QUASI-GRAPHIC MATROIDS

JIM GEELEN, BERT GERARDS, AND GEOFF WHITTLE

Abstract. Frame matroids and lifted-graphic matroids are two interesting generalizations of graphic matroids. Here we introduce a new generalization, *quasi-graphic matroids*, that unifies these two existing classes. Unlike frame matroids and lifted-graphic matroids, it is easy to certify that a matroid is quasi-graphic. The main result of the paper is that every 3-connected representable quasi-graphic matroid is either a lifted-graphic matroid or a frame matroid.

1. Introduction

Let G be a graph and let M be a matroid. For a vertex v of G we let $\text{loops}_G(v)$ denote the set of loop-edges of G at the vertex v. We say that G is a *framework* for M if

(1) $E(G) = E(M)$,
(2) $r_M(E(H)) \le |V(H)|$ for each component H of G, and
(3) for each vertex v of G we have $\text{cl}_M(E(G-v)) \subseteq E(G-v) \cup \text{loops}_G(v)$.

This definition is motivated by the following result that is essentially due to Seymour [1].

Theorem 1.1. *Let G be a graph with c components and let M be a matroid. Then M is the cycle matroid of G if and only if G is a framework for M and $r(M) \le |V(G)| - c$.*

We will call a matroid *quasi-graphic* if it has a framework. Next we will consider two classes of quasi-graphic matroids; namely "lifted-graphic matroids" and "frame matroids".

We say that a matroid M is a *lift* of a matroid N if there is a matroid M' and an element $e \in E(M')$ such that $M' \setminus e = M$ and $M'/e = N$.

Date: December 8, 2015.
1991 *Mathematics Subject Classification.* 05B35.
Key words and phrases. matroids, representation, graphic matroids, frame matroids.
This research was partially supported by grants from the Office of Naval Research [N00014-10-1-0851], NSERC [203110-2011], the Marsden Fund of New Zealand, and NWO (The Netherlands Organisation for Scientific Research).

If M is a lift of a graphic matroid, then we will call M a *lifted-graphic matroid.*

Theorem 1.2. *If G is a graph and M is a lift of $M(G)$, then G is a framework for M.*

We say that a matroid M is *framed* if it has a basis V such that for each element $e \in E(M)$ there is a set $W \subseteq V$ such that $|W| \leq 2$ and $e \in \mathrm{cl}_M(W)$. A *frame matroid* is a restriction of a framed matroid.

Theorem 1.3. *Every frame matroid is quasi-graphic.*

Our main result is that for matroids that are both 3-connected and representable, there are no kinds of quasi-graphic matroids other than those described above.

Theorem 1.4. *Let M be a 3-connected representable matroid. If M is quasi-graphic, then either M is a frame matroid or M is a lifted-graphic matroid.*

The representability condition in Theorem 1.4 is necessary; the Vámos matroid, for example, is quasi-graphic but it is neither a frame matroid nor a lifted-graphic matroid. However, for frameworks with loop-edges, we do not require representability.

Theorem 1.5. *Let G be a framework for a 3-connected matroid M. If G has a loop-edge, then M is either a frame matroid or a lifted-graphic matroid.*

Our proof of Theorem 1.5 uses results of Zaslavsky [2] who characterized frame matroids and lifted-graphic matroids using "biased graphs"; we review those results in Sections 4 and 5.

One attractive feature of frameworks is that they are easy to certify. That is, given a graph G and a matroid M one can readily check whether or not G is a framework for M. More specifically, there is a polynomial-time algorithm that given G and M (via its rank oracle) will decide whether or not G is a framework for M.

We conjecture that there is no general way for certifying that a matroid is a frame matroid, or a lifted-graphic matroid, using only polynomially many rank evaluations.

Conjecture 1.6. *For any polynomial $p(\cdot)$ there is a frame matroid M such that for any set $\mathcal{S}$ of subsets of $E(M)$ with $|\mathcal{S}| \leq p(|M|)$ there is a non-frame matroid M' such that $E(M') = E(M)$ and $r_{M'}(X) = r_M(X)$ for each $X \in \mathcal{S}$.*

Conjecture 1.7. *For any polynomial $p(\cdot)$ there is a lifted-graphic matroid M such that for any set $\mathcal{S}$ of subsets of $E(M)$ with $|\mathcal{S}| \leq p(|M|)$ there is a non-lifted-graphic matroid M' such that $E(M') = E(M)$ and $r_{M'}(X) = r_M(X)$ for each $X \in \mathcal{S}$.*

In stark contrast to these two negative conjectures, we conjecture that the problem of recognizing quasi-graphic matroids is tractable.

Conjecture 1.8. *There is a polynomial-time algorithm that given a matroid M, via its rank-oracle, decides whether or not M is quasi-graphic.*

2. Minors of quasi-graphic matroids

We will start by proving that the class of quasi-graphic matroids is minor-closed.

Lemma 2.1. *Let G be a framework for M. If H is a component of G, then H is a framework for $M|E(H)$.*

Proof. Note that conditions (1) and (2) are immediate. Condition (3) follows from the fact that for each flat F of M, the set $F \cap E(H)$ is a flat of $M|E(H)$. □

The following result is very easy, but it is used repeatedly.

Lemma 2.2. *Let G be a framework for M. If v is a vertex of G that is incident with at least one non-loop-edge, then $r_M(E(G-v)) < r(M)$. Moreover, if v has degree one, then $r_M(E(G-v)) = r(M) - 1$.*

Proof. This follows directly from (3). □

Lemma 2.3. *Let G be a connected framework for M and let H be a subgraph of G. Then $|V(H)| - r(M|E(H)) \geq |V(G)| - r(M)$.*

Proof. The result holds when H is trivial, so we may assume that $V(H) \neq \emptyset$. We can extend H to a spanning subgraph H^+ of G with $|E(H^+)| - |E(H)| = |V(G)| - |V(H)|$. Clearly $|V(H^+)| - r(E(H^+)) \geq |V(G)| - r(M)$. If $H \neq H^+$, then there is a vertex $v \in V(H^+) - V(H)$ that has degree one in H^+. By Lemma 2.2, $r(E(H^+ - v)) = r(E(H)) - 1$ and, hence, $|V(H^+ - v)| - r(E(H^+) - v) \geq |V(G)| - r(M)$. Now we obtain the result by repeatedly deleting vertices in $V(H^+) - V(H)$ in this way. □

If X is a set of edges in a graph G, then $G[X]$ is the subgraph of G with edge-set X and with no isolated vertices.

Lemma 2.4. *Let G be a framework for M and let $X \subseteq E(M)$. Then $G[X]$ is a framework for $M|X$.*

Proof. Condition (1) is clearly satisfied. Condition (2) follows from Lemmas 2.1 and 2.3. Condition (3) follows from the fact that for each flat F of M, the set $F \cap E(H)$ is a flat of $M|E(H)$. □

The following two results give sufficient conditions for independence and dependence, respectively, for a set in a matroid given only the structure in the framework.

Lemma 2.5. *Let G be a framework for M. If F is a forest of G, then $E(F)$ is an independent set of M.*

Proof. We may assume that $E(F)$ is non-empty and, hence, that F has a degree-one vertex v. By Lemma 2.2, $r_M(E(F)) = r_M(E(F-v))+1$. Now the result follows inductively. □

Lemma 2.6. *Let G be a framework for G. If H is a subgraph of G and $|E(H)| > |V(G)|$, then $E(H)$ is a dependent set of M.*

Proof. By Lemma 2.4 and (2), we have $r_M(E(H)) \le |V(H)|$. So, if $|E(H)| > |V(G)|$, then $E(H)$ is a dependent set of M. □

We can now prove Theorem 1.1.

Theorem (Theorem 1.1 restated)**.** *Let G be a graph with c components and let M be a matroid. Then M is the cycle matroid of G if and only if G is a framework for M and $r(M) \le |V(G)| - c$.*

Proof. By Lemma 2.5 and the fact that $r(M) \le |V(G)| - c$, we have $r(E(H)) = |V(H)| - 1$ for each component H of G. Hence we may assume that G is connected. By Lemma 2.5, the edge-set of each forest of G is independent in M. Therefore, it suffices to prove, for each circuit C of G, that $E(C)$ is dependent in M. By Lemma 2.3, $|V(C)| - r(E(C)) \ge |V(G)| - r(E(G)) = 1$. So $r(E(C)) < |V(C)| = |E(C)|$ and, hence, $E(C)$ is dependent as required. □

To prove that the class of quasi-graphic matroids is closed under contraction, we consider two cases depending on whether or not we are contracting a loop-edge of the framework.

Lemma 2.7. *Let G be a framework for M and let e be a non-loop-edge of G. Then G/e is a framework for M/e.*

Proof. Conditions (1) and (2) are clearly satisfied. Let u and v be the ends of e in G, and let f be an edge of G that is incident with u but not with v. To prove (3) it suffices to prove that that there exists a cocircuit C in M such that $f \in C$, $e \notin C$, and C contains only edges incident with either u or v.

By (3), there exist cocircuits C_e and C_f such that $e \in C_e$, that C_e contains only edges incident with v, that $f \in C_f$, and that C_f contains only edges incident with u. We may assume that $e \in C_f$ since otherwise we could take $C = C_f$. Since f is not incident with v, we have $f \notin C_e$. Then, by the strong circuit exchange axiom, there is a cocircuit C of M with $f \in C \subseteq (C_1 \cup C_2) - \{e\}$, as required. □

Lemma 2.8. *Let G be a framework for M, let e be a loop-edge of G at a vertex v and let H be the graph obtained by first, for each non-loop edge $f = vw$ incident with v adding f as a loop at w, and then for each loop-edge f of $G - e$ at v adding f as a loop on an arbitrary vertex. If e is not a loop of M, then H is a framework for M/e.*

Proof. Conditions (1) and (2) are clearly satisfied. By Lemma 2.4, we have $r_M(\text{loops}_G(v)) = 1$, so each element of $\text{loops}_G(v) - \{e\}$ is a loop in M/e. Each vertex $w \in V(G) - \{v\}$ is incident with the same edges in G as it is in H except for the elements in $\text{loops}_G(v)$. Moreover, $\text{cl}_M(E(G - w)) = \text{cl}_{M/e}(E(H - w)) \cup \{e\}$. Therefore (3) follows. □

We have proved the following:

Theorem 2.9. *The class of quasi-graphic matroids is closed under taking minors.*

3. Balanced circuits

Let G be a framework for a matroid M. If C is a circuit of G, then, by Lemmas 2.3 and 2.5, $E(C)$ is either independent in M or $E(C)$ is a circuit in M; we say that C is *balanced* if $E(C)$ is a circuit of M.

Lemma 3.1. *Let G be a framework for M. Then $M = M(G)$ if and only if each circuit of G is balanced.*

Proof. If $M = M(G)$, then each circuit of G is balanced. Conversely, suppose that each circuit of G is balanced. Let F be a maximal forest in G. Since each circuit is balanced, $E(F)$ is a basis of M. Then, by Theorem 1.1, $M = M(G)$. □

A *theta* is a 2-connected graph that has exactly two vertices of degree 3 and all other vertices have degree 2. Observe that there are exactly three circuits in a theta.

Lemma 3.2. *Let G be a framework for M and let H be a theta-subgraph of G. If two of the circuits in H are balanced, then so too is the third.*

Proof. If there are two balanced circuits in H then $r_M(E(H)) \leq |E(H)| - 2 = |V(H)| - 1$. So, by Theorem 1.1, $M|E(H) = M(H)$ and, by Lemma 3.1, all circuits of H are balanced. □

The following result describes the circuits of a matroid in terms of the framework; first we will give an unusual example to demonstrate one of the outcomes. If M consists of a single circuit and G is a graph with $E(G) = E(M)$ whose components are circuits, then G is a framework for M.

Lemma 3.3. *Let G be a framework for M and let C be a circuit in M. Then either*

- *$G[C]$ is a balanced circuit,*
- *$G[C]$ is a connected graph with minimum degree at least two, $|C| = V(G[C]) + 1$, and $G[C]$ has no balanced circuits, or*
- *$G[C]$ is a collection of vertex-disjoint non-balanced circuits.*

Proof. We may assume that $G[C]$ is not a balanced circuit, and, hence, that $G[C]$ contains no balanced circuit. Next suppose that $|C| \geq V(G[C]) + 1$. By Lemma 2.6, C is minimal with this property. Hence $G[C]$ is connected, the minimum degree of $G[C]$ is two, and $|C| = V(G[C]) + 1$. Now suppose that $|C| \leq V(G[C])$ and consider a component H of $G[C]$; it suffices to show that $G[C]$ is a circuit. By Lemma 2.6 and the argument above, we may assume that $|E(H)| \leq |V(H)|$. If H is not a circuit there is a degree-one vertex v of H. Moreover, the edge e that is incident with v is not a loop-edge. Then, by (3), the element e is a coloop of $M|C$, which contradicts the fact that C is a circuit. □

For a set X of elements in a matroid M we let

$$\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M).$$

Lemma 3.4. *Let G be a framework for M. If H is a component of G, then $\lambda_M(E(H)) \leq 1$.*

Proof. By Lemma 2.2, $r(E(M)-E(H)) \leq r(M)-(|V(H)|-1)$. Hence $\lambda_M(E(H)) = r_M(E(H)) + r_M(E(M) - E(H)) - r(M) \leq |V(H)| + (r(M) - (|V(H)| - 1)) - r(M) = 1$. □

The following result is an immediate consequence of Lemma 3.4.

Lemma 3.5. *If G is a framework for a 3-connected matroid M with $|M| \geq 4$ and G has no isolated vertices, then either*

- *G is connected, or*
- *G has two components one of which consists of a single vertex with a loop.*

Lemma 3.6. *Let M be a 3-connected matroid with $|E(M)| \geq 4$. If M is quasi-graphic, then M has a connected framework.*

Proof. Let G be a framework for M and suppose that G is not connected. We may assume that G has no isolated vertices. Then, by Lemma 3.5, G has two components, one of which consists of a single vertex v and a single edge e. Since e is not a coloop of M, $r(M) \leq |V(G)| - 1$. Let $w \in V(G) - \{v\}$. Now we construct a new graph G^+ by adding a new edge f with ends v and w and let M^+ be a matroid obtained from M by adding f as a coloop. Note that G^+ is a framework for M^+. Therefore G^+/f is a framework for M^+/f. Since f is a coloop of M^+, we have $M^+/f = M^+ \setminus f = M$. So G^+/f is a connected framework for M. □

Lemma 3.7. *Let M be a 3-connected matroid with $|E(M)| \geq 4$. If G is a connected framework for M, then G is 2-connected.*

Proof. Suppose otherwise. Then there is a pair (H_1, H_2) of subgraphs of G such that $G = H_1 \cup H_2$, $|V(H_1) \cap V(H_2)| = 1$, and $|V(H_1)|, |V(H_2)| \geq 2$. Note that H_1 and H_2 are both connected. Now $M(G)$ is not 3-connected, so, by Lemma 1.1, $r(M) = |V(G)|$. Therefore $\lambda_M(E(H_1)) \leq |V(H_1)| + |V(H_2)| - |V(G)| = 1$. Since M is 3-connected either $|E(H_1)| \leq 1$ or $|E(H_2)| \leq 1$; we may assume that $|E(H_1)| = 1$. Let $e \in E(H_1)$. Since H_1 is a connected and $|V(H_1)| \geq 2$, the edge e is not a loop. Therefore, by (3), e is a coloop of M. This contradicts the fact that M is 3-connected. □

The following two lemmas refine Lemma 3.3 in the case that M is 3-connected.

Lemma 3.8. *Let M be a 3-connected matroid with $|M| \geq 4$ and let G be a framework for M. If C_1 and C_2 are vertex-disjoint non-balanced circuits of G, then either*

- *$E(C_1) \cup E(C_2)$ is a circuit of M,*
- *$E(C_1) \cup E(C_2) \cup E(P)$ is a circuit of M for each minimal path P in G from $V(C_1)$ to $V(C_2)$.*

Moreover, if C_1 and C_2 are in distinct components of G, then $E(C_1) \cup E(C_2)$ is a circuit of M.

Proof. We may assume that $E(C_1) \cup E(C_2)$ is not a circuit. Let P be a minimal path in G from $V(C_1)$ to $V(C_2)$. By Lemma 2.6, $E(C_1 \cup C_2 \cup P)$ is dependent. Let $C \subseteq E(C_1 \cup C_2 \cup P)$ be a circuit of M. By Lemma 3.3, $C = E(C_1 \cup C_2 \cup P)$.

Finally, suppose that C_1 and C_2 are in distinct components of G. We may assume that G has no isolated vertices. Then, by Lemma 3.5, G has two components one of which has a single vertex, say v, and a single loop-edge, say e. Since M is 3-connected, e is not a coloop of

M. Then, by (3), $r(M) \leq |V(G)|$. We may assume that $E(C_1) = \{e\}$; let w be a vertex of C_2. Construct a graph G^+ from G by adding a new edge f with ends v and w and construct a new matroid M^+ by adding f as a coloop to M. Note that G^+ is a framework for M^+ and hence G^+/f is a framework for M^+/f. By Lemmas 2.6 and 3.3, $E(C_1) \cup E(C_2)$ is a circuit in M^+/f. Moreover, as f is a coloop of M^+, we have $M^+/f = M$, so $E(C_1) \cup E(C_2)$ is a circuit in M. □

Lemma 3.9. *Let M be a 3-connected matroid with $|M| \geq 4$ and let G be a framework for M. If C is a circuit for M, then $G[C]$ has at most two components.*

Proof. Suppose that $G[C]$ has more than two components. By Lemma 3.3, each component of G is a balanced circuit. By Lemma 3.5, two of these circuits are in the same component of G. Let P be a shortest path connecting two components of $G[C]$; let these components be C_1 and C_2. Since C is a circuit, $G[C_1 \cup C_2]$ is independent. Therefore, by Lemma 3.8, $E(C_1 \cup C_2 \cup P)$ is a circuit of M. Let $e \in E(P)$ and $f \in E(C_1)$. By the strong exchange property for circuits, there is a circuit C' of G with $e \in C' \subseteq (C \cup E(P)) - \{f\}$. However this is inconsistent with the outcomes of Lemma 3.3. □

4. Frame matroids

We start by proving Theorem 1.3.

Theorem (Theorem 1.3 restated)**.** *Every frame matroid is quasi-graphic.*

Proof. Let M be a frame matroid. Note that M is a quasi-graphic matroid if and only if $\text{si}(M)$ is a quasi-graphic matroid, so we may assume that M is simple. Recall that the class of quasi-graphic matroids is closed under taking minors, so we may further assume that M is framed; let V be a basis of M such that each element is spanned by a 2-element subset of V. We now construct a graph G with vertex-set V and edge-set $E(M)$ such that, for each $v \in V$ the edge v is a loop on the vertex v and for each $e \in E(M) - V$ the edge e has ends u and v where $\{e, u, v\}$ is the unique circuit of M in $V \cup \{e\}$. We claim that G is a framework for M.

By construction $E(G) = E(M)$ and, since V is a basis of M, for each component H of G we have $r(E(H)) = |V(H)|$. Finally, for each vertex v of G, the hyperplane of M spanned by $V - \{v\}$ is $E(G - v)$. Hence G is indeed a framework for M. □

Next we give an alternative characterization of frame matroids using frameworks; these results are effectively due to Zaslavsky [2].

Let G be a graph and let $\mathcal{B}$ be a subset of the circuits of G. We say that $\mathcal{B}$ satisfies the *theta-property* if there is no theta in G with exactly two of its three circuits in $\mathcal{B}$.

Theorem 4.1. *Let G be a graph and let $\mathcal{B}$ be a collection of circuits in G that satisfy the theta-property. Now let $M = (E(G), \mathcal{I})$ where a set $I \subseteq E(G)$ is contained in $\mathcal{I}$ if and only if there is no $C \in \mathcal{B}$ with $E(C) \subseteq I$ and $|E(H)| \leq |V(H)|$ for each component H of $G[I]$. Then M is a matroid.*

Proof. We call the circuits of G in $\mathcal{B}$ *balanced*. To prove that M is a matroid it suffices to check the following conditions, which are effectively a reformulation of the circuit axioms in terms of independent sets:

(a) $\emptyset \in \mathcal{I}$,
(b) for each $J \in \mathcal{I}$ and $I \subseteq J$, we have $I \in \mathcal{I}$, and
(c) for each set $I \in \mathcal{I}$ and $e \in E(M) - I$ either $I \cup \{e\} \in \mathcal{I}$ or there is a unique minimal subset C of $I \cup \{e\}$ that is not in $\mathcal{I}$.

Conditions (a) and (b) follow from the construction.

Let $I \in \mathcal{I}$ and $e \in E(M) - I$ with $I \cup \{e\} \notin \mathcal{I}$. Let C_1 and C_2 be minimal subsets of $I \cup \{e\}$ that are not in $\mathcal{I}$. Suppose for a contradiction that $C_1 \neq C_2$. By definition, for each $i \in \{1, 2\}$, we have $G[C_i - \{e\}]$ is connected, $e \in C_i$, and either $G[C_i]$ is a balanced circuit or $|C_i| > |V(G[C_i])|$. Consider $J = (C_1 \cup C_2) - \{e\}$. Since $J \subseteq I$, we have $J \in \mathcal{I}$. Since $G[C_1 - \{e\}]$ and $G[C_2 - \{e\}]$ are connected, $G[J]$ is connected. Therefore $|J| \leq |V(G[J])|$. It follows that $|C_1| \leq |V(G[C_1])|$ and $|C_2| \leq |V(G[C_2])|$. Hence $G[C_1]$ and $G[C_2]$ are balanced circuits. Now $G[C_1 \cup C_2]$ is a theta and $G[J]$ is a circuit. By the theta-property, $G[J]$ is balanced. However, this contradicts the fact that $J \in \mathcal{I}$. □

We denote the matroid M in Theorem 4.1 by $FM(G, \mathcal{B})$.

Theorem 4.2. *If G is a graph and $\mathcal{B}$ is a collection of circuits in G that satisfies the theta-property, then $FM(G, \mathcal{B})$ is a frame matroid.*

Proof. Let G^+ be obtained from G by adding a loop-edge e_v at each vertex of v. Now let $\mathcal{B}^+$ be obtained from $\mathcal{B}$ by adding the circuits $(G[\{e_v\}] : v \in V(G))$. Since we only added loops to $\mathcal{B}$, the collection $\mathcal{B}^+$ satisfies the theta-property. Let $M^+ = FM(G^+, \mathcal{B}^+)$ and $V = \{e_v : v \in V(G)\}$. By the definition of $FM(G^+, \mathcal{B}^+)$, the set V is a basis of M^+. For each non-loop edge e of G with ends u and v, the set $\{e_u, e, e_v\}$ is a circuit of M^+ and for each loop-edge e of G at v, the set $\{e, e_v\}$ is a circuit of M^+. Therefore M^+ is a framed matroid and hence $FM(G, \mathcal{B})$ is a frame matroid. □

Theorem 4.3. *A loopless matroid M is a frame matroid if and only if there is a graph G and a collection $\mathcal{B}$ of circuits of G satisfying the theta-property such that $M = FM(G, \mathcal{B})$.*

Proof. The "if" direction of the result follows from Theorem 4.2. For the converse we may assume that M is a framed matroid; let V be a basis of M such that each element is spanned by a 2-element subset of V. We now construct a graph G with vertex-set V and edge-set $E(M)$ such that, for each $v \in V$ the edge v is a loop on the vertex v and for each $e \in E(M) - V$ the edge e has ends u and v where $\{e, u, v\}$ is the unique circuit of M in $V \cup \{e\}$. By the proof of Theorem 1.3, G is a framework for M.

By Lemma 3.3, it suffices to prove that, if $C_1, \dots, C_k$ are disjoint non-balanced circuits of G, then $E(C_1 \cup \cdots \cup C_k)$ is independent. This follows from the fact that $V(C_1 \cup \cdots \cup C_k)$ is independent and that, for each $i \in \{1, \dots, k\}$, the sets $E(C_i)$ and $V(C_i)$ span each other. □

5. Lifted-graphic matroids

We start by proving Theorem 1.2.

Theorem (Theorem 1.2 restated)**.** *If G is a graph and M is a lift of $M(G)$, then G is a framework for M.*

Proof. Let e be an element of a matroid M' such that $M' \setminus e = M$ and $M'/e = M(G)$. Thus $E(M) = E(G)$. For each component H of G, $r_{M'/e}(E(H)) = |V(G)| - 1$ so $r_M(E(H)) = r_{M'}(E(H)) \le r_{M'/e}(E(H)) + 1 = |V(H)|$. For a vertex v of G, we have $\mathrm{cl}_M(E(G-v)) \subseteq \mathrm{cl}'_M(E(G-v) \cup \{e\}) - \{e\} = \mathrm{cl}_{M'/e}(E(G-v)) \subseteq E(G-v) \cup \mathrm{loops}_G(v)$. So G is a framework for M. □

Next we will give an alternative characterization of lifted-graphic matroids using frameworks; again, these results are effectively due to Zaslavsky [2].

Theorem 5.1. *Let G be a graph and let $\mathcal{B}$ be a collection of circuits in G that satisfy the theta-property. Now let $M = (E(G), \mathcal{I})$ where a set $I \subseteq E(G)$ is contained in $\mathcal{I}$ if and only if there is no $C \in \mathcal{B}$ with $E(C) \subseteq I$ and $G[I]$ contains at most one circuit. Then M is a matroid and G is a framework for M.*

Proof. We call the circuits of G in $\mathcal{B}$ *balanced*. To prove that M is a matroid it suffices to check the following conditions:

(a) $\emptyset \in \mathcal{I}$,
(b) for each $J \in \mathcal{I}$ and $I \subseteq J$, we have $I \in \mathcal{I}$, and

(c) for each set $I \in \mathcal{I}$ and $e \in E(M) - I$ either $I \cup \{e\} \in \mathcal{I}$ or there is a unique minimal subset C of $I \cup \{e\}$ that is not in $\mathcal{I}$.

Conditions (a) and (b) follow from the construction.

Let $I \in \mathcal{I}$ and $e \in E(M) - I$ with $I \cup \{e\} \notin \mathcal{I}$. Let C_1 and C_2 be minimal subsets of $I \cup \{e\}$ that are not in $\mathcal{I}$. Suppose for a contradiction that $C_1 \neq C_2$. By definition, for each $i \in \{1, 2\}$, either $G[C_i]$ is a balanced circuit, $G[C_i]$ is the union of two vertex disjoint non-balanced circuits, or $G[C_i]$ is 2-edge-connected and $|C_i| = |V(G[C_i])| + 1$. Consider $J = (C_1 \cup C_2) - \{e\}$. Since $J \subseteq I$, we have $J \in \mathcal{I}$ so either $G[J]$ is a forest or $G[J]$ contains a unique circuit.

For each $i \in \{1, 2\}$, there is a circuit A_i of $G[C_i]$ that contains e. Since $G[J]$ contains at most one circuit, either $A_1 = A_2$ or $A_1 \cup A_2$ is a theta.

First suppose that $A_1 = A_2$. Since $C_1 \neq C_2$, the circuit A_1 is non-balanced. Therefore, for each $i \in \{1, 2\}$, there is a non-balanced circuit B_i in $G[C_i - e]$. Since $G[J]$ contains a unique circuit $B_1 = B_2$. But then $C_1 = E(A_1 \cup B_1)$ and $C_2 = E(A_2 \cup B_2)$, contradicting the fact that $C_1 \neq C_2$.

Now suppose that $A_1 \cup A_2$ is a theta, and let C be the circuit in $(A_1 \cup A_2) - e$. Since J is independent, C is not balanced. By the theta-property and symmetry, we may assume that A_1 is not balanced. Then there is a non-balanced circuit B_1 in $G[C_1 - \{e\}]$. Since $G[J]$ has at most one circuit $C = B_1$. Therefore $C_1 = E(A_1 \cup A_2)$ and, hence, A_2 is non-balanced. Then there is a non-balanced circuit B_2 in $G[C_2 - \{e\}]$. Since $G[J]$ has at most one circuit $C = B_2$, however, this contradicts the fact that $C_1 \neq C_2$. □

We denote the matroid M in Theorem 5.1 by $LM(G, \mathcal{B})$.

Theorem 5.2. *If G is a graph and $\mathcal{B}$ is a collection of circuits in G that satisfies the theta-property, then $LM(G, \mathcal{B})$ is a lift of $M(G)$.*

Proof. Let G^+ be obtained from G by adding a loop-edge e at a vertex v and let $\mathcal{B}^+ = \mathcal{B} \cup \{G[\{e\}]\}$. Since we only added a loop to $\mathcal{B}$, the collection $\mathcal{B}^+$ satisfies the theta-property. Let $M^+ = LM(G^+, \mathcal{B}^+)$. By the definition of $LM(G^+, \mathcal{B}^+)$, for each circuit C of G, $\{e\} \cup E(C)$ is dependent in M^+. Hence $E(C)$ is a dependent set M^+/e. Similarly, by the definition of $LM(G^+, \mathcal{B}^+)$, for each forest F of G, the set $\{e\} \cup E(F)$ is independent in M^+ and, hence, $E(F)$ is independent in M^+/e. Thus $M^+/e = M(G)$ and, hence, M is a lift of $M(G)$. □

The following result is a converse to Theorem 5.2.

Theorem 5.3. *If G is a graph, M is a lift of $M(G)$, and $\mathcal{B}$ is the set of balanced circuits of (M, G), then $M = LM(G, \mathcal{B})$.*

Proof. It suffices to prove that if C_1 and C_2 are vertex disjoint circuits of G, then $E(C_1 \cup C_2)$ is dependent in M. Now $E(C_1 \cup C_2)$ has rank equal to $|E(C_1 \cup C_2)| - 2$ in $M(G)$ so its rank in M is at most $|E(C_1 \cup C_2)| - 1$. Thus $E(C_1 \cup C_2)$ is indeed dependent in M. □

6. Frameworks with loops

In this section we prove Theorem 1.5 which is am immediate consequence of the following two results.

Theorem 6.1. *Let G be a framework for a 3-connected matroid M, let $\mathcal{B}$ be the set of balanced circuits of G, and let e be a non-balanced loop-edge at a vertex v. If $e \in \mathrm{cl}_M(E(G - v))$, then $M = LM(G, \mathcal{B})$.*

Proof. It suffices to prove that if C_1 and C_2 are vertex-disjoint circuits of G, then $E(C_1 \cup C_2)$ is dependent in M. We may assume that C_1 and C_2 are non-balanced and, by Lemma 3.8, we may assume that C_1 and C_2 are in the same component of G.

First suppose that $C_1 = \{e\}$. Let P be a minimal path from $V(C_1)$ to $V(C_2)$. Let f be the edge of P that is incident with v. By (3) and the fact that $e \in \mathrm{cl}_M(E(G-v))$, there is a cocircuit C^* of M such that $C^* \cap E(C_1 \cup P \cup C_2) = \{f\}$. Therefore $E(C_1 \cup P \cup C_2)$ is not a circuit of M. So, by Lemma 3.8, $E(C_1 \cup C_2)$ is a circuit of M, as required.

Now we may assume that neither C_1 nor C_2 is equal to $G[\{e\}]$. By the preceding paragraph, both $E(C_1) \cup \{e\}$ and $E(C_2) \cup \{e\}$ are circuits of M. So, by the circuit-exchange property, $E(C_1 \cup C_2)$ is dependent, as required. □

Theorem 6.2. *Let G be a framework for a 3-connected matroid M, let $\mathcal{B}$ be the set of balanced circuits of G, and let e be a loop-edge at a vertex v. If $e \notin \mathrm{cl}_M(E(G - v))$, then $M = FM(G, \mathcal{B})$.*

Proof. By Lemmas 3.3, 3.8, and 3.9, it suffices to prove that, if C_1 and C_2 are vertex-disjoint non-balanced circuits of M, then $E(C_1 \cup C_2)$ is independent in M.

First suppose that $C_1 = G[\{e\}]$. Since $e \notin \mathrm{cl}_M(E(G - v))$, there is a cocircuit C^* of M that is disjoint from $E(C_2)$. Hence $E(C_1 \cup C_2)$ is independent as required.

Now suppose the neither C_1 nor C_2 is equal to $G[\{e\}]$. We may also assume that $E(C_1 \cup C_2)$ is dependent; by Lemma 3.3, $E(C_1 \cup C_2)$ is a circuit of M. Since $e \notin \mathrm{cl}_M(E(G-v))$ and M has no co-loops, $G[\{e\}]$ is not a component of G. Then, by Lemma 3.5, there is a path from v to $V(C_1 \cup C_2)$ in G; let P be a minimal such path. We may assume that P has an end in $V(C_1)$. By Lemma 3.8 and the preceding paragraph, $E(C_1 \cup P) \cup \{e\}$ is a circuit of M. Let $f \in E(C_1)$; by the circuit exchange property, there exists a circuit C in $(E(C_1 \cup C_2 \cup P) \cup \{e\}) - \{f\}$. By Lemma 3.3, $C = E(C_2) \cup \{e\}$. However this contradicts the fact that $e \notin \mathrm{cl}_M(E(G - v))$. □

7. Representable matroids

A framework G for a matroid M is called *strong* if G is connected and $r_M(E(G - v)) = r(M) - 1$ for each vertex v of G.

Lemma 7.1. *If M is a quasi-graphic matroid with $|M| \geq 4$, then M has a strong framework.*

Proof. By Lemma 3.6, M has a connected framework. Let G be a connected framework having as many loop-edges as possible. Suppose that G is not a strong framework and let $v \in V(G)$ such that $r_M(E(G) - v) < r(M) - 1$. Let C^* be a cocircuit of M with $C^* \cap E(G - v) = \emptyset$; if possible we choose C^* so that it contains a loop-edge of G. Since M is 3-connected, $|C^*| \geq 2$ and, by Lemma 2.6, there is at most one loop-edge at v. Therefore C^* contains at least one non-loop-edge. Let L denote the set of non-loop-edges of $G - C^*$ incident with v. By our choice of C^*, the set L is non-empty.

Let H be the graph obtained from G by replacing each edge $f = vw \in L$ with a loop-edge at w. By Lemma 3.7, H is connected. Note that H is framework for M. However, this contradicts our choice of G. □

We are now ready to prove Theorem 1.4.

Theorem (Theorem 1.4 restated)**.** *Let M be a 3-connected representable matroid. If M is quasi-graphic, then either M is a frame matroid or M is a lifted-graphic matroid.*

Proof. Let $M = M(A)$, where A is a matrix over a field $\mathbb{F}$ with linearly independent rows. We may assume that $|M| \geq 4$. Therefore, by Lemma 7.1, M has a strong framework G.

Claim. *There is a matrix $B \in \mathbb{F}^{V(G)\times E(G)}$ such that*

- *the row-space of B is contained in the row-space of A, and*
- *for each $v \in V(G)$ and non-loop edge e of G, we have $B[v, e] \neq 0$ if and only if v is incident with e.*

Proof of claim. Let $v \in V(G)$ and let $C^* = E(M) - \mathrm{cl}_M(E(G - v))$. By the definition of a strong framework, C^* is a cocircuit of M. Since $r(E(M) - C^*) < r(M)$, by applying row-operations to A we may assume that there is a row w of A whose support is contained in C^*. Since C^* is minimally co-dependent, the support of row-w is equal to C^*. Now we set the row-v of B equal to the row-w of A. □

Note that $M(B)$ is a frame matroid and G is a framework for $M(B)$. We may assume that $r(M(A)) > r(M(B))$ since otherwise $M(A)$ is a frame matroid. Since G is a connected framework for both $M(A)$ and $M(B)$, it follows that $r(M(B)) = |V(G)| - 1$ and that $r(M(A)) = |V(G)|$. Up to row-operations we may assume that A is obtained from b by appending a single row. By Lemma 1.1, $M(B) = M(G)$. Hence M is a lift of $M(G)$. □

References

[1] P. D. Seymour, Recognizing graphic matroids, Combinatorica **1** (1981), 75–78.
[2] T. Zaslavsky, Biased graphs. II. The three matroids, J. Combin. Theory Ser. B **47** (1989), 32-52.

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada

Centrum Wiskunde & Informatica, Amsterdam, The Netherlands

School of Mathematics, Statistics and Operations Research, Victoria University of Wellington, New Zealand