i^{\mathrm{T}}\mathbf{x} \leqq \alpha_i$ is not valid for $(P_i)^{(i)} \subseteq Q^{(i)} = Q_I$. Hence Q_I is not empty, which by (11) implies that $(P_i)_I$ is not empty for some $i \in \mathbf{N}$. This is a contradiction, so Case 1 cannot occur.

Case 2: For each $i \in \mathbf{N}$, let $\mathbf{x}_i \in P_i \cap \mathbf{Z}^n$ such that

$$\mathbf{w}_i^{\mathrm{T}}\mathbf{x}_i = \max\{\mathbf{w}_i^{\mathrm{T}}\mathbf{x} \mid \mathbf{x} \in P_i \cap \mathbf{Z}^n\}.$$

By translation, we may assume that, for each $i \in \mathbf{N}$, x_i is the all-zero vector $\mathbf{0} \in P_i$ and that $\alpha_i = 0$. Using the same arguments as used in Case 1 we may assume that $\mathbf{A}\mathbf{x} \leqq \mathbf{b}_i$ can be split into two subsystems $\mathbf{A}_1\mathbf{x} \leqq \mathbf{c}$ and $\mathbf{A}_2\mathbf{x} \leqq \mathbf{d}_i$, where $\mathbf{c}$ and $\mathbf{d}_i$ are as in Case 1 and satisfy (10) and (11). Again we define $Q := \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}_1\mathbf{x} \leqq \mathbf{c}\}$.

Before we proceed we construct a finite set L as follows. Choose an integral vector $\mathbf{y}_F$ from each minimal face F of Q_I. Moreover, choose a collection $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbf{Z}^n$ such that $\mathbf{v}_1, \dots,$ and $\mathbf{v}_k$ generate the cone $\{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{A}_1\mathbf{x} \leqq 0\}$. Because of $\mathbf{0} \in P_i$ for each i, $\mathbf{0} \in Q$. Therefore $\mathbf{c} \geq \mathbf{0}$, so that in fact $\mathbf{v}_1, \dots, \mathbf{v}_k \in Q \cap \mathbf{Z}^n$. Define

$$L := \{\mathbf{y}_F \mid F \text{ minimal face of } Q_I\} \cup \{\mathbf{v}_1, \dots, \mathbf{v}_k\}.$$

Let $t \in \mathbf{N}$ such that $Q^{(t)} = Q_I$ (t exists by (5)). For $i > t$ we have that $\mathbf{w}_i^{\mathrm{T}}\mathbf{x} \leqq 0$ is not valid for $(P_i)^{(i)} \subseteq Q^{(i)} = Q_I$. Hence there exists for each $i \in \mathbf{N}$ a vector $\mathbf{z}_i \in Q \cap \mathbf{Z}^n$ with $\mathbf{w}_i^{\mathrm{T}}\mathbf{z}_i > 0$. By standard linear programming theory, we may assume that $\mathbf{z}_i \in L$ for each $i \in \mathbf{N}$. By (10), (11), and the fact that L is bounded, there exists an $i \in \mathbf{N}$, such that $\mathbf{z}_i \in P_i$. As $\mathbf{z}_i \in \mathbf{Z}^n$, this contradicts our assumption that $\max\{\mathbf{w}_i^{\mathrm{T}}\mathbf{x} \mid \mathbf{x} \in P_i \cap \mathbf{Z}^n\} = \mathbf{w}_i^{\mathrm{T}}\mathbf{x}_i = \mathbf{w}_i^{\mathrm{T}}\mathbf{0} = 0$. So Case 2 cannot occur either.

Since neither case is possible, (6) follows. □

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CWI, AMSTERDAM, THE NETHERLANDS